

Functions with bounded n th differences

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Abstract. Let A be a commutative semigroup under addition and let Y be a Banach space. For each natural number n , there exists $k_n > 0$ with the property: if $\delta > 0$ and $f: A \rightarrow Y$ such that

$$\left| \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(x+kh) \right| \leq \delta \quad \text{for all } x, h \in A,$$

then there exists $g: A \rightarrow Y$ such that

$$|f(x) - g(x)| \leq k_n \delta$$

and

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} g(x+kh) = 0 \quad \text{for all } x, h \in A.$$

1. Introduction. Throughout this paper A denotes a commutative semigroup under addition and Y denotes a real Banach space with the norm of $y \in Y$ denoted by $|y|$. Let Y^A be the real vector space of all functions from A to Y . For $h \in A$, the linear difference operator $\Delta_h: Y^A \rightarrow Y^A$ is defined by $\Delta_h f(x) = f(x+h) - f(x)$ for $f \in Y^A$ and $x, h \in A$. The n th iterate of Δ_h , Δ_h^n , satisfies the identity

$$\Delta_h^n f(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(x+kh) \quad \text{for } x, h \in A \text{ and } f \in Y^A.$$

Whitney [9] has shown that if $Y = \mathbb{R}$ (the real numbers), $A = \mathbb{R}$ (or

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THEOREM A. *If $a: A^n \rightarrow Y$ is symmetric and n -additive, then for all $x, h_1, \dots, h_k \in A$,*

$$\Delta_{h_1} \dots \Delta_{h_k} a^*(x) = \begin{cases} n!a(h_1, \dots, h_n) & \text{if } k = n, \\ 0 & \text{if } k > n. \end{cases}$$

THEOREM B. *To each natural number n there correspond natural numbers s and p and integers m_1, \dots, m_p with the property: for every $h_1, \dots, h_n \in A$ there exist $u_1, \dots, u_p, v_1, \dots, v_p \in A$ such that*

$$(n!)^{2^s} \Delta_{h_1} \dots \Delta_{h_n} f(x) = \sum_{k=1}^p m_k \Delta_{u_k}^n f(x+v_k)$$

for all $x \in A$ and $f \in Y^A$.

For future reference, let $M_n = (n!)^{-2^s} \sum_{k=1}^p |m_k|$.

THEOREM C. *If n is natural number and $f: A \rightarrow Y$, then the following are equivalent:*

- (i) $\Delta_h^n f(x) = 0$ for all $x, h \in A$,
- (ii) $\Delta_{h_1} \dots \Delta_{h_n} f(x) = 0$ for all $x, h_1, \dots, h_n \in A$,
- (iii) *there exist $a_0 \in Y$ and symmetric, k -additive $a_k: A^k \rightarrow Y, 1 \leq k \leq n-1$, such that*

$$f(x) = a_0 + \sum_{k=1}^{n-1} a_k^*(x) \quad \text{for all } x \in A.$$

It is easy to check that $\Delta_{h+k} - \Delta_h - \Delta_k = \Delta_h \Delta_k = \Delta_k \Delta_h$ for all $h, k \in A$.

3. Main results. First we will generalize and give shorter proofs of the theorems of Hyers in [3] and [4]. The proof of the first theorem, on which our analysis rests, is essentially the same as the proof given by Hyers [3] but is short and included here for completeness.

THEOREM 1. *If $\delta > 0$ and $f: A \rightarrow Y$ such that $|f(x+y) - f(x) - f(y)| \leq \delta$ for all $x, y \in A$, then there exists a unique $a: A \rightarrow Y$ such that*

$$a(x+y) = a(x) + a(y)$$

and

$$|f(x) - a(x)| \leq \delta \quad \text{for all } x, y \in A.$$

Moreover, $a(x) = \lim_{k \rightarrow +\infty} f(kx)/k$ for all $x \in A$.

Proof. For every $x \in A$, $|\frac{1}{2}f(2x) - f(x)| = \frac{1}{2}|f(2x) - 2f(x)| \leq \frac{1}{2}\delta$. If n is a natural number and $x \in A$, then

$$\begin{aligned} \left| \frac{f(2^{n+1}x)}{2^{n+1}} - f(x) \right| &\leq \frac{1}{2^n} \left| \frac{f(2(2^n x))}{2} - f(2^n x) \right| + \left| \frac{f(2^n x)}{2^n} - f(x) \right| \\ &\leq \delta/2^{n+1} + \left| \frac{f(2^n x)}{2^n} - f(x) \right|. \end{aligned}$$

It follows by induction that for every natural number n and every $x \in A$,

$$\left| \frac{f(2^n x)}{2^n} - f(x) \right| \leq \left(\frac{1}{2} + \dots + \frac{1}{2^n} \right) \delta.$$

If n and p are natural numbers and $x \in A$, then

$$\left| \frac{f(2^{n+p}x)}{2^{n+p}} - \frac{f(2^n x)}{2^n} \right| = \frac{1}{2^n} \left| \frac{f(2^p(2^n x))}{2^p} - f(2^n x) \right| \leq \delta/2^n.$$

Hence, for every $x \in A$, $\left\{ \frac{f(2^n x)}{2^n} \right\}_{n=1}^{\infty}$ is a Cauchy sequence in Y and if we denote its limit by $a(x)$ we have

$$a(x+y) = a(x) + a(y) \quad \text{for all } x, y \in A$$

since

$$\left| \frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} \right| \leq \delta/2^n$$

for all $x, y \in A$ and every natural number n . Also, since

$$\left| \frac{f(2^n x)}{2^n} - f(x) \right| \leq \left(\frac{1}{2} + \dots + \frac{1}{2^n} \right) \delta,$$

it follows that

$$|f(x) - a(x)| \leq \delta \quad \text{for all } x \in A.$$

Since a is additive, for every natural number k and each $x \in A$,

$$\left| \frac{f(kx)}{k} - a(x) \right| = \frac{1}{k} |f(kx) - a(kx)| \leq \delta/k,$$

so that

$$a(x) = \lim_{k \rightarrow +\infty} \frac{f(kx)}{k} \quad \text{for each } x \in A.$$

This calculation also proves the uniqueness of a and the proof is complete.

We will need

Suppose the theorem is true for $n = m \geq 1$. Let $f: A \rightarrow Y$ such that

$$|\Delta_{h_1} \dots \Delta_{h_{m+1}} f(x)| \leq \delta \quad \text{for all } x, h_1, \dots, h_{m+1} \in A.$$

For each $h_1, \dots, h_m \in A$,

$$|\Delta_h (\Delta_{h_1} \dots \Delta_{h_m} f(x))| \leq \delta$$

or

$$(1) \quad |\Delta_{h_1} \dots \Delta_{h_m} f(x+h) - \Delta_{h_1} \dots \Delta_{h_m} f(x)| \leq \delta \quad \text{for all } x, h \in A.$$

But, for any $x, h'_1, h_1, h_2, \dots, h_m \in A$,

$$\begin{aligned} & |\Delta_{h_1+h'_1} \Delta_{h_2} \dots \Delta_{h_m} f(x) - \Delta_{h'_1} \Delta_{h_2} \dots \Delta_{h_m} f(x) - \Delta_{h_1} \Delta_{h_2} \dots \Delta_{h_m} f(x)| \\ &= |(\Delta_{h_1+h'_1} - \Delta_{h'_1} - \Delta_{h_1})(\Delta_{h_2} \dots \Delta_{h_m} f(x))| = |\Delta_{h'_1} \Delta_{h_1} \Delta_{h_2} \dots \Delta_{h_m} f(x)| \delta. \end{aligned}$$

Thus, for each $x \in A$, the mapping $(h_1, h_2, \dots, h_m) \rightarrow \Delta_{h_1} \Delta_{h_2} \dots \Delta_{h_m} f(x)$ of A into Y is "almost" additive in the first variable. But, since difference operators commute, for each $x \in A$ the mapping is symmetric and thus "almost" additive in each variable. Applying Theorem 2 we find that for each $x \in A$ there is a unique $a_x: A^m \rightarrow Y$ which is symmetric, m -additive and such that

$$(2) \quad |\Delta_{h_1} \dots \Delta_{h_m} f(x) - a_x(h_1, \dots, h_m)| \leq \delta$$

for all $h_1, \dots, h_m \in A$.

From (1) and (2) we find

$$(3) \quad |\Delta_{h_1} \dots \Delta_{h_m} f(x+h) - a_x(h_1, \dots, h_m)| \leq 2\delta$$

for all $x, h, h_1, \dots, h_m \in A$.

Now replace x by $x+y$ in (3) and then replace h by $y+h$ in (3) and compare the resulting inequalities to conclude that

$$(4) \quad |a_x(h_1, \dots, h_m) - a_{x+y}(h_1, \dots, h_m)| \leq 4\delta$$

for all $x, y, h_1, \dots, h_m \in A$. Thus, for every $x, y \in A$, $a_x - a_{x+y}$ is bounded and m -additive. It easily follows that $a_x = a_{x+y}$. Hence $a_x = a_y$ for all $x, y \in A$. Let $a_x = m!a_m$, $x \in A$, to conclude from (2) that

$$(5) \quad |\Delta_{h_1} \dots \Delta_{h_m} f(x) - m!a_m(h_1, \dots, h_m)| \leq \delta$$

for all $x, h_1, \dots, h_m \in A$.

Let $f_1 = f - a_m^*$. Then, by Theorem A,

$$\begin{aligned} |\Delta_{h_1} \dots \Delta_{h_m} f_1(x)| &= |\Delta_{h_1} \dots \Delta_{h_m} f(x) - \Delta_{h_1} \dots \Delta_{h_m} a_m^*(x)| \\ &= |\Delta_{h_1} \dots \Delta_{h_m} f(x) - m! a_m(h_1, \dots, h_m)| \leq \delta \end{aligned}$$

for all $x, h_1, \dots, h_m \in A$. By our inductive hypothesis, there exist symmetric k -additive $a_k: A^k \rightarrow Y$, $1 \leq k \leq m-1$, such that $|\Delta_h(f_1 - \sum_{k=1}^{m-1} a_k^*)(x)| \leq \delta$ for all $x, h \in A$. But $f - \sum_{k=1}^m a_k^* = f_1 - \sum_{k=1}^{m-1} a_k^*$ so we are done.

Theorem II of [4] follows from Theorem 3 by applying (b) of

LEMMA 1. Suppose $f: A \rightarrow Y$ and $\delta > 0$ such that $|\Delta_h f(x)| \leq \delta$ for all $x, h \in A$. Then

- (a) there exists $a_0 \in Y$ such that $|f(x) - a_0| \leq 2\delta$ for all $x \in A$;
- (b) if A has a zero (a member 0 of A such that $0 + x = x + 0 = x$ for all $x \in A$), then $|f(x) - f(0)| \leq \delta$ for all $x \in A$;
- (c) if for $x, y \in A$ there exists $h \in A$ such that either $y = x + h$ or $x = y + h$ and if $Y = \mathbb{R}$, then there exists $a_0 \in Y$ such that $|f(x) - a_0| \leq \delta/2$ for all $x \in A$.

Proof. (a) Fix $y_0 \in A$ and let $a_0 = f(y_0)$. For any $x \in A$

$$|f(x + y_0) - f(x)| = |\Delta_{y_0} f(x)| \leq \delta$$

and

$$|f(x + y_0) - f(y_0)| = |\Delta_x f(y_0)| \leq \delta$$

so

$$|f(x) - a_0| \leq 2\delta.$$

(b) For any $x \in A$, $|f(x) - f(0)| = |\Delta_x f(0)| \leq \delta$.

(c) Let $x, y \in A$ and suppose that $y = x + h$ for some $h \in A$. Then

$$|f(y) - f(x)| = |\Delta_h f(x)| \leq \delta.$$

Then f is bounded and we can let

$$a_0 = \{\sup_{x \in A} f(x) + \inf_{x \in A} f(x)\}/2.$$

We now use Theorem 3 to generalize the theorem of Whitney referred to in the introduction.

THEOREM 4. *Suppose n is a natural number, $\delta > 0$ and $f: A \rightarrow Y$ such that*

$$|\Delta_h^n f(x)| \leq \delta \quad \text{for all } x, h \in A.$$

Then

(i) *there exist symmetric, k -additive $a_k: A^k \rightarrow Y$, $1 \leq k \leq n-1$ such that*

$$|\Delta_h(f - \sum_{k=1}^{n-1} a_k^*)(x)| \leq M_n \delta \quad \text{for all } x, h \in A;$$

(ii) *if in addition A admits division by $n!$ (for every $k \in A$ there exists $h \in A$ such that $k = n!h$), then*

$$|\Delta_h(f - \sum_{k=1}^{n-1} a_k^*)(x)| \leq 2\delta \quad \text{for all } x, h \in A;$$

(iii) *if in addition A is a group and admits division by $n!$, then*

$$|\Delta_h(f - \sum_{k=1}^{n-1} a_k^*)(x)| \leq 2\delta / \sup_m \binom{n}{m} \quad \text{for all } x, h \in A.$$

Proof. By Theorem B,

$$|\Delta_{h_1} \dots \Delta_{h_n} f(x)| \leq M_n \delta \quad \text{for all } x, h_1, \dots, h_n \in A$$

and (i) follows from Theorem 3.

Let $f_1 = f - \sum_{k=1}^{n-1} a_k^*$ so that f_1 is bounded and $|\Delta_h^n f_1(x)| = |\Delta_h^n f(x)| \leq \delta$ for all $x, h \in A$. Let $x, k \in A$. Choose $h \in A$ such that $k = n!h$. Then the argument used by Whitney on pages 83 and 84 of [8], and attributed to A. Beurling (see [9] also) shows that

$$|\Delta_k f_1(x)| = |f_1(x + n!h) - f_1(x)| \leq 2\delta$$

in case (ii) and

$$|\Delta_k f_1(x)| \leq 2\delta / \sup_m \binom{n}{m}$$

in case (iii).

Notice that the assumptions of Theorem 4 are weaker than those of Theorem 3 but the estimate is not as good in case (i). Using Lemma 1, the result can be sharpened.

The next theorem generalizes Whitney's result.

THEOREM 5. *In addition to the assumptions of Theorem 4, suppose A is a cone in a normed linear space X with nonvoid interior and suppose f is bounded on a nonvoid open subset of A . Then a_1, \dots, a_{n-1} are continuous.*

Proof. Let $g = \sum_{k=1}^{n-1} a_k^*$. Then $\Delta_h^n g(x) = 0$ for all $x, h \in A$ and, by Theorem 4 (i), $\Delta_h(f-g)(x)$ is bounded for $x, h \in A$. By Lemma 1 (a), $f-g$ is bounded and so g is bounded on a nonvoid open subset of A .

Now $a_1: A \rightarrow Y$ is additive. Since A has nonvoid interior, $X = A - A = \{x-y \mid x, y \in A\}$. If $x_1, x_2, y_1, y_2 \in A$ and $x_1 - y_1 = x_2 - y_2$, then $x_1 + y_2 = x_2 + y_1$ so $a_1(x_1) + a_1(y_2) = a_1(x_2) + a_1(y_1)$ or $a_1(x_1) - a_1(y_1) = a_1(x_2) - a_1(y_2)$. Thus we may define $\tilde{a}_1: X \rightarrow Y$ by letting $\tilde{a}_1(x-y) = a_1(x) - a_1(y)$ for all $x, y \in A$. It is easy to check that \tilde{a}_1 is the unique additive extension of a_1 from A to X . Similarly for each $k = 1, 2, \dots, n-1$, there is a unique symmetric k -additive $\tilde{a}_k: X \rightarrow Y$ such that $\tilde{a}_k(x_1, \dots, x_n) = a_k(x_1, \dots, x_n)$ for all $x_1, \dots, x_n \in A$. Let $\tilde{g} = \sum_{k=1}^{n-1} \tilde{a}_k^*$ so that \tilde{g} extends g , \tilde{g} is bounded on a nonvoid open subset of X and

$$\Delta_h^n \tilde{g}(x) = 0 \quad \text{for all } x, h \in X.$$

From a theorem of Mazur and Orlicz [7] it follows that a_k is continuous for $1 \leq k \leq n-1$.

A similar argument, using a theorem of Kemperman [5] instead of the theorem of Mazur and Orlicz, can be applied to prove the following theorem, thereby partially answering a question raised in [9].

THEOREM 6. *In Theorem 4, suppose A is a cone in R^p with nonvoid interior and $Y = R$. If f is bounded on a subset of A having positive Lebesgue measure (in particular, if f is Lebesgue measurable), then a_k is continuous for $1 \leq k \leq n-1$.*

4. Related functional inequalities. For real valued functions on abelian groups, many of our results hold assuming only one-sided boundedness.

LEMMA 2. *Suppose n is a natural number, A is an abelian group and $f: A \rightarrow Y$ and $\delta > 0$. If $\Delta_{h_1} \dots \Delta_{h_n} f(x) \leq \delta$ for all $x, h_1, \dots, h_n \in A$, then*

$$|\Delta_{h_1} \dots \Delta_{h_n} f(x)| \leq \delta \quad \text{for all } x, h_1, \dots, h_n \in A.$$

If n is odd and $\Delta_h^n f(x) \leq \delta$ for all $x, h \in A$, then $|\Delta_h^n f(x)| \leq \delta$ for all $x, h \in A$.

Proof. For any $x, h \in A$ and any $g: A \rightarrow R$, $\Delta_h g(x) = -\Delta_{-h} g(x+h)$.

Thus, in the first case

$$\Delta_{h_1, h_2} \dots \Delta_{h_n} f(x) = -\Delta_{-h_1, h_2} \dots \Delta_{h_n} f(x+h_1) \geq -\delta \quad \text{for all } x, h_1, \dots, h_n \in A.$$

The second assertion follows from the fact that if n is odd, and $g: A \rightarrow R$, then $\Delta_h^n g(x) = -\Delta_{-h}^n g(x+nh)$ for any $x, h \in A$.

We now turn to inequalities associated with equations considered, for example, in [2].

THEOREM 7. *Suppose n is a natural number, $\delta > 0$ and $f, g: A \rightarrow Y$ such that*

$$|\Delta_h^n f(x) - g(h)| \leq \delta \quad \text{for all } x, h \in A.$$

Then there exist symmetric, k -additive $a_k: A^k \rightarrow Y$, $1 \leq k \leq n$, such that

$$|\Delta_h(f - \sum_{k=1}^n a_k^*)(x)| \leq 2M_{n+1}\delta$$

and

$$|g(h) - n!a_n^*(h)| \leq (1 + 2^n M_{n+1})\delta \quad \text{for all } x, h \in A.$$

Proof. For all $x, h \in A$,

$$|\Delta_h^{n+1} f(x)| = |\{\Delta_h^n f(x+h) - g(h)\} - \{\Delta_h^n f(x) - g(h)\}| \leq 2\delta.$$

Hence, by Theorem 4 (i), there exist symmetric, k -additive $a_k: A^k \rightarrow Y$, $1 \leq k \leq n$ such that, for all $x, h \in A$,

$$|\Delta_h(f - \sum_{k=1}^n a_k^*)(x)| \leq 2M_{n+1}\delta$$

and hence

$$|\Delta_h^n(f - \sum_{k=1}^n a_k^*)(x)| \leq 2^n M_{n+1}\delta.$$

But

$$\Delta_h^n(\sum_{k=1}^n a_k^*)(x) = n!a_n^*(h)$$

so that

$$|\Delta_h^n f(x) - n!a_n^*(h)| \leq 2^n M_{n+1}\delta$$

and hence

$$|g(h) - n!a_n^*(h)| \leq (1 + 2^n M_{n+1})\delta \quad \text{for all } x, h \in A.$$

COROLLARY. *If n is a natural number, $\delta > 0$ and $f: A \rightarrow Y$ such that*

$$|\Delta_h^n f(x) - n!f(h)| \leq \delta \quad \text{for all } x$$

there exists a symmetric, n -additive $a_n: A^n \rightarrow Y$ such that

$$|f(h) - a_n^*(h)| \leq (1 + 2^n M_{n+1})\delta/n! \quad \text{for all } h \in A.$$

In Theorem 7 and the corollary, the estimates could be improved, if stronger assumption are made on A and Y , by applying (ii) or (iii) of Theorem 4.

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