

FUNCTIONS WITH A SPECTRAL GAP

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Introduction. In harmonic analysis, it is important to know how various properties of a function on \mathbf{R}^n reflect themselves as restrictions on its *spectrum*, i.e., the support of its (distributional) Fourier transform. Thus, according to Paley and Wiener, a compact spectrum is characteristic of entire functions of exponential type. In this note we consider a milder restriction: it is only required of the spectrum that it be smaller than the whole n -space. Our results extend those of Levinson, Logan, Ehrenpreis and Malliavin; cf. also Boas [1]. Here we give only bare outlines of proofs; we employ standard vector notations: $t = (t_1, \dots, t_n)$ and $x = (x_1, \dots, x_n)$ are points of \mathbf{R}^n and (t, x) denotes $\sum_1^n t_j x_j$; $|t| = (t, t)^{1/2}$, and dt denotes Haar measure on \mathbf{R}^n .

1. A *gap* in a distribution on \mathbf{R}^n is a nonvoid open ball disjoint from its support. A *spectral gap* in a tempered distribution is a gap in its Fourier transform. In particular, an L^1 function f has a spectral gap if its Fourier transform $\hat{f}(x)$ vanishes on some nonvoid open set. Such f cannot decay too rapidly, by virtue of the following result of N. Levinson.

THEOREM A. *Let $f \in L^1(\mathbf{R})$, and suppose for some $\delta > 0$*

$$(1) \quad \int_0^\infty |f(t)|e^{\delta t} dt < \infty.$$

Then, if $\hat{f}(x)$ vanishes throughout any interval, it vanishes identically.

For the proof, one need only check [4, p. 74] that (1) implies that $\hat{f}(x)$ is the boundary value of a function holomorphic in a strip above the real axis. (Actually Levinson, *loc. cit.*, proves much deeper results, with (1) replaced by weaker hypotheses that do not force analyticity of $\hat{f}(x)$. An account of these, based on a new and simple method, will be given by me in a subsequent paper. The weaker Theorem A will serve as a basis for the present discussion.)

Theorem A admits a straightforward generalization to \mathbf{R}^n . Let us say that a convex cone K in \mathbf{R}^n (all cones will be supposed to have vertex at the origin) is *minor* if there exists a unit vector $t^0 \in \mathbf{R}^n$ such that $\inf(t^0, t)$; $t \in K, |t| = 1$, is positive. Thus, a half-line in \mathbf{R}^1 , or a sector of opening

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less than π in \mathbf{R}^2 , are minor. (It is easy to see that an open convex cone K is minor if and only if there exists a nonsingular linear transformation of \mathbf{R}^n carrying K onto the "first quadrant", i.e., the set K^+ of points of \mathbf{R}^n having all coordinates positive.) By an analyticity argument, as above, one proves easily

THEOREM A'. *Let f be a tempered function on \mathbf{R}^n and suppose for some minor cone K and $\delta > 0$*

$$\int_{\mathbf{R}^n K} |f(t)| e^{\delta|t|} dt < \infty.$$

Then, if f has a spectral gap, $f = 0$.

The condition that K be minor is essential, since for $n \geq 2$ there exist nontrivial $f \in L^1(\mathbf{R}^n)$ which vanish on a half-space and have a spectral gap [9, p. 172].

If $f \in L^1(\mathbf{R}^n)$ does not vanish identically, any $\phi \in L^\infty(\mathbf{R}^n)$ satisfying the convolution equation $f * \phi = 0$ has a spectral gap. Hence it is easy, by duality, to deduce from Theorem A' the following approximation theorem, as observed recently for $n = 1$ by D. J. Newman [6]:

Let K be a minor cone in \mathbf{R}^n , $\delta > 0$, and $w(t)$ a nonnegative measurable function on \mathbf{R}^n equal to 1 on K , and satisfying

$$\int_{\mathbf{R}^n K} w(t) e^{\delta|t|} dt < \infty.$$

Then, for any $f \in L^1(\mathbf{R}^n)$ not identically zero, the translates of f span $L^1(w dt)$.

In particular, the translates of f , restricted to K , span the integrable functions on K .

2. Logan, in a 1965 dissertation [5, p. 26, Theorem 5.2.1] proved

THEOREM B. *Let $f \in L^\infty(\mathbf{R})$ be nonnegative on \mathbf{R}^+ . Then, if f has a spectral gap containing 0, f vanishes identically.*

Observe that here (and in the next section) the *position* of the spectral gap (i.e. containing the origin) is essential. We sketch a proof, based on a new idea which suggests the correct generalization to \mathbf{R}^n . We may assume $f \in L^1(\mathbf{R})$ (for to reduce the general case to this, consider $f(t) \cdot (\sin \epsilon t)^2/t^2$ with sufficiently small ϵ), and that $\hat{f}(t) = 0$ for $|x| \leq 3$. A simple application of Parseval's formula gives for $m = 0, 1, \dots$

$$2\pi \int_0^\infty f(t) t^m e^{-t} dt = m! \int_{-\infty}^\infty \hat{f}(x) (1 - ix)^{-m-1} dx.$$

The integral on the right is bounded by $(\int_{|x| \geq 3} |x|^{-m-1} dx) \cdot \|\hat{f}\|_\infty = O(3^{-m})$, hence

$$\int_0^{\infty} f(t)e^t dt = \int_0^{\infty} f(t) \left(\sum_{m=0}^{\infty} (2t)^m/m! \right) e^{-t} dt < C \sum_{m=0}^{\infty} (2/3)^m < \infty.$$

Now Theorem A implies $f \equiv 0$. Q.E.D.

Let K, K' be closed cones in \mathbf{R}^n ; we say K' is *strongly enclosed* by K if $\{x \in K' : |x| = 1\}$ is in the interior of K . We now state our first main result:

THEOREM B'. *Let f be a tempered function on \mathbf{R}^n having a spectral gap containing 0, and nonnegative a.e. on the closed convex cone K . Let K' be any closed cone strongly enclosed by K . Then, for some $\delta = \delta(f; K') > 0$, $\int_K f(t)e^{\delta|t|} dt < \infty$.*

COROLLARY. *In the hypotheses of Theorem B', if the cone complementary to K is minor, then f vanishes identically.*

The proof of Theorem B' is in principle like that sketched for Theorem B, but complicated technically. The Corollary then follows using Theorem A'.

REMARK. The hypothesis of nonnegativity on K can be weakened to having range in a sector of opening less than π .

3. Logan (*loc. cit.*) also established a relation between a spectral gap about 0 and exponential decay of the Poisson integral [5, Theorems 6.2.3 and 6.3.1]:

THEOREM C. *Let $f \in L^\infty(\mathbf{R})$. The spectrum of f is disjoint from $(-a, a)$ if and only if the Poisson integral*

$$u(x; y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{y}{(x - \xi)^2 + y^2} d\xi$$

of f satisfies

$$(2) \quad |u(x; y)| \leq Ae^{-ay}, \quad y > 0,$$

where A is a positive constant independent of x .

This theorem readily implies that of Paley and Wiener. Logan's proof uses analytic functions. I gave [8, p. 152] a proof using only Fourier analysis, which can be extended to n dimensions. Letting $B(x^0; a)$ denote the open ball in \mathbf{R}^n with center x^0 and radius a , we have our second main result:

THEOREM C'. *Let f be a locally integrable function on \mathbf{R}^n such that*

$$(3) \quad \int (1 + |x|^2)^{-(n+1)/2} |f(x)| dx < \infty.$$

The spectrum of f is disjoint from $B(0; a)$ if and only if the Poisson integral

$$u(x; y) = c_n \int_{\mathbf{R}^n} f(\xi) \frac{y}{(|x - \xi|^2 + y^2)^{(n+1)/2}} d\xi$$

of f satisfies, for every $\varepsilon > 0$,

$$(4) \quad \int |u(x, y)|(1 + |x|^2)^{-(n+1)/2} dx \leq A(\varepsilon)e^{-(a-\varepsilon)y}, \quad y > 0,$$

where $A(\varepsilon)$ is a positive number independent of x . If (3) is replaced by the stronger condition $f \in L^\infty$, (4) is to be replaced by

$$(4a) \quad |u(x, y)| \leq A(\varepsilon)e^{-(a-\varepsilon)y}, \quad y > 0.$$

The proof requires estimates for “minimal extrapolations” from the interior, as well as the exterior, of a ball; these will be given elsewhere. As in the case $n = 1$, the “only if” part of the theorem can be strengthened when $f \in L^\infty$.

If, for $k \in L^1(\mathbf{R}^n)$, we denote by $k_{(y)}$ the “dilated function”:

$$k_{(y)}(x) = y^{-n}k(y^{-1}x); \quad x \in \mathbf{R}^n, y > 0,$$

then Theorem C' (in the case $f \in L^\infty$) may be written: Let

$$(5) \quad k(t) = c_n(1 + |t|^2)^{-(n+1)/2}$$

(so that $\hat{k}(x) = e^{-|x|}$); the condition

$$(6) \quad |(f * k_{(y)})(x)| \leq A(\varepsilon)\hat{k}((a - \varepsilon)y), \quad y > 0,$$

holds for all $\varepsilon > 0$, if and only if the spectrum of f is disjoint from $B(0; a)$.

Now, this proposition can be established for a large class of kernels $k(t)$ in place of (5), using exactly the same method; in particular, for $k(t) = e^{-|t|^2}$, a result obtained otherwise by Ehrenpreis and Malliavin; see [3, Corollary 5]. With this special choice of k , we may permit f in (6) to be any tempered distribution.

4. Assuming (4a) holds for a single value of x , we can nonetheless obtain spectral information about f . First, some notation: a locally integrable function on \mathbf{R}^n is *anti-radial* if its integral over $B(0; r)$ vanishes for every $r > 0$. Every locally integrable function admits an essentially unique decomposition into a radial and an anti-radial part (for $n = 1$, this is just the even-odd decomposition). We now state our third main result:

THEOREM D. *Let f be a bounded measurable function on \mathbf{R}^n whose Poisson integral $u(x; y)$ satisfies $|u(0; y)| \leq Ae^{-ay}$. Then, the radial part of f has spectrum disjoint from $B(0; a)$.*

Combining Theorems C' and D we deduce a proposition solely about harmonic functions: *If u is the Poisson integral of a radial function in*

$L^\infty(\mathbf{R}^n)$, (4a) holds for every $x \in \mathbf{R}^n$ if it holds for $x = 0$. For $n = 1$ this is a consequence of a classical Phragmén-Lindelöf theorem, but for $n > 1$ it appears to be new. Another corollary of Theorem D is: *If u is a bounded harmonic function on $\mathbf{R}^n \times \mathbf{R}^+$ satisfying $u(0; y) = O(e^{-ay})$ as $y \rightarrow +\infty$, for every $a > 0$, then $u(0, y)$ vanishes identically (or, what is the same thing, $u(x; 0+)$ is an anti-radial function on \mathbf{R}^n).*

Theorem D also yields a particularly simple proof of the existence of “lacunae” for the wave equation (cf. [3, p. 417] for terminology). Also, *the analog of Theorem D for the kernel $k(t) = e^{-|t|^2}$ is valid*; this is a refinement of a theorem in [3].

PROOF OF THEOREM D. Assume first $f \in L^1(\mathbf{R}^n)$. We may assume f radial, since the anti-radial part contributes nothing to $u(0; y)$. By assumption,

$$\left| \int_{\mathbf{R}^n} f(x) \cdot y(|x|^2 + y^2)^{-(n+1)/2} dx \right| \leq Ce^{-ay}.$$

Substituting here

$$y(|x|^2 + y^2)^{-(n+1)/2} = A_n \int e^{-y|t|} \cdot e^{-i(t,x)} dt$$

(where A_n depends only on n), and applying Fubini’s Theorem, yields (integrations are over \mathbf{R}^n):

$$A_n \left| \int \hat{f}(t) \cdot e^{-y|t|} dt \right| \leq Ce^{-ay}.$$

Writing $\hat{f}(t) = \phi(|t|)$, we have

$$\int_0^\infty s^{n-1} \phi(s) e^{-ys} ds = O(e^{-ay}), \quad y \rightarrow +\infty.$$

Now a simple argument shows that $\phi(s)$ must vanish for $s < a$, hence $\hat{f}(t) = 0$ for $|t| < a$, as we wished to show.

The general case, when f need not belong to L^1 , leads to serious complications; to be able to perform the crucial “Fubini” step, we replace the Fourier kernel $e^{-i(t,x)}$ by that of Bochner [2, p. 112, (5)], in an n -dimensional version, and then suitably extend the spectral analysis of Pollard [7]. (This is also applicable to f which satisfy only (3).)

5. Let G denote any l.c.a. group, \hat{G} its dual. Let E, \hat{E} be closed subsets of G, \hat{G} respectively. We say the pair (E, \hat{E}) is *interpolatory* if, for every $f \in L^1(G)$, there exists $f_0 \in L^1(G)$ supported in E such that $\hat{f}_0(x) = \hat{f}(x)$, $x \in \hat{E}$. This is equivalent to saying that every function in $L^1(G \setminus E)$ extends to an element of $L^1(G)$ whose Fourier transform vanishes on \hat{E} . For instance, one can show when $G = \mathbf{R}^n$, that *this is the case if $\mathbf{R}^n \setminus E$ and \hat{E} are compact*. On the other hand, Theorem B’ implies that (E, \hat{E}) is not inter-

polatory if $\mathbf{R}^n \setminus E$ contains a nonvoid open cone and \hat{E} has interior. Thus, the Fourier transforms of functions supported on proper subcones of \mathbf{R}^n are constrained in their local behavior, they possess "local structure". The detailed nature of this local structure was somewhat clarified in [9] for the analogous situation in $L^\infty(\mathbf{R}^n)$. For $n = 1$ one can show that the presence of arbitrarily long intervals in the complement of the spectrum already forces local structure.

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