PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 105(119) (2019), 65–80 DOI: https://doi.org/10.2298/PIM1919065P

FUNCTIONS WITH CONSTANT SUMS OVER A HYPERPLANE AND APPLICATIONS

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Abstract. Two functional equations exhibiting functions with constant sums over points lying in a hyperplane are solved, and the results are applied to characterize major trigonometric and hyperbolic functions.

1. Introduction

In a recent paper [**4**], the first and third authors solved a functional equation with n parameters, representing the angles of a convex n-gon, and used it to characterize the tangent function. This functional equation generalizes the one originally considered by Davison [**3**] and later proved by Benz [**2**] where the parameters are the three angles of a triangle. More specifically, the case $n = 3$ which is Davison-Benz's theorem states that the function $f\colon (0, \pi/2) \to (0, \infty)$ satisfying

$$f(x)f(y)f(z) = f(x) + f(y) + f(z) \quad (x, y, z \in (0, \pi/2)) \tag{1.1}$$

with

$$x + y + z = \pi \tag{1.2}$$

is of the form

$$f(x) = \tan\Big(kx + (1-k)\frac{\pi}{3}\Big) \quad (x \in (0, \pi/2)),$$

with an arbitrary constant $k \in [-1/2, 1]$. Since $\tan A \tan B \tan C = \tan A + \tan B + \tan C$ for any triangle ABC, the functional equation (1.1) indeed yields a characterization of the tangent function. One natural question is whether there are functional equations derived through generalizing the well-known trigonometric and hyperbolic identities

$$\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2, \tag{1.3}$$

$$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2, \tag{1.4}$$

$$\sinh(y_1 + y_2) = \sinh y_1 \cosh y_2 + \cosh y_1 \sinh y_2, \tag{1.5}$$

2010 *Mathematics Subject Classification:* Primary 39B22; Secondary 33B10.
Key words and phrases: functions with constant sums, hyperplane.
Communicated by Stevan Pilipović.

$$\cosh(y_1 + y_2) = \cosh y_1 \cosh y_2 + \sinh y_1 \sinh y_2 \tag{1.6}$$

that can be used to characterize the sine, cosine and other major hyperbolic functions. This question will be affirmatively confirmed as consequences of our main theorems here.

Analyzing the proof in [**4**], we see the following key steps:

- first, the functional equation, generalization of (1.1), is bijectively transformed into a new functional equation, henceforth referred to as a *constant sum functional equation* or CSFE for short, showing that the unknown function possesses a constant sum over a set of n parameters lying in a hyperplane, i.e., points subject to a condition generalizing (1.2), referred to as a *hyperplane condition* or HC for short;
- second, by suitable change of variables the CSFE and HC are simplified in order to determine all the possible solution functions;
- third, the modified CSFE for each possible solution function is strategically transformed into a Cauchy additive functional equation over restricted domains, and its shape is determined.

Note that this approach bears results resembling the following one in the book of Kannappan [**5**, Theorem 1.76, p. 58]: the functions $f_i\colon (0,1) \to \mathbb{R}$ satisfy the functional equation

$$\sum_{i=1}^{n} f_i(p_i) = 0, \quad 0 < p_i < 1 \ (i = 1, \dots, n), \ \sum_{i=1}^{n} p_i = 1,$$

for arbitrary (but fixed) $n \geqslant 3$, if and only if, there exists an additive function $A\colon \mathbb{R} \to \mathbb{R}$ such that

$$f_i(x) = A(x) + b_i, \quad x \in (0,1),$$

where b_i $(i = 1, \dots, n)$ are constants with $A(1) + \sum_{i=1}^{n} b_i = 0$.

In the present work, we push our earlier investigation further by solving two general CSFE's, one for a finite number of unknown functions and the other for a single unknown function, subject to two types of HC's extending the work in [**4**]. The results so obtained are then applied to characterize the sine, cosine and other major hyperbolic functions.

Our two main theorems are:

Theorem 1.1. *Let n be an integer $\geqslant 3$, and let I denote the closed interval $[a, b]$ with $b > a$. Then the functions $\phi_i\colon I \to \mathbb{R}$ $(i = 1, 2, \dots, n)$ satisfy the CSFE*

$$\sum_{i=1}^{n} \phi_i(x_i) = T_1, \quad x_i \in I \ (i = 1, 2, \dots, n), \tag{1.7}$$

subject to the HC

$$\sum_{i=1}^{n} x_i = T_2, \tag{1.8}$$

where T_1, T_2 are real constants with

$$\frac{n(2a+b)}{3} < T_2 < \frac{n(a+2b)}{3}, \tag{1.9}$$

if and only if, there exists an additive function $A\colon \mathbb{R} \to \mathbb{R}$ such that

$$\phi_i(x) = A(x) - A(T_2/n) + \gamma_i \quad (i = 1, 2, \ldots, n),$$

where the constants γ_i satisfy

$$\sum_{i=1}^{n} \gamma_i = T_1. \tag{1.10}$$

Theorem 1.2. *Let n be an integer $\geqslant 3$, and let $I_1 := (a,b)$, $I_2 := (c,d)$ be two open intervals with $b > a$, $d > c$. Then the function $\phi\colon I_1 \to I_2$ satisfies the CSFE*

$$\sum_{i=1}^{n} \phi(x_i) = U_1, \tag{1.11}$$

subject to the HC

$$\sum_{i=1}^{n} x_i = U_2, \tag{1.12}$$

where U_1, U_2 are real constants, if and only if,

$$\phi(x) = k\Big(x - \frac{U_2}{n}\Big) + \frac{U_1}{n}$$

for some fixed k lying in the range

$$\max\Big\{\frac{nc - U_1}{nb - U_2}, \frac{nd - U_1}{na - U_2}\Big\} < k < \min\Big\{\frac{nc - U_1}{na - U_2}, \frac{nd - U_1}{nb - U_2}\Big\}.$$

2. Proof of Theorem 1.1

From (1.9), we see that

$$a < \frac{2a+b}{3} < \frac{T_2}{n} < \frac{a+2b}{3} < b, \quad \text{and} \quad a < \frac{2a+b}{3} < \frac{a+b}{2} < \frac{a+2b}{3} < b. \tag{2.1}$$

We start by making a change of variables to simplify the HC (1.8). Let

$$J := [a - T_2/n, b - T_2/n],$$

which is not a singleton, and define new unknown functions $\psi_i\colon J \to \mathbb{R}$ $(i = 1, \ldots, n)$ by

$$\psi_i(y) = \phi_i\Big(y + \frac{T_2}{n}\Big). \tag{2.2}$$

Observe that if $y \in J$, then $y + T_2/n \in I = [a,b]$. Using (1.7), (1.8) and (2.2), we get

$$\sum_{i=1}^{n} \psi_i(y_i) = T_1, \quad \text{subject to a simplified HC} \quad \sum_{i=1}^{n} y_i = 0 \qquad (y_i \in J). \tag{2.3}$$

There are three disjoint cases to be considered according to the three possibilities on the sizes of $T_2/n - a$ and $b - T_2/n$, i.e.,

$$T_2/n - a > b - T_2/n, \quad T_2/n - a < b - T_2/n \quad \text{and} \quad T_2/n - a = b - T_2/n.$$

We give a detailed proof only for the first case as those for the other two are similar.

If $T_2/n - a > b - T_2/n$, i.e., $T_2/n > (a+b)/2$, then

$$H := [T_2/n - b, b - T_2/n] \subset J$$

is a closed interval, which is not a singleton, symmetric about the origin. Let $u, v \in H$ be such that $u + v \in H$. Using (2.3), we have $\psi_1(u) + \psi_2(v) + \psi_3(-u - v) + \sum_{i=4}^{n} \psi_i(0) = T_1$, i.e.,

$$\psi_1(u) + \psi_2(v) + \psi_3(-u - v) = M_1, \tag{2.4}$$

where $M_1 = T_1 - \sum_{i=4}^{n} \psi_i(0)$. Interchanging u and v, we get

$$\psi_1(v) + \psi_2(u) + \psi_3(-v - u) = M_1. \tag{2.5}$$

Subtracting (2.4) from (2.5) leads to $\psi_1(u) - \psi_2(u) = \psi_1(v) - \psi_2(v)$, showing that this expression must be a constant; call it c_1. Thus,

$$\psi_2(v) = \psi_1(v) - c_1. \tag{2.6}$$

Substituting (2.6) into (2.4), we get

$$\psi_1(u) + \psi_1(v) - c_1 + \psi_3(-u - v) = M_1 \quad (u, v, u + v \in H). \tag{2.7}$$

Next, let $p, q, r \in H$ be such that $p + q, p + r, q + r, p + q + r \in H$. Using (2.7), we have

$$\psi_1(p) + \psi_1(q + r) - c_1 + \psi_3(-p - q - r) = M_1 \tag{2.8}$$

$$\psi_1(p + q) + \psi_1(r) - c_1 + \psi_3(-p - q - r) = M_1 \tag{2.9}$$

$$\psi_1(q) + \psi_1(p + r) - c_1 + \psi_3(-p - q - r) = M_1. \tag{2.10}$$

Subtracting (2.9) and (2.8), we get $\psi_1(p + q) - \psi_1(p) = \psi_1(q + r) - \psi_1(r)$, which depends only on q; call it $D(q)$. Thus,

$$\psi_1(p + q) = \psi_1(p) + D(q). \tag{2.11}$$

Similarly, subtracting (2.9) and (2.10) yields $\psi_1(p+q) - \psi_1(q) = \psi_1(p+r) - \psi_1(r) =: D(p)$, and so

$$\psi_1(p + q) = \psi_1(q) + D(p). \tag{2.12}$$

Equating (2.11) and (2.12), we arrive at $\psi_1(p) - D(p) = \psi_1(q) - D(q)$, which must then be a constant; call it d_1. Thus, $D(q) = \psi_1(q) - d_1$. Replacing this last expression into (2.11), we get

$$\psi_1(p + q) = \psi_1(p) + \psi_1(q) - d_1,$$

i.e.,

$$\psi_1(p + q) - d_1 = (\psi_1(p) - d_1) + (\psi_1(q) - d_1) \quad (p, q, p + q \in H).$$

From the result mentioned in the Remark 1.73 of [**5**, p. 57], we deduce that

$$\psi_1(p) = A_1(p) + d_1 \quad (p \in H), \tag{2.13}$$

for some unique additive function $A_1\colon \mathbb{R} \to \mathbb{R}$.

We next extend the domain of ψ_1. From (2.1), we know that $(a-T_2/n, T_2/n-b)$ and $(2T_2/n-(b+a), b-T_2/n)$ are non-empty sets. Let $w \in [a-T_2/n, T_2/n-b]$, and choose $s \in [2T_2/n-(b+a), b-T_2/n) \subset H$ in such a way that $w+s \in [T_2/n-b, 0) \subset H$. Since $-w-s \in (0, b-T_2/n] \subset H$, using (2.3), we have

$$\psi_1(w) + \psi_2(s) + \psi_3(-w-s) + \sum_{i=4}^{n} \psi_i(0) = T_1,$$

i.e.,

$$\psi_1(w) + \psi_2(s) + \psi_3(-w-s) = M_1. \tag{2.14}$$

Proceeding similarly, we get

$$\psi_1(s) + \psi_2(w) + \psi_3(-s-w) = M_1. \tag{2.15}$$

Subtracting (2.14) and (2.15), we arrive at $\psi_1(w) - \psi_2(w) = \psi_1(s) - \psi_2(s) = c_2$, a constant, and so, using also (2.13),

$$\psi_2(w) = \psi_1(w) - c_2, \tag{2.16}$$

$$\psi_2(s) = \psi_1(s) - c_2 = A_1(s) + d_1 - c_2. \tag{2.17}$$

Repeating the process again with ψ_3 in place of ψ_1, we get

$$\psi_3(w) + \psi_2(s) + \psi_1(-w-s) = M_1, \tag{2.18}$$

$$\psi_3(s) + \psi_2(w) + \psi_1(-s-w) = M_1. \tag{2.19}$$

Subtracting (2.18) and (2.19), we arrive at $\psi_3(w) - \psi_2(w) = \psi_3(s) - \psi_2(s) = c_2'$, a constant, and so,

$$\psi_2(s) = \psi_3(s) - c_2'. \tag{2.20}$$

Combining (2.17) and (2.20), we obtain

$$\psi_3(s) = A_1(s) + d_1 - (c_2 - c_2'). \tag{2.21}$$

Substituting (2.21) and (2.16) into (2.19), using (2.13) and the additivity of the function A_1, we get

$$\psi_1(w) = A_1(w) - 2d_1 + 2c_2 - c_2' + M_1 \quad (w \in [a-T_2/n, T_2/n-b]). \tag{2.22}$$

Combining the two expressions in (2.13) and (2.22), the domain of the function ψ_1 has been extended and so

$$\psi_1(y) = A_1(y) + d_1 \quad (y \in J).$$

Deriving in the same manner, we deduce that

$$\psi_i(y) = A_i(y) + d_i \quad (y \in J,\ i = 1, 2, \ldots, n). \tag{2.23}$$

Keeping ψ_2 fixed, but varying 1 to be any index i, (2.6) and (2.16) become

$$\psi_2(y) = \psi_i(y) - t_i \quad (y \in J,\ i \neq 2) \tag{2.24}$$

for some constants t_i. Using (2.23) and (2.24), we have for all $i \neq 2$

$$A_i(y) + d_i - t_i = \psi_2(y),$$

which shows that for all i, j, we have

$$A_i(y) = A_j(y) + (d_j - d_i) - (t_j - t_i) \quad (y \in J). \tag{2.25}$$

Using (2.23) and (2.25), we deduce that there is an additive function $A\colon \mathbb{R} \to \mathbb{R}$ such that

$$\psi_i(y) = A(y) + \gamma_i \quad (y \in J,\ i = 1, 2, \dots, n) \tag{2.26}$$

where γ_i's are constants. Using (2.2) and (2.26), we see

$$\phi_i(x) = A(x) - A(T_2/n) + \gamma_i \quad (x \in I,\ i = 1, 2, \dots, n).$$

The shapes of the solution functions and the associated condition (1.10) are easily verified.

Remark. Technical condition (1.9) on the range of T_2 is needed in the process of choosing suitable variables in the necessity part of the proof, even though the shape of solution function works without such restriction.

3. Proof of Theorem 1.2

Note first that $na < U_2 < nb$, $nc < U_1 < nd$. As in the proof of Theorem 1.1, we begin with simplifying the HC (1.12). Let

$$J_1 := (a - U_2/n, b - U_2/n) \neq \phi,$$

and define $\psi\colon J_1 \to I_2$ by

$$\psi(y) = \phi\Big(y + \frac{U_2}{n}\Big) \quad (y \in J_1).$$

Functional equation (1.11) and the HC (1.12) become

$$\sum_{i=1}^{n} \psi(y_i) = U_1 \quad \text{subject to} \quad \sum_{i=1}^{n} y_i = 0 \qquad (y_i \in J_1). \tag{3.1}$$

Taking all $y_i = 0$ in (3.1), we have

$$\psi(0) = U_1/n. \tag{3.2}$$

Again, there are three possibilities, namely,

$$U_2/n - a > b - U_2/n, \quad U_2/n - a < b - U_2/n \quad \text{and} \quad U_2/n - a = b - U_2/n.$$

We treat only the first case and omit the proofs of the other two cases as they are similar. If $U_2/n - a > b - U_2/n$, then

$$H_1 := (U_2/n - b, b - U_2/n) \subset J_1$$

is a non-empty open interval symmetric about the origin, and so (3.1) gives

$$\psi(u) + \psi(-u) + \sum_{i=1}^{n-2} \psi(0) = U_1 \quad (u \in H_1).$$

Combining this last relation with (3.2), we get

$$\psi(-u) = \frac{2U_1}{n} - \psi(u) \quad (u \in H_1). \tag{3.3}$$

Next, substituting $u, v \in H_1$ with $u + v \in H_1$ into (3.1) gives

$$\psi(u) + \psi(v) + \psi(-(u+v)) + \sum_{i=1}^{n-3} \psi(0) = U_1.$$

Combining this with (3.2) and (3.3), we see that ψ is almost additive over H_1, i.e.,

$$\psi(u+v) = \psi(u) + \psi(v) - U_1/n \quad (u, v, u+v \in H_1). \tag{3.4}$$

Since $(a - U_2/n, U_2/n - b)$ and $(0, b - U_2/n)$ are nonempty open intervals, substituting

$$w \in (a - U_2/n, U_2/n - b], \ \ s \in (0, b - U_2/n) \subset H_1 \quad \text{with} \quad w + s \in (U_2/n - b, 0) \subset H_1,$$

into (3.1) and using (3.3), we have

$$\psi(w+s) = \psi(w) + \psi(s) - U_1/n. \tag{3.5}$$

Relations (3.4) and (3.5) suggest that the function ψ can be transformed into an additive function. To verify this, define $\beta\colon J_1 \to (c - U_1/n, d - U_1/n)$ by

$$\beta(y) = \psi(y) - U_1/n \quad (y \in J_1), \tag{3.6}$$

so that (3.2) and (3.6) yield $\beta(0) = 0$, while (3.3) and (3.6) yield

$$\beta(-u) = -\beta(u) \quad (u \in H_1).$$

From (3.4) and (3.6), we get

$$\beta(u+v) = \beta(u) + \beta(v) \quad (u, v, u+v \in H_1). \tag{3.7}$$

Now using Remark 1.73 of [**5**, p. 57], there exists a unique additive function $A\colon \mathbb{R} \to \mathbb{R}$ satisfying (3.7), which is an extension of β, i.e., $A \mid_{H_1} = \beta$. Since the additive function A is bounded on H_1, by [**1**, Corollary 5 on p. 15], we have $A(u) = ku$ $(u \in \mathbb{R})$, for some constant k. Thus,

$$\beta(u) = ku \quad (u \in H_1). \tag{3.8}$$

From (3.5), (3.6) and (3.8), for $w \in (a - U_2/n, U_2/n - b] \subset J_1$, $s \in (0, b - U_2/n) \subset H_1$ with $w + s \in (U_2/n - b, 0) \subset H_1$, we get

$$\beta(w) = \beta(w+s) - \beta(s) = k(w+s) - ks = kw$$

which yields $\beta(y) = ky$ $(y \in J_1)$. Since β is the map from $J_1 := (a - U_2/n, b - U_2/n)$ into $(c - U_1/n, d - U_1/n)$, we have

$$\max\Big\{\frac{nc - U_1}{nb - U_2}, \frac{nd - U_1}{na - U_2}\Big\} < k < \min\Big\{\frac{nc - U_1}{na - U_2}, \frac{nd - U_1}{nb - U_2}\Big\}.$$

Reverting back to the definitions of β and ψ, we conclude that

$$\psi(y) = ky + \frac{U_1}{n} \quad (y \in J_1), \qquad \phi(x) = k\Big(x - \frac{U_2}{n}\Big) + \frac{U_1}{n} \quad (x \in I_1).$$

The solution function is easily verified.

4. Applications

A special case of Theorem 1.2 has already been used to characterize the tangent function over a convex n-gon in [**4**]. In this section, we derive functional equations that can be used to characterize

- major hyperbolic functions through applications of Theorem 1.1, and
- the sine and cosine functions through applications of Theorem 1.2.

We begin with the hyperbolic tangent function.

LEMMA 4.1. *Let* $n \in \mathbb{N}$, $n \geqslant 3$, *let* $A_1, \dots, A_{n-1} \in \mathbb{R}$ *and let*

$$h_1(n) := \sum_{M=1}^{\lfloor\frac{n-1}{2}\rfloor} \sum_{1\leqslant i_1<i_2<\cdots<i_{2M}\leqslant n-1} \prod_{k=1}^{2M} \tanh A_{i_k}$$

$$h_2(n) := \sum_{M=0}^{\lfloor\frac{n-2}{2}\rfloor} \sum_{1\leqslant i_1<i_2<\cdots<i_{2M+1}\leqslant n-1} \prod_{k=1}^{2M+1} \tanh A_{i_k}.$$

If $1 + h_1(n) \neq 0$, *then*

$$\tanh(A_1 + \cdots + A_{n-1}) = \frac{h_2(n)}{1 + h_1(n)}.$$

PROOF. We prove the lemma by induction on $n \geqslant 3$. The case $n = 3$ follows at once from the identity

$$\frac{h_2(3)}{1 + h_1(3)} = \frac{\tanh A_1 + \tanh A_2}{1 + \tanh A_1 \tanh A_2} = \tanh(A_1 + A_2). \tag{4.1}$$

Assume that the assertion holds for $n(\geqslant 3)$ and we aim to show that it is true for $n + 1$. By the hyperbolic tangent-sum formula (4.1) and the induction hypothesis, we get

$$\begin{aligned}\tanh(A_1 + \cdots + A_{n-1} + A_n) &= \frac{\tanh(A_1 + \cdots + A_{n-1}) + \tanh A_n}{1 + \tanh(A_1 + \cdots + A_{n-1}) \tanh A_n} \\ &= \frac{h_2(n) + (1 + h_1(n)) \tanh A_n}{1 + h_1(n) + h_2(n) \tanh A_n}.\end{aligned}$$

We treat only the case of even n, as that of odd n is similar and is thus omitted. If n is even, then $\lfloor(n-1)/2\rfloor = (n-2)/2 = \lfloor(n-2)/2\rfloor$, and so

$$\begin{aligned} h_2(n) + (1 + h_1(n)) \tanh A_n &= \sum_{M=0}^{\frac{n-2}{2}} \sum_{1\leqslant i_1<i_2<\cdots<i_{2M+1}\leqslant n-1} \prod_{k=1}^{2M+1} \tanh A_{i_k} \\ &\quad + \tanh A_n + \sum_{M=1}^{\frac{n-2}{2}} \sum_{1\leqslant i_1<i_2<\cdots<i_{2M}\leqslant n-1} \left(\prod_{k=1}^{2M} \tanh A_{i_k}\right) \tanh A_n \\ &= \sum_{1\leqslant i_1\leqslant n} \tanh A_{i_1} + \sum_{M=1}^{\frac{n-2}{2}} \sum_{1\leqslant i_1<i_2<\cdots<i_{2M+1}\leqslant n-1} \prod_{k=1}^{2M+1} \tanh A_{i_k}\end{aligned}$$

$$+\sum_{M=1}^{\frac{n-2}{2}}\sum_{1\leqslant i_1<i_2<\cdots<i_{2M}\leqslant n-1}\left(\prod_{k=1}^{2M}\tanh A_{i_k}\right)\tanh A_n$$

$$=\sum_{1\leqslant i_1\leqslant n}\tanh A_{i_1}+\sum_{M=1}^{\frac{n-2}{2}}\sum_{1\leqslant i_1<i_2<\cdots<i_{2M+1}\leqslant n}\prod_{k=1}^{2M+1}\tanh A_{i_k}=h_2(n+1)$$

$$1+h_1(n)+h_2(n)\tanh A_n=1+\sum_{M=1}^{\frac{n-2}{2}}\sum_{1\leqslant i_1<i_2<\cdots<i_{2M}\leqslant n-1}\prod_{k=1}^{2M}\tanh A_{i_k}$$

$$+\sum_{M=0}^{\frac{n-2}{2}}\sum_{1\leqslant i_1<i_2<\cdots<i_{2M+1}\leqslant n-1}\left(\prod_{k=1}^{2M+1}\tanh A_{i_k}\right)\tanh A_n$$

$$=1+\sum_{M=1}^{\frac{n-2}{2}}\sum_{1\leqslant i_1<i_2<\cdots<i_{2M}\leqslant n-1}\prod_{k=1}^{2M}\tanh A_{i_k}$$

$$+\sum_{M=1}^{\frac{n}{2}}\sum_{1\leqslant i_1<i_2<\cdots<i_{2M-1}\leqslant n-1}\left(\prod_{k=1}^{2M-1}\tanh A_{i_k}\right)\tanh A_n$$

$$=1+\sum_{M=1}^{\frac{n-2}{2}}\sum_{1\leqslant i_1<i_2<\cdots<i_{2M}\leqslant n}\prod_{k=1}^{2M}\tanh A_{i_k}+\prod_{i=1}^{n}\tanh A_i=1+h_1(n+1).\qquad\square$$

Theorem 4.1. *Let $n\in\mathbb{N}$, $n\geqslant 3$, $I:=[a,b]$ with $b>a$. The functions $f_i\colon I\to(-1,1)$ $(i=1,\ldots,n)$ satisfying*

$$\sum_{i=1}^{n}f_i(x_i)=-\sum_{M=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{1\leqslant i_1<\cdots<i_{2M+1}\leqslant n}\prod_{k=1}^{2M+1}f_{i_k}(x_{i_k}),\quad x_i\in I\ (i=1,\ldots,n),\tag{4.2}$$

subject to the two conditions

$$\sum_{i=1}^{n}x_i=L\quad\text{and}\quad 1+\sum_{M=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n-1}\prod_{k=1}^{2M}f_{i_k}(x_{i_k})\neq 0,$$

where L is a constant belonging to the range $\frac{n(2a+b)}{3}<L<\frac{n(a+2b)}{3}$, are given by

$$f_i(x)=\tanh(A(x)-A(L/n)+d_i)\quad(i=1,\ldots,n),$$

where A is an additive function on $\mathbb{R}$, and the constants d_i satisfy $\sum_{i=1}^{n}d_i=0$.

Proof. For a suitable bijection (to be determined) $\phi_i\colon I\to\mathbb{R}$ $(i=1,\ldots,n)$, let

$$f_i(x)=\tanh(\phi_i(x))\quad(i=1,\ldots,n).$$

Substituting into (4.2), we get

$$\begin{aligned}\sum_{i=1}^{n-1}&\tanh(\phi_i(x_i))+\tanh(\phi_n(x_n))\\ &= -\sum_{M=1}^{\lfloor\frac{n-1}{2}\rfloor}\ \sum_{1\leqslant i_1<\cdots<i_{2M+1}\leqslant n-1}\ \prod_{k=1}^{2M+1}\tanh(\phi_{i_k}(x_{i_k}))\\ &\quad -\sum_{M=1}^{\lfloor\frac{n-1}{2}\rfloor}\ \sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n-1}\left(\prod_{k=1}^{2M}\tanh(\phi_{i_k}(x_{i_k}))\right)\tanh(\phi_n(x_n)),\end{aligned}$$

which yields

$$\frac{\sum_{M=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{1\leqslant i_1<\cdots<i_{2M+1}\leqslant n-1}\prod_{k=1}^{2M+1}\tanh(\phi_{i_k}(x_{i_k}))}{1+\sum_{M=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n-1}\prod_{k=1}^{2M}\tanh(\phi_{i_k}(x_{i_k}))}=\tanh(-\phi_n(x_n)).$$

We work out the case of even n and omit similar derivation of the case n odd. If n is even, then $\lfloor(n-1)/2\rfloor=(n-2)/2$, and so

$$\frac{\sum_{M=0}^{\frac{n-2}{2}}\sum_{1\leqslant i_1<\cdots<i_{2M+1}\leqslant n-1}\prod_{k=1}^{2M+1}\tanh(\phi_{i_k}(x_{i_k}))}{1+\sum_{M=1}^{\frac{n-2}{2}}\sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n-1}\prod_{k=1}^{2M}\tanh(\phi_{i_k}(x_{i_k}))}=\tanh(-\phi_n(x_n)).$$

By Lemma 4.1, we have $\tanh(\phi_1(x_1)+\cdots+\phi_{n-1}(x_{n-1}))=\tanh(-\phi_n(x_n))$. Since the real hyperbolic tangent function is injective, we deduce that $\phi_1(x_1)+\cdots+\phi_{n-1}(x_{n-1})+\phi_n(x_n)=0$, and Theorem 1.1 yields then that

$$\phi_i(x)=A(x)-A(L/n)+d_i\quad(x\in I;\ i=1,\ldots,n).\qquad\square$$

The sought after functional equations for the trigonometric and hyperbolic sine and cosine functions are guided by the following generalizations of the well-known identies (1.3), (1.4), (1.5) and (1.6).

Lemma 4.2. *Let n be an integer $\geqslant 2$.*
I. If $x_1,\ldots,x_n\in(0,\pi)$, then

$$\sin(x_1+\cdots+x_n)=\sum_{M=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^M\sum_{1\leqslant i_1<\cdots<i_{2M+1}\leqslant n}S_n(i_1,\ldots,i_{2M+1}),\tag{4.3}$$

where

$$S_n(i_1,\ldots,i_{2M+1}):=\left(\prod_{k=1}^{2M+1}\frac{\sin x_{i_k}}{\cos x_{i_k}}\right)\left(\prod_{j=1}^{n}\cos x_j\right)$$

and

$$\cos(x_1+\cdots+x_n)=\sum_{M=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^M\sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n}C_n(i_1,\ldots,i_{2M}),\tag{4.4}$$

where $C_n(i_1,\ldots,i_{2M}):=\begin{cases}\left(\prod_{k=1}^{2M}\frac{\sin x_{i_k}}{\cos x_{i_k}}\right)\left(\prod_{j=1}^{n}\cos x_j\right) & \textit{if } M\neq 0\\ \prod_{j=1}^{n}\cos x_j & \textit{if } M=0.\end{cases}$

II. If $y_1, \dots, y_n \in \mathbb{R}$, *then*

$$\sinh(y_1 + \cdots + y_n) = \sum_{M=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{1 \leqslant i_1 < \cdots < i_{2M+1} \leqslant n} \mathscr{S}_n(i_1, \dots, i_{2M+1}), \tag{4.5}$$

where

$$\mathscr{S}_n(i_1, \dots, i_{2M+1}) := \Big(\prod_{k=1}^{2M+1} \frac{\sinh y_{i_k}}{\cosh y_{i_k}} \Big) \Big(\prod_{j=1}^{n} \cosh y_j \Big)$$

and

$$\cosh(y_1 + \cdots + y_n) = \sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{1 \leqslant i_1 < \cdots < i_{2M} \leqslant n} \mathscr{C}_n(i_1, \dots, i_{2M}), \tag{4.6}$$

where $\mathscr{C}_n(i_1, \dots, i_{2M}) := \begin{cases} \big(\prod_{k=1}^{2M} \frac{\sinh y_{i_k}}{\cosh y_{i_k}} \big) \big(\prod_{j=1}^{n} \cosh y_j \big) & \text{if } M \neq 0 \\ \prod_{j=1}^{n} \cosh y_j & \text{if } M = 0. \end{cases}$

Proof. I. We prove both (4.3) and (4.4) simultaneously by induction on n; the starting case $n = 2$ follows at once from the identities (1.3) and (1.4). Assume that both (4.3) and (4.4) hold for n. We treat here only the case when n *is odd*, as the other case is quite similar. In this case, $\lfloor n/2 \rfloor = (n-1)/2$, $\lfloor (n+1)/2 \rfloor = (n+1)/2$. First, consider the right-hand side of (4.3) for $n+1$, we have

$$\begin{aligned} &\sum_{M=0}^{\frac{n-1}{2}} (-1)^M \sum_{1 \leqslant i_1 < \cdots < i_{2M+1} \leqslant n+1} S_{n+1}(i_1, \dots, i_{2M+1}) \\ &\quad = \Bigg\{ \sum_{M=0}^{\frac{n-1}{2}} (-1)^M \sum_{1 \leqslant i_1 < \cdots < i_{2M+1} \leqslant n} \Big(\prod_{k=1}^{2M+1} \frac{\sin x_{i_k}}{\cos x_{i_k}} \Big) \Big(\prod_{j=1}^{n} \cos x_j \Big) \Bigg\} \cos x_{n+1} \\ &\qquad + \Bigg\{ \sum_{M=0}^{\frac{n-1}{2}} (-1)^M \sum_{1 \leqslant i_1 < \cdots < i_{2M} \leqslant n} \Big(\prod_{k=1}^{2M} \frac{\sin x_{i_k}}{\cos x_{i_k}} \Big) \Big(\prod_{j=1}^{n} \cos x_j \Big) \Bigg\} \sin x_{n+1}. \end{aligned}$$

By induction hypothesis, this last right-hand expression is equal to

$$\sin(x_1 + \cdots + x_n) \cos x_{n+1} + \cos(x_1 + \cdots + x_n) \sin x_{n+1} = \sin(x_1 + \cdots + x_n + x_{n+1}).$$

Next, consider the right-hand side of (4.4) for $n+1$,

$$\begin{aligned} &\sum_{M=0}^{\frac{n+1}{2}} (-1)^M \sum_{1 \leqslant i_1 < \cdots < i_{2M} \leqslant n+1} C_{n+1}(i_1, \dots, i_{2M}) \\ &\quad = \Bigg\{ \sum_{M=0}^{\frac{n+1}{2}} (-1)^M \sum_{1 \leqslant i_1 < \cdots < i_{2M} \leqslant n} \Big(\prod_{k=1}^{2M} \frac{\sin x_{i_k}}{\cos x_{i_k}} \Big) \Big(\prod_{j=1}^{n} \cos x_j \Big) \Bigg\} \cos x_{n+1} \\ &\qquad + \Bigg\{ \sum_{M=0}^{\frac{n+1}{2}} (-1)^M \sum_{1 \leqslant i_1 < \cdots < i_{2M-1} \leqslant n} \Big(\prod_{k=1}^{2M-1} \frac{\sin x_{i_k}}{\cos x_{i_k}} \Big) \Big(\prod_{j=1}^{n} \cos x_j \Big) \Bigg\} \sin x_{n+1} \end{aligned}$$

$$= \left\{ \sum_{M=0}^{\frac{n-1}{2}} (-1)^M \sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n} \left(\prod_{k=1}^{2M} \frac{\sin x_{i_k}}{\cos x_{i_k}} \right) \left(\prod_{j=1}^{n} \cos x_j \right) \right\} \cos x_{n+1}$$
$$- \left\{ \sum_{M=0}^{\frac{n-1}{2}} (-1)^M \sum_{1\leqslant i_1<\cdots<i_{2M+1}\leqslant n} \left(\prod_{k=1}^{2M+1} \frac{\sin x_{i_k}}{\cos x_{i_k}} \right) \left(\prod_{j=1}^{n} \cos x_j \right) \right\} \sin x_{n+1},$$

where empty sums are defined to be 0. By induction hypothesis, the right-hand expression is equal to

$$\cos(x_1 + \cdots + x_n) \cos x_{n+1} - \sin(x_1 + \cdots + x_n) \sin x_{n+1}$$
$$= \cos(x_1 + \cdots + x_n + x_{n+1}).$$

II. We prove both (4.5) and (4.6) simultaneously by induction on n; the starting case $n = 2$ follows at once from identities (1.5) and (1.6). Assume that both (4.5) and (4.6) hold for n. We treat here only the case when n *is odd*, as the other case is quite similar. In this case, $\lfloor n/2 \rfloor = (n-1)/2$, $\lfloor (n+1)/2 \rfloor = (n+1)/2$. First, consider the right-hand side of (4.5) for $n+1$, we have

$$\sum_{M=0}^{\frac{n-1}{2}} \sum_{1\leqslant i_1<\cdots<i_{2M+1}\leqslant n+1} \mathscr{S}_{n+1}(i_1, \ldots, i_{2M+1})$$
$$= \left\{ \sum_{M=0}^{\frac{n-1}{2}} \sum_{1\leqslant i_1<\cdots<i_{2M+1}\leqslant n} \left(\prod_{k=1}^{2M+1} \frac{\sinh y_{i_k}}{\cosh y_{i_k}} \right) \left(\prod_{j=1}^{n} \cosh y_j \right) \right\} \cosh y_{n+1}$$
$$+ \left\{ \sum_{M=0}^{\frac{n-1}{2}} \sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n} \left(\prod_{k=1}^{2M} \frac{\sinh y_{i_k}}{\cosh y_{i_k}} \right) \left(\prod_{j=1}^{n} \cosh y_j \right) \right\} \sinh y_{n+1}.$$

By induction hypothesis, this last right-hand expression is equal to

$$\sinh(y_1 + \cdots + y_n) \cosh y_{n+1} + \cosh(y_1 + \cdots + y_n) \sinh y_{n+1}$$
$$= \sinh(y_1 + \cdots + y_n + y_{n+1}).$$

Next, consider the right-hand side of (4.6) for $n+1$,

$$\sum_{M=0}^{\frac{n+1}{2}} \sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n+1} \mathscr{C}_{n+1}(i_1, \ldots, i_{2M})$$
$$= \left\{ \sum_{M=0}^{\frac{n+1}{2}} \sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n} \left(\prod_{k=1}^{2M} \frac{\sinh y_{i_k}}{\cosh y_{i_k}} \right) \left(\prod_{j=1}^{n} \cosh y_j \right) \right\} \cosh y_{n+1}$$
$$+ \left\{ \sum_{M=0}^{\frac{n+1}{2}} \sum_{1\leqslant i_1<\cdots<i_{2M-1}\leqslant n} \left(\prod_{k=1}^{2M-1} \frac{\sinh y_{i_k}}{\cosh y_{i_k}} \right) \left(\prod_{j=1}^{n} \cosh y_j \right) \right\} \sinh y_{n+1}$$

$$= \left\{ \sum_{M=0}^{\frac{n-1}{2}} \sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n} \left(\prod_{k=1}^{2M} \frac{\sinh y_{i_k}}{\cosh y_{i_k}} \right) \left(\prod_{j=1}^{n} \cosh y_j \right) \right\} \cosh y_{n+1}$$
$$+ \left\{ \sum_{M=0}^{\frac{n-1}{2}} \sum_{1\leqslant i_1<\cdots<i_{2M+1}\leqslant n} \left(\prod_{k=1}^{2M+1} \frac{\sinh y_{i_k}}{\cosh y_{i_k}} \right) \left(\prod_{j=1}^{n} \cosh y_j \right) \right\} \sinh y_{n+1},$$

where empty sums are defined to be 0. As before, the desired result now follows from induction. □

Our final result is a characterization of the trigonometric and hyperbolic sine and cosine functions.

Theorem 4.2. *Let n be an integer $\geqslant 3$.*

I. The functions $f_1, g_1 \colon (0,\pi) \to [-1,1]$ satisfying

$$\sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^M \sum_{1\leqslant i_1<\cdots<i_{2M}\leqslant n} \mathcal{C}_n(f_1, g_1; i_1, \dots, i_{2M}) = (-1)^n, \tag{4.7}$$

where $\mathcal{C}_n(f_1, g_1; i_1, \dots, i_{2M}) := \begin{cases} \left(\prod_{k=1}^{2M} \frac{f_1(x_{i_k})}{g_1(x_{i_k})}\right)\left(\prod_{j=1}^{n} g_1(x_j)\right) & \text{if } M \neq 0 \\ \prod_{j=1}^{n} g_1(x_j) & \text{if } M = 0, \end{cases}$

subject to the two conditions

$$\begin{gathered} \sin^{-1} \circ f_1 = \cos^{-1} \circ g_1 \\ x_1 + \cdots + x_n = (n-2)\pi \end{gathered} \tag{4.8}$$

are given by

$$f_1(x) = \begin{cases} \sin\left(k_1\left(x - \frac{(n-2)\pi}{n}\right) + \frac{s\pi}{n}\right) & \text{for } n \text{ odd} \\ \sin\left(k_2\left(x - \frac{(n-2)\pi}{n}\right) + \frac{\ell\pi}{n}\right) & \text{for } n \text{ even} \end{cases}$$

and

$$g_1(x) = \begin{cases} \cos\left(k_1\left(x - \frac{(n-2)\pi}{n}\right) + \frac{s\pi}{n}\right) & \text{for } n \text{ odd} \\ \cos\left(k_2\left(x - \frac{(n-2)\pi}{n}\right) + \frac{\ell\pi}{n}\right) & \text{for } n \text{ even} \end{cases}$$

where $s \in \{1, 3, \dots, n-2\}$ is an odd integer, $\ell \in \{2, 4, \dots, n-2\}$ is an even integer, and k_1, k_2 are constants belonging to the ranges

$$\max\left\{ -\frac{s}{2}, \frac{s-n}{n-2} \right\} < k_1 < \min\left\{ \frac{s}{n-2}, \frac{n-s}{2} \right\},$$
$$\max\left\{ -\frac{\ell}{2}, \frac{\ell-n}{n-2} \right\} < k_2 < \min\left\{ \frac{\ell}{n-2}, \frac{n-\ell}{2} \right\}.$$

II. The functions $f_2, g_2 \colon (0,\pi) \to [-1,1]$ satisfying

$$\sum_{M=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^M \sum_{1\leqslant i_1<\cdots<i_{2M+1}\leqslant n} \mathcal{S}_n(f_2, g_2; i_1, \dots, i_{2M+1}) = 0, \tag{4.9}$$

where $\mathcal{S}_n(f_2, g_2; i_1, \ldots, i_{2M+1}) := \big(\prod_{k=1}^{2M+1} \frac{f_2(x_{i_k})}{g_2(x_{i_k})}\big)\big(\prod_{j=1}^{n} g_2(x_j)\big)$, *subject to the two conditions*

$$\sin^{-1} \circ f_2 = \cos^{-1} \circ g_2 \tag{4.10}$$

$$x_1 + \cdots + x_n = (n-2)\pi$$

are given by

$$f_2(x) = \sin\Big(k\Big(x - \frac{(n-2)\pi}{n}\Big) + \frac{s\pi}{n}\Big),$$

$$g_2(x) = \cos\Big(k\Big(x - \frac{(n-2)\pi}{n}\Big) + \frac{s\pi}{n}\Big),$$

where $s \in \{1, 2, 3, \ldots, n-2\}$, *and* k *belongs to the range*

$$\max\Big\{-\frac{s}{2}, \frac{s-n}{n-2}\Big\} < k < \min\Big\{\frac{s}{n-2}, \frac{n-s}{2}\Big\}.$$

III. Let $b > a$, *the functions* $f_j \colon [a, b] \to \mathbb{R}$ *and* $g_j \colon [a, b] \to [1, \infty)$ $(j = 1, \ldots, n)$ *satisfying*

$$\sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{1 \leqslant i_1 < \cdots < i_{2M} \leqslant n} \mathfrak{C}_n(f_j, g_j; i_1, \ldots, i_{2M}) = 1, \tag{4.11}$$

where $\mathfrak{C}_n(f_j, g_j; i_1, \ldots, i_{2M}) := \begin{cases} \Big(\prod_{k=1}^{2M} \frac{f_{i_k}(x_{i_k})}{g_{i_k}(x_{i_k})}\Big)\Big(\prod_{j=1}^{n} g_j(x_j)\Big) & \text{if } M \neq 0 \\ \prod_{j=1}^{n} g_j(x_j) & \text{if } M = 0, \end{cases}$

subject to the two conditions

$$\sinh^{-1} \circ f_j = \cosh^{-1} \circ g_j \quad (j = 1, \ldots, n) \tag{4.12}$$

$$\sum_{j=1}^{n} x_j = L_1,$$

where L_1 *is a constant belonging to the range* $\frac{n(2a+b)}{3} < L_1 < \frac{n(a+2b)}{3}$, *are given by*

$$f_j(x) = \sinh(A_1(x) - A_1(L_1/n) + d_j), \quad g_j(x) = \cosh(A_1(x) - A_1(L_1/n) + d_j),$$

where A_1 *is an additive function on* $\mathbb{R}$ *and the constants* d_j *satisfy* $\sum_{j=1}^{n} d_j = 0$.

IV. Let $b > a$. *The functions* $f_j \colon [a, b] \to \mathbb{R}$ *and* $g_j \colon [a, b] \to [1, \infty)$ $(j = 1, \ldots, n)$ *satisfying*

$$\sum_{M=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{1 \leqslant i_1 < \cdots < i_{2M+1} \leqslant n} \mathfrak{S}_n(f_j, g_j; i_1, \ldots, i_{2M+1}) = 0, \tag{4.13}$$

where $\mathfrak{S}_n(f_j, g_j; i_1, \ldots, i_{2M+1}) := \big(\prod_{k=1}^{2M+1} \frac{f_{i_k}(x_{i_k})}{g_{i_k}(x_{i_k})}\big)\big(\prod_{j=1}^{n} g_j(x_j)\big)$, *subject to the two conditions*

$$\sinh^{-1} \circ f_j = \cosh^{-1} \circ g_j \quad (j = 1, \ldots, n) \tag{4.14}$$

$$\sum_{j=1}^{n} x_j = L_2,$$

where L_2 *is a constant belonging to the range* $\frac{n(2a+b)}{3} < L_2 < \frac{n(a+2b)}{3}$*, are given by*

$$f_j(x) = \sinh(A_2(x) - A_2(L_2/n) + \ell_j), \quad g_j(x) = \cosh(A_2(x) - A_2(L_2/n) + \ell_j),$$

where A_2 *is an additive function on* $\mathbb{R}$ *and the constants* ℓ_i *satisfy* $\sum_{j=1}^{n} \ell_j = 0$.

PROOF. I. By (4.8), there exists $\phi\colon (0,\pi) \to (0,\pi)$ such that

$$f_1(x) = \sin(\phi(x)) \quad \text{and} \quad g_1(x) = \cos(\phi(x)) \qquad (x \in (0,\pi)).$$

Thus, (4.7) becomes

$$(-1)^n = \sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^M \sum_{1 \leqslant i_1 < \cdots < i_{2M} \leqslant n} \mathcal{C}_n(\sin\phi, \cos\phi; i_1, \ldots, i_{2M}).$$

Invoking upon (4.4) of Lemma 4.2, we obtain $\cos(\phi(x_1) + \cdots + \phi(x_n)) = (-1)^n$. For odd n, we deduce that

$$\phi(x_1) + \cdots + \phi(x_n) = s\pi,$$

where $s \in \{1, 3, \ldots, n-2\}$ is an odd integer. Theorem 1.2 implies then that the solution of this last functional equation together with (4.2) is

$$\phi(x) = k_1\Big(x - \frac{(n-2)\pi}{n}\Big) + \frac{s\pi}{n},$$

for some fixed k_1 belonging to the range $\max\big\{-\frac{s}{2}, \frac{s-n}{n-2}\big\} < k_1 < \min\big\{\frac{s}{n-2}, \frac{n-s}{2}\big\}$. Similarly, for n even, we deduce that

$$\phi(x) = k_2\Big(x - \frac{(n-2)\pi}{n}\Big) + \frac{\ell\pi}{n},$$

where $\ell \in \{2, \ldots, n-2\}$ is an even integer, and k_2 belonging to the range

$$\max\Big\{-\frac{\ell}{2}, \frac{\ell-n}{n-2}\Big\} < k_2 < \min\Big\{\frac{\ell}{n-2}, \frac{n-\ell}{2}\Big\}.$$

II. By (4.10), there exists $\psi\colon (0,\pi) \to (0,\pi)$ such that

$$f_2(x) = \sin(\psi(x)) \quad \text{and} \quad g_2(x) = \cos(\psi(x)) \qquad (x \in (0,\pi)).$$

Thus, (4.9) becomes

$$0 = \sum_{M=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^M \sum_{1 \leqslant i_1 < \cdots < i_{2M+1} \leqslant n} \mathcal{S}_n(\sin\psi, \cos\psi; i_1, \ldots, i_{2M+1}).$$

Using (4.3) of Lemma 4.2, we get $\sin(\psi(x_1) + \cdots + \psi(x_n)) = 0$, and so

$$\psi(x_1) + \cdots + \psi(x_n) = s\pi$$

for some $s \in \{1, 2, \ldots, n-2\}$. The desired result then follows immediately from Theorem 1.2.

III. By (4.12), there exist $\phi_j\colon [a,b] \to \mathbb{R}$ $(j = 1, \ldots, n)$ such that

$$f_j(x) = \sinh(\phi_j(x)), \quad g_j(x) = \cosh(\phi_j(x)) \qquad (j = 1, \ldots, n).$$

Using (4.11) and (4.6), we get $\cosh(\phi_1(x_1) + \cdots + \phi_n(x_n)) = 1$. Thus,

$$\phi_1(x_1) + \cdots + \phi_n(x_n) = 0.$$

By Theorem 1.1, there exists an additive function $A_1\colon \mathbb{R} \to \mathbb{R}$ such that

$$\phi_j(x) = A_1(x) - A_1(L/n) + d_j \quad (x \in [a, b]; j = 1, \ldots, n),$$

where the constants d_j satisfy $\sum_{j=1}^{n} d_j = 0$.

IV. By (4.14), there exist $\psi_j\colon [a, b] \to \mathbb{R}$ $(j = 1, \ldots, n)$ such that

$$f_j(x) = \sinh(\psi_j(x)), \quad g_j(x) = \cosh(\psi_j(x)) \qquad (j = 1, \ldots, n).$$

Using (4.13) and (4.5), we get $\sinh(\psi_1(x_1) + \cdots + \psi_n(x_n)) = 0$. Thus,

$$\psi_1(x_1) + \cdots + \psi_n(x_n) = 0.$$

By Theorem 1.1, there exists an additive function $A_2\colon \mathbb{R} \to \mathbb{R}$ such that

$$\psi_j(x) = A_2(x) - A_2(L_2/n) + \ell_j \quad (x \in [a, b], j = 1, \ldots, n),$$

where the constants ℓ_j satisfy $\sum_{j=1}^{n} \ell_j = 0$. ☐

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(Received 07 10 2015)

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