FUNCTIONS WHOSE DERIVATIVES AT ONE POINT FORM A FINITE SET[1]

BY

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1. **Introduction.** In this paper we consider families F_S of analytic functions $f(z)$ such that the derivatives $f^{(n)}(z_0)$; $n=0, 1, 2, \ldots$ all lie in a given finite set S of complex numbers. Since the value of z_0 is irrelevant we shall assume $z_0=0$. If S consists of a single element s then $F_S=\{se^z\}$. We shall therefore assume $|S|>1$. The family F_S clearly consists of entire functions of bounded exponential type. It is closed under differentiation and compact under the usual topology of uniform convergence on bounded sets. This topology is that of the space S^ω of sequences of elements of S; in other words, the topology for which the elements of a convergent family of sequences are ultimately constant. One could instead consider the Laplace transforms, thereby obtaining the family $\hat{F}_S$ of functions whose Taylor series expansion at 0 has only a finite number of different coefficients.

In an earlier paper [3] Sato and Straus proved that any function $f \in F_S$ whose derivatives at any point $z_0 \neq 0$ form a finite set satisfies a differential equation $f^{(n)}(z)=f^{(m)}(z)$, $n \neq m$ so that the set of derivatives at every point is finite. In §2 we investigate the value distributions of the functions in F_S under more general conditions. In particular we show that the closure $\langle f\rangle(z_0)$ of the set $\{f^{(n)}(z_0) \mid n=0, 1, 2, \ldots\}$, if infinite, has the same cardinality (which is either $\aleph_0$ or $2^{\aleph_0}$) at every point $z_0 \neq 0$ and that for all z_0, except for a totally disconnected perfect set, the continuous mapping from F_S to the set of values $F_S(z_0)$ is one-to-one.

In §3 we examine the structure of the sets $\langle f\rangle$ in greater detail, by looking at the successive derived sets and the order type beyond which the derived sets are the same. This investigation is essentially independent of function theory and could equally concern the behavior of the fractional parts of numbers $b^n x$ where $b>1$ is an integer and x a real number.

A rather striking result in Theorem 3.13 shows that denumerably infinite closed subsets of the unit interval which are closed under multiplication (mod 1) by an integer $b>1$ must contain an infinite closed set of rational numbers.

Throughout the paper we shall use the following notation:

$$(1.1) \qquad m = \min_{s \in S \setminus \{0\}} |s|, \qquad M = \max_{s \in S} |s|,$$

$$(1.2) \qquad \delta = \min_{s_1,\, s_2 \in S,\, s_1 \neq s_2} |s_1 - s_2|, \qquad \Delta = \max_{s_1,\, s_2 \in S} |s_1 - s_2|.$$

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1.3. Definition. Let G be a (closed) set, then for every ordinal α we define G_α inductively by $G_0=G$, $G_1=G'$, $G_{\alpha+1}=G'_\alpha$, the derived set of G_α and if λ is a limit ordinal then

$$G_\lambda = \bigcap_{\alpha<\lambda} G_\alpha.$$

1.4. Definition. The *topological order type* of a (closed) set G is the least ordinal $\alpha=\alpha(G)$ such that $G_{\alpha+1}=G_\alpha$.

1.5. Definition. If G is a set of functions then $G(z_0)=\{g(z_0) \mid g\in G\}$.

1.6. Definition. $N_S=\{z \mid f(z)=0 \text{ for some } f\in F_S, f\neq 0\}$.

1.7. Definition. A point z_0 is a *representation point* of F_S if the mapping $F_S\to F_S(z_0)$ is one-to-one.

1.8. Definition. If A, B are sets of numbers then $A-B=\{a-b \mid a\in A, b\in B\}$.

1.9. Definition. For every $f\in F_S$ the set $\langle f\rangle$ is the closure of the set $\{f^{(n)} \mid n=0, 1, 2, \ldots\}$.

2. The value distributions of functions in F_S.

2.1. Theorem. *Let $f\in F_S$, f not identically 0, and let n_0 denote the number of zeroes of f at 0 while $n(r)$ denotes the number of zeros, z, of f which satisfy $0<|z|<r$. Then*

$$n(r) \leqq \log_2\left(1+\frac{M}{m}(e^{2r}-1)\right) \tag{2.2}$$

and

$$\begin{aligned} n^*(r) &= n_0+n(r) \\ &\leqq n_0+\log_2\left(1+\frac{M}{m}\frac{2r}{n_0+1}\left(1+\frac{2r}{n_0+2}+\frac{4r^2}{(n_0+2)(n_0+3)}+\cdots\right)\right). \end{aligned} \tag{2.3}$$

Proof. Inequality (2.2) is a consequence of (2.3) and the fact that

$$\frac{2r}{n_0+1}\left(1+\frac{2r}{n_0+2}+\frac{4r^2}{(n_0+2)(n_0+3)}+\cdots\right) \leqq 2r\left(1+\frac{2r}{2}+\frac{4r^2}{2\cdot 3}+\cdots\right) = e^{2r}-1.$$

Now, if $f\in F_S$ has a zero of order n_0 at 0 then

$$f(z) = \sum_{n=n_0}^{\infty}\frac{c_n z^n}{n!} = \frac{z^{n_0}}{n_0!}\left(c_{n_0}+\frac{c_{n_0+1}}{n_0+1}z+\cdots\right) = \frac{z^{n_0}}{n_0!}g(z) \tag{2.4}$$

with $c_n\in S$, $c_{n_0}\neq 0$. Thus, if $z_1,\ldots,z_{n(r)}$ are the zeros of f—and hence of g—in the punctured disk $0<|z|<r$, then by Jensen's formula

$$\frac{1}{2\pi}\int_0^{2\pi}\log|g(2re^{i\theta})|\,d\theta = \log|g(0)|+\log\frac{(2r)^{n(2r)}}{|z_1\cdots z_{n(2r)}|} \tag{2.5}$$

or, as a result of (2.4),

$$\begin{aligned} \log 2^{n(r)} &\leqq \log \frac{(2r)^{n(r)}}{|z_1 \cdots z_{n(r)}|} \leqq \log \frac{(2r)^{n(2r)}}{|z_1 \cdots z_{n(2r)}|} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{g(2re^{i\theta})}{g(0)} \right| d\theta \qquad (2.6) \\ &\leqq \log \left(1 + \frac{M}{m} \left(\frac{2r}{n_0+1} + \frac{4r^2}{(n_0+1)(n_0+2)} + \cdots \right) \right), \end{aligned}$$

which yields (2.3).

Of course if $0 \notin S$ then $n_0 = 0$.

As a result of Theorem 2.1 we see that there exists a neighborhood, U, of 0 such that no nonzero function of F_S has a zero different from 0 in U. We can find the best possible bounds for the radius of U that can be expressed in terms of m and M.

2.7. Theorem. *If* $f \in F_S$, $f \not\equiv 0$ *then* $f(z) \neq 0$ *for*

$$0 < |z| < \log(1 + m/M). \tag{2.8}$$

If f has a zero of order n_0 at 0 then $f(z) \neq 0$ for $0 < |z| < r_S(n_0)$ where $r = r_S(n_0)$ is the positive solution of

$$m(r^{n_0}/n_0!) = M(e^r - 1 - r/1! - \cdots - r^{n_0}/n_0!). \tag{2.9}$$

In particular,

$$r_S(n_0) > \frac{m(n_0+1)(n_0+2)}{m(n_0+1) + M(n_0+2)}. \tag{2.10}$$

Proof. The equations $z \neq 0$ and

$$f(z) = \sum_{n=n_0}^{\infty} \frac{c_n z^n}{n!} = \frac{z^{n_0}}{n_0!} \left(c_{n_0} + \frac{c_{n_0+1}}{n_0+1} z + \cdots \right) = 0$$

lead to

$$\begin{aligned} m \leqq |c_{n_0}| &= \left| \frac{c_{n_0+1}}{n_0+1} z + \frac{c_{n_0+2}}{(n_0+1)(n_0+2)} z^2 + \cdots \right| \\ &\leqq M \left(e^{|z|} - 1 - \frac{|z|}{1!} - \cdots - \frac{|z|^{n_0}}{n_0!} \right) \cdot \frac{n_0!}{|z|^{n_0}}. \end{aligned} \tag{2.11}$$

Since the right side of (2.11) is an increasing function of $|z|$, it follows that $|z| \geqq r_S(n_0)$ where $r_S(n_0)$ is the solution of (2.9). To verify (2.10) it is clear that it holds when $r \geqq n_0 + 2$. If $r < n_0 + 2$ then

$$\begin{aligned} m &= M \left(e^r - 1 - \cdots - \frac{r^{n_0}}{n_0!} \right) \frac{n_0!}{r^{n_0}} = M \frac{r}{n_0+1} \left(1 + \frac{r}{n_0+2} + \frac{r^2}{(n_0+2)(n_0+3)} + \cdots \right) \\ &\leqq M \frac{r}{n_0+1} \sum_{k=0}^{\infty} \left(\frac{r}{n_0+2} \right)^k = M \frac{r}{n_0+1} \bigg/ \left(1 - \frac{r}{n_0+2} \right) \end{aligned}$$

leads to (2.10).

For $n_0=0$ equation (2.9) becomes $m=M(e^r-1)$ leading to

$$r_S(0) = \log\left(1+\frac{m}{M}\right) < \frac{m}{M},$$

and for each $n_0>0$ we have

$$\frac{m}{M} < \frac{m(n_0+1)(n_0+2)}{m(n_0+1)+M(n_0+2)} < r_S(n_0).$$

Thus $|z| \geqq r_S(n_0)$ for any n_0 implies $|z| \geqq r_S(0)$ which verifies (2.8).

2.12. COROLLARY. *If $f_n \in F_S$, $f_n \not\equiv 0$, $f_n \to 0$ and $f_n(z_n)=0$, $z_n \neq 0$, then $z_n \to \infty$.*

Proof. This is an immediate consequence of (2.10) since $r_S(n_0) \to \infty$ as $n_0 \to \infty$ and, as $f_n \to 0$, the multiplicity, n_0, of the zero of f_n at 0 tends to infinity.

2.13. COROLLARY. *For every value, a, there is a neighborhood U of 0 so that no $f \in F_S$ which is not identically equal to a attains the value a in $U\backslash\{0\}$. The radius of U can be chosen as*

$$\log(1+m(a)/M) \quad where \quad m(a) = \min\{|s-a| \mid s \in S,\ s \neq a\},$$

2.14. COROLLARY. *Every point z satisfying*

$$0 < |z| < \log(1+\delta/\Delta) \tag{2.15}$$

is a representation point of F_S.

Proof. A point z_0 fails to be a representation point of F_S only if $f(z_0)=g(z_0)$ for two different functions f, $g \in F_S$, in other words, if the nonzero function $f-g \in F_{S-S}$ has a zero at z_0. The corollary now follows from the application of Theorem 2.7 to F_{S-S}.

2.16. THEOREM. *The set N_S of zeros of the nonzero elements of F_S is a totally disconnected and nowhere dense closed set. If $0 \in S$ then 0 is an isolated point of N_S. In any case $N_S\backslash\{0\}$ is perfect.*

Proof. The fact that 0 is isolated follows from Theorem 2.7. If $z_n \in N_S$ and $z_n \to z_0 \neq 0$ then there exist nonzero functions $f_n \in F_S$ so that $f_n(z_n)=0$ and since F_S is compact there is a convergent subsequence $f_{n_i} \to f_0 \in F_S$. Since $\{z_n\}$ is bounded, it follows from Corollary 2.12 that $f_0 \neq 0$. On the other hand

$$f_0(z_0) = \lim f_{n_i}(z_0) = \lim (f_{n_i}(z_0) - f_{n_i}(z_{n_i})) = 0.$$

Thus $z_0 \in N_S$ and N_S is closed.

The set F_S is perfect so that for every $f_0 \in F_S\backslash\{0\}$ with $f_0(z_0)=0$, $z_0 \in N_S\backslash\{0\}$ there exists a sequence $f_n \in F_S$, $f_n \neq f_0$, $f_n \to f_0$. By Rouche's theorem all f_n sufficiently near to f_0 have a zero, z_n, arbitrarily close to z_0. On the other hand, $z_n \neq z_0$ for large n, since otherwise $f_n - f_0$ would have a zero at z_0 in contradiction to Corollary 2.12. Thus z_0 is a limit point of N_S and $N_S\backslash\{0\}$ is perfect.

Now assume that N_S contains a continuum C (compact connected set consisting of more than one point). If we divide C into $|S|^n$ (not necessarily disjoint) closed sets of zeros of functions $f \in F_S$ with fixed values of $f^{(\nu)}(0)$; $\nu = 0, 1, \ldots, n-1$ then at least one of these sets must contain a subcontinuum C_1. Let the corresponding subset of F_S be F_1.

Now according to Corollary 2.12 we can choose n so large that no two distinct functions which have the same first n derivatives can have the same value at any point of C_1, since otherwise their difference would approximate 0 and at the same time have a zero in the bounded set C_1. Thus for sufficiently large n the set F_1 must consist of a single function f_1 which vanishes on C_1 and therefore vanishes identically—a contradiction.

2.17. Corollary. *The correspondence $f \leftrightarrow f(z)$ is a homeomorphism between F_S and $F_S(z)$ whenever $z \notin N_{S-S}$. That is, F_S and $F_S(z)$ are homeomorphic for all z except for a nowhere dense totally disconnected perfect set of nonrepresentation points and the origin.*

Proof. The continuity of the mapping $F_S \to F_S(z)$ is obvious. Now assume $f_n(z) \to w_0$, then, by the compactness of F_S, there exists a subsequence $f_{n_i} \to f_0$ with $f_0(z) = w_0$. Now, unless $f_n \to f_0$, there would exist another subsequence $f_{n_i'} \to g_0 \neq f_0$. Then $g_0(z) = w_0$ and $(f_0 - g_0)(z) = 0$ so that $z \in N_{S-S}$ contrary to hypothesis.

2.18. Theorem. *The number of different functions of F_S which attain the same value at a point $z_0 \neq 0$ is less than*

$$\mathcal{N}(z_0, S) = |S|^{|z_0|(1+\Delta/\delta)-1}.$$

Proof. If the number of functions is at least $\mathcal{N}(z_0, S)$ then there must exist two functions $f, g \in F_S$ with $f(z_0) = g(z_0)$ and $f^{(\nu)}(0) = g^{(\nu)}(0)$ for $\nu = 0, 1, \ldots, n_0 - 1$ where $n_0 \geqq |z_0|(1+\Delta/\delta) - 1$ so that $(f-g)(z_0) = 0$ and

$$|z_0| \leqq \frac{\delta(n_0+1)}{\delta+\Delta} < \frac{\delta(n_0+1)(n_0+2)}{\delta(n_0+1)+\Delta(n_0+2)},$$

which contradicts (2.10) as applied to the set $S-S$, that is with m replaced by δ and M replaced by Δ.

2.19. Corollary. *If G is an infinite subset of F_S then $|G(z_0)| = |G|$ for every $z_0 \neq 0$.*

The continuous finite-to-one map $G \to G(z_0)$ of any $G \subset F_S$ preserves more than the cardinality of infinite G, it also preserves the topological order types of denumerable closed sets.

2.20. Lemma. *For any $G \subset F_S$ the derived sets satisfy $G_1(z_0) = (G(z_0))_1$ for any $z_0 \neq 0$.*

Proof. Assume $g_n(z_0) \to w \in (G(z_0))_1$ where $(g_n(z_0))$ is an infinite sequence with $g_n \in G$. Then there exists a convergent subsequence $g_{n_i} \to g_0 \in G_1$ so that $w = g_0(z_0) \in G_1(z_0)$. Thus $(G(z_0))_1 \subset G_1(z_0)$. Now assume $g_n \to g_0 \in G_1$ where (g_n) is an infinite sequence. Then, by Theorem 2.13, the sequence $(g_n(z_0))$ contains infinitely many distinct elements so that $\lim g_n(z_0) = g_0(z_0) \in (G(z_0))_1$. Thus $(G(z_0))_1 \supset G_1(z_0)$.

2.21. COROLLARY. *For each ordinal α and any $G \subset F_S$ we have $G_\alpha(z_0) = (G(z_0))_\alpha$ for any $z_0 \neq 0$.*

Proof. By Lemma 2.20 the corollary holds for $\alpha = 1$. If the corollary holds for α then

$$(G(z_0))_{\alpha+1} = ((G(z_0))_\alpha)_1 = (G_\alpha(z_0))_1 = (G_\alpha)_1(z_0) = G_{\alpha+1}(z_0).$$

If λ is a limit ordinal and the corollary holds for all $\alpha < \lambda$ then

$$(G(z_0))_\lambda = \bigcap_{\alpha<\lambda} (G(z_0))_\alpha = \bigcap_{\alpha<\lambda} G_\alpha(z_0) = \Big(\bigcap_{\alpha<\lambda} G_\alpha\Big)(z_0) = G_\lambda(z_0).$$

Thus the corollary holds by transfinite induction.

2.22. COROLLARY. *For every $G \subset F_S$ and every $z_0 \neq 0$ we have $\alpha(G(z_0)) \leqq \alpha(G)$.*

Proof. If $G_{\alpha+1} = G_\alpha$ then

$$(G(z_0))_{\alpha+1} = G_{\alpha+1}(z_0) = G_\alpha(z_0) = (G(z_0))_\alpha.$$

2.23. THEOREM. *For every $G \subset F_S$ the topological order type $\alpha(G)$ is denumerable and $G_{\alpha(G)}$ is the set of all condensation points of G, which is the maximal perfect set in the closure $\bar{G}$ of G. In particular, if the closure of G is denumerable, then $G_{\alpha(G)} = \varnothing$.*

Proof. Since $(G_{\alpha(G)})' = G_{\alpha(G)}$ it follows that the set is perfect. Since every perfect subset of $\bar{G}$ belongs to all derived sets of G it follows that it is contained in $G_{\alpha(G)}$. Thus $G_{\alpha(G)}$ is the maximal perfect subset of $\bar{G}$. The set of all condensation points of $\bar{G}$ is clearly perfect and every point of a nonempty perfect set is a condensation point. Now if $\alpha(G)$ were nondenumerable then the set $\bar{G} \backslash G_{\alpha(G)}$ would be nondenumerable. By the separability of F_S there would be a neighborhood U of $G_{\alpha(G)}$ so that $|\bar{G} \backslash U| > \aleph_0$. Hence $\bar{G} \backslash U$ would have condensation points not in $G_{\alpha(G)}$, a contradiction.

2.24. THEOREM. *If $G \subset F_S$ is closed and denumerable then $\alpha(G) = \alpha(G(z_0))$ for every $z_0 \neq 0$.*

Proof. If G is finite then trivially $\alpha(G) = \alpha(G(z_0)) = 1$. If $\beta + 1 < \alpha(G)$ then G_β is infinite and hence, according to Corollary 2.19, $G_\beta(z_0)$ is infinite. Since $G_\beta(z_0)$ is compact it follows that $(G(z_0))_{\beta+1} \neq \varnothing$ so that $\alpha(G(z_0)) > \beta + 1$. Now the topological order type of a denumerable compact set cannot be a limit ordinal since the intersection of a nested family of nonempty compact sets is nonempty. Hence we can write $\alpha(G(z_0))=\beta+1$, and $\beta+1<\alpha(G)$ would lead to the contradiction $\beta+1<\alpha(G(z_0))$.

3. **On denumerable sets $\langle f\rangle$.** In this section we wish to consider the closure of the set of all derivatives of a function f in F_S, in particular in the exceptional cases in which $\langle f\rangle$ is denumerable. It will be convenient to identify the function f with the sequence $(f^{(n)}(0))$ whose elements are in S. Such sequences in turn can be identified with numbers in the interval [0, 1] by introducing a basis $b=|S|$ and a mapping $\beta(s_\nu)=\nu-1$; $\nu=1, 2, \ldots, b$ of the elements of S onto the integers $0, 1, \ldots, b-1$. We then let

$$f\leftrightarrow(f^{(n)}(0))\to x(f) = \sum_{n=0}^{\infty}\beta(f^{(n)}(0))b^{-n-1}. \tag{3.1}$$

The last mapping is not quite one-to-one because of the ambiguity in the expansion of b-adic fractions. The differentiation operation for f corresponds to the shift operation for the sequence and to multiplication by b (mod 1) for the real number. Thus $\langle f\rangle$ corresponds to the closed set invariant under multiplication by b (mod 1) which is generated by a single point $x(f)$ of [0, 1].

3.2. Definition. To a set $G\subset F_S$ we assign the measure $\mu(G)=\mu(x(G))$, where $x(G)=\{x(g)\mid g\in G\}$, provided the latter measure exists.

3.3. Theorem. *For almost all (in the sense of μ) functions of F_S we have $\langle f\rangle=F_S$.*

Proof. If $x(f)$ is normal to the base b then $x(\langle f\rangle)=[0,1]=x(F_S)$ and $\langle f\rangle=F_S$. Of course $\langle f\rangle$ may equal F_S even when $x(f)$ is not normal.

It will be convenient to see that the structure of the sets of F_S which we wish to examine is the same as the structure of the sets we get for S with 2 elements.

3.4. Lemma. *Given $G\subset F_S$ then there exists a $G^*\subset F_{\{0,1\}}$ which is homeomorphic to G.*

Proof. To each $s\in S$ we associate the block $B(s)$ of length $\beta(s)+2$ consisting of $\beta(s)$ digits 0 and digits 1 at the ends. To each sequence $g\in G$ we now associate the sequence $g^*\in F_{\{0,1\}}$ obtained by replacing each element s of g by the block $B(s)$. This one-to-one association between F_S and $F_S^*\subset F_{\{0,1\}}$ is clearly a homeomorphism.

The correspondence established in Lemma 3.4 will, in general, associate a set G^* which is not closed under differentiation to a given $G\subset F_S$ which is closed under differentiation. However the behavior of G^* under differentiation can be determined.

3.5. Lemma. *If $G\subset F_S$ is closed under differentiation, then the closure under differentiation G^{**} of $G^*\subset F_{\{0,1\}}$ is*

$$G^{**}\subseteqq G^*_{(0)}\cup G^*_{(1)}\cup\cdots\cup G^*_{(b+1)} \tag{3.6}$$

where $G^*_{(0)}=G^*$, each $G^*_{(i)}$ is the union of a finite number of sets each homeomorphic *to a subset of G where the closures of these subsets are disjoint, and the closures of* $G^*_{(i)}$ *and* $G^*_{(j)}$ *are disjoint for* $i \neq j$.

Proof. Let $G^*_{(i)}$ for $i>0$ consist of the sequences in G^* preceded by a digit 1 which in turn is preceded by $i-1$ digits 0. These are clearly all the sequences of $F_{\{0,1\}}$ obtained by successive differentiation of the elements of G^*. The closures of $G^*_{(i)}$ and $G^*_{(j)}$ are disjoint for $i \neq j$ since the initial blocks of length b are distinct. Finally each $G^*_{(i)}$ corresponds to those elements in G whose initial digit is s_β with $\beta \geqq i$. In other words there are $b-i+1$ subsets of G with distinct initial digits, the union of whose homeomorphic images in $F_{\{0,1\}}$ is $G^*_{(i)}$.

3.7. THEOREM. *For each* $G \subset F_S$ *there is a* $G^* \subset F_{\{0,1\}}$ *so that* $\alpha(G)=\alpha(G^*)$. *If G is closed under differentiation then there is a* $G^{**} \subset F_{\{0,1\}}$ *so that* G^{**} *is closed under differentiation and* $\alpha(G)=\alpha(G^{**})$. *If G is also closed under integration (that is, if every element of G is the derivative of an element of G) then* G^{**} *is closed under integration. For every* $f \in F_S$ *there is an* $f^* \in F_{\{0,1\}}$ *so that* $\alpha(\langle f\rangle)=\alpha(\langle f^*\rangle)$.

Proof. The first statement is an immediate consequence of Lemma 3.4 since $\alpha(G)$ is a topological invariant. The second statement follows from Lemma 3.5 and

$$\alpha(G) = \alpha(G^*) \leqq \alpha(G^{**}) = \alpha(G^* \cup G^*_{(1)} \cup \cdots \cup G^*_{(b+1)})$$
$$= \max\{\alpha(G^*), \alpha(G^*_{(1)}), \ldots, \alpha(G^*_{(b+1)})\} = \alpha(G^*).$$

It is clear that G^{**} is closed under integration whenever G is.

Finally, if we define f^* by $\{f\}^*=\{f^*\}$ then $\langle f\rangle^{**}=\langle f^*\rangle$ so that the last statement of the theorem is a special case of $\alpha(G^{**})=\alpha(G)$.

3.8. LEMMA. *For every denumerable closed set* $G \subset F_{\{0,1\}}$ *there exists a closed* $G^\sim \subset F_{\{0,1,2\}}$, *which is closed under integration, so that*

$$\alpha(G^\sim) = \alpha(G)+1, \qquad G^\sim_{\alpha(G)} = \{2e^2\}.$$

If G is closed under differentiation then so is $G^\sim$.

Proof. Let $G^\sim$ consist of the sequences in G preceded by an arbitrary number of 2's and the element $2e^2$. Then for every $\beta \leqq \alpha(G)$ the set $G^\sim_\beta$ consists of the sequences in G_β preceded by an arbitrary number of 2's and the element $2e^2$. In particular $G^\sim_{\alpha(G)}=\{2e^2\}$ and $\alpha(G^\sim)=\alpha(G)+1$.

3.9. LEMMA. *For each sequence of denumerable closed sets* $G_n \subset F_{\{0,1\}}$ *there exists a closed* $G^\sim \subset F_{\{0,1,2\}}$ *such that*

$$\alpha(G^\sim) = \sigma+1 \quad \textit{where} \quad \sigma = \sup_n \alpha(G_n).$$

If the G_n *are closed under differentiation then there exists a closed* $G^\approx \subset F_{\{0,1,2\}}$ *so that*

$$\alpha(G^\approx) = \sigma + 2, \qquad G^\approx_{\sigma+1} = \{2e^2\}$$

and $G^\approx$ *is closed under differentiation and integration.*

Proof. If $\sigma=\alpha(G_m)$ for some m then the lemma results from Lemma 3.8. If not, let $G_n^{\sim}$ consist of the elements of G_n preceded by n digits 2. Then $G_n^{\sim}$ is obviously homeomorphic to G_n and $G_n^{\sim} \cap G_m^{\sim} = \varnothing$ for $n \neq m$. Let $G^{\sim} = \bigcup_n G_n^{\sim} \cup \{2e^z\}$. Then $G^{\sim}$ is closed, since $\lim g_{n_i} = 2e^z$ if $g_{n_i} \in G_{n_i}^{\sim}$ and $\lim n_i = \infty$. Now for every $\beta < \sigma$ we have $G_\beta^{\sim} = \bigcup_n (G_n^{\sim})_\beta \cup \{2e^z\}$ so that $G_\sigma^{\sim} = \{2e^z\}$ and $G_{\sigma+1}^{\sim} = \varnothing$.

To prove the second part of the lemma associate to each G_n the set $G^{\approx} \subset F_{\{0,1,2\}}$ consisting of $2e^z$ and the elements obtained from the sequences $g_n \in G_n$ by inserting n terms 2 between any two consecutive terms of g_n and an arbitrary number of terms 2 before the first term of g_n. Then $G_n^{\approx}$ is closed and closed under integration. Since G_n is closed under differentiation then so is $G_n^{\approx}$. Now

$$G_n^{\approx} = \{2e^z\} \cup G_{n0}^{\approx} \cup G_{n1}^{\approx} \cup \cdots$$

where the second index indicates the number of terms 2 at the beginning of the sequences. The $G_{ni}^{\approx}$ are disjoint and homeomorphic to G_n. Let $\alpha_n = \alpha(G_n)$. Then α_n is not a limit ordinal and $(G_n)_{\alpha_n - 1}$ consists of a finite number of (eventually periodic) sequences. Thus $(G_n^{\approx})_{\alpha_n - 1}$ consists of $2e^z$ and a finite number of (eventually periodic) sequences preceded by blocks of 2's of arbitrary length. Hence $(G_n^{\approx})_{\alpha_n} = \{2e^z\}$ and $\alpha(G_n^{\approx}) = \alpha(G_n) + 1$.

Now let $G^{\approx} = \bigcup_n G_n^{\approx} \cup H$ where H consists of all sequences of 2's with at most one digit 0 or 1 in an arbitrary place. It is clear that any convergent sequence (g_{n_i}) with $n_1 < n_2 < \cdots$, $g_{n_i} \in G_{n_i}^{\approx}$ must converge to an element of H and that every element of H is such a limit.

Now for each $\beta < \sup_n \alpha(G_n) = \sup_n \alpha(G_n^{\approx})$ we have

$$G^{\approx} = \bigcup_n (G_n^{\approx})_\beta \cup H$$

since for all sufficiently large n the set $(G_n)_\beta$ is infinite and contains elements whose initial block consists of an arbitrary number of 2's followed by 0 or 1 followed by n terms 2. If $\sigma = \sup_n \alpha(G_n)$ we get $(G_n^{\approx})_\sigma = \varnothing$ and $G_\sigma^{\approx} = H$. Now $G_{\sigma+1}^{\approx} = H_1 = \{2e^z\}$ so that $G_{\sigma+2}^{\approx} = \varnothing$ and $\alpha(G^{\approx}) = \sigma + 2$.

3.10. Lemma. *If G is a denumerable compact set then $\alpha(G)$ is not a limit ordinal. If, in addition, $G \subset F_S$ and G is closed under differentiation then $\alpha(G)$ is not the successor of a limit ordinal.*

Proof. If λ is a limit ordinal and $G_\beta \neq \varnothing$ for $\beta < \lambda$ then $G_\lambda = \bigcap_{\beta<\lambda} G_\beta \neq \varnothing$ since it is the intersection of a nested family of nonempty compact sets.

If $G \subset F_S$ and $\alpha(G) = \lambda + 1$, where λ is a limit ordinal, then G_λ is finite, consisting of a finite number of eventually periodic sequences. Let $f \in G_\lambda$ be purely periodic of period p. For each $\beta < \lambda$ the set G_β contains a sequence $f_{\beta n} \to f$ and $f_{\beta n} \notin G_\lambda$. By the periodicity of f we have $f_{\beta n}^{(pk+l)} \to f^{(l)}$ for each $l = 0, 1, \ldots, p-1$ and $k = 0, 1, 2, \ldots$. Pick pN so large that no two elements of G_λ have the same initial segment of length pN and choose $f_{\beta n}$ so that it has the same initial segment of length pN as f. Let the first term in $f_{\beta n}$ which differs from the corresponding term of f be the

$(pN+pk+l)$th one. Then $f_{\beta n}^{(pk+l)}$ agrees with $f^{(l)}$ precisely in the first pN terms and disagrees in $(pN+1)$st term. In other words, each G_β, $\beta<\lambda$ contains an f_β which agrees with some $f^{(l)}$ in the first pN terms but disagrees in the $(pN+1)$st term. Since the $f^{(l)}$ form a finite set there is one l which corresponds to a cofinal sequence of β's and hence to all $\beta<\lambda$. Similarly since there are only $|S|-1$ choices for the $(pN+1)$st term there is a cofinal sequence of β's with the same $(pN+1)$st digit. Since F_S is compact, these f_β have a convergent subsequence converging to $f^*\in G_\lambda$ whose initial block of $pN+1$ digits disagrees with the initial blocks of the elements of G_λ, a contradiction.

3.11. THEOREM. *For each denumerable ordinal α which is not a limit ordinal there exists a closed denumerable $G\subset F_{\{0,1\}}$ which is closed under integration such that $\alpha(G)=\alpha$. If α is not the successor of a limit ordinal then the set G can be chosen to be closed under differentiation. The restrictions on α are necessary. For each of these α we can pick $G\subset F_{\{0,1,2\}}$ so that $G_{\alpha-1}=\{2e^z\}$.*

Proof. We first construct $G\subset F_{\{0,1,2\}}$. For $\alpha=1$ we can pick $G=\{2e^z\}$. If the theorem holds for α then it holds for $\alpha+1$ by Lemma 3.8. If the theorem holds for all $\alpha<\lambda$, where λ is a limit ordinal, then it holds for $\lambda+1$ and $\lambda+2$ respectively by Lemma 3.9. Thus the theorem holds for all denumerable ordinals. The necessity of the restrictions on α is the result of Lemma 3.10. The fact that all constructions can be made in $F_{\{0,1\}}$ was proved in Theorem 3.7.

3.12. LEMMA. *For every $f\in F_S$ the set $\langle f\rangle_1$ is closed under differentiation and integration. If $\langle f\rangle$ is infinite then so is $\langle f\rangle_1$.*

Proof. Let $g\in\langle f\rangle_1$, then $g=\lim f^{(n_i)}$, where $n_1<n_2<\cdots$. Thus $g'=\lim f^{(n_i+1)}$ and the sequence $(f^{(n_i-1)})$ has a convergent subsequence which converges to an integral of g.

If $\langle f\rangle_1$ is finite it can be closed under integration only if all its elements are purely periodic. If $\langle f\rangle$ is infinite then f is not eventually periodic. Therefore f must contain infinitely many blocks of consecutive digits of some length, l, which are not initial blocks of any element of $\langle f\rangle_1$. Since the number of different blocks is finite, there must be some block of length l which occurs infinitely often in f but does not occur in any element of $\langle f\rangle_1$. This contradicts the compactness of $\langle f\rangle$.

3.13. THEOREM. *A denumerably infinite closed subset of F_S which is closed under differentiation contains an infinite closed subset of solutions of differential equations of the form $f^{(n+p)}=f^{(n)}$ with fixed p.*

Equivalently, a denumerably infinite closed set of sequences with elements in a finite set which is closed under the shift operation contains an infinite closed subset of eventually periodic sequences with fixed period p.

Equivalently, a denumerably infinite closed subset of the interval [0, 1], *which is closed under multiplication by an integer $b>1$ modulo* 1, *contains an infinite closed subset of rational points whose denominators are bounded multiples of powers of b.*

Proof. If we call the set in question G, then according to Theorem 3.10 we have $\alpha(G)=\beta+2$ where $G_{\beta+1}$ consists of a finite number of eventually periodic sequences. Let p be the common period of the elements of $G_{\beta+1}$ and let all elements of $G_{\beta+1}$ be periodic from the Nth term on.

Now $G_{\beta+1}$ contains $\langle g\rangle_1$ for every $g\in G_\beta$ and thus, by Lemma 3.12, the set $\langle g\rangle$ is finite for every $g\in G_\beta$. In other words all elements of G_β are eventually periodic. If $g\in G_\beta$ does not have period p then there exists an $M\geqq N$ so that $g^{(M+p)}(0)\neq g^{(M)}(0)$. Thus the initial block of length $N+p$ of $g^{(M-N)}$ does not agree with the initial block of any element of $G_{\beta+1}$. The number of elements of G_β whose initial block of a fixed length disagrees with the initial blocks of all elements of $G_{\beta+1}$ must be finite due to the compactness of G_β. Hence there exists a finite number $g_1, g_2, \ldots, g_n$ of elements of G_β such that every $g\in G_\beta$ of period different from p has derivative of some order equal to some g_i. Thus, if we let P be the least common multiple of p and the periods of $g_1, \ldots, g_n$, then every element of G_β has eventual period P. The set G_β satisfies the conditions of the theorem.

It is worth noting that closure and closure under differentiation without the hypothesis of denumerability do not imply the existence of any periodic elements. For example, there exist sequences $f\in F_{\{0,1,2\}}$ so that no two adjacent blocks of digits are equal [1]. Thus $\langle f\rangle$ can contain no periodic elements. According to Theorem 3.13 we can infer that $|\langle f\rangle|=2^{\aleph_0}$.

3.14. THEOREM. *For every denumerable ordinal α which is different from* 2, *a limit ordinal, or the successor of a limit ordinal, there exists an $f\in F_S$ so that $\langle f\rangle$ is denumerable and $\alpha(\langle f\rangle)=\alpha$. The restrictions on α are necessary.*

Proof. If $\alpha=1$ we can choose f periodic so that $\langle f\rangle$ is finite and $\langle f\rangle_1=\varnothing$. If $\alpha>1$ write $\alpha=\beta+2$. If β is finite we can pick, by Theorem 3.11, a denumerable closed $G\subset F_{\{0,1\}}$ which is closed under differentiation and integration and has $\alpha(G)=\beta$. If β is infinite but not a limit ordinal we can pick by Theorem 3.4 a denumerable closed $G\subset F_{\{0,1\}}$ which is closed under differentiation and integration and has $\alpha(G)=\beta+1$. If β is a limit ordinal we pick a sequence of denumerable closed $G_n\subset F_{\{0,1\}}$ which are closed under differentiation and satisfy $\sup \alpha(G_n)=\beta$. We then let $G\subset F_{\{0,1,2\}}$ be the set G^* constructed in Lemma 3.9.

Now let $G=\{g_n \mid n=1, 2, \ldots\}$ and let g_{nm} be the initial block of length m of g_n. We define the sequence of blocks B_n by $B_1=g_{11}$ and if $B_n=g_{ij}$ then

$$B_{n+1} = g_{i+1,j} \quad \text{if } i<j,$$
$$B_{n+1} = g_{1,j+1} \quad \text{if } i=j.$$

Thus $(B_1, B_2, B_3, \ldots)=(g_{11}, g_{12}, g_{22}, g_{13}, g_{23}, g_{33}, \ldots)$.

Now define f as the sequence consisting of the blocks B_n in succession with n digits 2 between B_n and B_{n+1}. Then $\langle f\rangle_1$ consists of the sequences of G preceded by an arbitrary number of 2's and infinite sequences of 2's preceded by an arbitrary block g_{ij}. The latter sequences occur in $\langle f\rangle_1$ since G is closed under integration and thus every g_{ij} occurs infinitely often as the terminal block of B_n. Thus $\langle f\rangle_2$ consists of the sequences of G_1 preceded by an arbitrary number of 2's, of G and of $2e^z$. Hence for each $1+\gamma<\alpha(G)$ the set $\langle f\rangle_{2+\gamma}$ consists of the sequences of $G_{1+\gamma}$ preceded by an arbitrary number of 2's, of G_γ and of $2e^z$.

In case β is finite we therefore have

$$\langle f\rangle_{\beta+1} = \langle f\rangle_{1+\beta} = \langle f\rangle_{2+(\beta-1)} = G_{\beta-1} \cup \{2e^z\}$$

which is finite nonempty so that $\alpha(\langle f\rangle)=\beta+2$.

In case β is infinite but not a limit ordinal we have

$$\langle f\rangle_{\beta+1} = \langle f\rangle_{2+(\beta+1)} = \{2e^z\}$$

so that again $\alpha(\langle f\rangle)=\beta+2$.

If β is a limit ordinal then

$$\langle f\rangle_{\beta+1} = G_{\beta+1} = \{2e^z\}$$

so that $\alpha(\langle f\rangle)=\beta+2$.

4. **Open questions and concluding remarks.** We have seen that there are functions in F_S which satisfy particularly simple differential equations of the form $f^{(m)}=f^{(n)}$, $m\neq n$. It is easy to see that every solution in F_S of a linear differential equation with constant coefficients has eventually periodic derivatives, since the solutions of a linear recurrence equation with elements in the finite set S form a periodic sequence [2]. It is easy to show that the result holds for functions of F_S which satisfy linear differential equations with polynomial coefficients. Probably more is true.

4.1. Conjecture. A solution in F_S of an algebraic differential equation with constant coefficients satisfies $f^{(m)}=f^{(n)}$ for some $m\neq n$.

It is possible to generalize these investigations to functions of several variables. The conditions for the theorems in §2 become more complicated.

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