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L-FUNCTIONS OF φ-SHEAVES AND DRINFELD MODULES

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0. Introduction

In this paper, we apply Dwork's p-adic methods to study the meromorphic continuation and rationality of various L-functions arising from π-adic Galois representations, Drinfeld modules and φ-sheaves. As a consequence, we prove some conjectures of Goss about the rationality of the local L-function and the meromorphic continuation of the global L-function attached to a Drinfeld module.

Let $\mathbb{F}_q$ be a finite field of q elements with characteristic p. Let π be a prime of the polynomial ring $A = \mathbb{F}_q[t]$. Let A_π be the completion of the ring A at π. This is an analogue of the classical ring $\mathbb{Z}_p$ of p-adic integers. Let X be an irreducible algebraic variety defined over $\mathbb{F}_q$ and let $\pi_1(X)$ be the arithmetic fundamental group of $X/\mathbb{F}_q$ with respect to some base point. The group $\pi_1(X)$ may be regarded as the Galois group of a separable closure of the function field of $X/\mathbb{F}_q$ modulo the inertia groups at the closed points of $X/\mathbb{F}_q$. Suppose now that we are given a continuous π-adic representation

$$\rho : \pi_1(X) \longrightarrow \mathrm{GL}_r(A_\pi).$$

We can then define the Artin L-function of the representation in a standard manner:

$$L(\rho, T) \;:=\; \prod_{x \in X_0} \frac{1}{\det(I - T^{\deg(x)}\rho(\mathrm{Frob}_x))},$$

where X_0 is the set of closed points on $X/\mathbb{F}_q$ and Frob_x is (the conjugacy class of) a Frobenius element of $\pi_1(X)$ at the closed point x. This is a formal power series with coefficients in A_π. A fundamental question is then the meromorphic continuation (as a function of π-adic variables) and rationality of such L-functions. We shall see that such L-functions arise naturally in the theory of Drinfeld modules. It is not easy to work directly with the representation ρ itself because one does not know what a typical representation looks like. There is a much more transparent category of what we call *smooth* (or *lisse*) *π-adic φ-sheaves* (§1), which is functorially equivalent to the category of π-adic representations so that the corresponding L-functions are the same. It is in the category of smooth π-adic φ-sheaves (more generally, a larger category of π-adic φ-sheaves) that one can see what the Frobenius image $\rho(\mathrm{Frob}_x)$ looks like and one can transport Dwork's trace formula to study

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meromorphic continuation and rationality of L-functions. The situation is analogous to the classical p-adic case, where one has a functorial equivalence between the category of continuous p-adic representations of $\pi_1(X)$ and the category of unit root F-crystals on X such that the corresponding L-functions are the same, and the Dwork-Reich-Monsky trace formula is the key to the meromorphic continuation of the L-function attached to an F-crystal.

Our first goal here is to extend the results in [W1, Section 7], established only for the torus and affine space cases, to an arbitrary X. How far the L-function of a π-adic φ-sheaf can be meromorphically extended depends on a certain convergence condition of the π-adic φ-sheaf. In particular, the L-function is meromorphic everywhere if the π-adic φ-sheaf is overconvergent (Corollary 4.2). The L-function is rational if the π-adic φ-sheaf is algebraic (Theorem 4.1 (ii); see §1 for the definition of overconvergence and algebraicity). Since a Drinfeld module gives rise to an algebraic φ-sheaf with all expected properties, these results are then used to prove some conjectures of Goss [G2], which we explain next.

Suppose that X is now a scheme of finite type over A and Φ is a Drinfeld A-module over X. We first consider the finite characteristic case (the local case), i.e., the image of the structure morphism $X \to \operatorname{Spec} A$ is one closed point of $\operatorname{Spec} A$. We think of Φ as a family of Drinfeld modules parametrized by $X/\mathbb{F}_q$. Fix a closed point π of $\operatorname{Spec} A$ which is different from the image of the structure morphism $X \to \operatorname{Spec} A$. Let $T_\pi(\Phi)$ be the π-adic Tate module of Φ and let $H^1(\Phi, A_\pi) = \operatorname{Hom}_{A_\pi}(T_\pi(\Phi), A_\pi)$ be the π-adic cohomology group of Φ. This is a free A_π-module on which $\pi_1(X)$ acts continuously. This group action gives a continuous π-adic representation ρ_Φ of $\pi_1(X)$. The local L-function $L(\Phi/X, T)$ attached to the Drinfeld module Φ/X is defined to be the L-function of the π-adic representation ρ_Φ. For example, in the case Φ is trivial and thus the representation ρ_Φ is trivial, the local L-function is just the reduction modulo p of the classical zeta function of the scheme $X/\mathbb{F}_q$, which is well known to be a rational function. Goss' local conjecture says that the local L-function $L(\Phi/X, T)$ is always a rational function in T. In terms of π-adic φ-sheaves, the representation ρ_Φ corresponds to a smooth algebraic π-adic φ-sheaf, since it comes from an "algebraic object" Φ. Thus, by our result on the rationality of L-functions, the local L-function $L(\Phi/X, T)$ is indeed rational.

Suppose again that X is a scheme of finite type over A and Φ is a Drinfeld A-module over X. We now consider the global case, that is, we do not assume that the image of the structure morphism $X \to \operatorname{Spec} A$ is a point. For each non-zero prime ideal $\mathfrak{p}$ of A, the fiber $\Phi_\mathfrak{p}$ of Φ over $\mathfrak{p}$ is a Drinfeld module over the fibre variety $X_\mathfrak{p}/\mathbb{F}_\mathfrak{p}$, where $\mathbb{F}_\mathfrak{p}$ is the residue field $A/\mathfrak{p}$. Its local L-function $L(\Phi_\mathfrak{p}/X_\mathfrak{p}, T)$ is rational. Following the definition of the classical Hasse-Weil zeta function of schemes over $\mathbb{Z}$, Goss defines the global L-function of Φ to be

$$L(\Phi/X, s) := \prod_{\mathfrak{p} \in (\operatorname{Spec} A)_0} \frac{1}{L(\Phi_\mathfrak{p}/X_\mathfrak{p}, \mathfrak{p}^{-s})}. \tag{0.1}$$

We now explain what the "complex variable" s is and how to define the exponentiation $\mathfrak{p}^{-s}$ in the above expression. We choose $\pi_\infty = 1/t$ as a uniformizer of $\mathbb{F}_q(t)$ at infinity. Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of $\mathbb{F}_q(t)$ at π_∞. The "complex plane" of Goss is the space $\mathbb{C}_\infty^\times \times \mathbb{Z}_p$. Any non-zero ideal $\mathfrak{a}$ of A has a unique monic generator $a(t)$. We set

$$\langle \mathfrak{a} \rangle := \pi_\infty^{\deg(\mathfrak{a})} a(t),$$

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where $\deg(\mathfrak{a}) = \deg(a(t))$ is the dimension of the $\mathbb{F}_q$-vector space $A/\mathfrak{a}$. The quantity $\langle\mathfrak{a}\rangle$ is an element in $\mathbb{F}_q[\![\pi_\infty]\!]^\times \subset \mathbb{C}_\infty^\times$ which is congruent to 1 modulo π_∞. For $s = (z, y) \in \mathbb{C}_\infty^\times \times \mathbb{Z}_p$ and $\mathfrak{a}$ a non-zero ideal of A, one defines

$$\mathfrak{a}^s \ := \ z^{\deg(\mathfrak{a})}\langle\mathfrak{a}\rangle^y.$$

This is a well defined element of $\mathbb{C}_\infty^\times$. If m is an integer and $s = (t^m, m)$ (this is the way how the rational integers are embedded into the s-plane), then $\mathfrak{a}^s$ is simply the m-th power $a(t)^m$ of the monic generator $a(t)$ of $\mathfrak{a}$. In the special case where Φ is trivial and X is Spec A itself, the global L-function is just the following analogue of the classical Riemann zeta function:

$$\zeta(A, s) \ = \ \prod_{\mathfrak{p}} \frac{1}{1 - \mathfrak{p}^{-s}} \ = \ \sum_{\mathfrak{a}} \frac{1}{\mathfrak{a}^s},$$

where $\mathfrak{p}$ runs over all non-zero prime ideals of A and $\mathfrak{a}$ runs over all non-zero ideals of A.

Goss' global conjecture says that the L-function $L(\Phi/X, s)$ is meromorphic and essentially algebraic. Namely, the L-function satisfies the following three properties: (i) For each fixed $y \in \mathbb{Z}_p$, the L-function $L(\Phi/X, s)$ viewed as a function of $1/z$ is π_∞-adically meromorphic everywhere. (ii) $L(\Phi/X, s)$, viewed as a family of functions parametrized by the second variable $y \in \mathbb{Z}_p$ of s, is ∞-continuous in y (see §5 for the precise meaning of this statement). (iii) At $y = -j$, a negative integer, the L-function $L(\Phi/X, s)$ is a rational function in $1/z$. Note that in Goss' original definition of continuity (or entireness; see [G2, 3.5.1]), he only required that the L-function be continuous in y. Recently, Goss has strengthened his formulation of continuity of the L-function. We use the strengthened form of continuity in this paper. We prove this conjecture for arbitrary X and Φ with $A = \mathbb{F}_q[t]$ (§8).

Goss [G1] proved that his global conjecture is true in the case where X is a curve finite over A and Φ has complex multiplication, using the Riemann-Roch theorem. The first interesting unknown case will then be the case of a Drinfeld A-module over Spec$A$ which may not have complex multiplication. If Φ is a Drinfeld A-module over an A-scheme X, we associate to Φ a matrix B with entries in $\mathcal{O}_X \otimes_{\mathbb{F}_q} A$, which can be explicitly calculated in terms of the defining equation of Φ as in (1.3). Then it turns out that the global L-function of Φ (for the specific case $X = \mathrm{Spec}A$) is expressed as

$$L(\Phi/A, s) \ = \prod_{\mathfrak{p} \in (\mathrm{Spec}A)_0} \frac{1}{\det(1 - B(\mathfrak{p})^{\sigma^{\deg(\mathfrak{p})-1}} \cdots B(\mathfrak{p})^\sigma B(\mathfrak{p})\mathfrak{p}^{-s})}, \tag{0.2}$$

where $B(\mathfrak{p})$ is the value at $\mathfrak{p}$ (or, the reduction modulo the ideal $\mathfrak{p}$) of the matrix B, and $B(\mathfrak{p})^{\sigma^i}$ means the entry-wise application to $B(\mathfrak{p})$ of the q^i-th power map of the finite field $\mathbb{F}_\mathfrak{p} = A/\mathfrak{p}$ (so the polynomial $\det(1 - B(\mathfrak{p})B(\mathfrak{p})^\sigma \cdots B(\mathfrak{p})^{\sigma^{\deg(\mathfrak{p})-1}}T)$ has coefficients in $\mathbb{F}_q \otimes_{\mathbb{F}_q} A = A$). Our results imply that the global L-function in (0.2) is meromorphic and essentially algebraic in the sense of Goss.

Some of the points of our proof of the above conjecture are as follows: The reason for (i) to be true is that for a fixed y, the character $\langle\mathfrak{p}\rangle^{-y}$ defines a rank one overconvergent π-adic φ-sheaf. Thus, the L-function for a fixed y is essentially the local L-function attached to the Drinfeld module Φ on $X/\mathbb{F}_q$ twisted by the overconvergent abelian character $\langle\mathfrak{p}\rangle^{-y}$. The reason for (ii) to be true is that the family of characters $\langle\mathfrak{p}\rangle^{-y}$ parametrized by y defines a continuous family of uniformly overconvergent π-adic φ-sheaves (§5). As for (iii), when y is a negative

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integer, the character $\langle\mathfrak{p}\rangle^{-y}$ defines a rank one algebraic φ-sheaf. Since Φ defines an algebraic φ-sheaf, its twist by a rank one algebraic φ-sheaf is still algebraic. This implies that the L-function is a rational function if y is a negative integer.

Sometimes our L-functions turn out to be *analytic*, rather than meromorphic; we give some remarks on this phenomenon. As Lemma 4.3 shows, the local L-function, raised to the $(-1)^{\dim(X)-1}$-st power, of an overconvergent (resp. algebraic) φ-sheaf is analytic (resp. a polynomial) if X is an affine space or a torus. This happens only in characteristic $p > 0$; in the characteristic 0 case ([W1]), the L-function is a ratio of two analytic functions. By reduction modulo p, one of them reduces to 1, and an analytic function remains. This implies that, if X is an affine space or a torus, then (i) the local L-function $L(\Phi/X, T)$, raised to the $(-1)^{\dim(X)-1}$-st power, of a Drinfeld module Φ is a polynomial with coefficients in A; and (ii) the global L-function $L(\Phi/X, s)$, raised to the $(-1)^{\dim(X/\mathbb{F}_q)-1}$-st power, is π-adically analytic and essentially algebraic. For example, the L-function $L(\Phi/A, s)$ in (0.2) is analytic. More generally, we conjecture the analyticity of these L-functions for *affine* X satisfying certain conditions; see Conjecture 4.4 for a precise formulation. The conjectured results would generalize the above (i) and (ii).

One can further twist our global L-function by an arbitrary abelian character of $\pi_1(X)$ of finite order. Since an abelian character of finite order will be shown to define a rank one algebraic φ-sheaf (§10), the global conjecture is true for this twisted L-function.

All of these results carry over to the v-adic global L-function for a finite prime v of A (§9), as conjectured also by Goss.

Goss defined the exponentiation $\mathfrak{a}^s$ of ideals and formulated his global conjecture for a general Dedekind domain A which is of the form $\Gamma(C - \infty, \mathcal{O}_C)$, where C is a smooth projective geometrically irreducible curve over $\mathbb{F}_q$; this has been done from the viewpoint of "A-coefficient theory" (i.e. in the spirit of Drinfeld A-modules) for a general A. We have not been able to prove Goss' conjecture in this generality because we have been unable to show that the character $\langle\mathfrak{P}\rangle^y$ defined in his way is overconvergent, where $\mathfrak{P}$ is a non-zero prime ideal in A, not necessarily principal; we could use any overconvergent character to define a meromorphic L-function. On the other hand, if we follow the classical definition of the Dedekind zeta function of a number field by taking the norm of an ideal all the way down to $\mathbb{Z}$, then we can define a different global L-function. For this purpose, we fix an inclusion of $\mathbb{F}_q[t]$ into A (to set up the context of an $\mathbb{F}_q[t]$-coefficient theory with complex multiplication by A). For a prime ideal $\mathfrak{P}$ of A lying over a prime ideal $\mathfrak{p} \neq \infty$ of $\mathbb{F}_q[t]$, the norm $\mathrm{N}(\mathfrak{P})$ is defined to be $\mathfrak{p}^{\deg(\mathfrak{P},\mathfrak{p})}$ with $\deg(\mathfrak{P},\mathfrak{p})$ being the extension degree of the field $A/\mathfrak{P}$ over $\mathbb{F}_q[t]/\mathfrak{p}$. We then define

$$L_A(\Phi/X, s) := \prod_{\mathfrak{P}} \frac{1}{L(\Phi_{\mathfrak{P}}/X_{\mathfrak{P}}, \mathrm{N}(\mathfrak{P})^{-s})}, \tag{0.3}$$

where $\mathfrak{P}$ runs over all non-zero prime ideals of A not lying over ∞ of $\mathbb{F}_q[t]$. Then the global conjecture is still true for $L_A(\Phi/X, s)$. In fact, the new function $L_A(\Phi/X, s)$ is just the same global L-function defined in (0.1) for $\mathbb{F}_q[t]$ if we view X as defined over $\mathbb{F}_q[t]$ instead of its extension A. At this point, we point out that Thakur [Th] defines several different zeta functions for general A and $X = \mathrm{Spec}A$. All definitions essentially agree for $A = \mathbb{F}_q[t]$.

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Finally, we add some remarks: (1) To prove the Goss conjecture on the meromorphy of the L-functions of Drinfeld modules, we need only overconvergent π-adic φ-sheaves and no more general objects. If we restricted ourselves to the overconvergent case, the exposition would be simpler. Nevertheless, we decided to include all results on α log-convergent π-adic φ-sheaves (which have a much weaker convergence condition; see §3) in the belief that they should appear inevitably in a deeper study of φ-sheaves. For an account of the relevance of the α log-convergent objects in characteristic 0, we refer the reader to [DS] and [W1].

(2) In the study of π-adic φ-sheaves and their L-functions (§§1–6), we deal only with the A_π-module version. By inverting π, the K_π-module version (K_π: the fraction field of A_π) is also possible. In fact, the K_π-module version may be more natural for our purpose, since the L-functions are isogeny-invariant. Also, in Section 8, we actually need to invert $\pi = \pi_\infty$, because of the "pole" at ∞ of the φ-sheaf arising from a Drinfeld module.

Notation. All schemes denoted X (the "base scheme") in this paper are of finite type over $\mathbb{F}_q$. X_0 denotes the set of closed points on a scheme X. Since we work on schemes over a finite field $\mathbb{F}_q$, we use the notation $\mathbb{F}_x$ to denote the residue field of a closed point x on a scheme. If A_π is a complete discrete valuation ring with a uniformizer π, then ord_π is the valuation, normalized by $\mathrm{ord}_\pi(\pi) = 1$, on any algebraic extension (and its completion) of the fraction field of A_π. Also, set $|?|_\pi := q^{-\mathrm{ord}_\pi(?)}$. If X is a scheme of finite type over $\mathbb{F}_q$, then for a closed point x on X, $\deg(x)$ denotes the degree $[\mathbb{F}_x : \mathbb{F}_q]$ of its residue field $\mathbb{F}_x$ over $\mathbb{F}_q$.

1. φ-SHEAVES

Let $\mathbb{F}_q$ be a finite field of q elements with characteristic p. In this paper, all rings and schemes are over $\mathbb{F}_q$; all tensor products and direct products without reference to the base are over $\mathbb{F}_q$. For a scheme X, we denote by Fr_X the q-th power Frobenius endomorphism of X relative to $\mathbb{F}_q$. Let $\mathcal{A}$ be a Noetherian commutative $\mathbb{F}_q$-algebra (this will be the "coefficient ring" of φ-sheaves). The map Fr_X induces an endomorphism of $X \times \mathrm{Spec}\mathcal{A}$, being the identity on the second factor. We denote this map also by Fr_X. Slightly abusing the notation, we write $\mathcal{O}_X \otimes \mathcal{A}$ for the structure sheaf of $X \times \mathrm{Spec}\mathcal{A}$.

Definition. A *φ-sheaf* on X over $\mathcal{A}$ is a locally free $(\mathcal{O}_X \otimes \mathcal{A})$-module $\mathcal{E}$ on $X \times \mathrm{Spec}\ \mathcal{A}$ of finite rank together with an $(\mathcal{O}_X \otimes \mathcal{A})$-linear map

$$\varphi : \mathrm{Fr}_X^* \mathcal{E} \ \to \ \mathcal{E}.$$

Equivalently, we may regard φ as a Frobenius semi-linear map

$$\varphi : \mathcal{E} \ \to \ \mathcal{E}.$$

(Frobenius semi-linearity means that it is additive and one has $\varphi(ae) = \mathrm{Fr}_X(a)\varphi(e)$ for all local sections $a \in \mathcal{O}_X \otimes \mathcal{A}$ and $e \in \mathcal{E}$.) A *morphism* of φ-sheaves is an $(\mathcal{O}_X \otimes \mathcal{A})$-module homomorphism which is compatible with the φ's.

The map φ will be called the *Frobenius* on $\mathcal{E}$.

Let C be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$ and let K be its function field. Let π be a closed point of C. We denote by A_π and K_π respectively the completion of the local ring of C at π with respect to the π-adic topology and the fraction field of A_π. The letter π will also denote a uniformizer of A_π. For any scheme X, set $\mathcal{X}_\pi = X \otimes A_\pi$, and $\mathcal{X}_\pi^\wedge = X \widehat{\otimes} A_\pi$, the formal

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completion of $X \times C$ along $X \times \{\pi\}$. The Frobenius endomorphism Fr_X of X induces endomorphisms of $X \times C$, $X \otimes K$, $\mathcal{X}_\pi$, and $\mathcal{X}_\pi^\frown$, being the identity on the second factor. We denote these maps again by Fr_X.

We will mainly be interested in φ-sheaves over A_π, i.e. in the case $\mathcal{A} = A_\pi$. A φ-sheaf over $\mathbb{F}_q$ will be called a finite φ-sheaf in Section 6.

Definition. A *π-adic φ-sheaf* on X is a locally free $\mathcal{O}_{\mathcal{X}_\pi^\frown}$-module $\mathcal{E}$ on $\mathcal{X}_\pi^\frown$ of finite rank together with an $\mathcal{O}_{\mathcal{X}_\pi^\frown}$-linear map

$$\varphi : \mathrm{Fr}_X^* \mathcal{E} \ \to \ \mathcal{E}.$$

A *morphism* of π-adic φ-sheaves is an $\mathcal{O}_{\mathcal{X}_\pi^\frown}$-linear map which is compatible with the φ's. An *isogeny* of π-adic φ-sheaves is an injective morphism whose cokernel is killed by a power of π. A π-adic φ-sheaf is said to be *smooth* (or *lisse*) if the map φ is an isomorphism. A π-adic φ-sheaf $\mathcal{E}$ is said to be *algebraic* if both $\mathcal{E}$ and φ are defined over $\mathcal{O}_{\mathcal{X}_\pi}$, i.e., obtained from an $\mathcal{O}_{\mathcal{X}_\pi}$-module and an $\mathcal{O}_{\mathcal{X}_\pi}$-linear map by the base change $\mathcal{O}_{\mathcal{X}_\pi} \to \mathcal{O}_{\mathcal{X}_\pi^\frown}$.

If $\mathcal{E}$ is a smooth π-adic φ-sheaf on X, its *dual* $\check{\mathcal{E}}$ is defined as follows: the underlying sheaf of $\check{\mathcal{E}}$ is $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}_\pi^\frown}}(\mathcal{E}, \mathcal{O}_{\mathcal{X}_\pi^\frown})$ and the Frobenius on $\check{\mathcal{E}}$ is given by $f \mapsto (\sigma \circ f \circ \varphi^{-1})$ for a section f of $\mathrm{Fr}_X^* \check{\mathcal{E}}$, where $\sigma : \mathrm{Fr}_X^* \mathcal{O}_{\mathcal{X}_\pi^\frown} \to \mathcal{O}_{\mathcal{X}_\pi^\frown}$ is the Frobenius map relative to A_π. If $\mathcal{E}$ is not smooth but $\mathcal{E} \otimes_{A_\pi} K_\pi$ is smooth (i.e., if $\varphi \otimes K_\pi$ is an isomorphism), then the dual $\check{\mathcal{E}}$ exists in the category of π-adic φ-sheaves modulo isogeny.

If $\mathcal{E}$ is algebraic and smooth, its dual $\check{\mathcal{E}}$ is also algebraic and smooth. If $\mathcal{E}$ is a φ-sheaf over A_π, it can be viewed in a natural way as an algebraic π-adic φ-sheaf. We shall use the terminology *algebraic φ-sheaves* to mean both algebraic π-adic φ-sheaves and φ-sheaves over A_π. The categories of algebraic φ-sheaves and π-adic φ-sheaves on X are stable under direct sum and tensor product. If Y is an X-scheme of finite type, then the base extension $\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_\pi^\frown}} \mathcal{O}_{\mathcal{Y}_\pi^\frown}$ of a π-adic φ-sheaf $\mathcal{E}$ on X is a π-adic φ-sheaf on Y ($\mathcal{Y}_\pi^\frown$ has the same meaning for Y as $\mathcal{X}_\pi^\frown$ has for X), which is algebraic if $\mathcal{E}$ is algebraic.

As noted in the Introduction, π-adic φ-sheaves are a function field analogue of F-crystals in the p-adic theory. Here we have suppressed the use of a connection since it is irrelevant for our purpose. It is natural to expect that many results on F-crystals carry over to π-adic φ-sheaves. Here we apply Dwork's p-adic methods to prove some general results on meromorphic continuation and rationality of L-functions of π-adic φ-sheaves. Since the (Teichmüller) lifting problem disappears in characteristic p, we can work with π-adic φ-sheaves on general base schemes.

Algebraic φ-sheaves arise from Drinfeld modules. To explain this, we first recall the definition of Drinfeld A-modules over an A-scheme X (here A is the ring of functions on C which are regular outside a fixed closed point ∞ of C). If $\mathcal{L}$ is a line bundle on X, then it can be regarded as a group scheme which is isomorphic, locally on X, to the additive group scheme $\mathbb{G}_a$. To describe the ring $\mathrm{End}_X(\mathcal{L})$ of endomorphisms of $\mathcal{L}$ as a group scheme over X, suppose $\mathcal{L}$ is isomorphic to $\mathbb{G}_a$ on an affine open $U = \mathrm{Spec}\ R$ of X. Then one can choose a coordinate function Z of

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$\mathcal{L}|_U$, so that $\mathcal{L}|_U = \text{Spec } R[Z]$. One has

$$\begin{aligned}\text{End}_R(\mathcal{L}|_U) &= \{\text{additive polynomials} \in R[Z]\}\\ &= \{\sum^{\text{finite}} x_i Z^{p^i};\ x_i \in R\}\\ &\simeq R\{\tau\},\end{aligned}$$

where τ is the p-th power map on $\mathcal{L}|_U$ (corresponding to $Z^p \in R[Z]$) and $R\{\tau\}$ is the "twisted" polynomial ring with commutation relation $\tau x = x^p\tau$ for all $x \in R$. The ring R acts on the group scheme $\mathcal{L}|_U$ and R contains $\mathbb{F}_q$, hence $\mathbb{F}_q$ also acts on $\mathcal{L}|_U$. The $\mathbb{F}_q$-linear endomorphisms of $\mathcal{L}|_U$ form the subring

$$\begin{aligned}\text{End}_{R,\mathbb{F}_q\text{-mod}}(\mathcal{L}|_U) &= \{\sum^{\text{finite}} x_i Z^{q^i};\ x_i \in R\}\\ &= R\{\sigma\},\end{aligned}$$

where σ is the q-th power map on $\mathcal{L}|_U$.

Now let γ be the structure morphism $A \to \mathcal{O}_X$ (so in particular we have a ring homomorphism $\gamma : A \to R$ for each affine open Spec R of X). Let r be a positive integer. A *Drinfeld A-module* $(\mathcal{L}, \Phi)$ over X of rank r is a pair of a line bundle $\mathcal{L}$ on X and an A-module structure Φ on $\mathcal{L}$ (i.e. a ring homomorphism $\Phi : A \to \text{End}_X(\mathcal{L})$; $a \mapsto \Phi_a$) which satisfies the following condition: there exists an affine open covering $\{U_i\}$ of X such that the line bundle $\mathcal{L}$ is trivialized on each $U_i = \text{Spec } R_i$ and, for all $a \in A$ and i, Φ_a restricted to U_i is of the form

$$\Phi_a = \gamma(a) + x_1\sigma + \cdots + x_n\sigma^n, \qquad x_j \in R_i,\ x_n \in R_i^\times,$$

where $n = n(a)$ is the integer such that $q^n = \#(A/aA)^r$ if $a \neq 0$ and $n = 0$ if $a = 0$. Note that, if $A = \mathbb{F}_q[t]$, the Drinfeld A-module Φ is determined by the action of t, i.e., by the equation

$$\Phi_t = \gamma(t) + x_1\sigma + \cdots + x_r\sigma^r.$$

In the following, we shall write simply Φ for a Drinfeld module in place of $(\mathcal{L}, \Phi)$ (so Φ may be thought of as a ring homomorphism or a line bundle).

Given a Drinfeld A-module Φ over X, set

$$\mathcal{E} := \underline{\text{Hom}}_{\mathbb{F}_q\text{-mod}}(\Phi, \mathbb{G}_a), \tag{1.1}$$

the Zariski sheaf on X of $\mathbb{F}_q$-module homomorphisms of Φ to $\mathbb{G}_a$, both regarded as $\mathbb{F}_q$-module schemes over X (cf. [An], §1). The action of the ring A on Φ induces an A-action on $\mathcal{E}$. Thus it becomes an $(\mathcal{O}_X \otimes A)$-module. It is known that it is locally free of finite rank. If $A = \mathbb{F}_q[t]$, which is the relevant case for us, the sheaf $\mathcal{E}$ will be described explicitly in the following Example (in particular, it is "X-locally free" in this case, i.e., there exists an open affine covering (U_i) of X such that $\mathcal{E}|_{U_i \otimes A}$ is free over $\mathcal{O}_{U_i} \otimes A$ for all i). The Frobenius endomorphism on $\mathbb{G}_a$ relative to X induces an $(\mathcal{O}_X \otimes A)$-linear map $\varphi : \text{Fr}_X^*\mathcal{E} \to \mathcal{E}$. Thus $(\mathcal{E}, \varphi)$ is a φ-sheaf on X over A. Its formal completion along $X \times \{\pi\}$ is an algebraic π-adic φ-sheaf. We call it the *π-adic completion* of the φ-sheaf $\mathcal{E}$. It is smooth if π is not in the image of the structure morphism $\gamma : X \to \text{Spec}A$.

Example. Let $A = \mathbb{F}_q[t]$. Let R be an A-algebra and set $X = \text{Spec } R$. Denote by θ the image of $t \in A$ in R. We fix a coordinate Z on the additive group $\mathbb{G}_a$ over X (so $\mathbb{G}_a = \underline{\text{Spec}}\ \mathcal{O}_X[Z]$). Let σ be the q-th power Frobenius endomorphism $Z \mapsto Z^q$

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on $\mathbb{G}_a$ relative to X. Consider the Drinfeld A-module Φ of rank r over X of which the underlying line bundle is $\mathbb{G}_a$ and the A-module structure on it is defined by

$$\Phi_t \;=\; \theta + x_1\sigma + \cdots + x_r\sigma^r, \tag{1.2}$$

where $x_i \in R$ and $x_r \in R^\times$. Then the φ-sheaf $\mathcal{E} = \underline{\mathrm{Hom}}_{\mathbb{F}_q}(\Phi, \mathbb{G}_a)$ on X over A corresponding to Φ is simply the non-commutative ring $(R \otimes A)\{\sigma\}$ (in which elements of A commute with σ) modulo the left ideal $(t - (\theta + x_1\sigma + \cdots + x_r\sigma^r))$. More precisely, it is described as follows:

$$\mathcal{E} \;\simeq\; (\mathcal{O}_X \otimes A)^{\oplus r},$$

with a basis $e_0, \cdots, e_{r-1}$ (where we think of $e_i : \mathbb{G}_a \to \mathbb{G}_a$ as the map $\sigma^i : Z \mapsto Z^{q^i}$), and the Frobenius φ on $\mathcal{E}$ is given by $\varphi(e_i) = e_{i+1}$ for $0 \le i \le r-2$ and

$$\varphi(e_{r-1}) \;=\; (x_r^{-1} \otimes 1)\left((1 \otimes t)e_0 - ((\theta \otimes 1)e_0 + (x_1 \otimes 1)e_1 + \cdots + (x_{r-1} \otimes 1)e_{r-1})\right).$$

Thus its matrix is

$$B \;=\; (x_r^{-1} \otimes 1)\begin{pmatrix} & & & 1 \otimes t - \theta \otimes 1 \\ x_r \otimes 1 & & & -x_1 \otimes 1 \\ & \ddots & & \vdots \\ & & x_r \otimes 1 & -x_{r-1} \otimes 1 \end{pmatrix}. \tag{1.3}$$

Let x be a closed point of $\mathrm{Spec}R$. The action of the Frobenius Frob_x on the π-adic Tate module $T_\pi(\Phi)$ of Φ can be explicitly described as follows in terms of this matrix: Let $B(x)$ be the value of B at x (i.e., the reduction B mod $\mathfrak{m}$ if $\mathfrak{m}$ is the maximal ideal of R corresponding to x). Then by (6.4), the action of Frob_x on $T_\pi(\Phi)$ is given by the transpose of the matrix $B(x)^{\sigma^{\deg(x)-1}} \cdots B(x)^\sigma B(x)$ with respect to a suitable basis of $T_\pi(\Phi)$.

A φ-sheaf does not have enough data to recover a Drinfeld module (in fact, it can correspond to a much wider class of objects, such as abelian t-modules ([An])). To recover a Drinfeld module, one needs the notion of F-sheaves ([Dr]). For our purpose, we could dispense with F-sheaves, but since we refer to some results in [Dr], we explain here the relation of our φ-sheaves and Drinfeld's F-sheaves.

A *right F-sheaf* on X is a diagram

$$\mathrm{Fr}_X^*\mathcal{E} \;\overset{\varphi}{\to}\; \mathcal{F} \;\overset{\iota}{\leftarrow}\; \mathcal{E}, \tag{1.4}$$

where (1) $\mathcal{E}$ and $\mathcal{F}$ are locally free $\mathcal{O}_{X\times C}$-modules of the same finite rank, (2) φ and ι are injective morphisms of $\mathcal{O}_{X\times C}$-modules, (3) $\mathrm{Coker}(\varphi)$ is supported on the graph Γ_α of a morphism $\alpha : X \to C$ and is locally free of finite rank as an $\mathcal{O}_{\Gamma_\alpha}$-module, and (4) $\mathrm{Coker}(\iota)$ is supported on the graph Γ_β of a morphism $\beta : X \to C$ and is locally free of finite rank as an $\mathcal{O}_{\Gamma_\beta}$-module. Moreover, $\mathcal{F}$ is minimal in some sense. In [Dr], it is expressed by the condition that $\mathrm{Coker}(\varphi)$ and $\mathrm{Coker}(\iota)$ are of rank one on their supports. (For our purpose, it will be enough to assume that there is no proper sub-$\mathcal{O}_{X\times C}$-module of $\mathcal{F}$ which both φ and ι factor through. Such a loosening may be compared to abelian t-modules' generalizing Drinfeld modules ([An]). This means that our method should be applicable to families of abelian t-modules, though in this paper we restrict ourselves to Drinfeld modules.) We say that α is the *zero* of the F-sheaf $\mathcal{E}$, and β is its *pole*. Drinfeld modules over an A-scheme X are in one-to-one correspondence with a certain class of right F-sheaves

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on X ([Dr], §1). To construct the corresponding right F-sheaf from a Drinfeld module, one starts with the $\mathcal{E}$ of (1.1) to be fit into the diagram (1.4).

The map ι is an isomorphism on $(X \times C) - \Gamma_\beta$, and we may think $\mathcal{F} = \mathcal{E}$ here by means of ι. For example, if $\infty := \mathrm{Im}(\beta)$ is a closed point on C, then $(X \times C) - \Gamma_\beta = X \times \mathrm{Spec}\ A$. If this is the case and if the F-sheaf comes from a Drinfeld A-module, then the restriction to $X \times \mathrm{Spec}\ A$ of the left half of the diagram (1.4) yields a φ-sheaf on X over A. Thus if we are given a Drinfeld module, then we have a φ-sheaf by way of F-sheaves. The two constructions of φ-sheaves from Drinfeld modules coincide – one by way of (1.1) and the other by way of F-sheaves (in fact, they coincide by construction).

2. L-functions of φ-sheaves

Let $\mathcal{E}$ be a φ-sheaf over the coefficient ring $\mathcal{A}$ (resp. π-adic φ-sheaf) on a scheme X of finite type over $\mathbb{F}_q$. For the sake of brevity, we write $\mathcal{A}$ in this section for either the general $\mathcal{A}$ or A_π. For a closed point x on X of degree d, the stalk $\mathcal{E}_x$ of $\mathcal{E}$ at x is a φ-sheaf on $x = \mathrm{Spec}\ \mathbb{F}_{q^d}$ over $\mathcal{A}$, that is, a projective $\mathbb{F}_{q^d} \otimes \mathcal{A}$-module of finite rank together with a Frobenius semi-linear map φ_x. The d-th iterate φ_x^d is then $\mathbb{F}_{q^d} \otimes \mathcal{A}$-linear. Recall that the characteristic polynomial $\det(I - T^d \varphi_x^d | \mathcal{E}_x)$ of φ_x^d on $\mathcal{E}_x$ is well defined. (To recall the definition of the characteristic polynomial of an endomorphism of a projective module, let A be a commutative ring, M a projective A-module of rank r, and ϕ an endomorphism of M. Write $M[T]$ for $A[T] \otimes_A M$, where $A[T]$ is the polynomial ring in one variable over A. Then $1 - T\phi$ is an endomorphism of the projective $A[T]$-module $M[T]$ of rank r, and it induces an endomorphism of the projective $A[T]$-module $\wedge^r_{A[T]} M[T]$ of rank one. Since $\mathrm{End}_{A[T]}(\wedge^r_{A[T]} M[T])$ is canonically isomorphic to $A[T]$, it determines an element of $A[T]$, called the (inverse) characteristic polynomial of ϕ on M.) It is a polynomial in T with coefficients in $\mathcal{A}$, whose constant term is 1. In terms of matrices, the characteristic polynomial is written as $\det(I - T^d B(x)^{\sigma^{d-1}} \cdots B(x)^\sigma B(x))$, where $B(x)$ is the matrix, with respect to some basis, of the map induced by φ_x on a suitable free module which is a localization of $\mathcal{E}_x$, and $B(x)^{\sigma^i}$ means the entry-wise application of $\sigma = (q\text{-th power Frobenius}) \otimes \mathrm{id}$ on $\mathbb{F}_{q^d} \otimes \mathcal{A}$. In the π-adic case, the characteristic polynomial depends only on its isogeny class of $\mathcal{E}$, since an isogeny changes φ_x^d only by a conjugation. Following Grothendieck [Gr] and Katz [K1], we define

$$L(\mathcal{E}/X, T) \ := \ \prod_{x \in X_0} \det(I - T^d \varphi_x^d | \mathcal{E}_x)^{-1}. \tag{2.1}$$

This is a power series in T with coefficients in $\mathcal{A}$. We will be concerned with the rationality of this L-function.

Suppose now that we are in the π-adic case. Let $\mathbb{C}_\pi$ be the π-adic completion of an algebraic closure of the fraction field of A_π. We consider $L(\mathcal{E}/X, T)$ as a function on $\mathbb{C}_\pi$. A formal power series $f(T) \in \mathbb{C}_\pi[\![T]\!]$ is said to be *entire* if it is convergent for all $T \in \mathbb{C}_\pi$. A formal power series $f(T) \in \mathbb{C}_\pi[\![T]\!]$ is said to be *meromorphic* if it can be written as the quotient of two entire functions. The L-function $L(\mathcal{E}/X, T)$ is in general not meromorphic for an arbitrary $\mathcal{E}$. How far $L(\mathcal{E}/X, T)$ can be meromorphically extended depends on a certain growth condition on $\mathcal{E}$ which we explain in the next section.

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The following trick will be convenient and frequently used in our proofs to reduce the situation to a simpler one.

Trick (2.2). Take any finite affine open covering (U_i) of X. Then $L(\mathcal{E}/X,T)$ is written as an alternating product of L-functions of the restrictions of $\mathcal{E}$ to $U_i \times \mathrm{Spec}\mathcal{A}$ and to their intersections. Thus we are reduced to the affine case; $X = \mathrm{Spec}R$. Then $\mathcal{E}$ is identifed with a projective $(R\otimes\mathcal{A})$-module. We can find a free $(R\otimes\mathcal{A})$-module $\mathcal{F}$ of finite rank into which $\mathcal{E}$ injects as a direct factor; $\mathcal{F} = \mathcal{E}\oplus\mathcal{G}$. Extending the Frobenius to $\varphi_{\mathcal{F}} := \varphi\oplus 0$ on $\mathcal{F}$, we obtain a φ-sheaf $\mathcal{F}$ which is a free $(\mathcal{O}_X\otimes\mathcal{A})$-module and whose L-function coincides with that of $\mathcal{E}$. Thus our proofs of rationality and meromorphy of L-functions reduce to the case where the base scheme X is affine and the φ-sheaf is free as an $\mathcal{O}_X\otimes\mathcal{A}$-module.

When we want to prove the entireness of the L-function assuming that the base scheme is affine, the trick "$\varphi\oplus 0$" again allows us to make an additional assumption that the φ-sheaf is free over $\mathcal{O}_X\otimes\mathcal{A}$.

3. Overconvergent and α log-convergent φ-sheaves

Let R be an $\mathbb{F}_q$-algebra of finite type. Put $\mathcal{R}_\pi := R\otimes A_\pi$ and $\mathcal{R}_\pi^{\widehat{}} := R\widehat{\otimes}A_\pi$, the completion of $\mathcal{R}_\pi$ with respect to the π-adic topology. Choose a finite set of generators $\{x_1,\cdots,x_n\}$ of R as an $\mathbb{F}_q$-algebra. Any element of $\mathcal{R}_\pi^{\widehat{}}$ can be written (non-uniquely) in the form

$$(3.1)\qquad \sum_k c_k x^k \qquad \text{with}\ \ c_k\in A_\pi\ \ \text{and}\ \ c_k\to 0\ (|k|\to\infty).$$

Here $k=(k_1,\cdots,k_n)$ is a multi-index with $k_i\ge 0$; we put $x^k = x_1^{k_1}\cdots x_n^{k_n}$ and $|k| = k_1+\cdots+k_n$.

Definition. (1) An element of $\mathcal{R}_\pi^{\widehat{}}$ is said to be *overconvergent* if there exists an expression as above in which

$$(3.2)\qquad \liminf_{|k|\to\infty}\frac{\mathrm{ord}_\pi c_k}{|k|} \ >\ 0.$$

(2) Let α be a non-negative real number. An element of $\mathcal{R}_\pi^{\widehat{}}$ is said to be α *log-convergent* if there exists an expression as above in which

$$(3.3)\qquad \liminf_{|k|\to\infty}\frac{\mathrm{ord}_\pi c_k}{\log_q|k|}\ \ge\ \alpha.$$

An element is said to be ∞ log-*convergent* if it is α log-convergent for all $\alpha\ge 0$.

These definitions do not depend on the choice of the generators x_i because, if $\{y_1,\cdots,y_m\}$ is another set of generators, each x_i can be expressed as a polynomial of the y_j's. We denote by $\mathcal{R}_\pi^\dagger$ (resp. $\mathcal{R}_\pi^\alpha$) the subset of $\mathcal{R}_\pi^{\widehat{}}$ consisting of overconvergent (resp. α log-convergent) elements. These subsets are in fact subrings of $\mathcal{R}_\pi^{\widehat{}}$. Fulton [Fu] showed that $\mathcal{R}_\pi^\dagger$ is noetherian. More generally, it is shown in [W2] that $\mathcal{R}_\pi^\alpha$ is noetherian for any α. Note that algebraicity (i.e., being in $\mathcal{R}_\pi$) implies overconvergence, overconvergence implies α log-convergence for any α and, if $\alpha\ge\beta$, α log-convergence implies β log-convergence. In other words, we have the inclusions

$$(3.4)\qquad \mathcal{R}_\pi\ \subset\ \mathcal{R}_\pi^\dagger\ \subset\ \mathcal{R}_\pi^\alpha\ \subset\ \mathcal{R}_\pi^\beta\ \subset\ \mathcal{R}_\pi^{\widehat{}}\qquad (\alpha\ge\beta).$$

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If S is another $\mathbb{F}_q$-algebra of finite type which is a localization of R, and $\mathcal{S}_\pi^\dagger$ (resp. $\mathcal{S}_\pi^\alpha$) is the overconvergent (resp. α log-convergent) subring of $\mathcal{S}_{\widehat{\pi}} = S\widehat{\otimes}A_\pi$, then we have $\mathcal{R}_\pi^\dagger \subset \mathcal{S}_\pi^\dagger$ (resp. $\mathcal{R}_\pi^\alpha \subset \mathcal{S}_\pi^\alpha$).

Let X be a scheme of finite type over $\mathbb{F}_q$. Let $\mathcal{O}_{\mathcal{X}_\pi}^\dagger$ (resp. $\mathcal{O}_{\mathcal{X}_\pi}^\alpha$) be the sheafification of the presheaf $U \mapsto (\mathcal{O}_{\mathcal{X}_{\widehat{\pi}}}(U))^\dagger = (\mathcal{O}_X(U)\widehat{\otimes}A_\pi)^\dagger$ (resp. the presheaf $U \mapsto (\mathcal{O}_{\mathcal{X}_{\widehat{\pi}}}(U))^\alpha = (\mathcal{O}_X(U)\widehat{\otimes}A_\pi)^\alpha$). These are subsheaves of $\mathcal{O}_{\mathcal{X}_{\widehat{\pi}}}$.

Definition. A π-adic φ-sheaf $(\mathcal{E}, \varphi)$ on X is said to be *overconvergent* (resp. α log-*convergent*) if it can be defined over $\mathcal{O}_{\mathcal{X}_\pi}^\dagger$ (resp. $\mathcal{O}_{\mathcal{X}_\pi}^\alpha$) (i.e., if there exists a pair of an $\mathcal{O}_{\mathcal{X}_\pi}^\dagger$-module (resp. $\mathcal{O}_{\mathcal{X}_\pi}^\alpha$-module) and a Frobenius semi-linear map on it which yields $(\mathcal{E}, \varphi)$ by extension of scalars to $\mathcal{O}_{\mathcal{X}_{\widehat{\pi}}}$). A π-adic φ-sheaf $\mathcal{E}$ is said to be ∞ log-*convergent* if it is α log-convergent for all $\alpha \geq 0$.

In down-to-earth terms, this means that there exists an affine open covering $\{U_i\}$ of X such that, for each i, $\mathcal{E}|_{\widehat{\mathcal{U}_{i,\pi}}}$ (where $\widehat{\mathcal{U}_{i,\pi}} = U_i\widehat{\otimes}A_\pi$) is free and the map $\varphi|_{\widehat{\mathcal{U}_{i,\pi}}}$ on it is represented, with respect to some basis of $\mathcal{E}|_{\widehat{\mathcal{U}_{i,\pi}}}$, by a matrix whose entries (*a priori* in $\mathcal{O}_{\widehat{\mathcal{U}_{i,\pi}}}$) are all overconvergent (resp. α log-convergent, resp. ∞ log-convergent). In particular, algebraic φ-sheaves are overconvergent. Clearly we have

Lemma 3.1. *The category of algebraic φ-sheaves (resp. overconvergent φ-sheaves, resp. α log-convergent φ-sheaves) is stable under direct sum and tensor product.*

4. Meromorphic continuation and rationality of L-functions

Let X be a scheme of finite type over $\mathbb{F}_q$. Our first main theorem is:

Theorem 4.1. (i) *Let $\mathcal{E}$ be a π-adic φ-sheaf on X. If $\mathcal{E}$ is α log-convergent, then the L-function $L(\mathcal{E}/X, T)$ is π-adically meromorphic in the open disk $\mathrm{ord}_\pi(T) > -\alpha$ of $\mathbb{C}_\pi$.*

(ii) *Let $\mathcal{A}$ be a commutative ring containing $\mathbb{F}_q$, and $\mathcal{E}$ a φ-sheaf on X over $\mathcal{A}$. Then the L-function $L(\mathcal{E}/X, T)$ is a rational function in T. In particular, the L-function of an algebraic π-adic φ-sheaf is rational.*

Corollary 4.2. *If $\mathcal{E}$ is an overconvergent or ∞ log-convergent π-adic φ-sheaf on X, then $L(\mathcal{E}/X, T)$ is π-adically meromorphic on the whole plane $\mathbb{C}_\pi$.*

Remark. This theorem is the function field analogue of a p-adic result in [DS] and [W1], except that here we work in a more general setting. In the function field case, we show that the case of general X can be reduced to the affine space case as studied in [W1]. We note that Theorem 4.1 is in general best possible as the counterexamples in [W1] show.

We now prove Theorem 4.1. For the brevity of exposition, we pretend as if $\mathcal{A}$ in (ii) were A_π since the proof for a general $\mathcal{A}$ is the same; then the two statements in the Theorem will be proved almost at the same time (for a φ-sheaf over a general $\mathcal{A}$, one should replace the infinite sum (4.1) below by a finite sum).

Since α log-convergence and algebraicity for $\mathcal{E}$ are local properties, we may assume (cf. Trick (2.2)) that X is affine, say $X = \mathrm{Spec}\ R$, and $\mathcal{E}$ is a free $\mathcal{R}_{\widehat{\pi}}$-module of rank r (where $\mathcal{R}_{\widehat{\pi}} = R\widehat{\otimes}A_\pi$). Write $R = \mathbb{F}_q[X_1, \cdots, X_n]/I$ with an ideal I of the polynomial ring. The map φ on $\mathcal{E}$ is represented, with respect to some basis, by an $r \times r$ matrix B with entries in $\mathcal{R}_\pi^\alpha$. The matrix B can be written as a power series in

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$X = (X_1, \cdots, X_n)$ modulo I (the context will distinguish the two uses of the same letter X, one for the scheme X and the other for the variable $X = (X_1, \cdots, X_n)$) with coefficients in $\mathrm{M}_{r\times r}(A_\pi)$. Namely,

$$B(X) = \sum_{k\in\mathbb{N}^n} b_k X^k \qquad \text{with } \ b_k \in \mathrm{M}_{r\times r}(A_\pi) \ \text{ and} \tag{4.1}$$

$$\liminf_{|k|\to\infty} \frac{\mathrm{ord}_\pi(b_k)}{\log_q |k|} \ \geq \ \alpha,$$

where $\mathrm{ord}_\pi(b_k)$ is defined to be the minimum of ord_π of the entries in b_k. Choosing such an expression of B in terms of $X = (X_i)$, we regard $\mathcal{E}$ as lifted to $\mathbb{A}^n$. Let $\{f_1, \cdots, f_h\}$ be a set of generators of I, i.e., a system of defining equations of the scheme X in the affine space $\mathbb{A}^n$. For a subset $S \subset \{1, \cdots, h\}$, we define

$$B_S(X) = \ B \cdot \prod_{i\in S} f_i(X)^{q-1} \tag{4.2}$$

(so $B_\varnothing = B$). Each $B_S(X)$ defines a π-adic φ-sheaf $\mathcal{E}_S$ on $\mathbb{A}^n$. All of them can be taken to be polynomials in X with coefficients in $\mathrm{M}_{r\times r}(A_\pi)$ if $\mathcal{E}$ is algebraic because a regular function on $\mathbb{A}^n$ is a polynomial in X.

For any point $x \in \mathbb{A}^n$, let $B_S(x)$ denote the value of B_S at x. Then we will show that $L(\mathcal{E}/X, T) = \prod_S L(\mathcal{E}_S/\mathbb{A}^n, T)^{(-1)^{|S|}}$, or, more precisely, that

$$\prod_{x\in X_0} \frac{1}{\det(I - T^{\deg(x)} B(x)^{\sigma^{\deg(x)-1}} \cdots B(x)^\sigma B(x))} \tag{4.3}$$

$$= \prod_{x\in(\mathbb{A}^n)_0} \prod_{S\subseteq\{1,\cdots,h\}} \left(\frac{1}{\det(I - T^{\deg(x)} B_S(x)^{\sigma^{\deg(x)-1}} \cdots B_S(x)^\sigma B_S(x))} \right)^{(-1)^{|S|}}.$$

This equality is essentially the multiplicative form of the inclusion-exclusion principle. Indeed, if the closed point $x \in (\mathbb{A}^n)_0$ is on the subscheme X, then $f_i(x) = 0$ for all $1 \leq i \leq h$. Thus, we deduce that $B_S(x) = 0$ for each non-empty S and $B_\varnothing(x) = B(x)$. In this case, the product over S in (4.3) is equal to the expected Euler factor at x. If the closed point x is not on X, we can assume that the first k $(1 \leq k < h)$ polynomials $f_1(X), \cdots, f_k(X)$ do not vanish at x and the last $h - k$ polynomials $f_{k+1}(X), \cdots, f_h(X)$ vanish at x. In this case, the matrix $B_S(x)$ is zero if S has a non-empty intersection with $\{k+1, \cdots, h\}$. Thus, in the product over S we can assume that S is a subset of $\{1, \cdots, k\}$. Under this assumption, we have that

$$\begin{aligned} &B_S(x)^{\sigma^{\deg(x)-1}} \cdots B_S(x)^\sigma B_S(x) \\ &= B(x)^{\sigma^{\deg(x)-1}} \cdots B(x)^\sigma B(x) \prod_{i\in S} f_i(x)^{q^{\deg(x)}-1} \\ &= B(x)^{\sigma^{\deg(x)-1}} \cdots B(x)^\sigma B(x). \end{aligned}$$

The multiplicity of the factor $1/\det(I - T^{\deg(x)} B(x)^{\sigma^{\deg(x)-1}} \cdots B(x)^\sigma B(x))$ occurring in the product over S of (4.3) is given by

$$\sum_{S\subseteq\{1,\cdots,k\}} (-1)^{|S|} \ = \ (1-1)^k \ = \ 0.$$

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We have thus shown that

$$L(\mathcal{E}/X,T) \;=\; \prod_{S\subseteq\{1,\cdots,h\}} L(\mathcal{E}_S/\mathbb{A}^n,T)^{(-1)^{|S|}}. \tag{4.4}$$

This reduces us to the affine space case, viz., the base scheme X is the affine space $\mathbb{A}^n$, the π-adic φ-sheaf $\mathcal{E}$ is free of rank r and the matrix $B(X)$ which represents the Frobenius φ on $\mathcal{E}$ is an α log-convergent power series in X (resp. a polynomial in X if $\mathcal{E}$ is algebraic) with coefficients in $\mathrm{M}_{r\times r}(A_\pi)$. We can now quote the following result [W1, Theorems 7.1 and 7.3] to conclude the proof of Theorem 4.1.

Lemma 4.3. (i) *If $B(X)$ is an α log-convergent power series with coefficients in $\mathrm{M}_{r\times r}(A_\pi)$, then the L-function $L(B/\mathbb{A}^n,T)^{(-1)^{n-1}}$ is π-adically analytic in the open disk* $\mathrm{ord}_\pi(T) > -\alpha$.

(ii) *If $B(X)$ is a polynomial in X with coefficients in $\mathrm{M}_{r\times r}(\mathcal{A})$ of degree d, then the L-function $L(B/\mathbb{A}^n,T)^{(-1)^{n-1}}$ is a polynomial in T whose degree is at most*

$$r\binom{[d/(q-1)]}{n}.$$

Note that the conclusion in Lemma 4.3 (i) is stronger than the conclusion in Theorem 4.1 (i). It says that the L-function $L(B/\mathbb{A}^n,T)^{(-1)^{n-1}}$ is not only meromorphic but also *analytic* in the disk $\mathrm{ord}_\pi(T) > -\alpha$. The question of entireness can be viewed as an analogue of the classical Artin conjecture over number fields. In our characteristic p function field case, it is unreasonable to expect that the L-function $L(\mathcal{E}/X,T)$ (or its reciprocal) should be entire for a general X. For example, when X is an ordinary elliptic curve over $\mathbb{F}_q$ and $\mathcal{E}$ is trivial, then $L(\mathcal{E}/X,T)$ is just the reduction modulo p of the classical zeta function of $X/\mathbb{F}_q$. Since X is ordinary, neither $L(\mathcal{E}/X,T)$ nor its reciprocal is a polynomial. However, if X is *affine*, we have the following conjecture:

Conjecture 4.4. Let X be an affine scheme of finite type over $\mathbb{F}_q$ and of equidimension n. Suppose further that X is a complete intersection in some smooth affine Y of finite type over $\mathbb{F}_q$. Then the L-function $L(\mathcal{E}/X,T)^{(-1)^{n-1}}$ is π-adically analytic on the open disk $\mathrm{ord}_\pi(T) > -\alpha$ if $\mathcal{E}$ is α log-convergent.

Perhaps X could be permitted to have less mild singularities than the one in the above Conjecture. To deal with such a general base scheme, one would need to develop a suitable cohomology theory. For the case where the φ-sheaf $\mathcal{E}$ is trivial (so that the L-function is nothing but the classical congruence zeta function modulo p), see [K3] and [De].

As a first evidence for Conjecture 4.4, we have the following:

Proposition 4.5. *Let X be the complement in $\mathbb{A}^n$ of a hypersurface defined by a polynomial $f(X)$. If $\mathcal{E}$ is an algebraic φ-sheaf on $X/\mathbb{F}_q$, then the L-function $L(\mathcal{E}/X,T)^{(-1)^{n-1}}$ is a polynomial in T.*

Proof. By Trick (2.2), we may assume $\mathcal{E}$ is free. The Frobenius matrix B of $\mathcal{E}$ is a polynomial in X and $1/f$. One can choose a sufficiently large positive integer k such that $Bf^{k(q-1)}$ is a polynomial in X. Now, the twisted Frobenius matrix $Bf^{k(q-1)}$ defines a free algebraic φ-sheaf on the total affine space $\mathbb{A}^n$. It is easy to check from the Euler product definition of L-functions that $L(B/X,T) = L(Bf^{k(q-1)}/\mathbb{A}^n,T)$. We can now apply Lemma 4.3 to conclude the proof. ∎

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5. Uniform variation of L-functions

For applications to global L-functions of Drinfeld modules, we need to consider the variation of a family of π-adic φ-sheaves and their L-functions parametrized by the p-adic integers $y \in \mathbb{Z}_p$.

For a family $(\mathcal{E}(y))_{y\in\mathbb{Z}_p}$ of π-adic φ-sheaves $\mathcal{E}(y)$ on X parametrized by $y \in \mathbb{Z}_p$, we require in the following that there exists an affine open covering $\{U_i\}$ of X such that, for each i, the restriction $\mathcal{E}(y)|_{\widehat{\mathcal{U}_{i,\pi}}}$ (where $\widehat{\mathcal{U}_{i,\pi}} = U_i \widehat{\otimes} A_\pi$) is free and the map $\varphi(y)|_{\widehat{\mathcal{U}_{i,\pi}}}$ on it is represented, with respect to some basis of $\mathcal{E}(y)|_{\widehat{\mathcal{U}_{i,\pi}}}$ which is independent of y, by a matrix of the form

$$B(X, y) \;=\; \sum_{u\in\mathbb{N}^n} b_u(y) X^u, \tag{5.1}$$

where $X = (X_1, \cdots, X_n)$ is a set of generators of the affine algebra of U_i over $\mathbb{F}_q$, and $b_u(y) \in \mathrm{M}_{r\times r}(A_\pi)$ for each y and $b_u(y)$ approaches zero uniformly in y as $|u|$ approaches infinity.

We shall say "the family $\mathcal{E}(y)$", simplifying the notation.

Definition. The family $\mathcal{E}(y)$ is said to be *uniformly overconvergent* if there exists, for all i, an expression of the Frobenius matrix as in (5.1) for which we have

$$\liminf_{|u|\to\infty} \frac{\inf_{y\in\mathbb{Z}_p} \mathrm{ord}_\pi b_u(y)}{|u|} \;>\; 0. \tag{5.2}$$

The family $\mathcal{E}(y)$ is said to be *uniformly* $\alpha\log$-*convergent* if there exists, for all i, an expression of the Frobenius matrix as in (5.1) for which we have

$$\liminf_{|u|\to\infty} \frac{\inf_{y\in\mathbb{Z}_p} \mathrm{ord}_\pi b_u(y)}{\log_q |u|} \;\geq\; \alpha. \tag{5.3}$$

The family $\mathcal{E}(y)$ is said to be *continuous* in y if the Frobenius matrix $B(X, y)$ is continuous in y, i.e., if each coefficient $b_u(y)$ is a continuous function from $\mathbb{Z}_p$ to $\mathrm{M}_{r\times r}(A_\pi)$. This continuity is automatically uniform in u since $b_u(y)$ goes to zero uniformly in y as $|u|$ goes to infinity.

As an analogue of Lemma 3.1, one sees immediately that the following holds.

Lemma 5.1. *The category of uniformly overconvergent (resp. uniformly $\alpha\log$-convergent, resp. continuous) families of π-adic φ-sheaves parametrized by $y \in \mathbb{Z}_p$ is stable under direct sum and tensor product.*

Given a family of π-adic φ-sheaves $\mathcal{E}(y)$ on X, the L-function $L(\mathcal{E}(y)/X, T)$ is well defined for each y. If this family $\mathcal{E}(y)$ is uniformly $\alpha\log$-convergent, then Theorem 4.1 shows that each L-function $L(\mathcal{E}(y)/X, T)$ is meromorphic in the disk $\mathrm{ord}_\pi(T) > -\alpha$ for each fixed y. Next we will show that $L(\mathcal{E}(y)/X, T)$ varies "nicely" if $\mathcal{E}(y)$ varies "nicely".

Let D be a subset of $\mathbb{C}_\pi$ (e.g., an open or closed disk of finite radius). A formal power series $f(T) = \sum_{n>-\infty} f_n T^n \in \mathbb{C}_\pi(\!(T)\!)$ is said to be (π-adically) analytic on D if it converges at every point in D. Let $f(T, y)$ be a family of formal power series $\sum_{n>-\infty} f_n(y) T^n$, $f_n(y) \in \mathbb{C}_\pi$, parametrized by $y \in \mathbb{Z}_p$. We say $f(T, y)$ is a family of analytic functions on D if, for each $y \in \mathbb{Z}_p$, the power series $f(T, y)$ is analytic on

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D. We say $f(T,y)$ is a family of meromorphic functions on D if it can be written as a quotient of two families of analytic functions on D.

Definition. Let β be a real number. Let $f(T,y)=\sum_{n>-\infty} f_n(y)T^n$ be a family of power series in T with coefficients in $\mathbb{C}_\pi$. We say that the family $f(T,y)$ is β-*continuous* (in y) if the following condition holds: For any $y_0\in\mathbb{Z}_p$ and $\epsilon>0$, there exists a $\delta>0$ such that

$$|f_n(y_0)-f_n(y_1)|_\pi \ < \ \epsilon q^{-\beta n}$$

for all $n>-\infty$ and $y_1\in\mathbb{Z}_p$ satisfying $|y_0-y_1|_p<\delta$.

If we define a norm $\|\cdot\|_\beta$ on the space $\mathbb{C}_\pi(\!(T)\!)$ by

$$\|\sum_{n>-\infty} f_nT^n\|_\beta \ := \ \sup_{n>-\infty}|f_n|_\pi q^{\beta n},$$

then the above condition is equivalent to saying that the map $\mathbb{Z}_p\to\mathbb{C}_\pi(\!(T)\!)$; $y\mapsto f(T,y)$ is continuous with respect to the topology on $\mathbb{C}_\pi(\!(T)\!)$ defined by this norm.

We say that the family $f(T,y)$ is ∞-*continuous* if it is β-continuous for all real numbers β.

Note that these notions are defined for families of any formal power series (not only for convergent series). In other words, they are concerned only with the "y-direction" (and not with the "T-direction").

We have the following strengthened form of Theorem 4.1.

Theorem 5.2. *Let $\mathcal{E}(y)$ be a continuous family of uniformly $\alpha\log$-convergent π-adic φ-sheaves. Then for any given $\epsilon>0$, the family of L-functions $L(\mathcal{E}(y)/X,T)$ is an $(\alpha-\epsilon)$-continuous family of meromorphic functions on the closed disk* $\operatorname{ord}_\pi(T)\geq-(\alpha-\epsilon)$.

The same reduction steps as in the proof of Theorem 4.1 reduce Theorem 5.2 to the following affine space case proved in [W1, Theorem 7.4].

Lemma 5.3. *Let*

$$B(X,y) \ = \ \sum_{u\in\mathbb{N}^n} b_u(y)X^u,$$

where $b_u(y)\in \mathrm{M}_{r\times r}(A_\pi)$ for each y. Assume that $B(X,y)$ is a continuous family of uniformly $\alpha\log$-convergent matrices over $X=\mathbb{A}^n$. Then the L-function $L(B(X,y)/\mathbb{A}^n,T)^{(-1)^{n-1}}$ is π-adically analytic in the open disk $\operatorname{ord}_\pi(T)>-\alpha$ for each y. Furthermore, for any given $\epsilon>0$, the family of L-functions $L(B(X,y)/\mathbb{A}^n,T)^{(-1)^{n-1}}$ is an $(\alpha-\epsilon)$-continuous family of analytic functions on the closed disk $\operatorname{ord}_\pi(T)\geq-(\alpha-\epsilon)$.

Note. If $\mathcal{E}(y)$ is a continuous family of uniformly $\infty\log$-convergent π-adic φ-sheaves, the theorem follows from the mod p reduction of the Dwork trace formula ([W1, §7]) together with standard properties of the Fredholm determinants of completely continuous operators (cf. [S2, Proposition 8]).

6. Smooth φ-sheaves and π-adic representations

As always, we assume that the base scheme X is of finite type over $\mathbb{F}_q$. In addition to that, we assume in this section that X is connected (note that the study, as in this paper, of L-functions for a general scheme X is reduced to the

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case where X is connected, so we do not need the assumption of connectivity in the following sections). We recall the functorial equivalence between smooth π-adic φ-sheaves and continuous π-adic representations of the arithmetic fundamental group $\pi_1(X)$ (cf. [Dr], [Ta]). The situation is exactly parallel to its p-adic counterpart, where one has a functorial equivalence between unit root F-crystals and continuous p-adic representations of the arithmetic fundamental group (see [K2, 4.1]).

A *finite φ-sheaf* on X is a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank together with an $\mathcal{O}_X$-module homomorphism $\varphi : \mathrm{Fr}_X^*\mathcal{E} \to \mathcal{E}$ (cf. [Dr, §2], where it was called a φ-sheaf). It is said to be *étale* if φ is an isomorphism. Note that a π-adic φ-sheaf of rank r can be regarded as a projective system $(\mathcal{E}_n, p_n)_{n\geq 0}$ of finite φ-sheaves in which each $\mathcal{E}_n$ is equipped with an A_π-action and is locally free of rank r as an $\mathcal{O}_X \otimes A_\pi/(\pi^n)$-module and the transition map $p_n : \mathcal{E}_{n+1} \to \mathcal{E}_n$ identifies $\mathcal{E}_n$ with $\mathrm{Coker}(\pi^n : \mathcal{E}_{n+1} \to \mathcal{E}_{n+1})$ (here we identify π with a uniformizer of A_π).

For a finite φ-sheaf $\mathcal{E}$, put

$$\mathrm{Gr}(\mathcal{E}) \ := \ \underline{\mathrm{Spec}}\left(\mathrm{Sym}^{\cdot}_{\mathcal{O}_X}\mathcal{E}/\mathcal{I}\right), \tag{6.1}$$

where $\mathcal{I}$ is the ideal of $\mathrm{Sym}^{\cdot}_{\mathcal{O}_X}\mathcal{E}$ generated by the local sections $(\mathrm{frob} - \varphi)(x)$, $x \in \mathcal{E} \subset \mathrm{Sym}^{\cdot}_{\mathcal{O}_X}\mathcal{E}$, and frob is the q-th power Frobenius map $\mathrm{Fr}_X^*\mathrm{Sym}^{\cdot}_{\mathcal{O}_X}\mathcal{E} \to \mathrm{Sym}^{\cdot}_{\mathcal{O}_X}\mathcal{E}$ relative to $\mathcal{O}_X$. Then $\mathrm{Gr}(\mathcal{E})$ is a finite subgroup scheme over X of the vector group $\underline{\mathrm{Spec}}\ \mathrm{Sym}^{\cdot}_{\mathcal{O}_X}\mathcal{E}$ with a natural $\mathbb{F}_q$-action induced by the $\mathbb{F}_q$-module structure of $\mathcal{E}$. In the terminology of [Ta], $\mathrm{Gr}(\mathcal{E})$ is a finite $\mathbb{F}_q$-module scheme. (By an $\mathbb{F}_q$-module scheme, we mean a commutative group scheme G together with an $\mathbb{F}_q$-action $\alpha : \mathbb{F}_q \to \mathrm{End}(G)$ such that the action of $\mathbb{F}_q$ on the co-Lie module $\mathrm{Lie}^*(G)$ induced by α coincides with the action of $\mathbb{F}_q$ as a subring of $\mathcal{O}_X$.)

If the map φ of a finite φ-sheaf is given, locally on X, by a matrix $B \in \mathrm{M}_{r\times r}(\mathcal{O}_X)$ with respect to an $\mathcal{O}_X$-basis $(x_i)_{1\leq i\leq r}$ of $\mathcal{E}$, then

$$\mathrm{Gr}(\mathcal{E}) \ = \ \underline{\mathrm{Spec}}\ \mathcal{O}_X[x]/(x^{(q)} - Bx),$$

where $x = {}^t(x_1, \cdots, x_r)$ and $x^{(q)} = {}^t(x_1^q, \cdots, x_r^q)$. This correspondence $\mathcal{E} \mapsto \mathrm{Gr}(\mathcal{E})$ is a contra-variant functor and gives an anti-equivalence of the category of finite φ-sheaves on X to the category of finite locally free $\mathbb{F}_q$-module schemes, which preserves étale objects ([Dr, Proposition 2.1], [Ta, §1]). The opposite direction of the correspondence is obtained, for a finite $\mathbb{F}_q$-module scheme G over X, by taking the Zariski sheaf

$$\mathcal{E}(G) \ := \ \underline{\mathrm{Hom}}_{\mathbb{F}_q\text{-mod}}(G, \mathbb{G}_a),$$

with φ induced by the Frobenius endomorphism of G. It is a subsheaf of the structure sheaf of G.

Example. Let $A = \mathbb{F}_q[t]$. We consider the affine case $X := \mathrm{Spec}\ R$ with R an A-algebra, so that sheaves of $\mathcal{O}_X$-modules can be regarded as R-modules. Let Φ be a Drinfeld A-module over X of rank r defined, as in the Example of §1, by

$$\Phi_t \ = \ \theta + x_1\sigma + \cdots + x_r\sigma^r,$$

with $x_i \in R$ and $x_r \in R^\times$. Here σ is the q-th power Frobenius endomorphism $Z \mapsto Z^q$ on $\mathbb{G}_a = \mathrm{Spec} R[Z]$ relative to R. For $a \in A - \{0\}$, the group scheme ${}_a\Phi$ of a-division points of Φ is defined by the equation

$$\Phi_a(Z) \ = \ 0$$

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in the affine line $\mathbb{G}_a$ over R. Thus

$$ {}_a\Phi \; = \; \text{Spec } R[Z]/(\Phi_a(Z)). $$

The finite φ-sheaf corresponding to ${}_a\Phi$ is then

$$ \begin{aligned} \mathcal{E}_a \; &= \; \mathcal{E}({}_a\Phi) \; = \; \mathrm{Hom}_{\mathbb{F}_q\text{-mod},R\text{-sch}}({}_a\Phi, \mathbb{G}_a) \\ &= \bigoplus_{i=0}^{\deg(a)-1} R \cdot Z^{q^i}. \end{aligned} $$

It is the "essential part" of the structure sheaf

$$ \begin{aligned} R[Z]/(\Phi_a(Z)) \; &= \; \mathrm{Hom}_{R\text{-sch}}({}_a\Phi, \mathbb{A}^1) \\ &= \bigoplus_{j=0}^{q^{\deg(a)}-1} R \cdot Z^j. \end{aligned} $$

It coincides with the cokernel of multiplication by $1 \otimes a \in R \otimes A$ on the φ-sheaf over A

$$ \begin{aligned} \mathcal{E} \; &= \; \mathrm{Hom}_{\mathbb{F}_q\text{-mod},R\text{-sch}}(\Phi, \mathbb{G}_a) \\ &= \bigoplus_{i \geq 0} R \cdot Z^{q^i} \end{aligned} $$

corresponding to Φ, which was described in the Example of §1.

Clearly the construction $\mathcal{E} \mapsto \mathrm{Gr}(\mathcal{E})$ commutes with any base change $Y \to X$. As is seen from (6.1), for any X-scheme Y, the Y-valued points of $G = \mathrm{Gr}(\mathcal{E})$ are

$$ \begin{aligned} G(Y) \; &= \; \mathrm{Hom}_{\mathcal{O}_X[\varphi]}(\mathcal{E}, \mathcal{O}_Y) \\ &:= \; \{f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_Y);\; f(\varphi(x)) = f(x)^q \\ &\qquad\qquad \text{for all local sections } x \text{ of } \mathcal{E}\}. \end{aligned} \tag{6.2} $$

This is an $\mathbb{F}_q$-vector space, which is of finite dimension if, e.g., Y/X is of finite type or the finite φ-sheaf $\mathcal{E}$ is étale. If Y/X is étale, $\pi_1(X)$ acts continuously on $G(Y)$. Thus the action of Frobenius on $G(Y)$ is described by φ. For π-adic φ-sheaves over a field, this will be made precise after the following proposition.

Proposition 6.1. (1) *The category $\Phi\mathrm{M}_X^{\text{ét}} \otimes A_\pi$ of finite étale φ-sheaves on X equipped with A_π-action is anti-equivalent to the category $\mathrm{Rep}_{A_\pi}(\pi_1(X))$ of A_π-modules of finite length with continuous $\pi_1(X)$-action.*

(2) *The category of smooth π-adic φ-sheaves on X is anti-equivalent to the category of free A_π-modules of finite rank with continuous $\pi_1(X)$-action.*

Proof. By the above results of Drinfeld, with A_π-action taken into account, $\Phi\mathrm{M}_X^{\text{ét}} \otimes A_\pi$ is anti-equivalent to the category of finite locally free étale $\mathbb{F}_q$-module schemes over X equipped with A_π-action. (1) follows then by the A_π-module version of the equivalence [SGA1, Exp.V, §7 (p.140)] of the category of étale X-schemes and the category of finite sets with continuous $\pi_1(X)$-action. (2) follows from (1) by taking the projective limit.

The Galois representation corresponding to a finite étale (resp. smooth π-adic) φ-sheaf $\mathcal{E}$ will be denoted by $V(\mathcal{E})$. Explicitly, if $\mathcal{E}$ is a smooth π-adic φ-sheaf over

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a field k containing $\mathbb{F}_q$, then the corresponding Galois representation is given as the A_π-module version of (6.2) for $Y = \operatorname{Spec} k^{\mathrm{sep}}$:

$$\begin{aligned}(6.3)\qquad V(\mathcal{E}) &= \operatorname{Hom}_{k\widehat{\otimes}A_\pi[\varphi]}(\mathcal{E}, k^{\mathrm{sep}}\widehat{\otimes}A_\pi)\\ &= \{f \in \operatorname{Hom}_{k\widehat{\otimes}A_\pi}(\mathcal{E}, k^{\mathrm{sep}}\widehat{\otimes}A_\pi);\ f(\varphi(x)) = f(x)^\sigma \quad \text{for all } x \in \mathcal{E}\},\end{aligned}$$

where σ is the endomorphism (q-th power Frobenius)$\widehat{\otimes}\mathrm{id}_{A_\pi}$ and the $\widehat{\ }$ means the π-adic completion. (Cf. Appendix to [G3]; but note that we use a contra-variant functor here, whereas a co-variant functor is used in that Appendix.) The above expression (6.3) says that, on $V(\mathcal{E})$, the map induced by the q-th power Frobenius of k^{sep} coincides with the one induced by the transpose ${}^t\varphi$ of φ. If k is a finite field $\mathbb{F}_{q^d}$, the action of the Frobenius $\mathrm{Frob}_{\mathbb{F}_{q^d}}$ is therefore given by the d-th iterate of ${}^t\varphi$:

$$(6.4)\qquad \mathrm{Frob}_{\mathbb{F}_{q^d}} = {}^t\varphi^d \qquad \text{on } V(\mathcal{E}).$$

In particular, the characteristic polynomials of $\mathrm{Frob}_{\mathbb{F}_{q^d}}$ on $V(\mathcal{E})$ and φ^d on $\mathcal{E}$ coincide:

$$(6.5)\qquad \det(1 - T^d\mathrm{Frob}_{\mathbb{F}_{q^d}}|_{V(\mathcal{E})}) = \det(1 - T^d\varphi^d|_{\mathcal{E}}).$$

If X is of finite type over $\mathbb{F}_q$, each closed point x of X has a finite residue field, so there is a unique (up to conjugation) Frobenius $\mathrm{Frob}_x \in \pi_1(X)$.

Definition. For a free A_π-module V with continuous $\pi_1(X)$-action, define

$$L(V/X, T) := \prod_{x\in X_0} \det(I - T^{\deg(x)}\mathrm{Frob}_x|V)^{-1}.$$

This is a power series in T with coefficients in A_π. Noticing (6.5), we have

Corollary 6.2. *If $\mathcal{E}$ is a smooth π-adic φ-sheaf on X and $V = V(\mathcal{E})$ is the corresponding $A_\pi[\pi_1(X)]$-module, then we have*

$$L(\mathcal{E}/X, T) = L(V/X, T).$$

For the later use, we next consider L-functions of π-adic φ-sheaves twisted by certain abelian characters. In this paper, a *character* of a group means a homomorphism into $\mathbb{C}_\pi^\times$.

Let $\chi_0 : Z_0(X) \to \mathbb{C}_\pi^\times$ be a character of the discrete group $Z_0(X)$ of zero-cycles on X. For simplicity, we assume that the image of χ_0 is contained in $A_\pi^\times$. For such a character χ_0 and a π-adic φ-sheaf $\mathcal{E}$ on X, define

$$(6.6)\qquad L(\mathcal{E}/X, \chi_0, T) := \prod_{x\in X_0} \det(1 - T^{\deg(x)}\chi_0(x)\varphi_x^{\deg(x)}|_{\mathcal{E}_x})^{-1},$$

where a closed point x of X is naturally regarded as an element of $Z_0(X)$ and $\chi_0(x)\varphi_x^{\deg(x)}$ is regarded as an endomorphism of the free $(\mathbb{F}_x \otimes A_\pi)$-module $\mathcal{E}_x$. Each Euler factor has coefficients in A_π and $L(\mathcal{E}/X, \chi_0, T)$ is a power series in T with coefficients in A_π.

Suppose now we are given a continuous character $\chi : \pi_1(X)^{\mathrm{ab}} \to A_\pi^\times$ of the abelian arithmetic fundamental group. Via the reciprocity map $\theta : Z_0(X) \to \pi_1(X)^{\mathrm{ab}}$ ($x \mapsto \mathrm{Frob}_x$), the character χ induces a character $\chi_0 := \chi\circ\theta : Z_0(X) \to A_\pi^\times$ so that we have, for each closed point x of X,

$$(6.7)\qquad \chi(\mathrm{Frob}_x) = \chi_0(x).$$

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By Proposition 6.1, the character χ corresponds to a smooth π-adic φ-sheaf $(\mathcal{F},\psi)$ of rank one on X. By (6.2), $\chi(\mathrm{Frob}_x)$, hence $\chi_0(x)$, coincides with $\psi_x^{\deg(x)}$ (here ψ_x is regarded as a Frobenius linear map; if it is regarded as an element of $(\mathbb{F}_x\otimes A_\pi)^\times$, then $\psi_x^{\deg(x)}$ must be read as $\psi_x^{\sigma^{\deg(x)-1}}\cdots\psi_x^\sigma\psi_x$). Hence twisting by the character χ_0 means nothing but tensoring with $\mathcal{F}$:

$$L(\mathcal{E}/X,\chi_0,T) \;=\; L(\mathcal{F}\otimes\mathcal{E}/X,T).$$

Thus an L-function twisted by a character χ_0 of $Z_0(X)$ can be interpreted as an L-function without twist if χ_0 extends to a continuous character of $\pi_1(X)^{\mathrm{ab}}$. An arbitrary character χ_0 may not extend continuously to $\pi_1(X)^{\mathrm{ab}}$; it does if and only if it is expressed, as above, as

$$\chi_0(x) \;=\; \psi_x^{\sigma^{\deg(x)-1}}\cdots\psi_x^\sigma\psi_x \tag{6.8}$$

by a local section ψ of $(\mathcal{O}_X\widehat{\otimes}A_\pi)^\times$ around each point $x\in X_0$. Cf. [S1, Chap.VI, §24, Th. 2].

7. Local L-functions of Drinfeld modules

In the rest of the paper, we always assume that $A=\mathbb{F}_q[t]$.

Let X be an A-scheme of finite type and let Φ be a Drinfeld A-module over X. We assume that there exists a closed point π of $\mathrm{Spec}A$ which is *not* in the image of the structure morphism $X\to\mathrm{Spec}A$. Let $T_\pi(\Phi)$ be the π-adic Tate module of Φ and let $H^1(\Phi,A_\pi)=\mathrm{Hom}_{A_\pi}(T_\pi(\Phi),A_\pi)$ be the π-adic cohomology group of Φ. This is a free A_π-module on which $\pi_1(X)$ acts continuously. Define the local L-function $L(\Phi/X,T)$ of Φ to be $L(H^1(\Phi,A_\pi),T)$ by means of Galois representation as in Section 6. It coincides with the L-function defined in [G2, 3.2.15] if X has finite characteristic over A (i.e., the image of the structure morphism is a closed point of $\mathrm{Spec}A$) which is different from π.

Given a Drinfeld module Φ on X as above, by Drinfeld ([Dr, §1]), to Φ is associated (1.4), a right F-sheaf $\mathcal{E}$ (or rather, a φ-sheaf (1.1) over A, which is the "left half" of the F-sheaf). Let $\mathcal{E}_\pi$ be its π-adic completion. This is a smooth algebraic φ-sheaf. The dual $\check{\mathcal{E}}_\pi$ of $\mathcal{E}_\pi$ is again smooth and algebraic. Its L-function $L(\check{\mathcal{E}}_\pi/X,T)$ is independent of the choice of the good prime π and has coefficients in A, because $\mathcal{E}_\pi$ and the Frobenius on it are actually defined over $\mathcal{O}_X\otimes A$. It is a rational function by Theorem 4.1, (ii).

In §2 of [Dr], it is explained that the π^n-division points ${}_{\pi^n}\Phi$ of Φ correspond to $\mathcal{E}_\pi/\pi^n\mathcal{E}_\pi$; that is, ${}_{\pi^n}\Phi\simeq\mathrm{Gr}(\mathcal{E}_\pi/\pi^n\mathcal{E}_\pi)$. Taking the projective limit of the Galois representations associated to them, we have canonically $T_\pi(\Phi)\simeq V(\mathcal{E}_\pi)$. Or, we can see this isomorphism from (6.3). Indeed, if Φ is a Drinfeld module over a field k (say, the function field of X if it is integral), we have

$$\begin{aligned}{}_{\pi^n}\Phi(k^{\mathrm{sep}}) &\simeq \mathrm{Hom}_{k\text{-alg}}(k[Z]/(\Phi_{\pi^n}(Z)),k^{\mathrm{sep}})\\ &\simeq \mathrm{Hom}_{k[\varphi]\text{-mod}}(\mathcal{E}_\pi/\pi^n\mathcal{E}_\pi,k^{\mathrm{sep}}) \qquad \text{(by (6.2)).}\end{aligned}$$

Via a residue map (cf. Section 3 of the Appendix to [G3]), the last module is isomorphic to

$$\mathrm{Hom}_{(k\otimes A/\pi^nA)[\varphi]}(\mathcal{E}_\pi/\pi^n\mathcal{E}_\pi,k^{\mathrm{sep}}\otimes A/\pi^nA) \;\simeq\; \mathrm{Hom}_{(k\widehat{\otimes}A_\pi)[\varphi]}(\mathcal{E}_\pi,k^{\mathrm{sep}}\widehat{\otimes}A_\pi/\pi^nA_\pi).$$

Taking the projective limit as $n\to\infty$, we obtain the expression in (6.3).

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By Corollary 6.2, we see that $L(\Phi/X,T) = L(\check{\mathcal{E}}_\pi/X,T)$. Now the following rationality theorem for the local L-function of Φ, which was conjectured in [G2], is clear.

Theorem 7.1. *The local L-function $L(\Phi/X,T)$ of a Drinfeld module Φ over X is a rational function in T with coefficients in A. It is independent of π as far as π is not in the image of the structure morphism $X \to \mathrm{Spec}A$.*

Remark 7.2. Let K be the fraction field of A. Suppose $\mathcal{E}$ is an algebraic φ-sheaf on X over A coming from a Drinfeld module Φ as above. Set $\mathcal{E}_K := \mathcal{E} \otimes_A K$; this is an algebraic φ-sheaf on X over K.

(1) Since the L-function is isogeny-invariant, the most intrinsic L-function is $L(\check{\mathcal{E}}_K/X,T)$; we have

$$L(\Phi/X,T) \;=\; L(\check{\mathcal{E}}_K/X,T).$$

(2) Let K_∞ be the completion of K at ∞, and set $\mathcal{E}_{K_\infty} := \mathcal{E}_K \otimes_K K_\infty$. Although $\mathcal{E}_{K_\infty}$ does not correspond to a Tate module of Φ, we have

$$L(\Phi/X,T) \;=\; L(\check{\mathcal{E}}_{K_\infty}/X,T).$$

(Note that $L(\check{\mathcal{E}}_K/X,T)$, $L(\check{\mathcal{E}}_\pi/X,T)$ and $L(\check{\mathcal{E}}_{K_\infty}/X,T)$ are defined by the "same" formula.) In the next section, we have to consider such an $\check{\mathcal{E}}_{K_\infty}$.

8. Global L-functions of Drinfeld modules

In this section, we consider the ∞-adic global L-function. Let $\pi_\infty = 1/t$ and $A_\infty = \mathbb{F}_q[\![\pi_\infty]\!]$. Let $U_\infty^{(1)}$ be the group of units of A_∞ which are congruent to 1 modulo π_∞. In this case, the Goss complex plane S_∞ is defined simply by

$$S_\infty \;:=\; \mathbb{C}_\infty^\times \times \mathbb{Z}_p$$

with the obvious topology, where $\mathbb{C}_\infty$ is the completion of an algebraic closure of the quotient field K_∞ of A_∞.

The L-functions below tend to behave better around (∞, y) (rather than $(0,\infty)$) on the complex plane S_∞. We shall look at the expansion of a function $f(z,y)$ ($(z,y) \in S_\infty$) in the variable z^{-1}. To keep the same notation as in §§4, 5 and 7, it will be convenient to write $T = z^{-1}$. Also, for a subset D of S_∞, put

$$D^* \;:=\; \{(z^{-1},y);\ (z,y) \in D\}.$$

For a function $f : D \to \mathbb{C}_\infty \cup \{\infty\}$, define a function $f^* : D^* \to \mathbb{C}_\infty \cup \{\infty\}$ by $f^*(z,y) := f(z^{-1},y)$.

For a real number α, put

$$D_\alpha \;:=\; (\text{punctured open disk } \infty > \mathrm{ord}_{\pi_\infty}(T) > -\alpha) \times \mathbb{Z}_p.$$

This is an open subset of S_∞. The following definition is essentially due to Goss.

Definition. A function $f : D \to \mathbb{C}_\infty \cup \{\infty\}$ on an open subset D of S_∞ is said to be *α-analytic* (resp. *α-meromorphic*) if the following conditions are satisfied: (0) one has $D_\alpha \subset D^*$; (1) the function f^* is expressed by a formal power series $f^*(T,y) = \sum_{n\geq 0} f_n(y)T^n$, with $f_n(y) \in \mathbb{C}_\infty$, for all $(T,y) \in D_\alpha$, and it is a family of analytic (resp. meromorphic) functions on the open disk $\mathrm{ord}_{\pi_\infty}(T) > -\alpha$ in the sense of §5; (2) for any $\epsilon > 0$, the function $f^*(T,y)$, viewed as a family of formal power series as in (1), is $(\alpha - \epsilon)$-continuous in the sense of §5.

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A function $f : S_\infty \to \mathbb{C}_\infty \cup \{\infty\}$ is said to be *entire* (resp. *meromorphic*) if $f^*|_{D_\alpha}$ is α-analytic (resp. α-meromorphic) for all real numbers α.

A function $f(z, y)$ on S_∞ is said to be *essentially algebraic* if, for any fixed negative integer $y = -j$, the function $f(z, -j)$ is a rational function in z.

To define the global L-function for a π_∞-adic φ-sheaf, we first recall the definition of the Goss exponentiation ([G2, 3.3]) in the case $A = \mathbb{F}_q[t]$. For any non-zero integral ideal $\mathfrak{a}$ of A, there is a unique monic generator $a(t)$ of $\mathfrak{a}$. Set

$$\langle \mathfrak{a} \rangle \ := \ t^{-\deg(a(t))} a(t) \quad \in U_\infty^{(1)}.$$

Given an element $s = (z, y) \in S_\infty$, we define the exponentiation of the ideal $\mathfrak{a}$ by

$$\mathfrak{a}^s \ := \ z^{\deg(\mathfrak{a})} \langle \mathfrak{a} \rangle^y.$$

This is a well-defined element in $\mathbb{C}_\infty^\times$.

Let $X \to \mathrm{Spec} A$ be an A-scheme of finite type and let $(\mathcal{E}, \varphi)$ be a π_∞-adic φ-sheaf on X. For each non-zero prime ideal $\mathfrak{p}$ of A (or, closed point $\mathfrak{p}$ of Spec A), the fibre $(\mathcal{E}_\mathfrak{p}, \varphi)$ of $(\mathcal{E}, \varphi)$ over $\mathfrak{p}$ defines a π_∞-adic φ-sheaf on the fibre variety $X_\mathfrak{p}$, whose local L-function $L(\mathcal{E}_\mathfrak{p}/X_\mathfrak{p}, T)$ is well defined as in Section 2. Note that the fibre variety $X_\mathfrak{p}$ is defined over the extension field $\mathbb{F}_\mathfrak{p}$, where $\mathbb{F}_\mathfrak{p}$ is the residue field $A/\mathfrak{p}$. For $s = (z, y) \in S_\infty$, the ∞-adic global L-function is then

$$L(\mathcal{E}/X, s) \ := \ \prod_{\mathfrak{p}} \frac{1}{L(\mathcal{E}_\mathfrak{p}/X_\mathfrak{p}, \mathfrak{p}^{-s})}, \tag{8.1}$$

where $\mathfrak{p}$ runs over all non-zero prime ideals of A.

We now explain why the global L-function is in fact a family of local L-functions parametrized by $y \in \mathbb{Z}_p$. For a closed point x of $X/\mathbb{F}_q$, let $\mathfrak{p}_x$ be the image of x in Spec$A$ under the structure morphism. Thus, the closed point x on X lies on the fibre $X_{\mathfrak{p}_x}$. We define a character $\mathrm{N} : Z_0(X) \to A_\pi^\times$ of the zero-cycles on X by setting $\mathrm{N}(x) = \langle \mathfrak{p}_x \rangle^{\deg(x, \mathfrak{p}_x)}$ for any closed point x on X , where $\deg(x, \mathfrak{p}_x)$ is defined to be the degree of x relative to the residue field $\mathbb{F}_{\mathfrak{p}_x}$. Thus, we have the relation $\deg(x) = \deg(x, \mathfrak{p}_x) \cdot \deg(\mathfrak{p}_x)$. The values $\mathrm{N}(x)$ are one-units in A_∞. For $s = (z, y)$, the above global L-function of $(\mathcal{E}, \varphi)/X$ can then be rewritten as

$$\begin{aligned} L(\mathcal{E}/X, s)^{(-1)} \ &= \ \prod_{x \in X_0} \frac{1}{\det(1 - \mathfrak{p}_x^{-s \times \deg(x, \mathfrak{p}_x)} \varphi_x^{\deg(x)} | \mathcal{E}_x)} \\ &= \ \prod_{x \in X_0} \frac{1}{\det(1 - (1/z)^{\deg(x)} \mathrm{N}(x)^{-y} \varphi_x^{\deg(x)} | \mathcal{E}_x)}, \end{aligned} \tag{8.2}$$

where we have to use the reciprocal L-function $L(\mathcal{E}/X, s)^{(-1)}$ because we used the reciprocal of the local L-functions in (8.1). Note that, for each $y \in \mathbb{Z}_p$, the map $x \mapsto \mathrm{N}(x)^{-y}$ defines an abelian character on the zero-cycles $Z_0(X)$. We will show that it actually extends continuously to $\pi_1(X)^{\mathrm{ab}}$. Writing $T = z^{-1}$ for the inverse of the first variable of $s = (z, y)$, one sees that for a fixed p-adic integer y, the reciprocal L-function $L(\mathcal{E}/X, s)^{(-1)}$ is just the local L-function in Section 2 of the π_∞-adic φ-sheaf $(\mathcal{E}, \varphi)$ twisted by the abelian character $\mathrm{N}(x)^{-y}$ of $\pi_1(X)^{\mathrm{ab}}$ (cf. §6). Thus, to understand the meromorphic continuation of $L(\mathcal{E}/X, s)$ for a fixed y, it suffices to know the convergence property of the abelian character $\mathrm{N}(x)^{-y}$ as a function on X.

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Theorem 8.1. (i) *Assume that the π_∞-adic φ-sheaf $\mathcal{E}$ is α log-convergent over the scheme X. Then the L-function $L(\mathcal{E}/X, s)$ is an α-meromorphic function on D^*_α. In particular, if $\mathcal{E}$ is overconvergent (resp. algebraic), then $L(\mathcal{E}/X, s)$ is a meromorphic function on the whole plane S_∞.*

(ii) *Assume that the π_∞-adic φ-sheaf $\mathcal{E}$ is algebraic. Then the L-function $L(\mathcal{E}/X, s)$ is an essentially algebraic meromorphic function on S_∞.*

Since $\mathcal{E} \otimes_{A_\infty} K_\infty$ is algebraic if it comes from a Drinfeld A-module, it follows from the Theorem together with Remark 7.2 that

Corollary 8.2 (Goss' global conjecture [G2, 3.5.4 and 3.7]). *The global L-function $L(\Phi/X, s)$ of a Drinfeld A-module Φ over X is essentially algebraic and meromorphic on S_∞.*

Proof. Let the notation be as before; so the closed point x is on the fibre $X_{\mathfrak{p}_x}$, and the ratio $\deg(x)/\deg(\mathfrak{p}_x)$ is the relative degree $\deg(x, \mathfrak{p}_x)$ of x over the residue field $\mathbb{F}_{\mathfrak{p}_x}$. The prime ideal $\mathfrak{p}_x$ is generated by a monic irreducible element $a(t)$ of A which splits into a product of linear factors in an extension field of $\mathbb{F}_q$:

$$a(t) \;=\; (t-\lambda)(t-\lambda^q)\cdots(t-\lambda^{q^{\deg(\mathfrak{p}_x)-1}}), \qquad \lambda \in \mathbb{F}_{\mathfrak{p}_x}.$$

Thus,

$$\langle \mathfrak{p}_x \rangle \;=\; (1-\pi_\infty\lambda)(1-\pi_\infty\lambda^q)\cdots(1-\pi_\infty\lambda^{q^{\deg(\mathfrak{p}_x)-1}}).$$

One then computes that

$$\mathrm{N}(x) = \langle \mathfrak{p}_x \rangle^{\deg(x,\mathfrak{p}_x)} = \;(1-\pi_\infty\lambda)(1-\pi_\infty\lambda^q)\cdots(1-\pi_\infty\lambda^{q^{\deg(x)-1}}),$$

which is in the same form as (6.8). It follows that the character $\mathrm{N}(x)$ corresponds to the rank one algebraic φ-sheaf on $X/\mathbb{F}_q$ defined by the linear polynomial $(1-\pi_\infty\lambda)$ in λ. The quantity λ as a function of $\bar{x}$ (a geometric point on $x \in X$) is exactly the regular function on X which is the structure morphism $X \to \mathrm{Spec}A$. To prove (i), it now suffices to prove that the family of rank one π-adic φ-sheaves corresponding to $\mathrm{N}(x)^{-y}$ parametrized by y is uniformly overconvergent and continuous. For (ii), it suffices to prove that the π-adic φ-sheaf corresponding to $\mathrm{N}(x)^{-y}$ is algebraic if y is a negative integer.

If y is a p-adic integer, the p-adic power $(1-\pi_\infty\lambda)^{-y}$ can be developed using the binomial theorem:

$$(1-\pi_\infty\lambda)^{-y} \;=\; \sum_{u\geq 0} \binom{-y}{u} (-\pi_\infty)^u \lambda^u.$$

This is a uniformly overconvergent series in λ with respect to the π_∞-adic topology since the coefficient of λ^u is uniformly divisible by π_∞^u for all y. It is clear that the family $(1-\pi_\infty\lambda)^{-y}$ is continuous in y since $\binom{-y}{u}$ is a continuous function from $\mathbb{Z}_p$ to $\mathbb{C}_\infty$. Thus, the family $(1-\pi_\infty\lambda)^{-y}$ parametrized by y defines a continuous family $\mathcal{F}(y)$ of uniformly overconvergent rank one π_∞-adic φ-sheaves. Now $L(\mathcal{E}/X, s)$ has turned out to be the L-function of the continuous family $\mathcal{E} \otimes \mathcal{F}(y)$ of uniformly α-convergent π_∞-adic φ-sheaves, which is α-meromorphic on D^*_α by Theorem 5.2.

If $y=-j$ is a negative integer, then $(1-\pi_\infty\lambda)^{-y} = (1-\pi_\infty\lambda)^j$ is a polynomial in λ of degree j and thus it defines an algebraic φ-sheaf. By Theorem 4.1, (ii), the L-function $L(\mathcal{E}/X, (T, -j))$ is a rational function in T. The proof is complete.

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If X is an affine scheme which is of finite type over $\mathbb{F}_q$, of equi-dimension n over $\mathbb{F}_q$, and is a complete intersection in some smooth affine Y of finite type over $\mathbb{F}_q$, then the global L-function $L(\mathcal{E}/X,s)^{(-1)^{n-1}}$ in Theorem 8.1 is implied by Conjecture 4.4 to be an entire function on S_∞.

Lemma 4.3 can be used to prove the entireness of the global L-function of Theorem 8.1 in some cases. For example, if X is the family of affine elliptic curves $X_1^2 = X_2(X_2-1)(X_2-t)$ over $A=\mathbb{F}_q[t]$ with $p>3$ and $\mathcal{E}$ is trivial, then the global L-function is entire. In fact, for each closed point $\mathfrak{p}$ of Spec A, the fibre $X_\mathfrak{p}$ is a curve defined over $\mathbb{F}_\mathfrak{p}$ and its zeta function $Z(X_\mathfrak{p},T)$ modulo p is given by the linear polynomial $1-H(\mathfrak{p})H(\mathfrak{p}^q)\cdots H(\mathfrak{p}^{q^{\deg(\mathfrak{p})-1}})T^{\deg(\mathfrak{p})}$, where $H(t)$ is the Hasse polynomial associated with the family $X_1^2=X_2(X_2-1)(X_2-t)$ and $H(\mathfrak{p})$ is the value of H at any geometric point in $\mathfrak{p}$. Thus, the global L-function $L(1/X,s)$ in this case is just the local L-function $L(H(x)(1-x\pi_\infty)^{-y}/\mathbb{A}^1,T)$ on the affine line. By Lemma 4.3, we conclude that $L(1/X,s)$ is entire in $T=z^{-1}$ for each fixed y. Since $H(x)(1-x\pi_\infty)^{-y}$ defines a continuous family of overconvergent π_∞-adic φ-sheaves on $\mathbb{A}$, $L(1/X,s)$ is an entire function on S_∞ by Theorem 8.1. The same argument proves the entireness of the L-function given in (0.2).

9. v-ADIC THEORY

In this section, we explain how our arguments for the ∞-adic version can be modified to treat the v-adic version of the global conjecture.

Let $v(t)$ be a monic irreducible polynomial in $A=\mathbb{F}_q[t]$. Let A_v be the completion of A at v. Let η be the unique root of the polynomial $v(t)$ in A_v such that $t-\eta$ is a uniformizer of A_v. Denote the uniformizer $t-\eta$ by π_v. Let $\mathbb{C}_v$ be the completion of an algebraic closure of the quotient field of A_v. Let $U_v^{(1)}$ be the multiplicative group of units in A_v which are congruent to 1 modulo v.

To define the v-adic global L-function for a π_v-adic φ-sheaf, we first recall the definition of the Goss v-adic exponentiation. The Goss v-adic complex plane S_v is essentially

$$S_v \ := \ \mathbb{C}_v^\times \times \mathbb{Z}_p.$$

Note that in Goss' original definition, there is another direct factor $(A/(v))^\times$. This extra factor is a finite abelian group. It contributes only to an abelian character of finite order in the exponentiation, which will be shown to be algebraic in the next section. Thus, we can just work with S_v defined in our way.

α-analyticity (resp. entireness, α-meromorphy, meromorphy) are defined for functions on (a subset of) S_v in the same way as in the ∞-adic case.

Let $s=(z,y)\in S_v$. For any non-zero integral ideal $\mathfrak{a}$ in A which is prime to v, we define

$$\mathfrak{a}^s \ := \ z^{\deg(\mathfrak{a})}\langle\mathfrak{a}\rangle^y,$$

where $\langle\mathfrak{a}\rangle\in U_v^{(1)}$ is defined by $\langle\mathfrak{a}\rangle = a(t)/a(\eta)$ and $a(t)$ is any generator of $\mathfrak{a}$.

Let $X\to \mathrm{Spec}A$ be an A-scheme of finite type and let $(\mathcal{E},\varphi)$ be a π_v-adic φ-sheaf on X. For each non-zero prime ideal $\mathfrak{p}$ of A (or, closed point $\mathfrak{p}$ of Spec A), the fibre $(\mathcal{E}_\mathfrak{p},\varphi)$ of $(\mathcal{E},\varphi)$ over $\mathfrak{p}$ is a π_v-adic φ-sheaf on the fibre variety $X_\mathfrak{p}$, whose local L-function $L(\mathcal{E}_\mathfrak{p}/X_\mathfrak{p},T)$ is well defined as in §2. Note that the fibre variety $X_\mathfrak{p}$ is defined over the extension field $\mathbb{F}_\mathfrak{p}$, where $\mathbb{F}_\mathfrak{p}$ is the residue field $A/\mathfrak{p}$. For

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$s = (z, y) \in S_v$, the v-adic global L-function is then

$$L_v(\mathcal{E}/X, s) \ := \ \prod_{\mathfrak{p} \neq v} \frac{1}{L(\mathcal{E}_{\mathfrak{p}}/X_{\mathfrak{p}}, \mathfrak{p}^{-s})}.$$

Here, we removed the Euler factor at v because $\mathfrak{p}^{-s}$ is not defined at $\mathfrak{p} = v$.

As in the last section, the global L-function is in fact a family of local L-functions parametrized by $y \in \mathbb{Z}_p$. For a closed point x of $X/\mathbb{F}_q$, let $\mathfrak{p}_x$ be the image of x in Spec A under the structure morphism. Thus, the closed point x on X lies on the fibre $X_{\mathfrak{p}_x}$. If x is a closed point such that $\mathfrak{p}_x \neq v$, we define $\mathrm{N}(x) := \langle \mathfrak{p}_x \rangle^{\deg(x, \mathfrak{p}_x)}$, where $\deg(x, \mathfrak{p}_x)$ is defined to be the degree of x relative to the residue field $\mathbb{F}_{\mathfrak{p}_x}$. It gives rise to a character $\mathrm{N} : Z_0(X - X_v) \to A_v^{\times}$. Its values $\mathrm{N}(x)$ are one-units in A_v. For $s = (z, y) \in S_v$, the above v-adic global L-function of $(\mathcal{E}, \varphi)/X$ can be rewritten as

$$\begin{aligned} L_v(\mathcal{E}/X, s)^{(-1)} \ &= \ \prod_{x \in (X - X_v)_0} \frac{1}{\det(1 - \mathfrak{p}_x^{-s \times \deg(x, \mathfrak{p}_x)} \varphi_x^{\deg(x)} | \mathcal{E}_x)} \\ &= \ \prod_{x \in (X - X_v)_0} \frac{1}{\det(1 - (1/z)^{\deg(x)} \mathrm{N}(x)^{-y} \varphi_x^{\deg(x)} | \mathcal{E}_x)}. \end{aligned}$$

Note that, for each $y \in \mathbb{Z}_p$, the map $x \mapsto \mathrm{N}(x)^{-y}$ defines an abelian character on the zero-cycles $Z_0(X - X_v)$. We will show that it actually extends continuously to $\pi_1(X - X_v)^{\mathrm{ab}}$. Writing $T = z^{-1}$ for the inverse of the first variable of $s = (z, y)$, one sees that for a fixed p-adic integer y, the reciprocal L-function $L_v(\mathcal{E}/X, s)^{(-1)}$ is just the local L-function in §2 of the π_v-adic φ-sheaf $(\mathcal{E}, \varphi)$ twisted by the abelian character $\mathrm{N}(x)^{-y}$ of $\pi_1(X)^{\mathrm{ab}}$ (cf. §6). Thus, to understand the meromorphic continuation of $L(\mathcal{E}/X, s)$ for a fixed y, it suffices to know the convergence property of the abelian character $\mathrm{N}(x)^{-y}$ as a function on $X - X_v$.

For a real number α, put $D_\alpha :=$ (punctured open disk $\infty > \mathrm{ord}_{\pi_v}(T) > -\alpha$) $\times$ $\mathbb{Z}_p$.

Theorem 9.1. (i) *Assume that the π_v-adic φ-sheaf $\mathcal{E}$ is α log-convergent over the scheme X. Then the L-function $L_v(\mathcal{E}/X, s)$ is an α-meromorphic function on D_α^*. In particular, if $\mathcal{E}$ is overconvergent (resp. algebraic), then $L_v(\mathcal{E}/X, s)$ is meromorphic on the whole plane S_v.*

(ii) *If the π_v-adic φ-sheaf $\mathcal{E}$ is algebraic, then the L-function $L_v(\mathcal{E}/X, s)$ is an essentially algebraic meromorphic function on S_v.*

Since $\mathcal{E}$ is algebraic if it comes from a Drinfeld A-module, it follows from the Theorem that

Corollary 9.2. *The v-adic L-function $L_v(\Phi/X, s)$ of a Drinfeld A-module Φ over X is essentially algebraic and meromorphic on S_v.*

Proof. Let the notation be as before. The prime ideal $\mathfrak{p}_x$ is generated by a monic irreducible element $a(t) \neq v(t)$ of A, which can be factorized in an extension field of $\mathbb{F}_q$:

$$a(t) \ = \ (t - \lambda)(t - \lambda^q) \cdots (t - \lambda^{q^{\deg(\mathfrak{p}_x) - 1}}), \qquad \lambda \in \mathbb{F}_{\mathfrak{p}_x}.$$

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Write $g(\lambda) = v(\lambda)/(\lambda-\eta)$. This is a polynomial in λ with coefficients in the residue field $\mathbb{F}_q[\eta]$ of A at v. This residue field is contained in A_v. Now, we compute that

$$t-\lambda = (t-\eta)+(\eta-\lambda) = (\eta-\lambda)(1-\frac{\pi_v}{\lambda-\eta}) = (\eta-\lambda)(1-\frac{g(\lambda)}{v(\lambda)}\pi_v).$$

Thus, as in (6.8),

$$\mathrm{N}(x) = (\frac{a(t)}{a(\eta)})^{\deg(x,\mathfrak{p}_x)} = (1-\frac{g(\lambda)}{v(\lambda)}\pi_v)(1-\frac{g(\lambda^q)}{v(\lambda^q)}\pi_v)\cdots(1-\frac{g(\lambda^{q^{\deg(x)-1}})}{v(\lambda^{q^{\deg(x)-1}})}\pi_v).$$

Since $1/v(\lambda)$ is a regular function on the scheme $(X-X_v)/\mathbb{F}_q$, the character $\mathrm{N}(x)$ corresponds (§6) to the rank one algebraic φ-sheaf on $(X-X_v)/\mathbb{F}_q$ defined by the polynomial $1-(g(\lambda)/v(\lambda))\pi_v$ in λ and $1/v(\lambda)$ with coefficients in A_v. To prove (i), it now suffices to prove that the family of the rank one π-adic φ-sheaves corresponding to $\mathrm{N}(x)^{-y}$ parametrized by y is uniformly overconvergent and continuous. For (ii), it suffices to prove that the π-adic φ-sheaf corresponding to $\mathrm{N}(x)^{-y}$ is algebraic if y is a negative integer.

For each p-adic integer $y \in \mathbb{Z}_p$, the p-adic power $(1-\frac{g(\lambda)}{v(\lambda)}\pi_v)^{-y}$ can be expanded using the binomial theorem:

$$(1-\frac{g(\lambda)}{v(\lambda)}\pi_v)^{-y} \;=\; \sum_{u\geq 0}\binom{-y}{u}(-\pi_v)^u(\frac{g(\lambda)}{v(\lambda)})^u.$$

As in the proof of Theorem 8.1, it is uniformly overconvergent and continuous in y, hence defines a continuous family $\mathcal{F}(y)$ of uniformly overconvergent rank one π_v-adic φ-sheaves on $X-X_v$. Now $L_v(\mathcal{E}/X,s)$ has turned out to be the L-function of the continuous family $\mathcal{E}\otimes\mathcal{F}(y)$ of uniformly α-convergent π_v-adic φ-sheaves, which is α-meromorphic on D_α^* by Theorem 5.1.

If $y=-j$ is a negative integer, then $(1-(g(\lambda)/v(\lambda))\pi_v)^{-y} = (1-(g(\lambda)/v(\lambda))\pi_v)^j$ is a polynomial in λ and $1/v(\lambda)$. The L-function $L_v(\mathcal{E}/X,(T,-j))$ is a rational function in T by Theorem 4.1, (ii). The proof is complete.

10. Twist by a finite abelian character

In this section, we show that a finite abelian character χ of the arithmetic fundamental group defines a rank one algebraic φ-sheaf. This immediately implies that Theorem 8.1 and Theorem 9.1 remain true for a global L-function twisted by a finite abelian character.

Since χ is of finite order, we may assume that χ is a character of the Galois group G of a finite unramified abelian covering Y/X. Let $\chi : G \to \mathbb{C}_\pi^\times$ be an abelian character of G. For a closed point x of $X/\mathbb{F}_q$, we define $\chi(x)$ to be $\chi(\mathrm{Frob}_x)$, where $\mathrm{Frob}_x \in G$ is the Frobenius element at x.

Lemma 10.1. *Over some finite extension $\mathbb{F}_{q^l}$ of $\mathbb{F}_q$, there exists a rational function $b(x)$ on $X\otimes\mathbb{F}_{q^l}$ such that $\chi(x) = b(\overline{x})b(\overline{x}^q)\cdots b(\overline{x}^{q^{d(x)-1}})$ for any closed point x of $X-(b)$, where $\overline{x}$ is a geometric point on x and (b) is the support of the divisor of b.*

Proof. Since χ is a product of characters of cyclic subgroups of G, we may assume G is cyclic. Since χ has values in $\mathbb{C}_\pi^\times$ and $\mathbb{C}_\pi^\times$ has no torsions of order p, we may assume G is of order prime to p. Then according to Reich's arguments in [Re, p. 848], if we identify G with a subgroup of $\mathbb{F}_{q^l}^\times$ for some $l\geq 1$ in a suitable way, there

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exists a rational function b on $X \otimes \mathbb{F}_{q^l}$ such that the Frobenius of a closed point x of $X - (b)$ is expressed as

$$\mathrm{Frob}_x \;=\; b(\overline{x})b(\overline{x}^q)\cdots b(\overline{x}^{q^{d(x)-1}}) \in \mathbb{F}_{q^l}^\times.$$

(This b comes from Kummer theory.) Since any character χ of G (G being regarded as a subgroup of $\mathbb{F}_{q^l}^\times$) is of the form $g \mapsto g^k$ for some integer k, the above shows that $\chi(x)$ is also written in a similar form. The lemma is proved.

Since the covering Y/X is unramified, for any given closed point x of X, one can choose such a function b so that $x \in X - (b)$.

Note added in proof

We have now proved Conjecture 4.4 in full generality.

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