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FUNCTIONS OF L^p-MULTIPLIERS

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1. **Introduction**. Let Γ be a locally compact non-compact abelian group and $B(\Gamma)$ be the space of all Fourier-Stieltjes transforms of bounded measures on the dual group G of Γ. Then it is known that a function Φ on the interval $[-1,1]$ is extended to an entire function if and only if $\Phi(f)\in B(\Gamma)$ for all f in $B(\Gamma)$ with the range contained in $[-1,1]$ (see, for example, [10: p.135]).

A function φ defined on Γ is called an L^p-multiplier if for every $f\in L^p(G)$ there exists a function g in $L^p(G)$ so that $\varphi\hat{f}=\hat{g}$, where $\hat{f}$ denotes the Fourier transform of f. The set of all L^p-multipliers will be written by $M_p(\Gamma)$ and the norm of $\varphi\in M_p(\Gamma)$ is defined by

$$\|\varphi\|_{M_p(\Gamma)}=\sup\{\|g\|_{L^p(G)}:\ \|f\|_{L^p(G)}=1\}.$$

If we define the product in $M_p(\Gamma)$ by the pointwise multiplication, it is a commutative Banach algebra with identity.

It is well-known that $M_1(\Gamma)$ coincides with $B(\Gamma)$ with the norm of measures and $M_2(\Gamma)=L^\infty(\Gamma)$ isometrically. If $1\leqq q\leqq p\leqq 2$, then $M_q(\Gamma)\subset M_p(\Gamma)$ and if $1/p+1/p'=1$, then $M_p(\Gamma)=M_{p'}(\Gamma)$ isometrically.

Our main theorem is the following:

THEOREM 1. *Let Γ be a locally compact non-compact abelian group. Assume $1\leqq p<2$ and Φ is a function on $[-1,1]$. Then $\Phi(\varphi)\in M_p(\Gamma)$ for all φ in $M_1(\Gamma)$ whose range is contained in $[-1,1]$, if and only if Φ is extended to an entire function.*

2. **Equivalence of multiplier transforms**. In this section we shall show the equivalence of multiplier transforms which will be needed later.

A measurable function φ on the real line $\boldsymbol{R}$ is said to be regulated if there exists an approximate identity u_ε not necessarily continuous such that

$$\lim_{\varepsilon\to 0}\varphi * u_\varepsilon(x)=\varphi(x)$$

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for all x.

K. de Leeuw proved the followings.

THEOREM A ([2]). *Let φ be a bounded measurable periodic function with period 2π and $1 \leqq p \leqq 2$. Then $\varphi \in M_p(\boldsymbol{T})$ if and only if $\varphi \in M_p(\boldsymbol{R})$. In this case we have*

$$\|\varphi\|_{M_p(R)} = \|\varphi\|_{M_p(T)},$$

where $\boldsymbol{T}$ denotes the circle group.

THEOREM B ([2]). *Let φ be a bounded regulated function on $\boldsymbol{R}$ and $1 \leqq p \leqq 2$. If $\varphi \in M_p(\boldsymbol{R})$, then $\varphi(\lambda n) \in M_p(\boldsymbol{Z})$ for all $\lambda > 0$ and*

$$\|\varphi(\lambda n)\|_{M_p(Z)} \leqq \|\varphi\|_{M_p(R)},$$

where $\boldsymbol{Z}$ is the set of integers.

The next theorem is the converse of Theorem B which is given in [7], but for the sake of convenience we shall state the complete proof.

THEOREM 2. *Suppose $1 \leqq p \leqq 2$ and φ is a function on $\boldsymbol{R}$ whose points of discontinuity are null. If $\varphi(\lambda n) \in M_p(\boldsymbol{Z})$ for all $\lambda > 0$ and* $\|\varphi(\lambda n)\|_{M_p(Z)}$ are *bounded, then $\varphi(\xi) \in M_p(\boldsymbol{R})$ and we have*

$$\|\varphi\|_{M_p(R)} \leqq \underline{\lim}_{\lambda \to 0} \|\varphi(\lambda n)\|_{M_p(Z)}.$$

Thus if φ is, furthermore, regulated, we have

$$\|\varphi\|_{M_p(R)} = \lim_{\lambda \to 0} \|\varphi(\lambda n)\|_{M_p(Z)}.$$

PROOF. Let g be an infinitely differentiable function with compact support and put $g_\lambda(x) = \lambda g(\lambda x)$ where λ is chosen so large that the support of g_λ is contained in $\boldsymbol{T} = [-\pi, \pi)$. We denote by the same notation g_λ the periodic extension of g_λ. Then we have

$$\begin{aligned}\left(\int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} \varphi\left(\frac{n}{\lambda}\right) \hat{g}_\lambda(n) e^{inx} \right|^p dx\right)^{1/p} &\leqq \left\| \varphi\left(\frac{n}{\lambda}\right) \right\|_{M_p(Z)} \left(\int_{-\pi}^{\pi} |g_\lambda(x)|^p dx\right)^{1/p} \\ &= \left\| \varphi\left(\frac{n}{\lambda}\right) \right\|_{M_p(Z)} \lambda^{1-1/p} \left(\int_{-\infty}^{\infty} |g(x)|^p dx\right)^{1/p},\end{aligned}$$

where $\hat{g}_\lambda(n)$ denotes the n-th Fourier coefficient:

$$\hat{g}_\lambda(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_\lambda(x)e^{-inx}dx.$$

Changing variable we see that the left hand side equals

$$\lambda^{1-1/p}\left(\int_{-\pi\lambda}^{\pi\lambda}\left|\frac{1}{\lambda\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\varphi\left(\frac{n}{\lambda}\right)\hat{g}\left(\frac{n}{\lambda}\right)e^{inx/\lambda}\right|^p dx\right)^{1/p},$$

where

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\, e^{-iy\xi}dy.$$

Since the sum multiplied by $(\lambda\sqrt{2\pi})^{-1}$ converges to

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(\xi)\,\varphi(\xi)\, e^{i\xi x}\, d\xi$$

for every x as $\lambda\to\infty$, we have by Fatou's lemma

$$\left(\int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(\xi)\,\varphi(\xi)\, e^{i\xi x}\, d\xi\right|^p dx\right)^{1/p}$$

$$\leqq \varliminf_{\lambda\to\infty}\left\|\varphi\left(\frac{n}{\lambda}\right)\right\|_{M_p(Z)}\left(\int_{-\infty}^{\infty}|g|^p\, dx\right)^{1/p}.$$

Thus we get the theorem.

The n-dimensional extensions of Theorems A, B and 2 are obvious.

Let $\Delta(r)$ be the direct sum of countably many copies of the cyclic group $Z(r)$ of order r and $D(r)$ be the dual to $\Delta(r)$. Every element x of $\Delta(r)$ or $D(r)$ has the expression $x = x_1\oplus x_2\oplus\cdots$, where $x_j = 0, 1, \cdots, r-1$ are the realization of $Z(r)$. With this realization to every $x = x_1\oplus x_2\oplus\cdots$ of $D(r)$ such that $x_j = 0$ except finite numbers of j there corresponds an element of $\Delta(r)$. Thus a function on $D(r)$ is considered as a function on $\Delta(r)$.

THEOREM 3. *Let φ be a continuous function on $D(r)$ and $1\leqq p\leqq 2$. Then $\varphi\in M_p(D(r))$ if and only if $\varphi\in M_p(\Delta(r))$. In this case we have*

$$\|\varphi\|_{M_p(D(r))} = \|\varphi\|_{M_p(\Delta(r))}.$$

PROOF. That $\varphi \in M_p(\Delta(r))$ is equivalent to say that

$$(1) \qquad \left(\int_{D(r)} \left|\sum_y \varphi(y)\, p(y)(x,y)\right|^p dx\right)^{1/p} \leqq B\left(\int_{D(r)} \left|\sum_y p(y)(x,y)\right|^p dx\right)^{1/p}$$

for all polynomial $\sum p(y)(x,y)$ on $D(r)$, where B is a constant and $(\cdot\,,y)$ denotes a character of $D(r)$. By the same way that $\varphi \in M_p(D(r))$ is equivalent to say that

$$(2) \qquad \left(\sum_v \left|\int_{D(r)} \varphi(u) f(u)(u,v)\, du\right|^p\right)^{1/p} \leqq C\left(\sum_v \left|\int_{D(r)} f(u)(u,v)\, du\right|^p\right)^{1/p}$$

for all continuous step function f on $D(r)$, where C is a constant.

We first deduce (1) from (2) with $B \leqq C$. Let $\sum_y p(y)(x,y)$ be a polynomial. We may assume that the y's run over all elements of the form $y = y_1 \oplus \cdots \oplus y_N \oplus 0 \oplus 0 \oplus \cdots$ for some fixed N. Put $f_M(u) = p(y)r^M$ if u is of the form $u = y_1 \oplus \cdots \oplus y_N \oplus 0 \oplus \cdots 0 \oplus u_{M+1} \oplus u_{M+2} \oplus \cdots$ and $f_M(u) = 0$ otherwise. Then we have

$$\int_{D(r)} f_M(u)(u,v)\, du = \sum_y p(y)(y,v)$$

for all $v = v_1 \oplus \cdots \oplus v_M \oplus 0 \oplus 0 \oplus \cdots$ and the integral vanishes otherwise. We remark that the right hand side does not depend on the $n(>M)$-th components of v.

Let U_M be the set of all u of the form $u = 0 \oplus \cdots \oplus 0 \oplus u_{M+1} \oplus u_{M+2} \oplus \cdots$. Then, since φ is continuous,

$$\begin{aligned}\lim_{M\to\infty} r^M \int_{U_M} \varphi(y+u)(u,v)\, du &= \lim_{M\to\infty} r^M \int_{U_M} \varphi(y+u)\, du \\ &= \varphi(y).\end{aligned}$$

Thus we have

$$\begin{aligned}\int_{D(r)} \varphi(u) f_M(u)(u,v)\, du &= \sum_y p(y)(y,v)\, r^M \int_{U_M} \varphi(y+u)(u,v)\, du \\ &= \sum_y p(y)(y,v)\, \varphi(y) + o(1)\end{aligned}$$

uniformly in v of the form as before when $M \to \infty$. Therefore

$$\left(\sum_v \left|\int_{D(r)} f_M(u)(u, v)\ du\right|^p\right)^{1/p} = r^{M/p}\left(\int_{D(r)} \left|\sum_y p(y)(v, y)\right|^p dv\right)^{1/p},$$

where we replaced (y, v) by (v, y) and

$$\left(\sum_v \left|\int_{D(r)} \varphi(u) f_M(u)(u, v)\ du\right|^p\right)^{1/p}$$

$$\geqq r^{M/p}\left(\int_{D(r)} \left|\sum_y p(y)\ \varphi(y)(v, y)\right|^p dv + o(1)\right)^{1/p}.$$

Thus we get (1) with $B \leqq C$.

Now we show that (1) implies (2) with $C \leqq B$. Assume φ is continuous and satisfies (1). Let f be a continuous step function so that $f(u)$ depends only on the first N-th components of $u = u_1 \oplus u_2 \oplus \cdots$. Define $p(y) = f(u)$ for $y = u_1 \oplus \cdots \oplus u_N \oplus 0 \oplus 0 \oplus \cdots$ and $p(y) = 0$ for y not of that form. We fix this $p(y)$.

For every $\varepsilon > 0$, there exists a continuous step function φ_ε converging uniformly to φ such that

$$\left(\int_{D(r)} \left|\sum_y \varphi_\varepsilon(y)\ p(y)(x, y)\right|^p dx\right)^{1/p} \leqq (B + \varepsilon)\left(\int_{D(r)} \left|\sum_y p(y)(x, y)\right|^p dx\right)^{1/p}.$$

Thus there exists an integer M so that $\varphi_\varepsilon(u)$ depends only on the first M-th components of u. We may assume $M > N$. Let Y be the set of u in $D(r)$ whose $n(> N)$-th components are zero and X the set of x's in $D(r)$ whose $n(> M)$-th components are zero. Then we have

$$\int_{D(r)} \varphi_\varepsilon(u) f(u)(u, v)\ du = r^{-M} \sum_{y \in Y} \varphi_\varepsilon(y)\ p(y)(y, v)$$

for $v \in X$ and the left hand side vanishes for v not in X. By the same way we have

$$\int_{D(r)} f(u)(u, v)\ du = r^{-M} \sum_{y \in Y} p(y)(y, v)$$

for v in X and zero for v not in X. Therefore

$$\left(\sum_v \left|\int_{D(r)} \varphi_\varepsilon(u) f(u)(u, v)\, du\right|^p\right)^{1/p}$$
$$= r^{-M(1-1/p)} \left(\int_{D(r)} \left|\sum_{y \in Y} \varphi_\varepsilon(y)\, p(y)(x, y)\right|^p dx\right)^{1/p}$$

and

$$\left(\sum_v \left|\int_{D(r)} f(u)(u, v)\, du\right|^p\right)^{1/p} = r^{-M(1-1/p)} \left(\int_{D(r)} \left|\sum_{y \in Y} p(y)(x, y)\right|^p dx\right)^{1/p}.$$

Therefore we get from (1)

$$\left(\sum_v \left|\int_{D(r)} \varphi_\varepsilon(u) f(u)(u, v)\, du\right|^p\right)^{1/p} \leqq (B + \varepsilon) \left(\sum_v \left|\int_{D(r)} f(u)(u, v)\, du\right|^p\right)^{1/p}.$$

Letting $\varepsilon \to 0$ we get (2).

3. Proof of Theorem 1.

LEMMA 1. *Let* Γ *be* $\boldsymbol{Z}$ *or* $\Delta(r)$. *Then for any* $1 \leqq p < 2$ *we have a constant* $K_p > 1$ *depending only on* Γ *and* p *such that*

$$\sup_\varphi \|e^{i\varphi}\|_{M_p(\Gamma)} \geqq K_p^a,$$

where φ *ranges over all real-valued functions in* $M_1(\Gamma)$ *satisfying* $\|\varphi\|_{M_1(\Gamma)} \leqq a$.

PROOF. Let G be the dual to Γ. For a function f on G define

$$\|f\|_{A_p(G)} = \left(\sum_{\gamma \in \Gamma} \left|\hat{f}(\gamma)\right|^p\right)^{1/p},$$

where $\hat{f}$ denotes the Fourier coefficient of f. Then we know [9] that there exists a constant $K_p > 1$ for which we have

$$\sup_Q \|e^{iQ}\|_{A_p(G)} > K_p^a,$$

where Q runs over all real polynomials on G with $\|Q\|_{A_1(G)} \leqq a$.

Since $\|Q\|_{A_1(G)} = \|Q\|_{M_1(G)}$ and $\|f\|_{A_p(G)} \leqq \|f\|_{M_p(G)}$, there exists a real polynomial φ on G such that $\|\varphi\|_{M_1(G)} \leqq a$ and

$$\|e^{i\varphi}\|_{M_p(G)} > K_p^a.$$

Assume $\Gamma = \boldsymbol{Z}$, then by Theorems A, B and 2 we have a real-valued continuous function φ on $\boldsymbol{T}$ such that

$$\|e^{i\varphi(\lambda n)}\|_{M_p(\boldsymbol{Z})} > K_p^a \quad \text{and} \quad \|\varphi\|_{M_1(\boldsymbol{T})} \leqq a$$

for sufficiently small $\lambda > 0$. Remark that $\|\varphi(\lambda n)\|_{M_1(\boldsymbol{Z})} \leqq \|\varphi\|_{M_1(\boldsymbol{R})} = \|\varphi\|_{M_1(\boldsymbol{T})} \leqq a$ and then we get the desired inequality for $\Gamma = \boldsymbol{Z}$,

For the group $\Delta(r)$ the result is obvious by Theorem 3.

LEMMA 2. *Let* Γ *be* $\boldsymbol{R}$ *or a discrete group and assume* $1 \leqq p < 2$. *If* $\Phi(\varphi) \in M_p(\Gamma)$ *for all* $\varphi \in M_1(\Gamma)$ *whose range is contained in* $[-1, 1]$, *then* Φ *is continuous in* $[-1, 1]$.

PROOF. First we assume Γ is a discrete group. If Φ is discontinuous at a point in $[-1, 1]$, there exists a sequence $\{a_j\}_{j=0}^{\infty}$ in $[-1, 1]$ and a finite number B satisfying:

$$B \neq \Phi(a), \quad a_i \neq a_j \ (i \neq j), \quad \sum_{j=0}^{\infty} \left| a_j - a \right| < \infty$$

and

$$\sum_{j=0}^{\infty} \left| \Phi(a_j) - B \right| < \infty.$$

We may assume $\Phi(a) = 0$.

Take a function f in $L^p(G)$ and a sequence $\{\varepsilon_j\}_{j=0}^{\infty}$, $\varepsilon_j = \pm 1$, such that $\sum_{j=0}^{\infty} \hat{f}(\gamma_j)\, \varepsilon_j(x, \gamma_j)$ does not belong to $L^p(G)$, where $f \sim \sum_{j=0}^{\infty} \hat{f}(\gamma_j)(x, \gamma_j)$ (see [3] or [11]). Thus if we set $\eta_j = \Phi(a_j)$ for $\varepsilon_j = 1$ and $\eta_j = 0$ for $\varepsilon_j = -1$, then $\sum_{j=0}^{\infty} \hat{f}(\gamma_j)\, \eta_j(x, \gamma_j) \notin L^p(G)$. In fact we have

$$\sum_{j=0}^{\infty} \hat{f}(\gamma_j)\, \eta_j(x, \gamma_j) = \frac{B}{2} \sum_{j=0}^{\infty} \hat{f}(\gamma_j)(x, \gamma_j) + \frac{B}{2} \sum_{j=0}^{\infty} \hat{f}(\gamma_j)\, \varepsilon_j(x, \gamma_j)$$

$$+ \sum_{\varepsilon_j = 1} \hat{f}(\gamma_j)\, [\Phi(a_j) - B]\, (x, \gamma_j).$$

The first and the third sums on the right hand side belong to $L^p(G)$ and the second does not by the assumption.

Put $\varphi(\gamma_j) = a_j$ for $\varepsilon_j = 1$ and $\varphi(\gamma) = a$ for other γ. Then for any g in $L^1(G)$ we have

$$\sum \hat{g}(\gamma)\,\varphi(\gamma)(x,\gamma) = \sum \hat{g}(\gamma)\,[\varphi(\gamma) - a]\,(x,\gamma) + a\sum \hat{g}(\gamma)(x,\gamma),$$

which also belongs to $L^1(G)$, that is, $\varphi \in M_1(\Gamma)$. On the other hand $\Phi(\varphi(\gamma_j)) = \eta_j$. Thus $\Phi(\varphi) \notin M_p(\Gamma)$ which contradicts our assumption.

Next we assume $\Gamma = \boldsymbol{R}$. First we show that there exist positive numbers δ and M such that if φ is a real-valued function in $M_1(\boldsymbol{R})$, the support of $\varphi \subset [0,1]$ and $\|\varphi\|_{M_1(R)} < \delta$, then $\|\Phi(\varphi)\|_{M_p(R)} \leqq M$.

To prove this we may assume $\Phi(0) = 0$. If this assertion is false, then we have a sequence $\{\varphi_j\}$ such that the suport of $\varphi_j \subset (2j, 2j+1)$, the range of $\varphi_j \subset [-1,1]$, $\|\varphi_j\|_{M_1(R)} < 2^{-j}$ but $\|\Phi(\varphi_j)\|_{M_p(R)} > j$. Put $\psi = \sum_{j=1}^{\infty} \varphi_j$. Then $\|\psi\|_{M_1(R)} \leqq 1$. Let ξ_j be the continuous function such that $\xi_j(x) = 1$ on $(2j, 2j+1)$, $= 0$ outside $(2j - 1/2, 2j + 3/2)$ and is linear otherwise. Then $\xi_j\Phi(\psi) = \Phi(\varphi_j)$. Thus

$$3\|\Phi(\psi)\|_{M_p(R)} \geqq \|\xi_j\Phi(\psi)\|_{M_p(R)} = \|\Phi(\varphi_j)\|_{M_p(R)} > j$$

which is impossible.

Suppose Φ is not continuous at a point a. Let $\{a_j\}$ be a sequence converging to a such that $\Phi(a_j)$ converge to $B \neq \Phi(a)$. We may assume $\Phi(a) = 0$ and $a = 0$. Let F be any closed set contained in $(1/4, 3/4)$ and $\{C_j\}$ be an increasing sequence of closed sets in $[0,1]\backslash F$, such that $m(F \cup C_j) \to 1$. Then we have a sequence $\{\chi_j\}$ of functions in $M_1(\boldsymbol{R})$ which equal 1 on F and 0 on $(-\infty, 0) \cup C_j \cup (1,\infty)$. Take a sequence $\{k_j\}$ such that $\|a_{k_j}\chi_j\|_{M_1(R)} < \delta$. Then we have $\|\Phi(a_{k_j}\chi_j)\|_{M_p(R)} < M$ for all $j = 1, 2, \cdots$, Since $\Phi(a_{k_j}\chi_j) = \Phi(a_{k_j})$ on F and 0 on $(-\infty, 0) \cup (1, \infty)$, $\Phi(a_{k_j}\chi_j) \to B\chi_F$ almost everywhere as $j \to \infty$ and $\|B\chi_F\|_{M_p(R)} \leqq M$, where χ_F is the characteristic function of F. This implies that every open set in (1/4, 3/4) is an L^p-multiplier, which is impossible (see, [8]).

LEMMA 3. *Suppose Γ is a locally compact, non-compact abelian group and $1 \leqq p < 2$. If Φ is a function on the real line possessing the property that $\Phi(\varphi) \in M_p(\Gamma)$ for all real valued function φ in $M_1(\Gamma)$, then Φ has the similar property for an infinite discrete group.*

PROOF. By the structure theorem Γ contains an open subgroup Γ_0 which is the direct sum of a compact group Λ and an N-dimensional euclidean space

$\boldsymbol{R}^N$. Let H be the annihilator of Γ_0. Then H is the dual to Γ/Γ_0 and a compact subgroup of $G=\hat{\Gamma}$.

(a) The case where $N>0$. First we observe that Φ maps $M_1(\Gamma_0)$ to $M_p(\Gamma_0)$.

In fact for $\varphi\in M_1(\Gamma_0)$ put $\tilde{\varphi}=\varphi$ on Γ_0 and 0 outside Γ_0. Then $\tilde{\varphi}\in M_1(\Gamma)$. For if $f\in L^1(G)$. then $f^*(x)=\int_H f(x+y)\,dm_H(y)$ belongs to $L^1(G/H)$ and $\hat{f}^*(\gamma)=\hat{f}(\gamma)$ on Γ_0, where dm_H denotes the Haar measure on H. Thus there exists a function g^* in $L^1(G/H)$ such that $\hat{g}^*=\varphi\hat{f}^*=\tilde{\varphi}\hat{f}$ on Γ_0. Let π be the natural homomorphism of G onto G/H, then $g=g^*\circ\pi\in L^1(G)$ and satisfies the relation $\hat{g}=\tilde{\varphi}\hat{f}$ on Γ.

On the other hand if $\Psi\in M_p(\Gamma)$ and $\Psi=0$ outside Γ_0, then $\Psi\in M_p(\Gamma_0)$. For if $f^*\in L^p(G/H)$, then the function $f=f^*\circ\pi\in L^p(G)$ and $\hat{f}=\hat{f}^*$ on Γ_0. Thus there exists a function g in $L^p(G)$ such that $\Psi\hat{f}=\hat{g}$. Put $g^*(x)=\int_H g(x+y)\ dm_H(y)$, then $g^*\in L^p(G/H)$, since H is compact. Furthermore we have $\Psi\hat{f}^*=\Psi\hat{f}=\hat{g}=\hat{g}^*$ on Γ_0.

Therefore we can conclude that Φ maps $M_1(\Gamma_0)$ into $M_p(\Gamma_0)$.

Since $\Gamma_0=\Lambda\oplus\boldsymbol{R}\oplus\cdots\oplus\boldsymbol{R}$, Φ maps also $M_1(\boldsymbol{R})$ into $M_p(\boldsymbol{R})$. Thus Φ is continuous by Lemma 2. Let φ be a real-valued function in $M_1(\boldsymbol{Z})$, then there exists a measure μ on $\boldsymbol{T}$ such that

$$\varphi(n)=\int_{-\pi}^{\pi}e^{-inx}\ d\mu(x).$$

Thus the function φ^* defined by

$$\varphi^*(\xi)=\int_{-\pi}^{\pi}e^{-i\xi x}\ d\mu(x)$$

is real-valued on $\boldsymbol{R}$ and $\varphi^*\in M_1(\boldsymbol{R})$. Thus $\Phi(\varphi^*)\in M_p(\boldsymbol{R})$. Since Φ is continuous, Theorem B implies $\Phi(\varphi^*(n))=\Phi(\varphi(n))\in M_p(\boldsymbol{Z})$. Therefore Φ maps $M_1(\boldsymbol{Z})$ into $M_p(\boldsymbol{Z})$.

(b) The case where $N=0$. We shall show that Φ maps $M_1(\Gamma/\Gamma_0)$ into $M_p(\Gamma/\Gamma_0)$.

For $\varphi\in M_1(\Gamma/\Gamma_0)$ we put $\varphi^*=\varphi\circ\sigma$ where σ is the natural homomorphism of Γ onto Γ/Γ_0. Let T_φ and T_{φ^*} be the corresponding multiplier transforms on $L^1(H)$ and $L^1(G)$ respectively. Every element z of G is written as $z=x+y$ where $x\in H$ and y is an element of a coset of H. Then we have

$$[T_{\varphi^*}f](z)=T_\varphi[f(y+\cdot)](x)$$

for all f in $L^1(G)$. In fact the Fourier transform of the right hand side is

$$\int_{G/H} dm_{G/H}(y) \int_H \overline{(x+y,\gamma)}\, T_\varphi[f(y+\cdot)](x)\ dm_H(x)$$

$$= \int_{G/H} \overline{(y,\gamma)}\ dm_{G/H}(y) \int_H \overline{(x,\gamma)}\, T_\varphi[f(y+\cdot)](x)\ dm_H(x)$$

$$= \int_{G/H} \overline{(y,\gamma)}\ dm_{G/H}(y) \int_H \overline{(x,\gamma)}\, \varphi^*(\gamma) f(y+x)\ dm_H(x)$$

$$= \varphi^*(\gamma)\hat{f}(\gamma).$$

The last term is the Fourier transform of $T_{\varphi^*}f$.

On the other hand if $\Psi \in M_p(\Gamma)$ and Ψ is constant on each coset of Γ_0, then Ψ considered as a function on Γ/Γ_0 belongs to $M_p(\Gamma/\Gamma_0)$. For if $f \in L^p(H)$ put $\tilde{f} = f$ on H and 0 otherwise. Then $\tilde{f} \in L^p(G)$ and $\|\tilde{f}\|_{L^p(G)} = \|f\|_{L^p(H)}$. $\hat{\tilde{f}}(\gamma)$ is constant on each coset of Γ_0 and $\Psi(\gamma)\hat{\tilde{f}}(\gamma) = \Psi(\gamma_1)\hat{f}(\gamma_1)$ where $\gamma_1 \in \Gamma/\Gamma_0$ and $\gamma \in \gamma_1$. Since $T_\Psi\tilde{f} = T_\Psi f$ on H and 0 otherwise, we get $T_\Psi f \in L^p(H)$, that is, $\Psi \in M_p(\Gamma/\Gamma_0)$.

Therefore Φ maps $M_1(\Gamma/\Gamma_0)$ into $M_p(\Gamma/\Gamma_0)$. We remark that Γ/Γ_0 is an infinite discrete group, since Γ is not compact.

We refer the following lemma to [5].

LEMMA C. (a) *Let* $\{\Omega_j\}, j = 1, 2, \cdots$, *be a sequence of finite subgroups of* $\Delta(r)(r \geqq 2)$. *Then there exists a sequence* $\{\gamma_j\}$ *of* $\Delta(r)$ *having the property: Let* Γ_j *be the group generated by* Ω_j *and* γ_j, *then no two of groups* Γ_j *have a non-zero element in common. Let* $\{f_j\}$ *be a sequence of polynomials* (*real-valued if* $r = 2$) *on* $D(r)$ *such that* $\hat{f}_j$ *has its support in* Ω_j, *then we have an element* x_0 *in* $D(r)$ *so that*

$$\|f_j\|_\infty \leqq 2\,\Re[(x_0,\gamma_j)f_j(x_0)], \quad j = 1, 2, \cdots,$$

(b) *Let* Γ *be an infinite discrete group of unbounded order and* G *is the dual to* Γ. *Let* $\{n_j\}$, $j = 1, 2, \cdots$, *be a sequence of positive integers. Then there exist a sequence* $\{m_j\}$ *of positive integers and a sequence* $\{\gamma_j\}$ *in* Γ *having the properties*:

(4) *The order of* γ_j *exceeds* $2m_j + 6n_j^2$.

(5) *The sets* $E_j = \{n\gamma_j : m_j - 2n_j \leqq n \leqq m_j + 2n_j\}$ *are disjoint.*

(6) *If* $\{f_j\}$ *is a sequence of polynomials on* $\boldsymbol{T}$ *such that* $\hat{f}_j$ *has its support in* $\{n\colon |n|\leqq 2n_j\}$, *then we have an element* x_0 *in* G *such that*

$$\|f_j\|_\infty \leqq 2\,\Re[(x_0,\ m_j\gamma_j)\sum_{-2n_j}^{2n_j}\hat{f}_j(n)(x_0,\ \gamma_j)],\quad j=1,2,\cdots.$$

LEMMA 4. *Let* Γ *be an infinite discrete group and* Φ *be a continuous periodic function. Suppose* $\Phi(\varphi)\in M_p(\Gamma)$ *for every real-valued multiplier* φ *in* $M_1(\Gamma)$. *Then for any positive number* a, *there exists a constant* C_a *such that*

$$\|\Phi(\varphi)\|_{M_p(\Lambda)} \leqq C_a \tag{7}$$

for all real-valued φ *in* $M_1(\Lambda)$ *such that* $\|\varphi\|_{M_1(\Lambda)}\leqq a$, *where* Λ *is a group* $\Delta(r)(r\geqq 2)$ *or* $\boldsymbol{Z}$.

PROOF. We may suppose $\Phi(0)=0$. If (7) is false, we can find polynomils p_j on $\boldsymbol{L}$ and real-valued multipliers φ_j satisfying

$$\begin{aligned} &\|p_j\|_{L^p(L)}\leqq 2^{-j},\\ &\|\varphi_j\|_{M_1(\Lambda)}\leqq a, \\ &\Big\|\sum_\gamma \Phi(\varphi_j(\gamma))\,\hat{p}_j(\gamma)(\cdot,\gamma)\Big\|_{L^p(L)} > j,\quad j=1,2,\cdots,\end{aligned} \tag{8}$$

where Λ indicates the groups $\Delta(r)$ $(r\geqq 2)$ or $\boldsymbol{Z}$, and $\boldsymbol{L}$ is the dual to Λ.

Here we can assume that the support of φ_j is finite. For let k_j be the polynomials on $\boldsymbol{L}$ so that $\|k_j\|_{L^1(L)}\leqq 3$ and $\hat{k}_j=1$ on the support of $\hat{p}_j$. Then $\|\hat{k}_j\varphi_j\|_{M_1(\Lambda)}\leqq 3a$ and

$$\sum_\gamma \Phi(\hat{k}_j(\gamma)\varphi_j(\gamma))\,\hat{p}_j(\gamma)(x,\gamma)=\sum_\gamma \Phi(\varphi_j(\gamma))\,\hat{p}_j(\gamma)(x,\gamma).$$

First we assume that Γ is a group of bounded order. Then we can write $\Gamma=\Delta(r)\oplus\Pi$ for some $r\geqq 2$. Therefore Φ has the same property for $\Delta(r)$ as in the lemma, so that we can assume $\Gamma=\Delta(r)$. We show (8) is impossible for $\Lambda=\Delta(r)$.

Let Ω_j be the subgroup generated by the support of φ_j, then Ω_j is a finite subgroup of $\Delta(r)$. Let X be the space of real-valued continuous functions f of the form

$$f(x) = \sum_{j=1}^{\infty} (x, \gamma_j) f_j(x),$$

where $\{\gamma_j\}$ is a sequence of (a) in Lemma C and the support of f_j is contained in Ω_j. Then this representation of f is unique and we have

$$\|f\|_\infty \leqq \sum_{j=1}^{\infty} \|f_j\|_\infty \leqq 2\|f\|_\infty .$$

Thus the functional defined on X by

$$Tf = \sum_{j=1}^{\infty} \int_{D(r)} f_j(-x) \sum_{\gamma} \varphi_j(\gamma)(x, \gamma)\ dx$$

is bounded. Therefore there exists a finite measure μ on $D(r)$ such that

$$Tf = \int_{D(r)} f(-x)\, d\mu(x) .$$

In particular $\hat{\mu}(\gamma + \gamma_j) = \varphi_j(\gamma)$. If $\hat{\mu}$ is not real-valued we replace $\hat{\mu}$ by its real part. Since $\hat{\mu} \in M_1(\varDelta(r))$ and $\left\|\sum_{j=1}^{\infty} p_j(\,\cdot\,, \gamma_j)\right\|_{L^p(D(r))} \leqq 1$, we have

$$\left\|\sum_{j=1}^{\infty} \sum_{\gamma} \Phi(\hat{\mu}(\gamma + \gamma_j))\, \hat{p}_j(\gamma)(\,\cdot\,, \gamma + \gamma_j)\right\|_{L^p(D(r))} \leqq \|\Phi(\hat{\mu})\|_{M_p(\varDelta(r))} < \infty .$$

Consider the characteristic function of $\Omega_j + \gamma_j$ which is a multiplier of norm one. Then

$$\begin{aligned}\|\Phi(\hat{\mu})\|_{M_p(\varDelta(r))} &\geqq \left\|\sum_{\gamma} \Phi(\hat{\mu}(\gamma + \gamma_j))\, \hat{p}_j(\gamma)(\,\cdot\,, \gamma + \gamma_j)\right\|_{L^p(D(r))} \\ &= \left\|\sum_{\gamma} \Phi(\varphi_j(\gamma))\, \hat{p}_j(\gamma)(\,\cdot\,, \gamma)\right\|_{L^p(D(r))} \\ &\geqq j,\end{aligned}$$

$j = 1, 2, \cdots$, which is impossible.

Next we treat the case where Γ is not of bounded order. Assume (8) holds for $\Lambda = \boldsymbol{Z}$.

We can suppose that the support of $\hat{p}_j \subset [-n_j, n_j]$ and the support of $\varphi_j \subset [-2n_j, 2n_j]$. Let $\{\gamma_j\}$, $\{E_j\}$ and $\{m_j\}$ be the sequences of (b) in Lemma C.

Let X be the space of continuous functions f on G of the form

$$f^*(x) = \sum_{j=1}^{\infty} (x, m_j\gamma_j) f_j^*(x),$$

where $f_j^*(x) = \sum_{-2n_j}^{2n_j} \hat{f}_j(n)(x, n\gamma_j)$. Then the representation is unique. For f^* put

$$f(\theta) = \sum_{j=1}^{\infty} e^{im_j\theta} f_j(\theta),$$

where $f_j(\theta) = \sum_{-2n_j}^{2n_j} \hat{f}_j(n)e^{in\theta}$. Then we have, by (b) of Lemma C,

$$\sum_{j=1}^{\infty} \|f_j\|_\infty \leqq 2\|f^*\|_\infty.$$

We define a functional T on X by

$$Tf = \sum_{j=1}^{\infty} \int_{-\pi}^{\pi} f_j(-\theta) \sum_{-2n_j}^{2n_j} \varphi_j(n)\, e^{in\theta}\, d\theta.$$

Then this is bounded on X. Thus there exists, by extension theorem, a finite measure μ on G such that

$$Tf = \int_G f^*(-x)\, d\mu(x).$$

In particular $\hat{\mu}(m_j\gamma_j + n\gamma_j) = \varphi_j(n)$ for $|n| \leqq 2n_j$, $j = 1, 2, \cdots$. As above we may assume $\hat{\mu}$ is real-valued.

Now for the polynomial q on $\boldsymbol{T}$ of order $\leqq n_j$, put

$$q^*(x) = \sum_{-n_j}^{n_j} \hat{q}(n)(x, n\gamma_j), \quad x \in G.$$

If γ_j is of infinite order, then $\|q^*\|_{L^p(G)} = \|q\|_{L^p(T)}$. If γ_j has order d, say, then

$$\|q^*\|_{L^p(G)} = \left[\sum_{k=1}^{d} \frac{1}{d} \left| q\left(\frac{2\pi k}{d}\right)\right|^p\right]^{1/p}.$$

This differs from

$$\|q\|_{L^p(T)} = \left[\sum_{k=1}^{d} \frac{1}{2\pi} \int_{(2k-1)\pi/d}^{(2k+1)\pi/d} |q(\theta)|^p d\theta\right]^{1/p}$$

by at most

$$\frac{\pi}{d} \|q'\|_\infty \leqq \frac{5\pi}{d} n_j^2 \|q\|_{L^1(T)} \leqq \frac{1}{2} \|q\|_{L^1(T)} \leqq \frac{1}{2} \|q\|_{L^p(T)}.$$

Thus we have

$$2\|q\|_{L^p(T)} \geqq \|q^*\|_{L^p(G)} \geqq \frac{1}{2} \|q\|_{L^p(T)}.$$

Therefore from (8) we get

$$\|p_j^*\|_{L^p(G)} \leqq 2^{-j+1},$$

(9)

$$\left\| \sum_{-n_j}^{n_j} \Phi(\varphi_j(n)) \hat{p}_j(n)(\cdot, n\gamma_j) \right\|_{L^p(G)} \geqq \frac{1}{2} j,$$

$j = 1, 2, \cdots$. Since $\hat{\mu} \in M_1(\Gamma)$ and $\left\| \sum p_j^* \right\|_{L^1(G)} \leqq 2$,

$$\left\| \sum_{j=1}^{\infty} \sum_{-n_j}^{n_j} \Phi(\hat{\mu}(m_j\gamma_j + n\gamma_j)) \hat{p}_j(n)(\cdot, m_j\gamma_j + n\gamma_j) \right\|_{L^p(G)} \leqq 2\|\Phi(\hat{\mu})\|_{M_p(\Gamma)} < \infty.$$

If we put $\hat{K}_j(\gamma) = \min(1, 2 - |n|/n_j)$ for $\gamma = m_j\gamma_j + n\gamma_j$, $|n| \leqq 2n_j$ and $\hat{K}_j = 0$ otherwise, then $\|\hat{K}_j\|_{M_p(\Gamma)} \leqq 3$. Thus

$$\begin{aligned} 6\|\Phi(\hat{\mu})\|_{M_p(\Gamma)} &\geqq \left\| \sum_{-n_j}^{n_j} \Phi(\hat{\mu}(m_j\gamma_j + n\gamma_j))\hat{p}_j(n)(\cdot, m_j\gamma_j + n\gamma_j) \right\|_{L^p(G)} \\ &= \left\| \sum_{-n_j}^{n_j} \Phi(\varphi_j(n)) \hat{p}_j(n)(\cdot, n\gamma_j) \right\|_{L^p(G)}, \end{aligned}$$

which contradicts (9). Thus the lemma is proved.

Now we proceed to the proof of Theorem 1. Let Φ be the function in the theorem. Considering $\Phi(\sin t)$ and $\Phi(\varepsilon \sin t)\,(0<\varepsilon<1)$, it is sufficient to show that Φ is entire under the additional assumption that Φ is defined on the whole line and periodic. By Lemmas 2 and 3 Φ is continuous and maps real-valued functions in $M_1(\Lambda)$ into $M_p(\Lambda)$ where Λ is an infinite discrete group. We have

$$e^{in\varphi}\,\hat{\Phi}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(\varphi+x)\,e^{-inx}\,dx,$$

where $\varphi\in M_1(\Lambda)$ and $\Lambda=\Delta(r)(r\geqq 2)$ or $\boldsymbol{Z}$. Hence by Lemma 4

$$|\hat{\Phi}(n)|\ \ \|e^{in\varphi}\|_{M_p(\Lambda)}\leqq C_a$$

for any φ such that $\|\varphi\|_{M_1(\Lambda)}\leqq a$. Therefore by Lemma 1 we get $|\hat{\Phi}(n)|\leqq C_a K_p^{-an}$ for any $a>0$. Therefore Φ is extended to an entire function.

REMARK. Let Γ be a compact abelian group and $1\leqq p<2$. If Φ is a function on $[-1,1]$ and $\Phi(\varphi)\in M_p(\Gamma)$ for all φ in $M_1(\Gamma)$ with the range contained in $[-1,1]$, then Φ is the restriction of a function analytic in a neighborhood of $[-1,1]$.

In fact $M_1(\Gamma)=A_1(\Gamma)$ and $M_p(\Gamma)\subset A_p(\Gamma)$, so that this follows from a theorem of Rudin in [9].

4. Some consequences of Theorem 1. Let $1\leqq p<2$ and $m_p(\Gamma)$ be the space of continuous functions in $M_p(\Gamma)$. Since $\|\varphi\|_\infty\leqq\|\varphi\|_{M_p(\Gamma)}$, $m_p(\Gamma)$ is a closed subalgebra of $M_p(\Gamma)$ and each point of Γ is identified with a maximal ideal of $m_p(\Gamma)$.

THEOREM 4. *Let $1\leqq p<2$ and Γ be a locally compact non-compact abelian group. Then for any complex number z there exist a real-valued function φ in $m_p(\Gamma)$ and a homomorphism h of $m_p(\Gamma)$ such that $h(\varphi)=z$.*

PROOF. Otherwise the function $\Phi(x)=(x-z)^{-1}$ would carry the real-valued functions in $m_p(\Gamma)$ to $M_p(\Gamma)$, which is impossible since $M_1(\Gamma)\subset m_p(\Gamma)$.

COROLLARY 5. *Under the conditions in Theorem 4 the algebra $m_p(\Gamma)$*

is asymmetric and not regular.

PROOF. By Theorem 4, Γ is not dense in the maximal ideal space $\mathfrak{M}$ of $m_p(\Gamma)$. Therefore $m_p(\Gamma)$ is not regular. Let φ be a function in $m_p(\Gamma)$ such that the Fourier-Gelfand transform $\hat{\varphi}$ is real-valued on Γ but not on $\mathfrak{M}$. If for some $\psi \in m_p(\Gamma)$ we have $\hat{\psi} = \overline{\hat{\varphi}}$ on $\mathfrak{M}$, then $\psi(\gamma) = \varphi(\gamma)$ for all $\gamma \in \Gamma$, that is, $\hat{\varphi}$ is real-valued. Thus $\overline{\hat{\varphi}} \notin \widehat{m_p(\Gamma)}$.

THEOREM 6. *Under the conditions in Theorem* 4 *there exists a real-valued function* φ *in* $M_1(\Gamma)$ *such that* $\varphi(\gamma) \geqq 1$ *but* $1/\varphi \notin M_p(\Gamma)$.

PROOF. It suffices to consider the function $\Phi(x) = 1/(x^2 + 1)$.

This will be interesting in connection with the inversion theorem of the singular integral operators; see Calderón-Zygmund [1].

From Theorem 1 and Remark in § 3 we have the following result which is proved partially by Hörmander [6] and Figà-Talamanca [4] in the case $\Gamma = \boldsymbol{R}$.

THEOREM 7. *Let* Γ *be a locally compact abelian group and* $1 \leqq p < 2$. *Then the contraction does not operate on* $M_p(\Gamma)$ *and* $m_p(\Gamma)$.

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