FUNCTORIAL RESOLUTION BY TORUS ACTIONS

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Abstract. We show a simple and fast embedded resolution of varieties and principalization of ideals in the language of torus actions on ambient, smooth varieties with SNC divisors. The canonical functorial resolution of varieties in characteristic zero is given by the introduced here operations of *cobordant blow-ups* with smooth weighted centers. The centers are defined by the geometric invariant measuring the singularities on smooth schemes with SNC divisors.

As the result of the procedure, we obtain a smooth variety with a torus action and the exceptional divisor having simple normal crossings. Moreover, its geometric quotient is birational to the resolved variety, has abelian quotient singularities, and can be desingularized directly by combinatorial methods. The paper is based upon the ideas of the joint work with Abramovich and Temkin [ATW19] and a similar result by McQuillan [McQ19] on resolution in characteristic zero via stack-theoretic weighted blow-ups.

As an application of the method, we show the resolution of certain classes of singularities in positive and mixed characteristic.

Contents

1. Introduction

The significance of G_m-actions in birational geometry and their connection with Mori theory has already been identified by Reid, Thaddeus, and various other scholars (refer to [Tha94a], [Tha94b], [Tha96], [Rei], [DH98]). Particularly, Reid [Reid02] highlighted the role of weighted blowings up and flips in birational geometry utilizing G_m-actions. This notion was also reflected in the proof of the Weak Factorization theorem, which relied on the concept of *birational cobordism* and a pivotal role of G_m-action ([Wło00], [Wło03], [AKMW02]).

In the present paper, we propose approaching embedded resolution problems and the existing resolution algorithms from the point of view of torus actions. This is an alternative and, to a great extent, equivalent approach to the one pursued in a paper [McQ19] of McQuillan-Marzo, and a recent paper of Quek [Que20], and

most notably, in the series of papers by Abramovich-Temkin-Włodarczyk [ATW17], [ATW20],[ATW19], where all kind of stack-theoretic blow-ups were considered to simplify resolution procedure or to adapt the algorithm to the relative situation of morphisms. The introduced here definition of *cobordant blow-up* interprets the stack-theoretic weighted blow-ups, logarithmic Kummer blow-ups, and more general birational transformations of stacks in terms of torus actions. Such interpretation leads to the presentation of the weighed blow-ups and other transformations as smooth birational cobordisms. The operation is done in the language of smooth schemes with torus actions in terms of simple transformation rules. In fact, only one chart is needed to describe the cobordant blow-up, which makes it a very simple tool for algorithmic computations and possible theoretical implementations. The method can be applied to many existing and potential embedded resolution algorithms originally relying on smooth centers, regardless of characteristic. It also avoids many problems associated with smooth centers and studied intensely in [Abh67, Moh87, Moh96, Hau98, Wło08, CP08, CP09, Cut11, BVU13, KM16, HP19]. Moreover, it is significantly faster and eliminates unnecessary blow-ups. Furthermore, the algorithm in characteristic zero has a very simple structure with the centers defined by a geometric invariant related to the weighted normal cone (see Section 4.1.1). The smooth centers are replaced with more general smooth weighted centers. In fact, the present paper, similarly to [ATW19], contains a highly simplified version of the strong Hironaka algorithm in characteristic zero. Recall that the standard Hironaka algorithm uses a rather complicated invariant where the geometry of singularities is mixed with the combinatorics of the monomials. In the present paper, similarly to [ATW19] and [McQ19], the algorithm is purely geometric, with the combinatorial part hidden in the torus action or respectively in the stack-theoretic structure as, in the two papers above.

1.1. **Cobordant blow-ups.**
Unlike the formalism in [ATW19], or in [McQ19], the resolution here is given in the language of schemes with torus actions and does not use stack-theoretic language, although, if needed, it can be converted into such. The algorithm is carried in the language of *cobordant blow-ups* of weighted centers on regular schemes. A similar operation in the language of stacks was independently studied in the paper of Quek-Rydh [QR22].

The cobordant blow-ups of regular varieties have many advantages even when compared to the standard blow-ups of smooth blow-ups. They are described by particularly simple transformation rules with one chart only defined by the affine morphism. They carry important additional information about singularities which is contained in the *vertex* of the transformation. The singularities along the vertex after the blow-up are to a great extent equivalent to the singularities at the center, and they are removed as typically the worst singularities in the resolution process. Moreover the torus action contains additional additional important information for the resolution process.

All of this is especially beneficial for all kinds of generalizations of the resolution to classes of singularities in positive and mixed characteristic, foliations and others. In particular in Section 4 we show how to apply cobordant blow-ups with weighted centers to introduced here, class of *almost homogenous singularities* in any characteristic (see Theorems 4.2.2, 4.2.3). This class includes, in particular, the isolated

singularities whose generalized weighted tangent cone also has an isolated singularity. Moreover, the approach can be generalized to the case where the equimultiple locus of the subscheme and its weighted tangent cone coincide (see Lemma 4.3.1).

The language and the formalism of the cobordant blow-ups are related to the notion of *birational cobordism* introduced as the main tool in the proof of the Weak factorization theorem ([Wło00], [Wło03], [AKMW02]). On the other hand when looking from a more general perspective of Cox rings of morphisms in the subsequent paper [Wło23] one shows that any proper birational morphism of normal noetherian schemes has such a cobordant presentation called *cobordization*, where the considered here cobordant blow-up of weighted center is the cobordization of the standard weighted blow-up. Moreover the theorems on the resolution from Chapter 4 are extended in [Wło23] to generalized cobordant blow-ups of locally monomial ideals.

1.2. **Rees algebras.** Another novel of the paper is the use of rational Rees algebras, giving a fast algorithm with automatic uniqueness. One shall mention that the rational Rees algebras was also briefly used in the first version of [ATW19] to interpret the algebra of the center.

Recall that there are presently two approaches to the uniqueness and glueing of the embedded resolutions. The first method is known as "Hironaka's trick" which uses an equivalence relation defined by permissible blow-ups. It was used in the original Hironaka's approach, and in the subsequent papers of Bierstone-Milman and Villamayor and others (see [Hir64],[Vil89],[BM97],[EH02], [EV03]). The other method is known as "homogenization-tuning" trick was introduced by the author in [Wło05], and then used in the papers of Kollar [Kol07], Bierstone-Milman [BM08], and others. This approach is also used in [ATW17] and [ATW19]. It relies on the existence of local automorphisms of the homogenized ideals and functoriality for étale morphisms. Both approaches hinge on the induction on the dimension of ambient variety. The inductively built centers living on different hypersurfaces of maximal contact are compared via equivalence relations or étale maps.

With our approach the centers are unique despite being constructed using non-canonical itermediate steps. Moreover what is important no comparison between the intermediate steps is needed. Instead all the operations are related directly to the constructed canonical resolution center.

Recall that the integrally graded Rees algebras have been used for some time in the resolution algorithms([EV07],[Vil08],[BV11],[CP09],[KM16]). They were also used recently in the approach by Quek for the weighted logarithmic resolution [Que20]. They are also considered in the paper by Quek and Rydh [QR22]. There are however several advantages of the Rees algebras with rational gradations. The very first one is that the resolution center and the resolution invariant can be conveniently defined in this language as the natural generalization of the order.

1.2.1. *Definition of the invariant.* Let us start with the definition of the order of an ideal $\mathcal{I}$ at a point $p \in X$:

$$\mathrm{ord}_p(\mathcal{I}) = \max\{a \in \mathbb{Z}_{>0} \mid \mathcal{I} \subset m_p^a\},$$

where $m_p \subset \mathcal{O}_{X,p}$ is the maximal ideal of the point p in the local ring $\mathcal{O}_{X,p}$. This can be written as

$$\mathrm{ord}_p(\mathcal{I}) = \max\{a \in \mathbb{Q}_{>0} \mid \mathcal{I}t^a \subset \mathcal{O}_X[m_p t]\},$$

where $R = \mathcal{O}_X[m_p t]$ is the Rees algebra generated by the first gradation $m_p t$, where t is the dummy variable. Equivalently one writes the latter as

$$\mathrm{ord}_p(\mathcal{I}) = \max\{a \in \mathbb{Q}_{>0} \mid \mathcal{I}t \subset \mathcal{O}_X[m_p t^{1/a}]\},$$

If the divisors are not present that the resolution invariant of $\mathcal{I}$ at p can be written simply as

$$(1) \qquad \mathrm{inv}_p(\mathcal{I}) := \max\{(a_1, \ldots, a_k) \mid \mathcal{I}t \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}\},$$

where $a_1 \leq \ldots \leq a_k$ are rational numbers ordered lexicographically and the maximum is considered over all the partial coordinate systems $x_1, \ldots, x_k$. Here "int" stands for the corresponding integral closure. The present definition is equivalent to the one from [ATW19], which was considered in the different language. Note that the powers of t indicate the weights of the elements. In particular the weight of x_i is $1/a_i$ with the weight 1 assigned to $\mathcal{I}$.

1.2.2. *Rational Rees algebras.* One of the benefits of Rees algebras is that their generators lie in the different rational gradations which often leads to a simpler presentation. Additional advantage of the rational Rees algebras is that no rescaling is needed and we can always get the simplest possible set of generators by extending the gradation. This makes the computations of the centers very straightforward.

This could be illustrated by the Veronese embedding. When passing from the graded algebra $R = k[x_1, \ldots, x_k] = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} R_d$ to the subgradation $R^{(n)} = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} R_{nd}$, as in the n-tuple Veronese embedding we obtain the graded algebra $R^{(n)}$ with a very cumbersome set of generators given by all monomials of degree n and describing the same $\mathrm{Proj}(R^{(n)}) = \mathrm{Proj}(R)$. Now when we start from $\mathbb{Z}$-graded algebra $R^{(n)}$ written as $R^{(n)} = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} R_{nd} t^d$ for the dummy variable t then extending the gradation and passing to the integral closure $R^{(n)}[t^{1/n}]^{\mathrm{int}}$ one obtains back the algebra $R = k[x_1 t^{1/n}, \ldots, x_k t^{1/k}]$ in the graded form with respect to $t^{1/k}$ and with the nice set of generators, and giving a nicer presentation of the same object $\mathrm{Proj}(R)$.

The process avoids various multiple rescalings which often leads to the increased computational complexity. Since no rescaling is needed the construction of the center can be obtained in a minimalistic way by a series of transformations called *milling.*

1.2.3. *Milling.* The concept behind *milling* is to systematically decompose the elements of the ideal $\mathcal{I}$, as represented by the Rees algebra $R = \mathcal{O}_X[\mathcal{I}t]$, step by step, into elements of smaller gradations. This process is done using graded differential operators and is carried out with the objective of obtaining a smooth weighted center. A similar idea was considered in the first version of [ATW19] to illustrate a different construction method. Note that in our process only the final stage is canonical, and all the intermediate steps rely on the choices made for the convenient minimalistic computations.

Consider the admissibility inclusion

$$(2) \qquad\qquad \mathcal{I}t \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$$

as in the above definition (1) of the resolution invariant $\mathrm{inv}_p(\mathcal{I})$. The Rees algebra of the form

$$\mathcal{A} := \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$$

is called the *Rees center*. If the inclusion (2) holds then we say that the center $\mathcal{A}$ is *admissible* for $\mathcal{I}$ at p. If additionally $\mathrm{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k)$ as in (1) then the Rees center $\mathcal{A}$ on the right will be called a *maximal admissible center*. This terminology is to a great extent equivalent to the one introduced in [ATW19] in the language of $\mathbb{Q}$-ideals.

The algorithm uses the basic concepts of the standard Hironaka algorithm like the admissibility, coefficient ideal, maximal contact, from [Hir64],[Vil89], [BM97], [Wło05], [EH02], [EV03], [Kol07], which are redefined from the perspective of rational Rees algebras and cobordant blow-ups.

1.2.4. *The graded differential operators.* The idea of *milling* is to construct the maximal admissible center through a step-by-step process that involves enlarging the initial Rees algebra $R_1 := \mathcal{O}_X[\mathcal{I}t] \subset \mathcal{A}$, while keeping $\mathcal{A}$ unchanged. To achieve this, we perform actions on both sides of the admissibility inclusion using graded differential operators. These operators preserve the right side $\mathcal{A}$ of the inclusion while expanding the left side. They eventually lead to a situation where both sides become equal. As a result, the maximal admissible center is attained through a recursive process. Importantly, this process is dependent solely on $\mathcal{I}$, establishing the uniqueness of the maximal admissible center and the canonical invariant.

To illustrate the action of graded operators consider the inclusion

$$(3) \qquad \mathcal{I}t^a \subset \mathcal{O}_X[m_p t],$$

where $a = \mathrm{ord}_p(\mathcal{I})$. Here we can act on both sides of the inclusion by the differential operators $t^{-1}D_X$ for $\mathcal{D}_X$ being the sheaf of the differential operators on smooth variety X over a field k, and $a \in \mathbb{N}$. The right side is preserved by this action as $t^{-1}D_X(m_p t)^a \subset (m_p t)^{a-1}$ while we obtain new elements on the left side. In particular, we derive the inclusion of t-gradations $\mathcal{D}_X^{a-1}(\mathcal{I})t \subset m_p t$.

When rescaling (3) we obtain the inclusion

$$\mathcal{I}t \subset \mathcal{O}_X[m_p t^{1/a}],$$

with the action of the differential operators $t^{-1/a}D_X$ to get

$$\mathcal{D}_X^{a-1}(\mathcal{I})t^{1/a} \subset m_p t^{1/a}.$$

Similarly if $\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$ is maximal admissible for $\mathcal{I}$ then a_1 is necessarily the order of $\mathcal{I}$ at p. We can act by the operators $t^{-1/a}D_X$ on both sides of the admissibility relation

$$\mathcal{I}t \subset \mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$$

and obtain the inclusion of t^{1/a_1} gradations

$$(4) \qquad D_X^{a_1-1}(\mathcal{I})t^{1/a_1} \subset \mathcal{A}_{1/a_1} t^{1/a_1}$$

where $\mathcal{A} = \bigoplus_a \mathcal{A}_a t^a$.

1.2.5. *Maximal contacts.* The *maximal contacts* of $\mathcal{I}$ at p in our approach should be thought of as simply the graded coordinates $x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}$ in a certain presentation of a maximal admissible center $\mathcal{A} \subset \mathcal{I}t$ at p. Unlike the usual definition this approach makes sense in a nonzero characteristic. The maximal contacts are defined in terms of the recursively constructed algebras on the left side of the

admissibility inclusion. In particular, in view of (1) the first maximal contact is defined by a coordinate

$$x_1 \in T^{1/a_1}(R_1) := D_X^{a_1-1}(\mathcal{I}) \subset \mathcal{A}_{1/a_1}$$

in gradation t^{1/a_1}.

Observe that the notions of the *admissibility*, the *order*, the *invariant* and the *maximal contact* can be considered in the general context of Rees algebra R replacing $\mathcal{I}t$ with R in the relevant definitions.

Consequently, the notion of i-th maximal contact x_i for any Rees algebra shall be understood as the coordinate in t^{1/a_i}-gradation of its maximal admissible center $\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\text{int}} \supset \mathcal{I}t$ such that x_i is independent of the previous maximal contacts $x_1, \ldots, x_{i-1}$.

1.2.6. *Coefficient ideals.* The *coefficient ideal* of an algebra R is a way to produce larger Rees algebras containing the relevant maximal contacts.

Given any Rees algebra $R = \bigoplus R_a t^a$, and any graded coordinate $xt^{1/a}$ the coefficient ideal $C_{xt^{1/a_1}}(R)$ of R with respect to $xt^{1/a}$ is generated by $xt^{1/a}$ and all the derivations obtained by the action of the graded derivations $D_{xt^{1/a}} := t^{-1/a_1}\partial_x$. Consequently, by the above if $R \subset \mathcal{A}$ and $x_1 t^{1/a_1}$ is the first maximal contact of R then $C_{x_1 t^{1/a_1}}(R) \subset \mathcal{A}$.

The computation of the coefficient ideal is even simpler when we pass to the completion of the local ring with respect to (x) or maximal ideal m_p so that the splitting gives the inclusion $\widehat{\mathcal{O}}_{X,p}/(x) \subset \widehat{\mathcal{O}}_{X,p}$. In such a case $\widehat{\mathcal{O}}_{X,p} \cdot C_{xt^{1/a_1}}(R)$ is generated by the maximal contact and the coefficients in the graded form of the generators of R with respect to the powers $(xt^{1/a})^i$. These coefficients by splitting live on $\widehat{\mathcal{O}}_{X,p} \cdot R_{|V(x)} \subset \widehat{\mathcal{O}}_{X,p} \cdot R$. (See Lemma 3.5.7, Examples 3.2.2, 3.2.4, 3.2.6)

1.3. **Construction of the center.** Going back to the milling process we put

$$R_1 := \mathcal{O}_X[\mathcal{I}t] \subset \mathcal{A},$$
$$R_2 := C_{x_1 t^{1/a_1}}(R_1) \subset \mathcal{A} = C_{x_1 t^{1/a_1}}(\mathcal{A})$$
$$\ldots$$
$$R_{i+1} := C_{x_i t^{i/a_i}}(R_i) \subset \mathcal{A} = C_{x_i t^{1/a_i}}(\mathcal{A})$$
$$\ldots$$
$$R_{k+1} = \mathcal{A}.$$

The process will continue until $R_{k+1|V(x_1,\ldots,x_k)} = 0$ and thus $R_{k+1} = \mathcal{A}$ which completes the procedure. In this recursion, $a_i = \text{ord}_p(R_{i|V(x_1,\ldots,x_{i-1})})$ and the i-the maximal contact $x_i t^{1/a_i}$ is defined so that the restriction $x_{i|V(x_1,\ldots,x_{i-1})}$ is a local parameter in the t^{1/a_i}-gradation of the algebra obtained by the action of $t^{-1/a_i}D_{x_i}, \ldots, t^{-1/a_i}D_{x_n}$ on $R_{i|V(x_1,\ldots,x_{i-1})}$. In the process we change the coordinate system in such a way that gradually step by step all the maximal contacts are included. These changes however do not affect $\mathcal{A}$ just its representation. (See Examples 3.2.2, 3.2.4, 3.2.6). Again the above process in a non-SNC setting shall be compared to the construction of the algebra associated with the center in the first version of [ATW19].

The whole algorithm is independent of the a priori chosen maximal admissible center $\mathcal{A}$. At the same time it is done inside of $\mathcal{A}$ which is then uniquely determined as the terminal algebra of the recursive process.

1.4. **Rees centers and cobordant blow-up.** The computed maximal admissible Rees center $\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$ determines the extended Rees algebra

$$\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w_A}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}],$$

where $w_A = \mathrm{lcm}(a_1, \ldots, a_k)$ is the least common multiple of the rational numers $a_1, \ldots, a_k$.

Now the rescaled integral extended Rees algebra algebra

$$\mathcal{O}_B = \mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}],$$

with $w_i = w_A/a_i$ defines the desired transformation of the *full cobordant blow-up*

$$B = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]) \to X.$$

Here the new coordinates are $t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}$, where t^{-1} describes the exceptional divisor $D := V_B(t^{-1})$, and the *vertex of the cobordant blow-up* is defined as $V := V_B(x_1 t^{w_1}, \ldots, x_k t^{w_k})$.

1.5. **Controlled transforms.** Rescaling the admissibility inclusion $\mathcal{I}t \subset \mathcal{A}^{\mathrm{ext}}$ leads to $\mathcal{I}t^{w_A} \subset \mathcal{O}_B$. Thus the full transform $\mathcal{O}_B \cdot \mathcal{I}$ of $\mathcal{I}$ is divisible by the power $(t^{-1})^{w_A}$ of the exceptional divisor $D = V_B(t^{-1})$. The result of this factorization $\sigma^c(\mathcal{I}) := \mathcal{O}_B \mathcal{I}t^{w_A}$ is called the *controlled transform* of $\mathcal{I}$. Thus the admissibility condition turns into the condition for the controlled transform.

Also the new coordinates $x_1' := x_1 t^{w_1}, \ldots, x_k' := x_k t^{w_k}$ are simply the *controlled transforms* of $x_1, \ldots, x_k$. Consequently the controlled transform $\sigma^c(\mathcal{A})$ of the center $\mathcal{A}$ can be written at the vertex V as $\sigma^c(\mathcal{A}) = \mathcal{O}_B[x_1' t^{1/a_1}, \ldots, x_k' t^{1/a_k}]$. Similarly the derivations $D_{x_i'} = t^{-w_i} D_{x_i} = \sigma^c(D_{x_i})$ on B are the controlled transforms of D_{x_i}.

As a result the coefficient ideals $C_{x_i t^{1/a_i}}(R_i)$ commute with the operation of the controlled transform: $\sigma^c(C_{x_i t^{1/a_i}}(R_i)) = C_{x_i' t^{1/a_i}}(\sigma^c(R_i))$, and the entire milling algorithm commutes at the points of the vertex V converging from $\sigma^c(R_1) = \mathcal{O}_X[\sigma^c(\mathcal{I})t]$ to the maximal admissible center $\sigma^c(R_{k+1}) = \sigma^c(\mathcal{A})$ and giving the same value of the invariant

$$\mathrm{inv}(\sigma^c(R)) = (a_1, \ldots, a_k) = \mathrm{inv}(R)$$

for $\sigma^c(R)$ at the vertex V as for $\mathcal{I}$ at the center $V(\mathcal{A})$.

1.6. **Resolution principle for the resolution in characteristic zero.** The resolution algorithm in characteristic zero can be stated as follows:

For any ideal $\mathcal{I}$ and a point p consider a unique maximal admissible center $\mathcal{A}$. Then

(1) The vanishing locus $V(\mathcal{A})$ describes the locus of the points q where $\mathrm{inv}_q(\mathcal{I}) = \mathrm{inv}_p(\mathcal{I})$ is constant and attains its maximal value in a certain neighborhood U of p. Thus $\mathrm{inv}_p(\mathcal{I})$ is upper semicontinuous and attains finitely many values along the maximal admissible centers.

(2) The full cobordant of B at $\mathcal{A}$ corresponds to the rescaled algebra $\mathcal{O}_B$ of $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X$.

(3) Consider the Rees center $\mathcal{A}$ associated with the maximum $\mathrm{maxinv}_X(\mathcal{I})$ of the invariant inv on X. After the full cobordant blow-up B of X at $\mathcal{A}$ the invariant inv of the controlled transform $\sigma^c(\mathcal{I}) := \mathcal{O}_B \cdot \mathcal{I}t^{w_A}$ attains its maximum at the vertex V from B. This maximum is equal to to

$$\mathrm{maxinv}_B(\sigma^c(\mathcal{I})) = \mathrm{maxinv}_X(\mathcal{I}) = \mathrm{inv}_{p'}(\sigma^c(\mathcal{I}))$$

for any $p' \in V$.

(4) The invariant "inv" drops for the *cobordant blow-up* $B_+ := B \smallsetminus V$ after removing the vertex V, that is

$$\mathrm{maxinv}_B(\sigma^c(\mathcal{I})) < \mathrm{maxinv}_X(\mathcal{I})$$

leading to the resolution of singularities.

(5) The maximal admissible center for the ideal $\sigma^c(\mathcal{I})$ at V is given by the controlled transform $\sigma^c(A)$ of the center $\mathcal{A}$ so we have maximal admissibility inclusion $\sigma^c(\mathcal{I}) \subset \sigma^c(\mathcal{A})$.

1.7. Resolution principle in general. When considering maximal admissible centers in characteristic zero for an ideal $\mathcal{I}$ we see a strong correlation between the singularities at the vertex and the center. The singularities along the maximal admissible center $V(\mathcal{A})$ are the same as the singularities at the vertex V in the sense they have the same invariant.

This principle is a part of a more general picture and can be pursued in a nonzero characteristic which leads to various types of resolution and reduction theorems. It relies on the following observations:

(1) Let $D = V_B(t^{-1})$ be the exceptional divisor. The vertex V consists of two parts:
$$V \smallsetminus D \subset B_- := B \smallsetminus D = X \times \mathbb{A}^1$$
is isomorphic to $V \smallsetminus D \simeq V(\mathcal{A}) \times \mathbb{A}^1$, showing that the singularities of $V(\mathcal{I})$ at $V(\mathcal{A})$ are equivalent up to smooth projection to the singularities of *the strict transform* $V(\sigma^s(\mathcal{I}))$ at $V \smallsetminus D$. Here the strict transform $\sigma^s(\mathcal{I})$ is obtained by the factoring out the maximal powers of the exceptional divisors from the functions in the full transform $\mathcal{O}_B \cdot \mathcal{I}$.

(2) The exceptional divisor $D = V_B(t^{-1})$ is isomorphic to the normal weighted bundle $N_{X/V(\mathcal{A})}$ of X at $V(\mathcal{A})$. (Lemma 4.1.5) The singularities of the intersection
$V(\sigma^s(\mathcal{I})) \cap D$ at $V_D := V \cap D \subset D$ are isomorphic to the singularities of the normal weighted cone $C_{V(\mathcal{A})}(V(\mathcal{I})) \subset N_{X/V(\mathcal{A})}$ of the $V(\mathcal{I})$ at $V(\mathcal{A})$. (Lemma 4.1.7)

This allows to resolve or improve singularities if they are controlled along the center and when the singularities of its normal weighted cone are controlled at the center as well. The method works for some classes of singularities. In general, in a nonzero characteristic we may have some automorphisms of the normal cone which prevent an apparent improvement of singularities.

1.8. SNC divisors. Another novelty of this paper is the direct treatment of the exceptional SNC divisors without any additional structures. In [ATW19], or in [McQ19], this problem is omitted and the simple resolution is obtained without any conditions on the exceptional divisors. In the classical Hironaka approach the algorithm requires the exceptional divisor must have SNC crossings, and the process is controlled by a certain inductively defined filtration on the set of the exceptionals which is arguably the most cumbersome part of the proof. In this paper no structure is needed and a geometric invariant is a simple function of the subvariety or the ideal and the set of the exceptional divisors. Consequently, the embedded and nonembedded resolution of varieties and principalization of ideals is obtained in

the stronger form where the centers have SNC with the strict transforms, and the inverse image of the ideal is given by an effective SNC divisor.

The definition of the invariant $\mathrm{inv}_p(\mathcal{I})$ and the maximal admissible centers are adapted to the regular schemes with SNC divisor. In such a case we only consider the coordinate systems where each divisorial component at a given point is represented by one of the coordinates. We shall also think of the divisorial coordiantes as infinitesimally heavier than the other ones.

Consequently in the admissibility relation

$$\mathcal{I}t \subset \mathcal{A} = \mathcal{O}_X = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$$

we associate with the any free, that is non-divisorial coordinate x_i the element $b_i = a_i$ and with divisorial coordinate x_j the slightly "heavier" symbol $b_j = a_{j+}$. We assume here that $a_+ > a$ for any $a \in \mathbb{R}$ and if $b > a$ for $a, b \in \mathbb{R}$ then $b > a_+$ This gives a vector $(b_1, \ldots, b_k)$ and we assume that $b_1 \leq \ldots \leq b_k$. The invariant in such a case can be written as

$$\mathrm{inv}_p(\mathcal{I}) = \max\{(b_1, \ldots, b_k) \mid \mathcal{I}t \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}\}.$$

The main point of this construction is that we obtain still upper semicontinuous invariant, which drops after a cobordant blow-up at the associated maximal admissible center. Moreover, the whole algorithm is only marginally affected by this modification of the invariant. The only essential change lies in the construction of maximal contact allowing additional divisorial coordinates tangential to $T^{1/a_i}(R_i)$.

1.9. **Comparing to existing strategies and ideas.** The ideas of this paper stem mostly from [ATW19] and the joint project with Abramovich and Temkin. The main difference here is the use of cobordant blow-ups, rational Rees algebras for the construction and uniqueness of the centers, and the use of the SNC divisors giving stronger results on resolution and principalization without changes in the complexity of the algorithm. Moreover, the paper extends some ideas to a more general context of schemes in any characteristic. In the case where the exceptional divisors are ignored we obtain the same centers as in [ATW19].

The cobordant blow-ups were considered independently by Quek and Rydh in [QR22] from a slightly different angle. They provide a convenient presentation of the stack theoretic weighted blow-ups. Again the ideas were born from the method in [ATW19]. One shall stress that using stack-theoretic quotients does not give here any particular benefit. Instead, some of the information is lost in the process. It is mostly used in the historic context of the established notion.

In [ATW19] the construction of the center was achieved by a different inductive argument in a different language and its uniqueness was proven by using the idea of *homogenization ideals* from [Wło05]. The centers were represented by the $\mathbb{Q}$-ideals $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. In the very first version of [ATW19] this construction was related to the rescaled algebra of the center $\mathcal{A}_{\mathcal{J}} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$ for an ideal $\mathcal{I}$ obtained by applying the saturated weighted derivatives $\mathcal{D}_{x_i t^{1/a_1}}$ to $\mathcal{O}_X[\mathcal{I}t]$, and the center $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ in the $\mathbb{Q}$-ideal form with $w_1 a_1 = \ldots = w_k a_k$. Such an interpretation is valid when there are no SNC divisors. This was done as an illustration of our method, was not used in the context of the resolution strategy and thus was omitted in the next versions of [ATW19].

The method in this paper is designed to derive directly in an efficient way the invariant and the unique maximal center $\mathcal{A}$ from $\mathcal{I}$ using the minimalistic transformations determined solely by $\mathcal{I}$ without any references to prior constructions. The transformations are done in SNC setting and applied to generators only. As a consequence, all the intermediate steps are easy to compute but non-canonical and only the final center is automatically unique, since the process is independent of the maximal admissible center containing all the transitional algebras (see Examples 3.2.2, 3.2.4, 3.2.6).

The compact formula for the coefficient ideal which is generated by the coefficients and maximal contact can be linked to the Bierstone-Milman presentation with the important distinction that it involves no factorization of monomials and the coefficient presentation is done on the completion of the local ring of the original variety in the language of Rees algebras. The Bierstone-Milman coefficient ideal lives on a hypersurface of maximal contact and is generated solely by the adjusted coefficients ([BM91],[BM97]).

This type of Bierstone-Milman approach to the coefficient ideal is much more efficient for computations and practical implementations. Recall that most of the resolution strategies rely on the Villamayor approach to the coefficient ideal with rescaling of the ideals (see [Vil89], [Wło05], [EH02], [EV03], [Kol07],[BM08]. This strategy was also used in [ATW19]. Each coefficient ideal typically increases the grading from n to $n!$ which severely restricts the implementability of the algorithm. This is due to the simple fact that already $(5!)!$ exceeds 10^{97} which is according to the Standard Model of particles the assumed maximal estimated number of the subatomic particles in the universe. In practice, computing even the first coefficient ideal could be tedious for the relatively small gradings say $n \geq 7$. Here the grading n is typically given by the order of the ideal $\mathcal{I}$ at a point and is estimated by the degrees of its generators.

In their paper [QR22], Rydh and Quek observed that the canonical projection $B \to \mathbb{A}^1$ determines the *deformation to the normal weighted cone.* (see Remark 4.1.11). They also noticed that weighted blow-up can be interpreted in terms of the toric combinatorial construction of Cox rings.

The approach was pursued in the joint work [AQ21] of Abramovich-Quek based on the Satriano combinatorial method from [Sat13]. Finally in the subsequent paper [Wło23], we give a general formula for the realization of any proper birational morphism by the torus action using the Cox rings. In the paper [Wło23] we also extend the methods and results of the *weighted normal cone* from Section 4 (Theorems 4.2.2, 4.2.3, 4.3.1) to the context of the blow-ups of any locally monomial ideals.

1.10. The output of the algorithm.
The resolution algorithm outputs a smooth scheme with a torus action and admitting a geometric quotient having abelian quotient singularities and birational to the original scheme. The quotient can be directly resolved by canonical combinatorial methods in any characteristic as in [Wło20, Theorem 7.17.1], or in [BR19] by the destackification method of Bergh and Rydh.

1.11. **Main results in characteristic zero.** The following theorems extend the results and the resolution method by stack-theoretic weighted blow-ups from [ATW19, Section 1.2] in the language of torus actions and SNC divisors.

1.11.1. *Functorial principalization.*

Theorem 1.11.2. *Let X be a smooth variety over a field k of characteristic zero, and E be a simple normal crossing (SNC) divisor and $\mathcal{I}$ be an ideal sheaf on X. There exists a canonical principalization of $\mathcal{I}$, that is a sequence of cobordant blow-ups (Definition 2.4.7) at smooth weighted centers*

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_k = X', \quad (*)$$

such that:

   (1) *The torus $T_i = G_m^i$ acts on X_i, with finite stabilizers, where $T_0 = 1$, $T_{i+1} = T_i \times G_m$, ans $T := T_k$ so that the geometric quotient (space of orbits) X_i/T_i exists.*
   (2) *Set $\mathcal{I}_0 := \mathcal{I}$. For $i \geq 1$, the ideals $\mathcal{I}_i \subset \mathcal{O}_{X_i}$, defined as $\mathcal{I}_i := \mathcal{O}_{X_i} \cdot \mathcal{I}_{i-1}$ are T_i-stable.*
   (3) *Set $E_0 := E$, and let E_i, for $i \geq 1$, be the total transform of E_{i-1}. The divisors E_i on X_i are T_i-stable and have SNC.*
   (4) *The smooth weighted centers $\mathcal{J}_i$ of the cobordant blow-ups $X_i \leftarrow X_{i+1}$ are compatible with E_i (Definition 3.1.2), and are T_i-stable. Moreover $V(\mathcal{J}_i) \subseteq V(\mathcal{I}_i)$.*
   (5) *$\mathcal{O}_{X'} \cdot \mathcal{I} = \mathcal{I}_{D'} := \mathcal{O}_{X'}(-D')$ is the ideal of an SNC divisor D' whose components are the strict transforms of the components of the total transform E' of E. Moreover there is a T-equivariant isomorphism over $X \smallsetminus V(\mathcal{I})$:*

$$X' \smallsetminus D' \simeq (X \smallsetminus V(\mathcal{I})) \times T.$$

   (6) *The sequence (*) defines the sequence of weighted blow-ups on the induced geometric quotients $X = X/T_0 \leftarrow X_1/T_1 \leftarrow \ldots \leftarrow X_k/T_k = X'/T$, such that $\mathcal{O}_{X'/T'} \cdot \mathcal{I} = \mathcal{I}_{D''}$ is the ideal of a locally principal divisor $D'' := D'/T \subset X'/T'$, where D' is a T-stable SNC divisor on X', and $\mathcal{O}_X \cdot \mathcal{I}_{D''} = \mathcal{I}_{D'}$.*
   (7) *The sequence (*) defines the sequence of stack-theoretic weighted blow-ups on the induced smooth stack-theoretic quotients*

$$X = X_0 = [X/T_0] \leftarrow [X_1/T_1] \leftarrow \ldots \leftarrow [X_k/T_k] = [X'/T'],$$

   such that $\mathcal{O}_{[X'/T']} \cdot \mathcal{I}$ is the ideal of an SNC divisor on $[X'/T']$.
   (8) *The sequence (*) is functorial for smooth morphisms, field extensions, and group actions preserving $(\mathcal{I}, E)$.*

1.11.3. *Embedded desingularization.*

Theorem 1.11.4. *Let Y be a reduced closed subscheme of a smooth scheme X over a field k of characteristic zero. Let E be an SNC divisor on E. There exists a sequence of cobordant blow-ups at smooth weighted centers*

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_k = X' \quad (*)$$

and the induced sequence of the strict transforms

$$Y = Y_0 \leftarrow Y_1 \leftarrow \ldots \leftarrow Y_k = Y',$$

such that

(1) *The torus $T_i = G_m^i$ acts on X_i, with finite stabilizers, where $T_0 = 1$, $T_{i+1} = T_i \times G_m$ and $T := T_k$ so that the geometric quotient X_i/T_i exists.*

(2) *The closed subschemes $Y_i \subset X_i$ are T_i-stable. In particular, they admit geometric quotients Y_i/T_i.*

(3) *Set $E_0 := E$, and let E_i, for $i \geq 1$, be either the total or the strict transform of E_{i-1}. The divisors E_i on X_i are T_i-stable and have SNC.*

(4) *The smooth weighted centers of the cobordant blow-ups $X_i \leftarrow X_{i+1}$ are compatible with E_i. Moreover, they are T_i-stable and contained in the locus $\operatorname{Sing}_{E_i}(Y_i)$ of the points where either Y_i is singular, or Y_i is smooth but not transversal to E_i.*

(5) *Y' is a smooth subvariety of X and have SNC with $E' := E_k$. If E' is the total transform of E then the exceptional divisor of the induced morphism $Y' \to Y$ is also SNC.*

(6) *The sequence (*) defines the sequence of weighted blow-ups on the induced geometric quotients $X = X/T_0 \leftarrow X_1/T_1 \leftarrow \ldots \leftarrow X_k/T_k = X'/T$, such that $Y'/T \subset X'/T$ admit abelian quotient singularities,*

(7) *The sequence (*) defines the sequence of stack- theoretic weighted blow-ups on the induced smooth stack-theoretic quotients*

$$X = X_0 = [X/T_0] \leftarrow [X_1/T_1] \leftarrow \ldots \leftarrow [X_k/T_k] = [X'/T'],$$

such that $[Y'/T'] \subset [X'/T']$ is a smooth substack.

(8) *The sequence (*) is functorial for smooth morphisms, field extensions, and group actions preserving Y.*

Remark 1.11.5. The total transform of the divisor E_i is the union of the strict transforms and the exceptional divisors. Using the total transforms gives better control over the induced exceptional divisor of the resolution morphism $X' \to X$ which is SNC and has simple normal crossings with $Y' \subset X'$. The strict transforms of E_i give a slightly faster and simpler algorithm without control of the exceptional divisors.

In particular, if $E_0 = 0$ and the strict transforms are considered we get $E_i = 0$, and no divisors are present in the algorithm. In this case we obtain the resolution procedure similar to the one in [ATW19]. Moreover, when applying the stack-theoretic quotients as in (5), we reprove the main result of [ATW19, Corollary 1.2.3].

1.11.6. *Nonembedded desingularization.*

Theorem 1.11.7. *Let Y be a reduced scheme of finite type over a field k of characteristic zero. There exists a sequence of cobordant blow-ups of regular weighted centers (as in Definition 2.4.7):*

$$Y = Y_0 \leftarrow Y_1 \leftarrow \ldots \leftarrow Y_k = Y', \quad (**)$$

such that

(1) *Y' is a smooth variety.*

(2) *The torus $T_i = G_m^i$ acts on Y_i, with finite stabilizers, where $T_0 = 1$, $T_{i+1} = T_i \times G_m$ and $T := T_k$, so that the geometric quotient Y_i/T_i exists.*

(3) *The inverse image of the exceptional locus is a T'-stable SNC divisor E' on Y'*

(4) *The inverse image of the nonsingular locus $Y^{ns} \subseteq Y$ is equal to $Y^{ns} \times T$, with the natural action of T on the second component.*

(5) *The sequence (**) defines the sequence of weighted blow-ups on the induced geometric quotients $Y = Y/T_0 \leftarrow Y_1/T_1 \leftarrow \ldots \leftarrow Y_k/T_k = Y'/T$, such that Y'/T has an abelian quotient singularities.*

(6) *The sequence (**) defines the sequence of stack-theoretic weighted blow-ups on the induced stack-theoretic quotients*

$$Y = Y_0 = [Y/T_0] \leftarrow [Y_1/T_1] \leftarrow \ldots \leftarrow [Y_k/T_k] = [Y'/T],$$

such that $[Y'/T]$ is a smooth stack.

(7) *The sequence (**) is functorial for smooth morphisms, group actions, and field extensions.*

2. Cobordant blow-ups

2.1. $\mathbb{Q}$-ideals and rational powers of ideals.
In [ATW19] we introduced the notion of the *valuative $\mathbb{Q}$-ideals $\mathcal{J}$* or simply *$\mathbb{Q}$-ideals $\mathcal{J}$*. In the simplest case they could be directly related to the notion of *rational powers of ideals* considered by Huneke-Swanson ([HS06, Section 10.5]). They naturally generalize the ideals and can be used for the compact description of the centers. In this paper, for the most part and for computations in characteristic zero, we predominantly focus on the rational Rees algebras.

Definition 2.1.1. ([ATW19]) By a *Q-ideal $\mathcal{J}$* on a irreducible noetherian scheme X we mean the equivalence classes of the formal expressions $\mathcal{J}^{1/n}$, where $\mathcal{J}$ is the ideal on X and $n \in \mathbb{N}$ is a natural number. We say that $\mathcal{J}^{1/n}$ and $\mathcal{I}^{1/m}$ are *equivalent* if the integral closures of $\mathcal{J}^m$ and $\mathcal{I}^n$ are equal:

$$(\mathcal{J}^m)^{\text{int}} = (\mathcal{I}^m)^{\text{int}}.$$

Remark 2.1.2. Note that if $\mathcal{J}^{1/n}$ and $\mathcal{I}^{1/m}$ represent the same $\mathbb{Q}$-ideal for some ideals $\mathcal{I}$, and $\mathcal{J}$ and $m, n \in \mathbb{N}$ then the ideals $\mathcal{I}$ and $\mathcal{J}$ are called *projectively equivalent* wth respect to the coefficient m/n. ([Rush07])

One can associate with a $\mathbb{Q}$-ideal $\mathcal{J} = \mathcal{I}^{1/n}$ the unique graded Rees algebra

$$\mathcal{A}_{\mathcal{J}} := (\mathcal{O}_X[\mathcal{I}t^n])^{\text{int}} \subset \mathcal{O}_X[t]$$

which is the integral closure of

$$\mathcal{O}_X[\mathcal{I}t^n] = \mathcal{O}_X \oplus \mathcal{I}t^n \oplus \mathcal{I}^2 t^{2n} \oplus \ldots$$

in $\mathcal{O}_X[t]$. Conversely, it is worth noting that any integrally closed Rees algebra of the form

$$\mathcal{O}_X \oplus \mathcal{I}_1 t \oplus \ldots \oplus \mathcal{I}_n t^n \oplus \ldots$$

can be expressed as the integral closure of $\mathcal{O}_X[\mathcal{I}t^n]^{\text{int}}$ for a sufficiently large n, where $\mathcal{I} = \mathcal{I}_n$. This observation was made, in particular, by Quek:

Proposition 2.1.3. [Que20, Theorem 2.2.5] *There is a bijective correspondence between the $\mathbb{Q}$-ideals $\mathcal{J} = \mathcal{I}^{1/n}$ on a normal scheme X and the integrally closed algebras $\mathbb{Z}$-graded Rees algebras on X, given by*

$$\mathcal{J} \mapsto \mathcal{A}_{\mathcal{J}} := \mathcal{O}_X[\mathcal{I}t^n]^{\text{int}}.$$

Under this correspondence the ordinary ideal $\mathcal{I}$ corresponds to $\mathcal{O}_X[\mathcal{I}t]^{\text{int}}$. ♣

With any $\mathbb{Q}$-ideal $\mathcal{J} = \mathcal{I}^{1/n}$ on X one can associate *the ideal of sections*

$$(\mathcal{I}^{1/n})_X = \mathcal{J}_X := \{f \in \mathcal{O}_X \mid f^{mn} \in \mathcal{I}^m \mid m \in \mathbb{N}\} = \{f \in \mathcal{O}_X \mid f^n \in \mathcal{I}^{\mathrm{int}} \mid m \in \mathbb{N}\}.$$

. It is exactly the t-gradation of the algebra $\mathcal{A}_\mathcal{J} = \mathcal{O}_X[\mathcal{I}t^n]^{\mathrm{int}}$. In particular, if $\mathcal{J} = \mathcal{I}$ is an ideal then $\mathcal{J}_X = \mathcal{I}^{\mathrm{int}}$ is it is integral closure.

This terminology is the strictly related to the notion of *rational powers* of ideals which was coined by Huneke-Swanson [HS06, Section 10.5].

Definition 2.1.4. ([HS06]) For any ideal $\mathcal{I}$ and a $\mathbb{Q}$-ideal $\mathcal{I}^{m/n}$ for $m, n \in \mathbb{N}$ the ideal of sections $(\mathcal{I}^{m/n})_X$ is called the m/n-th *rational power* of ideal $\mathcal{I}$.

Using the correspondence between the $\mathbb{Q}$-ideals and Rees algebras one can associate with the formal sum $\sum_{i=1}^{k} \mathcal{I}_i^{a_i/n_i}$, of $\mathbb{Q}$-ideals $\mathcal{I}_i^{a_i/n_i}$ on X the Rees algebra

$$\mathcal{O}_X[\mathcal{I}_1^{a_1} t^{n_1}, \ldots, \mathcal{I}_k^{a_{n_k}} t^{n_k}]^{\mathrm{int}},$$

corresponding to a certain $\mathbb{Q}$-ideal on X.

This leads to the equality of $\mathbb{Q}$-ideals:

$$\sum \mathcal{I}_i^{1/n} = (\sum \mathcal{I}_i)^{1/n} \quad \text{and} \quad \mathcal{I}^{1/n} = (\mathcal{I}^{1/mn})^m$$

and determines the operations of addition and multiplication on the $\mathbb{Q}$-ideals on X. For any $\mathbb{Q}$-ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ on X written in a general form as $\mathcal{J}_1 := \sum \mathcal{I}_i^{1/n_i}$ and $\mathcal{J}_2 := \sum \mathcal{J}_j^{1/n_j}$ their sum is given by

$$\mathcal{J}_1 + \mathcal{J}_2 = \sum \mathcal{I}_i^{1/n_i} + \sum \mathcal{J}_j^{1/n_j}$$

and their product by

$$\mathcal{J}_1 \cdot \mathcal{J}_2 = \sum \mathcal{I}_i^{1/n_i} \cdot \sum \mathcal{J}_j^{1/n_j} = \sum_{i,j} \mathcal{I}_i^{1/n_i} \cdot \mathcal{J}_j^{1/n_j} =$$

$$= \sum_{i,j} (\mathcal{I}_i^{n_j} \cdot \mathcal{J}_j^{n_i})^{\frac{1}{n_i n_j}} = (\sum_{i,j} \mathcal{I}_i^{n_j} \cdot \mathcal{J}_j^{n_i})^{\frac{1}{n_i n_j}}$$

and they both define $\mathbb{Q}$-ideals on X.

Note that these operations extend the standard notion of the sum and the product of ideals. Similarly, we can extend the inclusion relation. Let $\mathcal{J}_1 = \mathcal{I}_1^{1/n_1}$, and $\mathcal{J}_2 = \mathcal{I}_2^{1/n_2}$ be two $\mathbb{Q}$-ideals on X. We say that $\mathcal{J}_1 \subseteq \mathcal{J}_2$ if we have the inclusion of ideals $(\mathcal{I}_1^{N/n_1})^{\mathrm{int}} \subseteq (\mathcal{I}_2^{N/n_2})^{\mathrm{int}}$ for sufficiently divisible N.

2.1.5. *Graded algebras of $\mathbb{Q}$-ideals.* One can associate with the $\mathbb{Q}$-ideal $\mathcal{J} = \mathcal{I}^{1/n}$, the corresponding graded algebra of $\mathbb{Q}$-ideals

$$\mathcal{O}_X[\mathcal{J}t] := \bigoplus (\mathcal{J}^i) t^i,$$

where t is a dummy variable. This, in turn, determines the graded Rees algebra of ideals

$$\mathcal{A}_J = (\mathcal{O}_X[\mathcal{J}t])_X := \bigoplus (\mathcal{J}^i)_X t^i$$

on X.

2.1.6. *Functoriality of $\mathbb{Q}$-ideals.* If $X' \to X$ is any morphism of integral schemes, and $\mathcal{I}^{1/n}$ is a $\mathbb{Q}$-ideal on X then $(\mathcal{O}_{X'} \cdot \mathcal{I})^{1/n}$ is the *preimage of $\mathcal{I}^{1/n}$*, which is a $\mathbb{Q}$-ideal on X'.

2.1.7. *Monomial valuation.*

Definition 2.1.8. Let $u_1, \ldots, u_k$ be a partial local system of parameters on a regular irreducible scheme X. We say that a valuation ν such that $\nu(u_i) = w_i \in \mathbb{Z}_{\geq 0}$ of $K(X)$ is *monomial at a point* $p \in X$, if

$$\mathcal{I}_{\nu,a,p} := \{f \in \mathcal{O}_{X,p} \mid \nu(f) \geq a\} = \mathcal{O}_{X,p} \cdot (u_1^{a_1} \cdot \ldots \cdot u_k^{a_k} \mid a_1 w_1 + \ldots + a_k w_k \geq a).$$

Then ν is a *monomial valuation* on X if it is monomial at all $p \in V(u_1, \ldots, u_k)$.

Lemma 2.1.9. *Let X be a any regular scheme, and $p \in X$ be a point.*
- (1) *If $p \in V(u_1, \ldots, u_k)$ then assigning weights $w(u_i) := w_i \in \mathbb{Z}_{\geq 0}$ determines a monomial valuation ν on $\mathrm{Spec}(\mathcal{O}_{X,p})$ with the ideals $\mathcal{I}_{\nu,a,p}$ as above.*
- (2) *If $V(u_1, \ldots, u_k)$ is irreducible on X then assigning weights $w(u_i) := w_i \in \mathbb{Z}_{\geq 0}$ determines a unique monomial valuation ν on X.*

Proof. (1) We need to show that if $f \in \mathcal{I}_{\nu,a,p} \smallsetminus \mathcal{I}_{\nu,a+1,p}$ and $g \in \mathcal{I}_{\nu,b,p} \smallsetminus \mathcal{I}_{\nu,b+1,p}$ then $fg \in \mathcal{I}_{\nu,a+b,p} \smallsetminus \mathcal{I}_{\nu,a+b+1,p}$. Suppose otherwise $fg \in \mathcal{I}_{\nu,a+b+1,p}$. By removing the extra terms we can assume that $f = \sum_{\nu(u^\alpha)=a} c_\alpha u^\alpha$ and $g = \sum_{\nu(u^\alpha)=b} d_\alpha u^\alpha$, where all c_α, d_α are invertible. Since $\mathcal{O}_{X,p}$ is regular then for the ideal $\mathcal{I} = (u_1, \ldots, u_k)$ we have that $\mathrm{gr}_I(\mathcal{O}_{X_p}) = \mathcal{O}_{X_p}/(\mathcal{I})[u_1, \ldots, u_k]$ is a domain, and assigning weights $\nu(u_i) = w_i$ determines a unique monomial valuation ν_0 on $\mathrm{gr}_I(\mathcal{O}_{X_p}) = \mathcal{O}_{X_p}/(\mathcal{I})[u_1, \ldots, u_k]$ with the ideals $\mathcal{I}_{\nu_0,a} = \mathrm{in}(\mathcal{I}_{\nu,a,p})$. Then considering the initial terms we get $\nu_0(\mathrm{in}(fg)) = \nu_0(\mathrm{in}(f)) + \nu_0(\mathrm{in}(g))$, which implies that $\mathrm{in}(fg) \notin \mathcal{I}_{\nu_0,a+b+1}$. But by the assumption $\mathrm{in}(fg)) \in \mathrm{in}(\mathcal{I}_{\nu,a+b+1,p}) = \mathcal{I}_{\nu_0,a+b+1}$.

(2) The valuation ν is centered at $V(u_1, \ldots, u_k)$ and uniquely determined for every point. Moreover it is the same when passing to the generic point of $V(u_1, \ldots, u_k)$. ♣

2.1.10. *Regular weighted centers.* For the purpose of the resolution one considers $\mathbb{Q}$-ideals on a regular scheme X, called *centers*, which can be locally presented in the form $(u_1^{a_1}, \ldots, u_k^{a_k})$ where $u_1, \ldots, u_k$ is a partial family of local parameters and $a_i \in \mathbb{Q}_{>0}$. We shall always assume that $a_1 \leq \ldots \leq a_k$. Note that one can write it in an equivalent form $(u_1^{a_1}, \ldots, u^{a_k}) = (u_1^{na_1}, \ldots, u^{na_k})^{1/n}$, where $(u_1^{na_1}, \ldots, u^{na_k})$ is an ordinary ideal for a sufficiently divisible n. Then $(u_1^{a_1}, \ldots, u_k^{a_k})_X$ will denote the ideal of sections.

One can describe these ideals in a few different equivalent ways.

Lemma 2.1.11. *(see also [ATW19],[Que20]) Let X be a regular X scheme. Let $w_1 \geq w_2 \geq \ldots \geq w_k$ be positive integers, and $u_1, \ldots, u_k$ are corresponding local parameters on X such that $V(u_1, \ldots, u_k)$ is irreducible. There is a natural bijective correspondence between*
- (1) *$\mathbb{Q}$-ideals $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$.*
- (2) *Graded Rees algebras*
$$\mathcal{A}_J = \mathcal{O}_X[u_1 t^{w_1}, \ldots, u_k t^{w_k}]^{\mathrm{int}} = \mathcal{O}_X[u_1 t^{c_1}, \ldots, u_k t^{a_k} \mid c_1 \leq w_1].$$
- (3) *Monomial valuations ν_J, such that $\nu_J(u_i) = w_i$, and*
$$\mathcal{I}_{\nu_J,a} := \{f \in \mathcal{O}_X \mid \nu_J(f) \geq a\} = (u^\alpha \mid \nu_J(u^\alpha) \geq a).$$
Moreover we get
$$\mathcal{A}_J = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} \mathcal{I}_{\nu_J,a} t^a = (\mathcal{O}_X[\mathcal{J} \cdot t])_X.$$

First we prove a more explicit form of this correspondence for the centers:

Lemma 2.1.12. *(see also* [ATW19],[Que20]*) Let* $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$ *be a regular center on a regular* X, *where* $w_i \in \mathbb{N}$. *Let* ν_J *be a monomial valuation such that* $\nu_J(u_i) = w_i$, *and for any* $a > 0$, $\mathcal{I}_{\nu_J,a} := \{f \in \mathcal{O}_X \mid \nu_J(f) \geq a\}$ *is equal to* $(u^\alpha \mid \nu_J(u^\alpha) \geq a)$. *Then for any* $a \in \mathbb{Q}_{>0}$:

$$((u_1^{1/w_1}, \ldots, u_k^{1/w_k})^a)_X = \mathcal{I}_{\nu_J,a} = \{f \in \mathcal{O}_X \mid \nu_J(f) \geq a\} =$$

$$(u_1^{c_1} \cdot \ldots \cdot u_k^{c_k} \mid c_i \in \mathbb{N},\ c_1 w_1 + \ldots c_k w_k \geq a).$$

In particular

$$((u_1^{a_1}, \ldots, u_k^{a_k}))_X = (u_1^{c_1} \cdot \ldots \cdot u_k^{c_k} \mid\ c_i \in \mathbb{N},\ c_1/a_1 + \ldots + c_k/a_k \geq 1).$$

Proof. Note that for any monomial $u^c = u_1^{c_1} \cdot \ldots \cdot u_1^{c_k}$ with integral c_i and for $a \in \mathbb{Q}_{>0}$ we have :

$$u^c \in \mathcal{I}_{\nu_J,a} \iff c_1 w_1 + \ldots + c_k w_k \geq a.$$

Let n be now any positive integer which is divisible by $\text{lcm}(w_1, \ldots, w_k)$, and write $a_i w_i = n$, for the relevant integers a_i. Then, the power $(u^c)^n = (u_1^{a_1})^{c_1 w_1} \cdot \ldots \cdot (u_k^{a_k})^{c_k w_k}$ is an element of an (ordinary) ideal $(u_1^{a_1}, \ldots, u_k^{a_k})^n$ if and only if $c_1 w_1 + \ldots + c_k w_1 \geq n$.

Thus if $u^c \in \mathcal{I}_{\nu_J,n}$ then $c_1 w_1 + \ldots + c_k w_k \geq n$ and $(u^c)^n \in (u_1^{a_1}, \ldots, u_k^{a_k})^n$, whence $u^c \in ((u_1^{a_1}, \ldots, u_k^{a_k})^{int} = (u_1^{a_1}, \ldots, u_k^{a_k})_X$. This shows that $\mathcal{I}_{\nu_J,n} \subseteq ((u_1^{a_1}, \ldots, u_k^{a_k})_X$. On the other hand $(u_1^{a_1}, \ldots, u_k^{a_k}) \subseteq \mathcal{I}_{\nu_J,n}$, and since $\mathcal{I}_{\nu_J,n}$ is integrally closed $(u_1^{a_1}, \ldots, u_k^{a_k})^{int} \subseteq \mathcal{I}_{\nu_J,n}$. This gives $\mathcal{I}_{\nu_J,n} = (u_1^{a_1}, \ldots, u_k^{a_k})_X$, for any integral n, and a_i such that $n = a_i w_i$. In general, let $a = \frac{l}{m} \in \mathbb{Q}_{>0}$, with l, m integral, and and let n be as above with $n = a_i w_i$. Then, by the definition, and the above $f \in ((u_1^{1/w_1}, \ldots, u_k^{1/w_k})^a)_X$ if and only if $f^{mn} \in ((u_1^{a_1}, \ldots, u_k^{a_k})^l)_X = \mathcal{I}_{\nu_J,l}$, which is equivalent to $\nu_J(f^{mn}) \geq ln \iff \nu_J(f) \geq a$.

For the second part, consider a sufficiently divisible integer $a > 0$ such that $w_i := a/a_i$ are integral. Then we can write

$$(u_1^{a_1}, \ldots, u_k^{a_k}) = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})^a$$

Thus, by the previous part $u^c \in (u_1^{a_1}, \ldots, u_k^{a_k})_X$ is equivalent to $c_1 w_1 + \ldots c_k w_k \geq a$, which translates into $c_1/a_1 + \ldots + c_k/a_k \geq 1$.

♣

Proof. (1) ⇔ (3) By Lemma 2.1.12, we can write the Rees algebra $(\mathcal{O}_X[\mathcal{J}t])_X$ as $(\mathcal{O}_X[\mathcal{J}t])_X = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} \mathcal{I}_{\nu_J,a} t^a$. Thus the valuation ν_J is determined uniquely. Conversely given a monomial valuation ν with the weights $w_1, \ldots, w_k$ one can recover a $\mathbb{Q}$-ideal $\mathcal{J}$ from N-th gradation $(\mathcal{J}^N)_X = \mathcal{I}_{\nu_J,N}$ of $(\mathcal{O}_X[\mathcal{J}t])_X$ for $N = w_1 \cdot \ldots \cdot w_k$ as $\mathcal{J} = (\mathcal{I}_{\nu_J,N})^{1/N}$.

(2) ⇔ (3) Given a monomial valuation ν_J, one can form the integrally closed algebra

$$\mathcal{A}_J := \bigoplus_{a \in \mathbb{Z}_{\geq 0}} \mathcal{I}_{\nu_J,a} t^a = \{f \cdot t^a \mid f \in \mathcal{O}_X, \mid a \leq \nu_J(f)\} = \mathcal{O}_X[u_1 t^{c_1}, \ldots, u_k t^{c_k} \mid c_1 \leq w_1].$$

Conversely, any algebra $\mathcal{O}_X[u_1 t^{c_1}, \ldots, u_k t^{c_k} \mid c_1 \leq w_1]$ determines a monomial valuation defined as follows: For any $f \in \mathcal{O}_X$, we put $\nu_J(f) = \max\{a : t^a \cdot f \in \mathcal{A}_J\}$.

♣

2.1.13. *Blow-ups of $\mathbb{Q}$-ideals on X.* Any $\mathbb{Q}$-ideal of the form $\mathcal{J} = \mathcal{I}^{1/n}$ on a normal scheme X determines a unique blow-up

$$Y = \mathcal{P}roj(\mathcal{A}_{\mathcal{J}}) \to X$$

of $\mathcal{J}$ on X which can be understood as the normalized blow-up of $\mathcal{I}$ on X. It transforms $\mathcal{J} = \mathcal{I}^{1/n}$ into a $(\mathcal{O}_X \cdot \mathcal{I})^{1/n} = (\mathcal{O}_X(-E))^{1/n}$ for a Cartier exceptional divisor E.

2.1.14. *Weighted blow-ups of centers.* In particular, the center $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ on a regular scheme X determines the weighted blow-up

$$Y = \mathcal{P}roj(\mathcal{A}_{\mathcal{J}}) = \mathcal{P}roj(\mathcal{O}_X[t^{w_1}x_1, \ldots, t^{w_k}x_k]^{\text{int}}).$$

2.1.15. *Stack-theoretic weighted blow-ups.* The above formulas of the standard blow-up can be natural extended to the stack-theoretic setting. In [ATW19, Section 3.1] the stack-theoretic blow-up of any $\mathbb{Q}$- ideal $\mathcal{J}$ is defined to be the stack-theoretic quotient of the scheme (using our notation):

$$[(\mathrm{Spec}_X(\mathcal{O}_X[\mathcal{J}t])_X \smallsetminus V((\mathcal{J}t)_X))/G_m],$$

with the natural action of the multiplicative group $G_m := \mathrm{Spec}\, k[t, t^{-1}]$.

In particular, the *stack-theoretic weighted blow-up of $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$* is defined to be the stack-theoretic quotient

$$= [(\mathrm{Spec}_X(\mathcal{O}_X[t^{w_1}x_1, \ldots, t^{w_k}x_k]^{\text{int}}) \smallsetminus V(t^{w_1}x_1, \ldots, t^{w_k}x_k))/G_m] =$$
$$= [(\mathrm{Spec}_X(\mathcal{O}_X[t^{a_1}x_1, \ldots, t^{a_k}x_k \mid c_i \leq w_i]) \smallsetminus V(t^{w_1}x_1, \ldots, t^{w_k}x_k)/G_m].$$

Observe that the space $\mathrm{Spec}_X(\mathcal{O}_X[t^{a_1}x_1, \ldots, t^{a_k}x_k \mid c_i \leq w_i])$ is usually non regular even for an ordinary blow-up of a regular center.

2.2. **Rational Rees algebra.** In this paper we are going to consider Rees algebras on a scheme X with gradations given by a finitely generated additive subsemigoups Γ of $\mathbb{Q}_{\geq 0}$.

Definition 2.2.1. By a *rational Rees algebra* or simply *Rees algebra* we mean a finitely generated $\mathcal{O}_X$-algebra which can be written of the form:

$$R = \bigoplus_{a \in \Gamma} R_a t^a \subset \mathcal{O}_X[t^{1/w_R}],$$

where $w_R \in \mathbb{Q}_{>0}$ is the smallest rational number such that $\Gamma \subseteq (1/w_R) \cdot \mathbb{Z}_{\geq 0}$, and the ideals $R_a \subseteq \mathcal{O}_X$ satisfy

(1) $R_0 = \mathcal{O}_X$
(2) $R_a \cdot R_b \subseteq R_{a+b}$

If R is a rational Rees algebra on X, and w is a multiple of w_R then $R^{\text{ext}} := R[t^{-1/w}]$ will be called an *extended Rees algebra*.

By the *integral closure R^{int} of a rational Rees algebra* $R = \bigoplus_{a \in \Gamma} R_a t^a$ we shall mean its integral closure in $\mathcal{O}_X[t^{1/w_R}]$. The integral closure in $\mathcal{O}_X[t^{1/w}]$, where $w = n w_R$ is a relevant integral multiple of w_R will be denoted by R^{Int}.

By the *vertex $V(R)$ of R* (or $V(R^{\text{ext}})$ of R^{ext}) we mean the vanishing locus

$$V(R) = V(\sum_{a>0} R_a)$$

(respectively $V(R^{\text{ext}}) = V(\sum_{a>0} R_a^{\text{ext}})$).

Remark 2.2.2. We do not assume here that $R_a \subseteq R_b$ if $a \geq b$ for $a, b \in \Gamma$. However this condition is satisfied if $R = R^{\mathrm{int}}$ is integrally closed in $\mathcal{O}_X[t^{1/w_R}]$.

2.2.3. *Rational Rees algebras and ideals.* With any ideal $\mathcal{I}$ one associates the $\mathbb{Z}$-graded Rees algebra $\mathcal{A}_\mathcal{I} := \mathcal{O}_X[\mathcal{I}t] = \bigoplus \mathcal{I}^n t^n$, and the extended Rees algebra

$$\mathcal{A}_\mathcal{I}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1}, \mathcal{I}t].$$

As we mentioned before the main idea of the rational Rees algebras is to enlarge the gradation so we obtain a simpler presentation of the graded algebra, and thus a nicer presentation of the blow-up of $\mathcal{I}$.

2.2.4. *Rees centers and $\mathbb{Q}$-ideals.* By a *Rees center* we mean a Rees algebra $\mathcal{A}$ locally of the form

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}},$$

for some a local partial system of coordinates(local sytem of parameters) $x_1, \ldots, x_k$ and some positive rational numbers $a_1, \ldots, a_k$, where the integral closure is considered in $\mathcal{O}_X[t^{1/w_A}]$, where $w_A := \mathrm{lcm}(a_1, \ldots, a_k)$ is the smallest positive rational such that $w_A/a_1, \ldots, w_A/a_k$ are all integers.

By the *extended center* which is also called *center* we shall mean the extended Rees algebra

$$\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}],$$

where w is a multiple of w_A, so w/a_i are all positive integers.

2.2.5. *Rescaling.*

Definition 2.2.6. Given $w_0 \in \mathbb{Q}_{>0}$, by the $t \mapsto t^{w_0}$ *rescaling* of the rational Rees algebra $R = \bigoplus_{a \in \Gamma} R_a t^a \subset \mathcal{O}_X[t^{1/w_R}]$ we mean the Rees algebra

$$R^{w_0} = \bigoplus_{a \in \Gamma} R_a t^{w_0 a} \subseteq \mathcal{O}_X[t^{w_0/w_R}],$$

Lemma 2.2.7. *There is a natural isomorphism $R \to R^{w_0}$ $ft^a \mapsto ft^{w_0 a}$. In particular, R is integrally closed in $\mathcal{O}_X[t^{1/w_R}]$ iff R^w is integrally closed in $\mathcal{O}_X[t^{w_0/w_R}]$.* ♣

Lemma 2.2.8. *Let $\mathcal{A} := \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{int}}$ and*

$$\mathcal{A}^{\mathrm{ext}} := \mathcal{O}_X[t^{-1/w}, \overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}],$$

where w is a multiple of w_A. Then the integral closure $\mathcal{A}^{\mathrm{Int}}$ of $\mathcal{A}$ in $\mathcal{O}_X[t^{1/w}]$ is equal to $\mathcal{A}^{\mathrm{Int}} := \mathcal{A}_{\geq 0}^{\mathrm{ext}}$.

Moreover if $V(\mathcal{A})$ is irreducible then there is a unique monomial associated valuation ν_A such that $\nu_A(x_i) = 1/a_i$, and

$$\mathcal{A}_a = \{f \in \mathcal{O}_X \mid \nu_A(f) \geq a\}.$$

Proof. It is a direct consequence of Lemma 2.1.11, obtained by rescaling $t \mapsto t^{1/w}$.
♣

2.3. **Regular centers vs Rees centers.** One can identify a regular $\mathbb{Q}$-ideal center locally described by $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$ with $w_i \in \mathbb{N}$, with the Rees center $\mathcal{A}_{\mathcal{J}} = \mathcal{O}_X[u_1 t^{w_1}, \ldots, u_k t^{w_k}]^{\mathrm{int}}$, or the extended Rees center $\mathcal{A}_{\mathcal{J}}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1}, u_1 t^{w_1}, \ldots, u_k t^{w_k}]$.

Per analogy with the correspondence $\mathcal{J} \mapsto \mathcal{A}_{\mathcal{J}}$ we associate with arbitrary $\mathbb{Q}$-ideal $\mathcal{J} = (x_1^{a_1}, \ldots, x_k^{a_k})$, with positive rational a_i the rational Rees algebra

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}},$$

and the extended center

$$\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}],$$

where $w = \mathrm{lcm}(a_1, \ldots, a_k)$.

Conversely any extended center $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$, with $\mathrm{lcm}(a_1, \ldots, a_k)|w$ determines a $\mathbb{Q}$-ideal $\mathcal{J} = (u_1^{a_1/w}, \ldots, u_k^{a_k/w})$, with

$$\mathcal{A}_{\mathcal{J}} = (\mathcal{A}^{\mathrm{ext}})^w = \mathcal{O}_X[t^{-1}, x_1 t^{w/a_1}, \ldots, x_k t^{w/a_k}].$$

Thus there is a bijective correspondence between

- $\mathbb{Q}$-ideals centers $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$ with $w_i \in \mathbb{N}$
- the extended integral Rees algebras $\mathcal{A}_{\mathcal{J}} = \mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]$, and
- the extended rational Rees algebras $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$, with $w_i = w/a_i \in \mathbb{N}$, defined up to rescaling.

2.4. **Cobordant blow-ups.**

2.4.1. *Good and geometric quotient.* We consider here a relatively affine action of $T = \mathrm{Spec}(\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_k, t_k^{-1}])$ on a scheme X over $\mathbb{Z}$. By the *good quotient (or GIT quotient)* we mean an affine T-invariant morphism $\pi : X \to Y = X /\!\!/ T$ such that the induced morphism of the sheaves $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)$ defines the isomorphism onto the subsheaf of invariants $\mathcal{O}_Y \simeq \pi_*(\mathcal{O}_X)^T$.

Then $\pi : X \to Y = X/T$ will be called a *geometric quotient* if additionally every fiber $X_{\overline{y}}$ of π over a geometric point $\overline{y} : \mathrm{Spec}(\overline{k}) \to Y$ defines a single orbit of the action of $T_{\overline{k}} = T \times_k \mathrm{Spec}(\overline{k}) = \mathrm{Spec}(\overline{k}[t_1, t_1^{-1}, \ldots, t_k, t_k^{-1}])$ on $X_{\overline{y}}$.

2.4.2. *Birational cobordisms.* Since the space $\mathrm{Spec}_X(\mathcal{O}_X[t^{w_1} x_1, \ldots, t^{w_k} x_k]^{\mathrm{int}}) = \mathrm{Spec}(\mathcal{A}_J)$ in the definition of the stack-theoretic quotient is singular even for the standard smooth blow-up with weights $w_i = 1$ we consider here an alternative approach which uses the concept of *birational cobordisms*. The notion was developed over a field, but it can be extended to the schemes with the action of the group scheme $G_m := \mathrm{Spec}(\mathbb{Z}[t, t^{-1}]$. One shall mention that the theory of cobordisms was also considered in the language of schemes in [AT19, Definition 4.3], but their definition of cobordism is far more restrictive.

The following definition is motivated for the analogous definition for the varieties with G_m-action and is adapted to the schemes.

Definition 2.4.3. Let X be an integral scheme with the action of G_m. Let $p \in X$ be a point. We say that $\lim_{t \to 0}(tp)$ *exists*, (respectively $\lim_{t \to \infty}(tp)$ *exists*) if there is an open neighborhood U of p such that the morphism $G_m \times U \to X$ extends to the $\mathrm{Spec}(\mathbb{Z}[t]) \times U \to X$ (respectively to $\mathrm{Spec}(\mathbb{Z}[t^{-1}]) \times U \to X$).

The following definition extends the original definition over a field.

Definition 2.4.4. [Wło00, Definition 2] By a birational cobordism representing a birational map $\phi : X_1 \dashrightarrow X_2$ of schemes we mean a scheme with an action of $T = G_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$ such that the sets

$$\begin{aligned}
B_- &:= \{p \in B : \ \lim_{t \to 0} (tp) \text{ does not exist in } B\} \\
\text{and } B_+ &:= \{x \in B : \ \lim_{t \to \infty} (tp) \text{ does not exist in } B\}
\end{aligned}$$

are nonempty Zariski open subsets of B, and the geometric quotients $\alpha_1 : B_+/T \simeq X_1$, and $\alpha_2 : B_-/T \simeq X_2$ exists, with the natural birational map $\psi : B_+/T \to B_-/T$ defined by open inclusions maps $(B_- \cap B_+)/T \to B_\pm/T$, such that

$$\phi \circ \alpha_1 = \alpha_2 \circ \psi.$$

Example 2.4.5. [Wło00, Example 2] Let T act on $B = \mathbf{A}_k^{n+1} = \mathrm{Spec}(k[x_0, \ldots, x_n])$ by $t(x_0, x_1, \ldots, x_n) = (t^{-1}x_0, t^{w_1}x_1, \ldots, t^{w_k}x_k)$, where $w_1 \geq w_2 \geq \ldots \geq w_k > 0$. Then $B_- = B \smallsetminus V(x_0)$, $B_+ = B \smallsetminus V(x_1, \ldots, x_n)$.

By using toric geometry, we will see that the induced morphism $B_+/T \to B_-/T = B /\!\!/ T$ is the weighted blow-up at $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$.

First, the cobordism B corresponds to the regular cone

$$\sigma = \langle e_0, \ldots, e_n \rangle := \sum_{i=0}^{n} \mathbb{Q}e_i$$

in $\mathbb{Q}^{n+1}$ with the basis $\{e_0, \ldots, e_n\}$. The action of T is determined by the vector $v := -e_0 + \sum w_i e_i$, representing the weights. Then the good quotient

$$B \to B /\!\!/ T = \mathrm{Spec}(\mathcal{O}(B)^T)$$

corresponds to the map $\sigma \to \pi(\sigma) = \langle e_1, \ldots, e_n \rangle$, defined by the projection

$$\pi : \mathbb{Q}^{n+1} \to \mathbb{Q}^{n+1}/\mathrm{span}(v) \simeq \mathbb{Q}^n$$

where $\mathbb{Q}^n$ is spanned by $\{e_1, \ldots, e_n\}$. It can be described as $\pi(x_0, x_1, \ldots, x_n) = (x_1 + w_1, \ldots, x_n + w_n)$. Note that face $\sigma_0 := \langle e_1, \ldots, e_n \rangle$ of σ can be identified with its image $\pi(\sigma_0) = \pi(\sigma)$.

The cone σ with a vector v and the projection π is an example of the *combinatorial cobordism* in the sense of Morelli [Mor96]. The "lower boundary" B_- of B corresponds to the "lower boundary" σ_- of σ, which is the complex consisting of the faces of σ visible from below with respect to v. It contains σ_0 and all of its faces. The geometric quotient $B_- \to B_-/T$ corresponds to the restriction $\sigma_0 \to \pi(\sigma_0) = \pi(\sigma)$ of $\pi : \mathbb{Q}^{n+1} \to \mathbb{Q}^n$ mapping isomorphically σ_0 to $\pi(\sigma_0)$. So both quotients: the geometric quotient B_-/T and the good quotient $B /\!\!/ T = \mathrm{Spec}(\mathcal{O}(B))^T$ can be identified as they correspond to $\pi(\sigma_0) = \pi(\sigma) = \sigma_0 \subset \mathbb{Q}^n$.

The "upper boundary" B_+ corresponds to the subcomplex σ_+ with the maximal faces $\langle e_0, e_1, \ldots, \check{e}_i, \ldots, e_n \rangle$, for $i = 1, \ldots, n$, "visible from above". Their projection $\pi(\sigma_+)$ corresponds to the star subdivision of σ_0 at the vector $w := \pi(e_0) = (w_1, \ldots, w_n)$ with maximal faces $\langle w, e_1, \ldots, \check{e}_i, \ldots, e_n \rangle$.

Consequently, the morphism

$$B_+/T \to B_-/T = B /\!\!/ T = \mathrm{Spec}(k[x_0, \ldots, x_k])^T = \mathrm{Spec}(k[u_1, \ldots, u_k]),$$

where $u_i := x_i x_0^{w_i}$, is the weighted blow-up corresponding to the star subdivision $\pi(\sigma_+)$ of $\pi(\sigma) = \sigma_0$. The $\mathbb{Q}$-ideal $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$ defines a piecewise linear convex function

$$F_{\mathcal{I}} := \min\{(1/w_i) \cdot x_i \mid i = 1, \ldots, n\}$$

on σ_0 such that $F_\mathcal{I}(e_i) = 0$, and $F_\mathcal{I}(w) = 1$, and which is linear exactly on the faces $\langle w, e_1, \ldots, \check{e}_i, \ldots, e_n \rangle$. Thus we see that the blow-up of $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$ corresponds to the star subdivision of σ at v, and can be identified with $B_+/T \to B_-/T = B \,/\!/\, T$.

One can describe the morphism $B_+/T \to B_-/T = B \,/\!/\, T$ explicitly. Set $t := x_0^{-1}$. Then we can write

$$\mathcal{B}_J := \mathcal{O}(B) = k[x_0, x_1, \ldots, x_k] = k[t^{-1}, t^{w_1} u_1, \ldots, t^{w_k} u_k] \quad \text{and}$$

$$B_+/T = (\mathrm{Spec}(\mathcal{B}_\mathcal{J}) \smallsetminus V(t^{w_1} u_1, \ldots, t^{w_k} u_k))/T \to$$
$$\to \quad B/T = B_-/T = \mathrm{Spec}(k[u_1, \ldots, u_k]).$$

Moreover the algebra

$$\mathcal{A}_J = (k[u_1, \ldots, u_k])[T^{1/a_1} u_1, \ldots, t^{a_k} u_k \mid a_i \leq w_i] = (k[t^{-1}, t^{w_1} u_1, \ldots, t^{w_k} u_k])_{\geq 0}$$

is exactly the nonnegative part of the algebra B_J. Thus the natural homomorphism $A_J \subset B_J$ defines the isomorphisms of the localizations $(A_J)_{t^{w_i} u_i} \to (B_J)_{t^{w_i} u_i}$ which gives the isomorphism of quasi-affine schemes

$$\mathrm{Spec}(B_J) \smallsetminus V(t^{w_1} u_1, \ldots, t^{w_k} u_k) \to \mathrm{Spec}(A_J) \smallsetminus V(t^{w_1} u_1, \ldots, t^{w_k} u_k),$$

and their quotients $B_+/T \to Proj_X(\mathcal{A}_\mathcal{J})$ over B/T.

2.4.6. *Cobordant blow-ups.* The idea of *cobordant blow-up* is to represent a weighted blow-up by a smooth cobordism.

Let $T := \mathbb{G}_m = Spec(\mathbb{Z}[t, t^{-1}])$.

Definition 2.4.7. Let X be a scheme. *A trivial cobordant blow-up* of X is simply the natural projection $X \times T \to X$ from the product $X \times T = \mathrm{Spec}\,\mathcal{O}_X[t, t^{-1}]$.

By the *full cobordant blow-up* of X at an extended Rees center

$$\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$$

we mean the T-invariant morphism $\sigma : B \to X$ defined by the birational cobordism

$$B := \mathrm{Spec}_X((\mathcal{A}^{\mathrm{ext}})^w) = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, t^{w_1} x_1, \ldots, t^{w_k} x_k]),$$

where $w = \mathrm{lcm}(a_1, \ldots, a_k)$, and $w_i = w/a_i$.

The upper and lower boundaries are equal to

$$B_+ = B \smallsetminus V((\mathcal{A}^{\mathrm{ext}})) = B \smallsetminus V(t^{w_1} x_1, \ldots, t^{w_k} x_k) = B \smallsetminus \mathrm{Vert}(B)$$

$$B_- = B \smallsetminus V(t^{-1}) = \mathrm{Spec}_X(\mathcal{O}_X[t, t^{-1}]).$$

By the *cobordant blow-up* we mean the T-invariant morphism $\sigma_+ : B_+ \to X$. Here $\sigma_- : B_- \to X$ is the trivial cobordant blow-up.

Equivalently B, B_+ will be called *full cobordant blow-up* (resp. *cobordant blow-up*) of a $\mathbb{Q}$-ideal center $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$.

We shall call the closed subscheme

$$V = \mathrm{Vert}(B) := V(t^{w_1} x_1, \ldots, t^{w_k} x_k) = V((\mathcal{A}^{\mathrm{ext}})^w) = V(\mathcal{O}_B \cdot \mathcal{J}) = B \smallsetminus B_+$$

the *vertex* of B (per analogy to the vertex of the affine cone over a projective scheme.)

Thus, as we shall see B is in fact the birational cobordism representing the weighted blow-up of $\mathcal{J}$, with $B_+ = B \smallsetminus V$, and $B_- = B \smallsetminus D = B \times G_m$, where $D := V_B(t^{-1})$ is the exceptional divisor and such that

$$B_+/G_m = B /\!\!/ G_m = \mathcal{P}roj(\mathcal{O}_X[t^{w_1}x_1, \ldots, t^{w_k}x_k]^{\text{int}}),$$

and $B_-/G_m = X$.

Remark 2.4.8. The author learned recently from Dan Abramovich that the algebras $\mathcal{O}_X[t^{-1}, t^{w_1}x_1, \ldots, t^{w_k}x_k]$ were also considered in the context of stack-theoretic blow-ups by Quek and Rydh in their upcoming paper [QR22]. In fact, Rydh noticed independently that such algebras gives a nice and natural description of stack-theoretic weighted blow-ups, which was part of his original goal for [QR22]. On the other hand, cobordant blow-ups are a different point of view on the same algebra, and they are motivated by the interpretation of the weighted blow-ups in terms of torus action and smooth birational cobordisms to avoid the use of the Artin stacks in positive and mixed characteristic. The cobordant blow-ups and the full cobordant blow-ups carry additional useful information and are often simpler for computations. One shall mention that all these ideas stem primarily from the paper [ATW19].

On the other hand, one can link this definition to the *extended Rees algebras* $\mathcal{O}_X[t^{-1}, \mathcal{I}t]$ introduced by Rees and considered by Swanson and Huneke in [HS06, Definition 5.1], in particular, in the context of blow-ups of smooth centers.

Remark 2.4.9.   (1) By Lemma 5.2.6, the geometric quotient $B_+/\mathbb{G}_m \to X$ exists. Moreover, by recalling the considerations in the second part of Example 2.4.5, it can be naturally identified with the usual weighted blow-up:

$$B_+/\mathbb{G}_m \simeq \mathrm{Spec}(\mathcal{O}_X[t^{-1}, t^{w_1}x_1, \ldots, t^{w_k}x_k] \smallsetminus V(t^{w_1}x_1, \ldots, t^{w_k}x_k)))/T \simeq$$
$$\mathcal{P}roj(\mathcal{A}_\mathcal{J}) \simeq (\mathcal{P}roj(\mathcal{O}_X[t^{w_1}x_1, \ldots, t^{w_k}x_k]))^{nor} \to X = B /\!\!/ \mathbb{G}_m = B_-/\mathbb{G}_m.$$

(2) The stack-theoretic quotient of smooth varieties over a field k of characteristic zero $[B_+/\mathbb{G}_m] \to X$ is equivalent to the definition considered in [ATW19, Section 3.1] .

(3) Similarly to the standard blow-up, the cobordant blow-up is trivial over the complement of the geometric locus of the center $V(\mathcal{J}) = V(x_1 \ldots, x_k)$:

$$\sigma_\pm : B_+ \cap B_- = \sigma^{-1}(X \smallsetminus V(\mathcal{J})) \simeq (X \smallsetminus V(\mathcal{J})) \times T \to X \smallsetminus V(\mathcal{J}).$$

(4) The vertex $V = \mathrm{Vert}(B)$ plays a crucial role in the resolution process. It typically represents the locus of points on B that exhibit the most severe singularities. The singularities at $\mathrm{Vert}(B)$ usually coincide with the singularities along the center $\mathcal{J}$ and are eliminated when transitioning to B_+.

2.5. **Exceptional divisor.** The following result explains the definition of the *exceptional divisor* $D := V_B(t^{-1})$ in the language of $\mathbb{Q}$-ideals:

Lemma 2.5.1. *The cobordant blow-up B_+ transforms the center $\mathcal{J} = (x_1^{\frac{1}{w_1}}, \ldots, x_k^{\frac{1}{w_k}})$ into the (ordinary) ideal of the exceptional divisor D on B_+ generated by a global invariant parameter t^{-1}:*

$$\mathcal{I}_D := \mathcal{J} \cdot \mathcal{O}_{B_+} = t^{-1}\mathcal{O}_{B_+}$$

Proof. We have
$$\mathcal{J} \cdot \mathcal{O}_{B_+} = (\mathcal{J} \cdot t) t^{-1} \cdot \mathcal{O}_{B_+}$$
Note however that
$$(\mathcal{J} \cdot t) = ((x_1 t^{w_1})^{1/w_1}, \ldots, (x_k t^{w_k})^{1/w_k}) = \mathcal{O}_{B_+}$$
is a trivial $\mathbb{Q}$-ideal on B_+. ♣

2.5.2. *Cobordant blow-up: local equations.* Let $x_1, \ldots, x_n$ be a system of local parameters (or coordinates) at a point p on a regular X, and let $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ be a center. Then the full cobordant blow-up $B \to X$ of $\mathcal{J}$, can be represented as
$$B = \operatorname{Spec}(\mathcal{O}_X[t^{-1}, x_1', \ldots, x_n']/((x_1' t^{-w_1} - x_1), \ldots, (x_k' t^{-w_1} - x_k)].$$
Thus $B \subset X \times \mathbb{A}^{n+1}$ is locally a closed regular subscheme of $X \times \mathbb{A}^{n+1}$. Moreover it has system of local T-semiinvariant parameters $t^{-1}, x_1', \ldots, x_k', x_{k+1}, \ldots, x_n$ at the relevant $p' \in V(t^{-1}, x_1', \ldots, x_k', x_{k+1} \ldots, x_n)$ mapping to p. Consequently, the full cobordant blow-up $B \to X$ at $(x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ can be described by a single chart with the following coordinates:

- t^{-1} is the inverse of the coordinate t representing the action of torus $\mathbb{G}_m = \operatorname{Spec}(\mathbb{Z}[t, t^{-1}])$.
- $x_i' = x_i \cdot t^{w_i}$ for $1 \le i \le k$, and
- $x_j' = x_j$ for $j > k$.

Remark 2.5.3. For some computations and considerations it is more natural to replace t^{-1} with s so that $x_i' = x_i/s^{w_i}$ in the above formulas. On the other hand the construction of the center and the majority of the computations are consistent with the graded Rees algebras and thus with our notation.

The cobordant blow-up $B_+ = B \smallsetminus V(x_1', \ldots, x_k')$ can be covered by the open subsets $(B_+)_{x_i'} = B \smallsetminus V(x_i')$, associated with x_i' producing several more specific "charts" similarly to the standard blow-up.

2.5.4. *Cobordant blow-ups of toric varieties.* By definition, the full cobordant blow-up of a smooth toric variety at a weighted toric center is again a smooth toric variety. Let $X_\sigma = \operatorname{Spec} k[x_1, \ldots, x_n]$ be the affine toric variety associated with $\sigma \subset N_R := \mathbb{R}_{\ge 0}^n$. The full cobordant blow-up $B = X_\tau \to X_\sigma$ of the $\mathbb{Q}$ ideal $(x_1^{1/w_1}, \ldots, x_n^{1/w_n})$ is the morphism of toric varieties corresponding to the inclusion of algebras:
$$k[x_1, \ldots, x_n] \subseteq k[x_1 t^{w_1}, \ldots, x_n t^{w_n}, t^{-1}]$$
The standard weighted blow-up at $\mathcal{J}$ corresponds to the star subdivision of σ with the center $v := w_1 e_1 + \ldots + w_n e_n$. The full cobordant blow-up B is described by the regular cone $\tau := \langle e_1, \ldots, e_n, v + e_{n+1} \rangle$ with the natural projection onto $\sigma = \langle e_1, \ldots, e_n \rangle$ along the vector e_{n+1}. The cone τ is a cobordism in the sense of Morelli [Mor96]. The upper boundary τ_+ of τ has maximal faces $\langle e_1, \ldots, e_i^\vee, \ldots, e_n, v + e_{n+1} \rangle$ which are "visible from above" and which project exactly to the star subdivision of σ at at v. The lower boundary τ_- is isomorphic to σ.

The above construction extends to any smooth toric variety $X = X_\Sigma$, and $v \in \operatorname{int}(\sigma)$, $\sigma \in \Sigma$. The fan Σ_B associated with B is the union of cones in Σ and those spanned by the vector $v + e_{n+1}$ and the cones in the closed star $\overline{\operatorname{Star}(\tau, \Sigma)}$.

3. Resolution of singularities in characteristic zero by weighted centers

3.1. Resolution invariant on regular schemes with SNC divisors.

3.1.1. *Compatibility with SNC divisors.*

Definition 3.1.2. A coordinate system (or a partial system of local parameters) $x_1, \ldots, x_n$ on an open subset U of a regular scheme X *is compatible with E* if the restriction $E_{i|U}$ of any component E_i in E intersecting U is of the form $V(x_i)$. If $p \in V(x_i)$, where $V(x_i)$ determines a component in $E_{|U}$ then we say that x_i is *divisorial* at p. Otherwise, we say that x_i is *free* at p. We say that a Rees center $\mathcal{A}$ is *is compatible with E* at p or on U if it can be written as

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}} \quad \text{or} \quad \mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$$

where $x_1, \ldots, x_k$ is compatible with E at p or respectively on U.

Remark 3.1.3. This definition can be linked to the logarithmic language. The coordinate system defines at every point $p \in U$ a regular map

$$U \to \mathrm{Spec}(\mathbb{Z}[x_1, \ldots, x_n]) = \mathrm{Spec}(\mathbb{Z}[P]),$$

where the divisorial coordinates x_i at p generate a monoid $P = \overline{\mathcal{M}}_p$, and induce the chart $U \to \mathrm{Spec}(\mathbb{Z}[P])$. The free coordinates x_j determine a coordinate system on the stratum $s := V(P \smallsetminus 0)$ through p defined by the divisorial parameters.

3.1.4. *The centers compatible with an SNC divisor.* Consider the total order on the set of symbols

$$\mathbb{Q}_+ := \mathbb{Q} \amalg \{a_+ \mid a \in \mathbb{Q}\},$$

by putting $a_+ > a$, and $a_+ < b$ if $a < b$. Similarly we define $\mathbb{Z}_+ := \mathbb{Z} \cup \{a_+ \mid a \in \mathbb{Z}\}$. The following operations are defined on $\mathbb{Q}_+$:

- addition
- subtraction of the elements in $\mathbb{Q}$ from an element in $\mathbb{Q}_+$.
- multiplication of elements in $\mathbb{Q}_+$ by the positive rational elements in $\mathbb{Q}_{>0}$.

Given a center $\mathcal{A}$ compatible with E and its local presentation

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$$

we associate with a divisorial coordinate $x_i \in E_p$ the symbol $b_i := a_{i+}$ and with a free coordinate x_i simply $b_i := a_i$. We shall always assume that in the presentation:

$$b_1 \leq \ldots \leq b_k.$$

For any such a center with the presentation $\mathcal{A}$ compatible with E we can define

$$\mathrm{inv}(\mathcal{A}) := (b_1, \ldots, b_k)$$

with $b_1 \leq \ldots \leq b_k$.

Remark 3.1.5. Recall that the logarithmic order introduced in [ATW17, Section 3.6] associates with divisorial unknowns the infinity weight, and with free unknowns the weight 1. This language is compatible with the logarithmic derivations and leads to the logarithmic resolution. Here we assign the weight 1_+ to the divisorial unknowns. This is the "minimal weight" greater than the free unknowns weight, which is still equal to 1. The notion is compatible with the standard derivations and the standard transformation rules and gives an SNC resolution. It prioritizes the divisorial unknowns without essentially changing their weights.

3.1.6. *Canonical invariant.* In [ATW19] we introduced the resolution invariant used for the desingularization. The invariant is defined by the inductive procedure, and can be described in terms of $\mathbb{Q}$-ideals. We shall modify this invariant below so it can be used in the SNC setting in the context of Rees algebra.

Consider the set

$$((\mathbb{Q}_+)_{\geq 0})^{\leq n} \quad := \quad \bigsqcup_{k \leq n} ((\mathbb{Q}_+)_{\geq 0})^k.$$

with lexicographic order. We compare the sequences of different lengths lexicographically by placing a sequence of ∞ at the end.

Definition 3.1.7. Let X be a regular scheme with an SNC divisors E, By the *canonical invariant* $\operatorname{inv}_p(\mathcal{I})$ of $\mathcal{I}$ at a point p we shall mean

$$\operatorname{inv}_p(\mathcal{I}) := \max\{(b_1, \ldots, b_k) \mid \mathcal{I}t \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\operatorname{int}} \mid (b_1 \leq \ldots \leq b_k)\}.$$

where $(x_1, \ldots, x_k)$ are the compatible with E at p. Then $\operatorname{inv}_p(\mathcal{I}) \in (\mathbb{Q}_{\geq 0}^+)^{\leq n}$ if it exists. The Rees center

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\operatorname{int}}$$

such that $\mathcal{I}_p \subseteq \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\operatorname{int}}$ and If $\operatorname{inv}_p(\mathcal{I}) = \operatorname{inv}(\mathcal{A}) = (b_1, \ldots, b_k)$ will be called *a maximal admissible center at p compatible with E.*

Remark 3.1.8. It is not clear a priori, that $\operatorname{inv}_p(\mathcal{I})$ is well defined since the maximum may not be attained. It will be proven in Section 3.7.1 that there exists unique maximal admissible algebra $\mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\operatorname{int}}$ for which the maximum $(\overline{b}_1, \ldots, \overline{b}_k)$ is attained.

The invariant $\operatorname{inv}_p(\mathcal{I})$ was introduced in [ATW19] in a different language in the case where no divisors are present giving the embedded desingularization without the divisorial SNC structure.

On the other hand $\operatorname{inv}_p(\mathcal{I})$ can be also associated with the part of the Hironaka resolution invariant developed in earlier papers on the resolution of singularities, in particular, Bierstone and Milman's [BM97], Villamayor's [Vil89], Encinas-Hauser [EH02] and Włodarczyk's [Wło05], in the particular situation when no exceptional divisors are present, so at the beginning of the process.

3.1.9. *Presentation of centers.* One can write centers in a more compact form:

$$\mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\operatorname{int}},$$

where $0 < a_1 < a_2 \ldots < a_k$, and $\overline{x}_1, \ldots, \overline{x}_k$ is a partial system of local parameters compatible with E on open U intersecting $V(\overline{x}_1, \ldots, \overline{x}_k)$ and each $\overline{x}_i := (x_{i1}, \ldots, x_{ik_i})$ is a subsystem of coordinates.

Then we associate with each block $\overline{x}_i t^{1/a_i}$ the invariant $\overline{b}_i = (b_{i1}, \ldots, b_{ik_i})$, where $b_{i1} \leq \ldots \leq b_{ik_i}$, such that $b_{ij} = a_i$ if x_{ij} is free on U or $b_{ij} = a_{i+}$ if x_{ij} is divisorial on U. Consequently we can rewrite the definition of the resolution invariant in the form which is more convenient for computations and presentation.

$$\operatorname{inv}_p(\mathcal{I}) = \max\{(\overline{b}_1, \ldots, \overline{b}_k) \mid \mathcal{I}t \subset \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\operatorname{int}}\}$$

3.1.10. *Order of ideal revisited.* One can see that the number a_1 in the maximal admissible center is simply the order of ideal $a_1 = \mathrm{ord}_p(\mathcal{I})$. In particular we have:

Lemma 3.1.11.

$$\mathrm{ord}_p(\mathcal{I}) = \max\{a_1 \in \mathbb{Q}_{>0} \mid \quad \mathcal{I}t \subset \mathcal{O}_X[m_p t^{1/a_1}]\}$$

Proof. We can write:

$$\mathrm{ord}_p(\mathcal{I}) = \max\{a_1 \in \mathbb{N} \mid \mathcal{I}_p \subset m_p^{a_1}\} = \max\{a_1 \in \mathbb{Q}_{>0} \mid \mathcal{I}_p \subset m_p^{\lceil a_1 \rceil}\}$$

On the other hand the condition by Lemma 2.2.8, $\mathcal{I}_p t \subset \mathcal{O}_X[m_p t^{1/a_1}]$ means that $\mathcal{I}_p \subset m_p^{\lceil a_1 \rceil}$. ♣

3.1.12. *Admissibility condition for ideals.*

Lemma 3.1.13. *The following conditions are equivalent in a neighborhood of $p \in X$ for an ideal $\mathcal{I}$:*

(1) $\mathcal{I}t \subset \mathcal{A} := \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$
(2) $\mathcal{I}t \subset \mathcal{A}^{\mathrm{ext}} := \mathcal{O}_X[t^{-1/w_A}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$
(3) $\mathcal{I}t^{w_A} \subset \mathcal{A}^{w_A} = \mathcal{O}_X[x_1 t^{w_1}, \ldots, x_k t^{w_k}]^{\mathrm{int}}$, *where* $w_i = w_A/a_i$
(4) $\mathcal{I}t^{w_A} \subset (\mathcal{A}^{\mathrm{ext}})^{w_A} = \mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]$

Proof. The condition (3) and (4) are obtained by rescaling the conditions in (1) and (2). On the other hand, by Lemma 2.1.11,

$$(\mathcal{A}^{\mathrm{ext}})_{\geq 0} = \mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]_{\geq 0} = \mathcal{O}_X[x_1 t^{w_1}, \ldots, x_k t^{w_k}]^{\mathrm{int}} = \mathcal{A}.$$

This implies that condition (1) and (2) are equivalent. ♣

Definition 3.1.14. We say that the center $\mathcal{A} := \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$ or $\mathcal{A}^{\mathrm{ext}}$ is *admissible* for $\mathcal{I}$ if one of the four equivalent conditions above holds.

3.1.15. *Admissibility of Rees algebras.* We extend the admissibility definition to Rees algebras:

Definition 3.1.16. We say that the Rees center $\mathcal{A} = \mathcal{O}_X[x_1 t^{w_1}, \ldots, x_k t^{w_k}]^{\mathrm{int}}$ is *admissible* for a Rees algebra R or simply R-admissible if one of the equivalent condition holds:

(1) $R \subseteq \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{Int}}$, where the inclusion is considered with the integral closure "Int" taken in the smallest ring $\mathcal{O}_X[t^{1/w_{R,A}}]$ containing both algebras, or, equivalently in $\mathcal{O}_X[t^{1/w}]$ for any multiple $w = n \cdot w_{R,A}$ or for a sufficiently divisible w.
(2) $R \subset \mathcal{O}_X[t^{-1/w_{R,A}}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$.
(3) $R \subset \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$, where $w = n \cdot w_{R,A}$.

Remark 3.1.17. Note that the inclusion is independent of choice of sufficiently divissible w.

3.1.18. *Resolution invariant of Rees algebras.* We generalize Definition 3.1.7 to the Rees algebras:

Definition 3.1.19. By the *resolution invariant* of a Rees algebra R at a point p on a regular scheme X with an SNC divisor E we mean

$$\mathrm{inv}_p(R) := \max\{(\overline{b}_1, \ldots, \overline{b}_k) \mid R \subset \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{Int}}\}$$

We show in Section 3.7.1 that there exists unique center $\mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\text{int}}$ for which the maximum $(\overline{b}_1, \ldots, \overline{b}_k)$ is attained. We shall call it the *maximal admissible center* for R at p.

Remark 3.1.20. In the actual resolution process of the ideals all the occurring Rees algebras R satisfy $w_{R,A} = w_A$ and thus the integral closure "Int" and "int" coincide.

One can also define a version of the resolution invariant used for the concept of maximal contact in Section 3.6.10.

$$\text{inv}^1_p(R) := \max\{\overline{b}_1 \mid R \subset \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\text{Int}}\}$$

In particular

Lemma 3.1.21. *If* $R \subset \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\text{Int}}$ *(with* $a_1 < a_2 \ldots < a_k$*) then* $\text{inv}^1_p(R) \geq \overline{b}_1$.

3.1.22. *Order of Rees algebras.* Consequently one generalizes the notion of the order of ideals to Rees algebras.

Definition 3.1.23. Let X be a regular scheme with and SNC divisor E. By the *order* of the Rees algebra $R = \bigoplus_{a \in \Gamma} R_a t^a$ at the point $p \in X$, we mean

$$\text{ord}_p(R) := \min_{a \in \Gamma \smallsetminus 0} \{\text{ord}_p(R_a)/a\}$$

Since R is finitely generated $\mathcal{O}_X$-algebra the order is attained for a certain homogenous element $ft^p \in R_a t^a$, so that

$$\text{ord}_p(R) = \text{ord}_p(ft^a) = \text{ord}_p(f)/a.$$

This definition is a generalization of the order of an ideal:

$$\text{ord}_p(\mathcal{I}) = \text{ord}_p(\mathcal{O}_X[\mathcal{I}t]).$$

Moreover, if $R = \mathcal{O}_X[\mathcal{I}t]$ then $\text{ord}_p(\mathcal{I}) = \text{ord}_p(R)$.

Lemma 3.1.24. *If* $R \subseteq \mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\text{Int}}$ *then* $a_1 \leq \text{ord}_p(R)$. *In particular,*

$$\text{ord}_p(R) = \max\{a_1 \in \mathbb{Q} \mid R \subset \mathcal{O}_X[m_p t^{1/a_1}]^{\text{Int}}\}.$$

Moreover, if $\mathcal{A}$ *is maximal admissible for* R *at* p *then* $\text{ord}_p(R) = a_1$.

We shall use the following

Lemma 3.1.25.

$$\text{ord}_p(R^w) = w \cdot \text{ord}_p(R)$$

♣

Proof. (of Lemma 3.1.24) By the assumption,

$$R = \bigoplus R_a t^a \subseteq \mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\text{int}} \subseteq \mathcal{O}_X[m_p t^{1/a_1}]^{\text{Int}}.$$

Rescaling $t \mapsto t^{a_1}$ gives $R^{a_1} = \bigoplus R_a t^{aa_1} \subseteq \mathcal{O}_X[m_p t]^{\text{Int}}$. Thus $R_a \subseteq m_p^{\lceil aa_1 \rceil}$ whence, by definition, for any $a \in \Gamma_R$,

$$(1/a_1)\text{ord}_p(R_a t^a) = \text{ord}_p(R_a t^{aa_1}) \geq \text{ord}_p(m_p^{\lceil aa_1 \rceil} t^{aa_1}) \geq 1,$$

and $\text{ord}_p(R) \geq a_1$.

On the other hand, if $\mathrm{ord}_p(R) = a_1$ then for any $a \in \Gamma \smallsetminus \{0\}$ we have $\mathrm{ord}_p(R_a) \geq aa_1$, so $R_a \subseteq m_p^{\lceil aa_1 \rceil}$ or

$$R_a t^a \subseteq m_p^{\lceil aa_1 \rceil} t^a \in \mathcal{O}_X[m_p t^{1/a_1}]^{\mathrm{Int}}.$$

♣

3.1.26. *The invariants associated with Rees centers.* Per analogy to the resolution invariant one can associate with the centers

$$\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \dots, \overline{x}_k t^{1/a_k}]^{\mathrm{int}}$$

the invariant

$$\mathrm{inv}(\mathcal{A}) = (\overline{b}_1, \dots, \overline{b}_k), \quad \mathrm{inv}^1(\mathcal{A}) := (\overline{b}_1)$$

3.1.27. *Uniqueness of presentation of the invariant of centers.* The following Lemma shows that $\mathrm{inv}(\mathcal{A})$ is well-defined and independent upon presentation. We shall need the following result:

3.1.28. *Replacement Lemma.*

Lemma 3.1.29. [ATW19](*non-divisorial case, $i = 1$*).
Let $\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \dots \overline{x}_k t^{1/a_k}]^{\mathrm{int}}$ *be a center and $p \in V(\mathcal{A})$ be a point. Let* $\overline{x}_1, \dots, \overline{x}_{i-1}, \overline{x}'$ *be a system of local parameters compatible with E at a point p, for some $i \leq k$, such that*

$$\overline{x}' t^{a_i} \in \mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \dots \overline{x}_k t^{1/a_k}]^{\mathrm{int}}$$

in a neighborhood of p then one can find the coordinates $\overline{x}_1, \dots, \overline{x}_{i-1}, \overline{x}'_i, \overline{x}_{i+1} \dots, \overline{x}_k$ such that $\overline{x} \subset \overline{x}'_i$ and

$$\mathcal{A} = \mathcal{O}_X[\overline{x_1} t^{1/a_1}, \dots, \overline{x}_{i-1} t^{1/a_{i-1}}, \overline{x}'_i t^{1/a_i}, \overline{x}_{i+1} t^{1/a_{i+1}} \dots, \overline{x}_k t^{1/a_k}]^{\mathrm{int}}$$

in a neighborhood of $p \in X$.

Proof. By Lemma 2.2.8, $\mathcal{A}_{a_i} \subset (\overline{x}_1, \dots, \overline{x}_i) + m_p^2$. Thus upon the coordinate change of $\overline{x}_i$ compatible with E in image in $m_p/(m_p^2 + (\overline{x}_1, \dots, \overline{x}_{i-1}))$, of the set of coordinates $\overline{x}$ is a subset of the image of coordinates in $\overline{x}_i$. So one can extend $\overline{x}$, and assume that $\overline{x}$ and $\overline{x}_i$ define the same images in $m_p/(m_p^2 + (\overline{x}_1, \dots, \overline{x}_{i-1}))$. Then in the completion $\widehat{\mathcal{O}_{X,p}}$ we can write equality of the vectors of the coordinates

$$\overline{x}'_i = \overline{x}_i + \overline{g}, \quad \overline{x}'_j = \overline{x}_j, \quad j \neq i,$$

where the coordinates of vector $\overline{g}$ are in $\mathcal{A}_{a_1} \cap (m_p^2 + (\overline{x}_1, \dots, \overline{x}_{i-1}))$. This determines an automorphism of $\widehat{\mathcal{O}_{X,p}}$ which takes $\widehat{\mathcal{O}_{X,p}} \cdot \mathcal{A}$ onto $\widehat{\mathcal{O}_{X,p}} \cdot \mathcal{A}$, and determines the desired coordinate change. ♣

Corollary 3.1.30. [ATW19] *Assume that a center $\mathcal{A}$ has two different presentations:*

$$\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \dots \overline{x}_k t^{1/a_k}]^{\mathrm{int}} = \mathcal{O}_X[\overline{x}'_1 t^{1/a'_1}, \dots \overline{x}'_{k'} t^{1/a_{k'}}]^{\mathrm{int}}$$

then the associated invariants $(\overline{b}_1, \dots, \overline{b}_k) = (\overline{b}'_1, \dots, \overline{b}'_{k'})$ are the same.

Proof. By Lemma 3.1.24, $a_1 = a'_1 = \mathrm{ord}_p(\mathcal{A})$, which can be verifed for generators. By Lemma 3.1.29, applied to both presentations, we can assume that $\overline{x_1} = \overline{x'_1}$. Restricting both algebras to $V(\overline{x}_1) = V(\overline{x'}_1)$ we get the equality by the inductive assumption. ♣

3.2. **Motivating examples.**

3.2.1. *Simple isolated singularities in characteristic zero.*

Example 3.2.2. This is the main motivating example as it illustrates the invariant in simple terms. The example is generalized in Sections 4.2, Example 4.2.4 and in Section 4.5, Example 4.5.3 in any characteristic in two different ways. Consider the isolated singularity defined at the origin p by

$$f = x_1^{c_1} + \ldots + x_n^{c_n},$$

on $X = \mathbb{A}_k^n = \operatorname{Spec} k[x_1, \ldots, x_k]$, where $c_1 \leq \ldots \leq c_n$. Find the partition of $(c_1, \ldots, c_n)$:

$$c_1 = \ldots = c_{i_1} < c_{i_1+1} = \ldots = c_{i_2} < \ldots < c_{i_{k-1}+1} = \ldots = c_{i_k} = c_n,$$

and put $\overline{x}_1 = (x_1, \ldots, x_{i_1})$, $\overline{x}_2 := (x_{i_1+1}, \ldots, x_{i_2}), \ldots,$ $\overline{x}_k = (x_{i_{k-1}+1}, \ldots, x_{i_k})$, $a_j := c_{i_{j-1}+1} = \ldots = c_{i_j}$. By a slight abuse of notations we shall write:

$$f = \overline{x}_1^{a_1} + \ldots + \overline{x}_k^{a_k},$$

where

$$\overline{x}_j^{a_j} := x_{i_{j-1}+1}^{a_j} + \ldots + x_{i_j}^{a_j}.$$

The goal is to find the maximal admissible Rees center $\mathcal{A} := \mathcal{O}_X[\overline{x}_1' t^{1/a_1'}, \ldots, \overline{x}_k' t^{1/a_k'}]^{\mathrm{int}}$ such that:

$$R := \mathcal{O}_X[(ft)] \subseteq \mathcal{A} := \mathcal{O}_X[\overline{x}_1' t^{1/a_1'}, \ldots, \overline{x}_k' t^{1/a_k'}]^{\mathrm{Int}},$$

for a certain k and $a_1' < a_2' < \ldots < a_k'$. The center $\mathcal{A}$ will be constructed in the process of transformation of the algebra R and recursive adjoining the generators $\overline{x}_1' t^{1/a_1'}$.

We put $R_1 = R$. By Lemma 3.1.24,

$$\operatorname{ord}_p(R_1) = \operatorname{ord}_p(ft) = a_1 = a_1'.$$

Applying iteratively the differential operators $t^{-1/a_1}\mathcal{D}_X$ to the homogenous elements of positive degree of both sides of the admissibility inclusion $\mathcal{R} \subset \mathcal{A}$ we see that left side $\mathcal{A}$ will be preserved, while the right side R_1 will be enlarged so that it contains so called *cotangent ideal* $T^{1/a_1}(\mathcal{R}_1) \cdot t^{1/a_1}$ that is the ideal $T^{1/a_1}(\mathcal{R}_1)$ in gradation t^{1/a_1} of obtained from R by applying to $\mathcal{I}t$ the iterated differential operators $t^{-1/a_1}\mathcal{D}_X$, $(a-1)$-times. More specifically:

$$T^{1/a_1}(\mathcal{R}_1) \cdot t^{1/a_1} := (t^{-1}\mathcal{D}_X)^{a_1-1}(\mathcal{I}t) = \mathcal{D}_X^{a_1-1}(\mathcal{I})t^{1/a_1} \subseteq \mathcal{A}_{1/a_1}t^{1/a_1} \supseteq \overline{x}_1' t^{1/a_1}.$$

Since $\operatorname{ord}_p(\mathcal{I}) = a_1$, we can see that $\operatorname{ord}_p(T^{1/a_1}) = \operatorname{ord}_p(\mathcal{D}_X^{a_1-1}(\mathcal{I})) = 1$, and thus it contains a partial set of local parameters called *maximal contact*. In our case

$$T^{1/a_1}(R_1) = \mathcal{D}_X^{a_1-1}(f) = (x_1, \ldots, x_{i_1}, x_{i_1+1}^{a_2-a_1}, \ldots, x_k^{a_k-a_1})$$

contains a maximal contact $\overline{x}_1 = (x_1, \ldots, x_{i_1})$. In general, by Lemma 3.6.20, we can assume that using a simple transformation one can change the presentation of the algebra $\mathcal{A}$ so that from now on $\overline{x}_1' = \overline{x}_1$ is a maximal contact of R_1. This will not affect the algebra $\mathcal{A}$ on the right just will change its presentation.

Next we construct R_2 as the coefficient ideal

$$R_2 := C_{\overline{x}_1 t^{1/a_1}}(R_1) \subset \mathcal{A}$$

In our case we get

$$R_2 = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1})^{\text{Int}}, ft] = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, ft].$$

Here the part $\mathcal{O}_X[\,\overline{x}t^{1/a_1}]^{\text{Int}}$, means the integral closure of $\mathcal{O}_X[\,\overline{x}t^{1/a_1}]$ in the smallest subalgebra $\mathcal{O}_X[t^{1/w}] = \mathcal{O}_X[t^{1/a_1}]$ containing all the generators in $C_{\overline{x}t^{1/a_1}}(R_1)$.

In this example the splitting condition hold in the sense that we have inclusion

$$\mathcal{O}_{H_1} := \mathcal{O}_X/(\overline{x}_1) = K[\overline{x}_2, \ldots, \overline{x}_k] \subset \mathcal{O}_X,$$

with $D_{\overline{x}_1}(\mathcal{O}_{H_1}) = 0$. In general such a splitting is possible in the formal (or étale) coordinate system at a point $p \in X$. In such a case we can write

$$R_2 = C_{\overline{x}_1 t^{1/a_1}}(R_1) = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1})^{\text{Int}}, C_{\overline{x}_1 t^{1/a_1}}(R_1)_{|H_1}].$$

In fact the part $C_{\overline{x}_1 t^{1/a_1}}(R_1)_{|H_1}$ is generated by the graded coefficients as in Lemma 3.5.6.

To illustrate the construction in our case we can write

$$ft = (\overline{x}_1 t^{1/a_1})^{a_1} + (\overline{x}_2^{a_2} + \ldots + x_k^{a_k})t.$$

in the graded coefficient form with respect to the graded coordinate $\overline{x}_1 t^{1/a_1}$. So the only graded coefficient is $f_{|V(x_1)} \cdot t = (\overline{x}_2^{a_2} + \ldots + x_n^{a_n})t$. Consequently using presentation in Lemma 3.5.6 we simply write

$$R_2 = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1})^{\text{Int}}, (\overline{x}_2^{a_2} + \ldots + x_n^{a_n})t].$$

Moreover the maximal admissibility condition is satisfied with $R_2 \subseteq \mathcal{A}$.

Moreover we say that R_2 is *strictly nested* at $H_1 = V(\overline{x}_1)$, if it has the presentation

$$R_2 = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1})^{\text{Int}}, R_{2|H_1}],$$

where, in our case $R_{2|H_1} = (\overline{x}_2^{a_2} + \ldots + \overline{x}_n^{a_n})t$. (Definition 3.5.13)

Then, by Lemma 3.5.11,

$$\mathcal{A}_{|H_1} = \mathcal{O}_{H_1}[\overline{x}'_{2|H_1} t^{1/a'_2}, \ldots, \overline{x}'_{k|H_1} t^{1/a'_k}]^{\text{int}}$$

is a maximal admissible center, for $R_{2|H_1}$ and thus the restricted Rees algebra

$$R_{2|H_1} = (\overline{x}_2^{a_2} + \ldots + x_n^{a_n})t$$

is of order a_2. This implies that

$$a_2 = a'_2 = \text{ord}_p(R_{2|H_1}).$$

We compute the maximal contact of $R_{2|H_1} = (\overline{x}_2^{a_2} + \ldots + \overline{x}_n^{a_n})t$ to be the maximal subsystem of local parameters

$$\overline{x}_2 \subset T^{1/a_2}(R_{2|H_1}) = \mathcal{D}_{\overline{x}_2,\ldots,\overline{x}_k}^{a_2-1}(f) = \mathcal{D}_{\overline{x}_2,\ldots,\overline{x}_k}^{a_2-1}(\overline{x}_2^{a_2} + \ldots + \overline{x}_k^{a_k}) \supseteq (\overline{x}_2).$$

Here $T^{1/a_2}(R_2)$ is obtained from R_2 by applying the differential operators $t^{-(a_2-1)/a_2}\mathcal{D}_{\overline{x}_2,\ldots,\overline{x}_k}^{a_2-1}$ to the generator ft of R_2 in the gradation t, or in the split form to $ft_{|H_1}$.

Since $T^{1/a_2}(R_2) \subset \mathcal{A}_{1/a}$ we conclude that $\overline{x}'_2 = \overline{x}_2$, after a change of the coordinate presentation of $\mathcal{A}$.

Then, as before the coordinate $\overline{x}_2$ splits and computing the coefficients of the only generator

$$f_{|H_1} = (\overline{x}_2^{a_2} + \ldots + \overline{x}_n^{a_n})t = (\overline{x}_2 t^{1/a_2})^{a_2} + (\overline{x}_3^{a_3} + \ldots + \overline{x}_n^{a_n})t$$

of $R_{2|H_1}$ gives the split form of the coefficient ideal as in Lemma 3.5.22:

$$R_3 := C_{\overline{x}_2 t^{1/a_1}}(R_2) = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \overline{x}_2 t^{1/a_2})^{\mathrm{Int}}, f_{|H_2}] =$$
$$= \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \overline{x}_2 t^{1/a_2})^{\mathrm{Int}}, (\overline{x}_3^{a_3} + \ldots + x_n^{a_n})t] \subset \mathcal{A},$$

where $H_2 := V(\overline{x}_1, \overline{x}_2)$. We maintain the maximal admissibility condition $R_3 \subset \mathcal{A}$, while enlarging R_2. At the same time R_3 is in the strictly nested form for $\overline{x}_1 t^{1/a_1}, \overline{x}_2 t^{1/a_2}$ (Definition 3.5.13), which means exactly

$$R_3 = \mathcal{O}_X([\overline{x}_1 t^{1/a_1}, \overline{x}_2 t^{1/a_2})^{\mathrm{Int}}, R_{3|H_2}] = \mathcal{O}_X([\overline{x}_1 t^{1/a_1}, \overline{x}_2 t^{1/a_2})^{\mathrm{Int}}, ft_{|H_2}] \subset \mathcal{A}$$

We continue the recursive process so that

$$R_{i+1} = C_{\overline{x}_i t^{1/a_i}}(R_i) = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_i t^{1/a_1})^{\mathrm{Int}}, ft_{|H_i}] \subset \mathcal{A},$$

where $H_i = V(\overline{x}_1, \ldots, \overline{x}_i)$ until $R_{k+1|H_k} = 0$. The latter implies, after reverting to our original notation that

$$R_{k+1} = \mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{Int}} = \mathcal{O}_X[x_1 t^{1/c_1}, \ldots, x_n t^{1/c_n}]^{\mathrm{int}}.$$

Consequently, $\mathrm{inv}_p(\mathcal{I}) = (c_1, \ldots, c_n)$. Since $\mathcal{A}$ is an a priori chosen maximal $\mathcal{I}$-admissible center, which remains unchanged, we conclude that the process is independent of choices and thus canonical.

The full cobordant blow-up B of X at

$$\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, \overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_i t^{1/a_k}],$$

where $w = \mathrm{lcm}(a_1, \ldots, a_k)$ is defined by the rescaled algebra

$$\mathcal{O}_B = (\mathcal{A}^{\mathrm{ext}})^w = \mathcal{O}_X[t^{-1}, \overline{x}_1 t^{w_1}, \ldots, \overline{x}_k t^{w_k}],$$

where $w_i := w/a_i$.

Equivalently one can write the center in the $\mathbb{Q}$-ideal form $(\overline{x}_1^{1/w_1}, \ldots, \overline{x}_k^{1/w_k})$ representing formally t-gradation of $\mathcal{O}_B$.

Observe that

$$B = \mathrm{Spec}(\mathcal{O}_X[t^{-1}, \overline{x}_1 t^{w_1}, \ldots, \overline{x}_i t^{w_k}]) = \mathrm{Spec}((K[t^{-1}, \overline{x}_1 t^{w_1}, \ldots, \overline{x}_i t^{w_k}])$$

is an affine space with coordinates t^{-1} and $x_i' = x_i t^{w_i}$.

Now, the admissibility condition $ft \subset \mathcal{A}^{\mathrm{ext}}$ implies that $ft^w \subset \mathcal{O}_B$. Here $ft^w \in \mathcal{O}_B$ is so called controlled transform $\sigma^c(f)$ of f is obtained by factoring the inverse image by the power $(t^{-1})^w$, of the exceptional divisor t^{-1}.

Solving for x_i and substituting into f gives

$$f = t^{-w}((x_1')^{a_1} + \ldots + (x_n')^{a_n}).$$

After clearing the exceptional divisor we obtain the formula for

$$\sigma^c(f) = t^{a_1 w_1} f = (x_1')^{a_1} + \ldots + (x_n')^{a_n}$$

in the new unknowns $x_1', \ldots, x_n'$ on B which is identical to the original formula for f on X. Thus the singularity locus on B at the vertex $V = V(x_1', \ldots, x_n')$ is described by the same formula, and the cobordant blow-up $B_+ = B \setminus V(x_1', \ldots, x_n')$ is smooth so the resolution is obtained by the single cobordant blow-up.

3.2.3. *Generalizations.* The previous example can be easily generalized without essential changes in computations.

Example 3.2.4. For the coordinate system $\overline{x}_1, \ldots, \overline{x}_k$, consider forms $F_i(\overline{x}_i)$ of degree a_i such that $a_1 < \ldots < a_k$, and for which $(\overline{x}_i) = \mathcal{D}_{\overline{x}_i}^{a_i-1}(F_i)$. Then for $f = F_1 + \ldots + F_k$ and $\mathcal{I} = (f)$ and $R = R_1 = \mathcal{O}_X[ft]$, As before we obtain the recursive formula

$$R_{i+1} = C_{\overline{x}_i t^{1/a_i}}(R_i) = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_i t^{1/a_1})^{\mathrm{Int}}, f_{|H_i} \cdot t] \subset \mathcal{A},$$

where $H_i = V(\overline{x}_1, \ldots, \overline{x}_i)$, $f_{|H_i} = F_{i+1} + \ldots + F_k$ and $\overline{x}_{i+1} t^{1/a_{i+1}}$ is a maximal contact for $R_{i+1|H_i} = \mathcal{O}_{H_i}[f_{|H_i} t]$. In the inductive step we write

$$f_{|H_i} \cdot t = F_{i+1}(\overline{x}_{i+1} t^{1/a_{i+1}}) + (F_{i+2} + \ldots + F_k)t = F_{i+1}(\overline{x}_{i+1} t^{1/a_{i+1}}) + f_{|H_{i+1}} \cdot t,$$

leading to the split formula for the $R_{i+2} = C_{\overline{x}_{i+1} t^{1/a_i}}(R_{i+1})$. Consequently

$$R_{k+1} = \mathcal{A} = \mathcal{O}_X[(\overline{x}_1 t^{1/\overline{a}_1}, \ldots, \overline{x}_k t^{1/a_k})]^{\mathrm{Int}} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_n t^{1/a_k}]^{\mathrm{int}}$$

is a maximal admissible center.

The full cobordant blow-up B of X at

$$\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, \overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_i t^{1/a_k}],$$

where $w = \mathrm{lcm}(a_1, \ldots, a_k)$ is defined by the rescaled algebra

$$\mathcal{O}_B = (\mathcal{A}^{\mathrm{ext}})^w = \mathcal{O}_X[t^{-1}, \overline{x}_1 t^{w_1}, \ldots, \overline{x}_k t^{w_k}],$$

where $w_i = w/a_i$.

It transforms f into its controlled transform

$$\sigma^c(f) := t^w f = F_1(\overline{x}_1') + \ldots + F_k(\overline{x}_k'),$$

For the new coordinates $\overline{x}_i' = \sigma^c(\overline{x}_i') = t^{w_i} \overline{x}_i$. Removing the vertex of the transfromation we obtain $B_+ = B \setminus V(\overline{x}_1', \ldots, \overline{x}_k')$, where the order of one of the forms $F_i(\overline{x}_i')$ drops, which causes the invariant $\mathrm{inv}_p(\mathcal{I})$ drop.

In particular if

$$f = x_1 x_2 x_3 + x_4^4 + x_5^2 x_6^2$$

then $R_1 = \mathcal{O}_X[ft]$. The first (multiple) maximal contact is (x_1, x_2, x_3) for $a_1 = \mathrm{ord}_p(f) = 3$.

Since $H_1 = V((x_1, x_2, x_3))$ splits we can write

$$R_2 = C_{(x_1, x_2, x_3) t^{1/3}}(R_1) = \mathcal{O}_X[(x_1, x_2, x_3) t^{1/3}, (x_4^4 + x_5^2 x_6^2) t]$$

for the graded coefficient decomposition for $(x_1, x_2, x_3) t^{1/3}$:

$$ft = x_1 t^{1/3} x_2 t^{1/3} x_3 t^{1/3} + (x_4^4 + x_5^2 x_6^2)t$$

Then

$$a_2 = \mathrm{ord}_p(f_{|H_1}) \cdot t = \mathrm{ord}_p((x_4^4 + x_5^2 x_6^2)t) = 4/1 = 4$$

with maximal contact $(x_4, x_5, x_6) t^{1/4}$, giving the maximal admissible center

$$R_3 = C_{(x_4, x_5, x_6) t^{1/4}}(R_2) = \mathcal{A} = \mathcal{O}_X[(x_1, x_2, x_3) t^{1/3}, (x_4, x_5, x_6) t^{1/4}]^{\mathrm{int}},$$

and the invariant $\mathrm{inv}_p(f) = (3, 3, 3, 4, 4, 4)$. The extended center

$$\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/12}, (x_1, x_2, x_3) t^{1/3}, (x_4, x_5, x_6) t^{1/4}])$$

determines the full cobordant blow-up

$$B = \mathrm{Spec}(\mathcal{O}_X[t^{-1}, (x_1, x_2, x_3) t^4, (x_4, x_5, x_6) t^3].$$

Alternatively one can write the center as the $\mathbb{Q}$-ideal

$$((x_1, x_2, x_3)^{1/4}, (x_4, x_5, x_6)^{1/3})$$

formally representing t-gradation of $\mathcal{O}_B$.

3.2.5. *Varieties with a divisorial SNC- structure.*

Example 3.2.6. Let $f = (x_1 + x_2^2)^2 + x_3^7$, where x_1, x_2 are divisorial at the origin at x_3 is free. In this case the process is similar except finding maximal contact is slightly different as it depends on the divisorial structure. Set $R = R_1 = \mathcal{O}_X[ft]$. Then the order of f and of R_1 at the origin is equal to 2. The computation of the cotangent ideal is the same $T^{1/2}(R) = D(f) = ((x_1 + x_2^2), x_3^6)$.

The maximal contact for the Rees algebra of order a_1 is given by the free coordinates in $T^{1/a_1}(R)$ and the divisorial unknowns occurring in the linear parts of $T^{1/a_1}(R)$. In our case the maximal contact $\overline{u} := (x_1)$ for $R_1 = \mathcal{O}_X[ft]$ is determined by the only divisorial coordinate x_1 tangential to $T^{1/2}(R) = ((x_1 + x_2^2), x_3^6)$ at p such that $\overline{u} + m_p^2 \supset T^{1/2}(R)$. (Definitions 3.6.2, 3.6.11)

Write $ft = ((x_1 + x_2^2)^2 + x_3^7)t$ in the coefficient form

$$ft = ((x_1 + x_2^2)^2 + x_3^7)t = (x_1 t^{1/2})^2 + (x_1 t^{1/2})2x_2^2 t^{1/2} + (x_2^4 + x_3^7)t.$$

with respect to the graded maximal contact $x_1 t^{1/2}$. We form the coefficient ideal

$$R_2 := C_{x_1 t^{1/2}}(R_1) = \mathcal{O}_X[(x_1 t^{1/2})^{\text{Int}}, x_2^2 t^{1/2}, (x_2^4 + x_3^7)t].$$

Then R_2 is in a strictly nested form: $R_2 = \mathcal{O}_X[(x_1 t^{1/2})^{\text{Int}}, R_{2|H_1}]$, where $H_1 = V(x_1)$.

The order $\text{ord}_p(R_{2|H_1}) = 4$, with

$$T^{1/4}(R_2) = \mathcal{D}_{(x_2,x_3)}(x_2^2) + \mathcal{D}_{(x_2,x_3)}^3((x_2^4 + x_3^7)) = (x_2, x_3^4)$$

consisting of all elements in gradation $t^{1/4}$ of the algebra obtained by applying iterated operators $\mathcal{D}_{(x_2,x_3)t^{1/4}}$ to R_2. Thus the maximal contact is equal to x_2 in gradation $t^{1/4}$ (Definition 3.6.2). Write $(x_2^4 + x_3^7)t \in R_2$ in the coefficient form with respect to $x_2 t^{1/4}$:

$$(x_2^4 + x_3^7)t = (x_2 t^{1/4})^2 + x_3^7 t.$$

Thus

$$R_3 := C_{x_2 t^{1/4}}(R_3) = \mathcal{O}_X[(x_1 t^{1/2}, x_2 t^{1/4})^{\text{Int}}, x_3^7 t].$$

The order of $R_{3|H_2} = \text{ord}_0(x_3^7 t) = 7$. The maximal contact is $x_3 \in T^7(R_3) = \mathcal{D}_{x_3}^6(x_3^7)$ in gradation $t^{1/7}$. Then

$$R_4 := C_{x_3 t^{1/7}}(R_3) = \mathcal{O}_X[x_1 t^{1/2}, x_2 t^{1/4}, x_3 t^{1/7}]^{\text{int}}.$$

Since $R_{4|H_3} = 0$, for $H_3 = V(x_1, x_2, x_3)$, we conclude that $\mathcal{A} = R_4$ is a maximal admissible center for $(x_1 + x_2^2)^2 + x_3^7$) t . Since $w = \text{lcm}(2, 4, 7) = 28$ we get the formula for the extended Rees center

$$\mathcal{A}^{\text{ext}} = \mathcal{O}_X[t^{-1/28}, x_1 t^{1/2}, x_2 t^{1/4}, x_3 t^{1/7}]$$

Rescaling defines the cobordant blow-up $B = \mathcal{O}_X[t^{-1}, x_1 t^{14}, x_2 t^7, x_3 t^4]$ at $\mathcal{A}^{\text{ext}}$. Since x_1, x_2 are divisorial and x_3 is free, and $\mathcal{A} = \mathcal{O}_X[x_1 t^{1/2}, x_2 t^{1/4}, x_3 t^{1/7}]^{\text{int}}$ is a maximal admissible center one obtains

$$\text{inv}_0(f) = (2_+, 4_+, 7).$$

The effect of the full cobordant blow-up is exactly

$$f = t^{-28}((x_1' + (x_2')^2)^2 + (x_3')^7).$$

Let us examine this effect on the cobordant blow-up $B_+ = B \smallsetminus V = B \smallsetminus V(x_1', x_2', x_3')$ after removing the vertex $V = V(x_1', x_2', x_3')$.

On $B \smallsetminus V(x_1', x_2')$ we write the equation $f' := t^{28}f$ as $f' = u^2 + (x_3')^7$, where

$$u := x_1' + (x_2')^2$$

is a free coordinate. Thus $\mathcal{A}' = \mathcal{O}_{X'}[u^{1/2}, (x_3')^{1/7}]^{\mathrm{int}}$ is maximal admissible center for (f') at $V(u, x_3')$ with the invariant

$$\mathrm{inv}_{p'}(f') = (2, 7) < \mathrm{inv}_0(f) = (2_+, 4_+, 7)$$

for $p' \in V(u, x_3')$.

The next cobordant blow-up at $\mathcal{A}' = \mathcal{O}_{X'}[ut^{1/2}, (x_{1/3}')t^{1/7}]^{\mathrm{int}}$ resolves the singularity.

On $B \smallsetminus V(x_3')$, the derivative

$$\mathcal{D}_{x_3'}(f') = \mathcal{D}_{x_3'}(x_1' + (x_2')^2)^2 + (x_3')^7) = (7x_3')^6$$

is invertible and f' is nonsingular.

In case there are no divisors on X, the process is faster as we can write immediately $f = u^2 + x_3^7$, where $u = x_1 + x_2^2$. So $\mathrm{maxinv}(f) = (2, 7)$, and by the previous example, $\mathcal{A}' = \mathcal{O}_{X'}[u^t 1/2, (x_3')^t 1/7]^{\mathrm{int}}$ is the maximal admissible center. The cobordant blow-up of $\mathcal{J} = (u^{1/7}, x_3^{1/2})$ resolves the singularity.

3.3. **Splitting of derivations and compatibility.** Splitting of derivations on regular schemes allow for the coefficient representations of functions and simplifies the computations of the coefficient ideals. A similar concept in a different language was also used in [ATW19].

Definition 3.3.1. We say that a closed subscheme H of a scheme X *splits* if the closed embedding $i : H \hookrightarrow X$ splits, so there is an affine morphism $\pi : X \to H$ which is a left inverse of i with $\pi i = \mathrm{id}_H$.

In other words for any open U, and any function $f \in \mathcal{O}_X(\pi^{-1}(U))$ its *restriction*

$$f_{|H} \in \mathcal{O}_H(U) \subset \mathcal{O}_X(\pi^{-1}(U)).$$

The splitting condition implies that there is an injective morphism $\pi^{-1}(\mathcal{O}_H) \hookrightarrow \mathcal{O}_X$ of sheaves on X with the left inverse $\mathcal{O}_X \hookrightarrow i_*(\mathcal{O}_H)$ given by the restriction.

The notion can be extended to derivations corresponding to a system of local parameters.

Definition 3.3.2. Consider a set of derivations $D_{\overline{x}} = (D_{x_1}, \ldots, D_{x_k})$ of $\mathcal{D}_X$ on a regular scheme X for a certain partial system of coordinates $\overline{x} = (x_1, \ldots, x_k)$ such that $D_{x_i}(x_j) = \delta_{ij}$, we say that $D_{\overline{x}}$ *splits* in $\mathcal{O}_X$, if $H = V(\overline{x}) \subset X$ splits and the derivations in $D_{\overline{x}}$ vanish on $\pi^{-1}(\mathcal{O}_H) \subset \mathcal{O}_X$.

Example 3.3.3. Let X be a variety smooth over a field K. Let $x_1, \ldots, x_n \in m_p$ be a complete coordinate system at a closed point p. Any regular subscheme

$$H = V(x_1, \ldots, x_k) \subset \mathrm{Spec}(\widehat{\mathcal{O}}_{X,p}) = \mathrm{Spec}(K_p[[x_1, \ldots, x_n]])$$

for any natural $k \leq n$, splits. Here $K_p = \mathcal{O}_{X,x}/m_{p,X}$ is the residue field at $p \in X$. Indeed, we can always write

$$\mathcal{O}_{H,p} \simeq K_p[[x_{k+1}, \ldots, x_n]] \subset K_p[[x_1, \ldots, x_n]].$$

Moreover the system of derivations $D_{x_1}, \ldots, D_{x_k}$ on $\mathrm{Spec}(\hat{\mathcal{O}}_{X,p})$ split as they vanish exactly on $K_p[[x_{k+1}, \ldots, x_n]]$.

Both notions are strictly related:

Lemma 3.3.4. *Let $H = V(\overline{x})$ be a smooth subvariety on a smooth variety X. If H splits on variety X, and $\pi : X \to H$ is a splitting morphism, then there is a unique system of derivatives $D_{\overline{x}}$ which splits in $\mathcal{O}_X$, so that $D_{x_i}(x_j) = \delta_{ij}$ and $D_{\overline{x}}$ vanishes on $\pi^{-1}(\mathcal{O}_H)$. Conversely, if $(H, D_{\overline{x}})$ splits then it defines a unique splitting morphism $\pi : X \to H$ for the closed embedding H. Moreover $D_{\overline{x}} = \partial_{\overline{x}}$ is computed by any coordinate coordinate system $(\overline{x}, \overline{y})$, where $\overline{y} = \overline{y}_{|H}$.*

Proof. If H splits in X we extend $\overline{x}$ to a complete system of coordinates $(\overline{x}, \overline{y})$ by adjoining a system of coordinates $\overline{y} \subset \pi^{-1}(\mathcal{O}_H) \subset \mathcal{O}_X$. Then $D_{\overline{x}} = \partial_{\overline{x}}$ is defined with respect to the coordinate system $(\overline{x}, \overline{y})$. Conversely if $(H, D_{\overline{x}})$ splits then $\pi^{-1}(\mathcal{O}_H)$ is determined exactly by vanishing $D_{\overline{x}}$. ♣

Recall a well known fact in our setting:

Lemma 3.3.5. *If $(\overline{x}, D_{\overline{x}})$ splits on a smooth variety X then for any open affine subset U, and $f \in \mathcal{O}_X(\pi^{-1}(U)$ there is a decomposition of f up to (x^k) for any $k \in \mathbb{N}$.*

$$f \equiv \sum_{|\alpha|<k} c_\alpha x^\alpha (mod\ (x^k)),$$

where

$$c_\alpha = \frac{1}{\alpha!} D_{x^\alpha}(f)_{|H} \in \mathcal{O}_H(U) \subset \mathcal{O}_X(\pi^{-1}(U)).$$

Proof. Let $H = V(\overline{x})$. Observe that $f - f_{|H} \in (\overline{x})$.

Note that

$$\frac{1}{\alpha!} D_{x^\alpha}(f - \sum_{|\alpha|<k} c_\alpha x^\alpha)_{|H} = c_\alpha - c_\alpha \frac{1}{\alpha!} D_{x^\alpha}(x^\alpha) = 0.$$

Suppose that $g := (f - \sum_{|\alpha|<k} c_\alpha x^\alpha) \notin \overline{x}^k$. Then there is a $l < k$, such that $g \in \overline{x}^l \setminus \overline{x}^{l+1}$. Then $g = \sum_{|\alpha|=l} b_\alpha x^\alpha$, where at least for one α_0 we have $b_{\alpha_0} \notin (\overline{x})$. Then

$$\frac{1}{\alpha_0!} D_{x^{\alpha_0}}(g)_{|H} = \sum (\frac{1}{\alpha_0!} D_{x^{\alpha_0}} b_\alpha x^\alpha)_{||H} \neq 0,$$

which is a contradiction. ♣

Definition 3.3.6. *Let $\overline{x}_1 = (x_{11}, \ldots, x_{1s_1})$ be a partial system of coordinates on a a regular scheme X with a set of derivations $D_{\overline{x}_1} := (D_{x_{11}}, \ldots, D_{x_{1s_1}})$ such that $D_{x_{1j}}(x_{1i}) = \delta_{ij}$. We say that a system of derivations $D_{\overline{x}_1}$ is compatible with a Rees center $\mathcal{A}$ on X if there is a certain presentation*

$$A = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x_k} t^{1/a_k}]^{\mathrm{int}},$$

such that so that $D_{\overline{x}_1}(\overline{x}_i) = 0$ for $i \geq 2$.

Lemma 3.3.7. *Let $A = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x_k} t^{1/a_k}]^{\mathrm{int}}$ be a center. If $(\overline{x}, D_{\overline{x}_1})$ splits in $\mathcal{O}_X$ then it is compatible with the center $\mathcal{A}$.*

Proof. Set $\overline{x}_i' := \overline{x}_{i|H_1}$. Then $\overline{x}_i' - \overline{x}_i \in (\overline{x}_1)$, and since $a_i > a_1$, we have that $\overline{x}_1 t^{1/a_i} \in \mathcal{A}$. So

$$\overline{x}_i' t^{1/a_i} - \overline{x}_i t^{1/a_i} \in (\overline{x}_1 t^{1/a_i}) \subset \mathcal{A}.$$

Hence $\overline{x}_i' t^{1/a_i} \in \mathcal{A}$ and we have inclusion of the centers $\mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x_k} t^{1/a_k}]^{\mathrm{int}} \subseteq \mathcal{A}$. By symmetry we have the reverse inclusion. ♣

Corollary 3.3.8. *If X is smooth over a field K then any derivation $D_{\overline{x}_1}$ splits in $\widehat{\mathcal{O}_{X,p}}$ so it is compatible with any center of the form*

$$A = \widehat{\mathcal{O}_{X,p}}[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{int}}.$$

♣

3.4. **Derivations on the Rees centers.** For an ideal $\mathcal{I}$ of order $a \in \mathbb{N}$ at a point $p \in X$ on a smooth variety X we can place it in t^a-gradation of a certain $\mathbb{Z}$-graded Rees algebra $R = \mathcal{O}_X[\mathcal{I}t^a]$ so that we have the admissibility condition

$$R = \mathcal{O}_X[\mathcal{I}t^a] \subset \mathcal{O}_X[m_p t],$$

with respect to maximal ideal $m_p \subset \mathcal{O}_{X,p}$. Here we naturally associate with the coordinates gradation t, and since the derivatives D_{x_i} lower the order of ideals by 1 and we associate with with gradation t^{-1}. Consequently the graded derivations $t^{-1} D_{x_i}$ act on the elements of $R \subset \mathcal{O}_X[t]$ with positive gradations. Moreover they preserve $\mathcal{O}_X[m_p t]$.

In our approach we place the ideal $\mathcal{I}$ in gradation t. Then the order of the Rees algebra $R = \mathcal{O}_X[\mathcal{I}t]$ at a point p is $a_1 = \mathrm{ord}_p(\mathcal{I})$ and thus the corresponding coordinates and derivations shall be rescaled accordingly and we consider the action of $t^{-1/a_1} D_{x_i}$. A similar concept was used in the first version of [ATW19] to interpret the algebra of the center.

In general, let $R = \bigoplus R_a \subset \mathcal{O}_X[t^{1/w}]$ be any Rees algebra on a smooth X. For any local coordinate system $\overline{x} = (x_1, \ldots, x_n)$ compatible with E on a smooth variety X, and a given $a_1 \in \mathbb{Q}$ (which is usually the order of the Rees algebra R) one considers graded derivations

$$D_{x_i t^{1/a_1}} := t^{-1/a_1} D_{x_i}$$

acting on the elements of R_a of the gradations $a \geq 1/a_1$.

Consequently, if $ft^a \in \mathcal{O}_X[t^{1/w}]$, where $a \geq 1/a_1$ then

$$D_{x_i t^{1/a_1}}(ft^a) = D_{x_i}(f)t^{a-1/a_1} \in \mathcal{O}_X[t^{1/w}].$$

The idea is the following if

$$R \subseteq \mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x_k} t^{1/a_k}]^{\mathrm{Int}}$$

then when applying the differential operators $D_{\overline{x}_1 t^{1/a_1}}$ to the both sides we preserve the right side while enlarging the left side by adding new elements. This way we step by step enlarge R which at some point of the algorithm becomes equal to the center $\mathcal{A}$ on the right side.

More generally, by composing $D_{\overline{x}_1 t^{1/a_1}}$ we consider differential operators:

$$D^{\alpha}_{\overline{x} t^{1/a_1}} := t^{-|\alpha|/a_i} D_{\overline{x}^{\alpha}}, \quad \text{where}$$

$$D_{\overline{x}^{\alpha}} := \frac{1}{\alpha_1! \ldots \alpha_k!} \cdot \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n.$$

acting on elements $ft^a \in \mathcal{O}_X[t^{1/w}]$ and $a \geq |\alpha|/a_1$:

$$D^{\alpha}_{\overline{x}t^{1/a_1}} ft^a = D_{\overline{x}^{\alpha}}(f)t^{a-|\alpha|/a_1}$$

Example 3.4.1. If $ft = (x^3 + x^2y + y^3)t$, and $a_1 = 3$ then

$$D_{xt^{1/3}}((x^3 + x^2y + y^3)t) = (3x^2 + 2xy)t^{2/3}.$$

3.4.2. *Differential operators preserving centers.* A main consequence of the condition $a_1 \leq \ldots \leq a_n$ is that the center A is preserved by the action of $D_{x_i t^{1/a_1}}$ for $i = 1, \ldots, k$.

Lemma 3.4.3. *Let $A = \mathcal{O}_X[x_1t^{1/a_1}, \ldots, x_kt^{1/a_k}]^{\mathrm{int}}$, and $\overline{x} = (x_1, \ldots, x_n)$, where $n \geq k$, be a complete coordinate system on X extending $(x_1, \ldots, x_k)$. If $ft^a \in A_at^a$, and $|\alpha|/a_1 < a$ then*

$$D^{\alpha}_{\overline{x}t^{1/a_1}}(ft^a) \in A_{a-(|\alpha|/a_1)}t^{a-(|\alpha|/a_1)},$$

Proof. The property can be verified on monomials $(x_1^{b_1} \cdot \ldots \cdot x_k^{b_k})t^a \in A_at^a$ and on the differential operators $D_{x_i t^{1/a_1}}$ for $i = 1, \ldots, k$. If $x_1^{b_1} \cdot \ldots \cdot x_k^{b_k} \in A_a$ then $b_1/a_1 + \ldots b_k/a_k \geq a$. So

$$b_1/a_1 + \ldots + (b_i - 1)/a_i + \ldots + b_k/a_k \geq a - (1/a_1)$$

and

$$D_{x_i t^{1/a_1}}(x_1^{b_1} \cdot \ldots \cdot x_k^{b_k} \cdot t^a) \sim (x_1^{b_1} \cdot \ldots \cdot x_i^{b_i-1} \cdot \ldots \cdot x_k^{b_k} \cdot t^{a-(1/a_1)}) \in A_{a-(1/a_1)}$$

♣

3.5. Coefficient ideal of Rees algebra. The origin of the coefficient ideals concept can be traced back to the work of Abhyankhar and Hironaka, as seen in [Hir64]. Various definitions of this notion have been explored in multiple studies, including [Vil89], [BM97], [Wło05], and [Kol07], among others.

In the context of our methodology, the coefficient ideal is used to derive a better approximation of a maximal admissible center. As demonstrated in Section 3.7.1, the maximal admissible center $\mathcal{A}$ can be obtained by the recursive application of coefficient ideals to a specified Rees algebra $R = \mathcal{O}_X[\mathcal{I}t]$. The coefficient ideal, as per our definition, is a non-canonical concept that is relatively straightforward to compute in the completion $\widehat{\mathcal{O}}_{X,p}$ in what we refer to as a 'split form' (as discussed in Section 3.5.4).

This version of the coefficient ideal simplifies and streamlines computations and can be likened to the approach used by Bierstone-Milman in their resolution algorithm, as referenced in [BM91] and [BM97], albeit in a differing framework.

Remark 3.5.1. If $D_{\overline{x}}$ splits in $\mathcal{O}_X$ then any $f \in \mathcal{O}_X$ can be written in the coefficient form

$$f = \sum c_{\alpha}\overline{x}^{\alpha},$$

where $c_{\alpha} \in \mathcal{O}_{V(\overline{x})} \subset \mathcal{O}_X$ so we can obtain a particularly nice description of the center on X. This can be always done in the completion $\widehat{\mathcal{O}}_{X,p}$.

Definition 3.5.2. Let $a_1 \in \mathbb{Q}_{>0}$, and $R = \mathcal{O}_X[f_jt^{b_j}]_{j=1,\ldots,s}$ be a Rees algebra generated by $f_jt^{b_j}$ for $j = 1, \ldots, s$ and let $\overline{x}$ be any partial system of local coordinates compatible with E at p, and set $H := V(\overline{x})$. By the *coefficient ideal with respect to* $\overline{x}t^{1/a_1}$ we mean the Rees algebra:

$$C_{\overline{x}t^{1/a_1}}(R) := \mathcal{O}_X[(\,\overline{x}t^{1/a_1})^{\mathrm{Int}}, D^{\alpha}_{\overline{x}t^{1/a_1}}(f_jt^{b_j}), \quad |\alpha| < b_ja_1, \; j = 1, \ldots, s\,]$$

Recall that the part $\mathcal{O}_X[\,\overline{x}t^{1/a_1}]^{\mathrm{Int}}$, means the integral closure of $\mathcal{O}_X[\,\overline{x}t^{1/a_1}]$ in the smallest subalgebra $\mathcal{O}_X[t^{1/w}]$ containing all the generators in $C_{\overline{x}t^{1/a_1}}(R)$. Thus

$$\mathcal{O}_X[\,\overline{x}t^{1/a_1}]^{\mathrm{Int}} = \mathcal{O}_X[\overline{x}t^{1/w}, \ldots, \overline{x}t^{w_1/w}],$$

where $w_1 = w/a_1$, so $\overline{x}t^{w_1/w} = \overline{x}t^{1/a_1}$.

Remark 3.5.3. The key concept of the coefficient ideal as well as the role of the graded local parameters $\overline{x}_1 t^{1/a_1}$ are encapsulated by Lemma 3.5.11. The coefficient ideal maintains admissibility property, while containing already some generators $\overline{x}_1 t^{1/a_1}$ from a to be constructed maximal admissible center $\mathcal{A}$. Moreover after splitting it is generated by $(\overline{x}_1 t^{1/a_1})^{\mathrm{Int}}$ and the part $C_{\overline{x}_1 t^{1/a_1}}(R)_{|V(\overline{x}_1}$ which is independent of the coordinates $\overline{x}_1$.

3.5.4. *Coefficient ideal in the split form.* The process of computing the coefficient ideal becomes remarkably straightforward when utilizing splitting derivations $D_{\overline{x}}$ in $\mathcal{O}_X$. This can be always done by passing to the completion $\widehat{\mathcal{O}}_{X,p}$ or to a corresponding étale neighborhood. Assume that $(\overline{x}, D_{\overline{x}})$ splits on X. For any element $f_j t^{b_j} \in R_{b_j}$, one can write f_j in $\mathcal{O}_X$ we can write

$$f_j \equiv \sum c_{j\alpha}\overline{x}^\alpha = \sum_{|\alpha|<b_j a_1} c_{j\alpha}\overline{x}^\alpha + \sum_{|\alpha|\geq b_j a_1} c_{j\alpha}\overline{x}^\alpha \in \widehat{\mathcal{O}}_X, p$$

where $c_{j\alpha} \in \widehat{\mathcal{O}}_H \subset \mathcal{O}_X$ for $H := V(\overline{x})$. Then, for $|\alpha| < b_j a_1$ we have

$$D^\alpha_{\overline{x}t^{1/a_1}}(f_j t^{b_j})_{|H} = c_{j\alpha} t^{b_j - |\alpha|/a_1}$$

Thus the restricted coefficient ideal can be represented in the format which is simpler for computations and justifies the name "coefficient":

Lemma 3.5.5.

$$C_{\overline{x}t^{1/a_1}}(R)_{|H} := \mathcal{O}_H[\,D^\alpha_{\overline{x}t^{1/a_1}}(f_j t^{b_j})_{|H}, |\alpha| < b_j a_1] = \mathcal{O}_H[\,c_{j\alpha}t^{b_j - |\alpha|/a_1}, |\alpha| < b_j a_1\,]$$

As a corollary from the above we obtain:

Lemma 3.5.6. *Suppose $D_{\overline{x}}$ splits in $\mathcal{O}_X$. Then, with the above notations and the assumptions the coefficient ideal $C_{\overline{x}t^{1/a_1}}(R)$ can be written in the split form:*

$$C_{\overline{x}t^{1/a_1}}(R) = \mathcal{O}_X[(\overline{x}t^{1/a_1})^{\mathrm{Int}}, C_{\overline{x}t^{1/a_1}}(R)_{|H}] =$$
$$= \mathcal{O}_X[(\overline{x}t^{1/a_1})^{\mathrm{Int}}, c_{j\alpha}t^{b_j - |\alpha|/a_1}, |\alpha| < b_j a_1\,]$$

where $c_{j\alpha} \in \mathcal{O}_H \subset \mathcal{O}_X$, for $H = V(\overline{x})$.

Proof. Let $R = \mathcal{O}_X[f_j t^{b_j}]$. In the presentation

$$f_j = \sum c_{j\alpha}\overline{x}^\alpha = \sum_{|\alpha|<a a_1} c_{j\alpha}\overline{x}^\alpha + \sum_{|\alpha|\geq a a_1} c_{j\alpha}\overline{x}^\alpha \in \mathcal{O}_X$$

the term

$$(\sum_{|\alpha|\geq b_j a_1} c_{j\alpha}\overline{x}^\alpha)t^{b_j} = (f_j - \sum_{|\alpha|<b_j a_1} c_{j\alpha}\overline{x}^\alpha)t^{b_j} \in \mathcal{O}_X[\overline{x}t^{1/a_1}]^{\mathrm{Int}}$$

Then we have

$$f_j^0 t^{b_j} := \sum_{|\alpha|<b_j a_1} c_{j\alpha}\overline{x}^\alpha t^{b_j} \in C_{\overline{x}t^{1/a_1}}(R)$$

Thus for any maximal α_j^0 in the above presentation we have

$$\frac{1}{\alpha_0!} D_{\overline{x}t^{1/a}}^{\alpha_j^0}(f_j^0 t^{b_j}) = c_{j\alpha_j^0} t^{b_j - |\alpha_j^0|/a_1} \in C_{\overline{x}t^{1/a_1}}(R).$$

Thus

$$c_{j\alpha_j^0} x^{\alpha_0} t^{b_j} = c_{j\alpha_j^0} t^{b_j - |\alpha_j^0|/a_1} \cdot x^{\alpha_j^0} t^{|\alpha_j^0|/a_1} \in C_{\overline{x}t^{1/a_1}}(R).$$

Next consider

$$f_j^1 t^{b_j} := f_j^0 t^{b_j} - c_{j\alpha_j^0} x^{\alpha_0} t^{b_j} \in C_{\overline{x}t^{1/a_1}}(R),$$

and repeat the process for maximal α_j^1 in f_j^1, and continue inductively for α_j^i in f_j^i until $f_j^k = 0$. ♣

In particular, we have

Corollary 3.5.7.

$$\widehat{\mathcal{O}_{X,p}} \cdot C_{\overline{x}t^{1/a_1}}(R) = \widehat{\mathcal{O}_{X,p}}[(\overline{x}_1 t^{1/a_1})^{\mathrm{Int}}, C_{\overline{x}_1 t^{1/a_1}}(R)_{|H}] =$$
$$= \widehat{\mathcal{O}}_{X,p}[(\overline{x}t^{1/a_1})^{\mathrm{Int}}, c_{j\alpha} t^{b_j - |\alpha|/a_1}, |\alpha| < b_j a_1].$$

3.5.8. *Split vs non-split form of the coefficient ideal.* The split form of the coefficient ideal is highly efficient for computations. The subsequent example provides a comparison between two methods used for computing the coefficient ideal:

Example 3.5.9. Let $a_1 = 3$ then $H = V(x)$ splits in $X = \mathrm{Spec}(K[x, y, z]$ and

$$ft = (x^3 + x^2 yw + z^n)t = (xt^{1/3})^3 + (xt^{1/3})^2 ywt^{1/3} + z^n t$$

has coefficients $ywt^{1/3}$, and $z^n t$. Thus the split form of the coefficient ideal :

$$C_{xt^{1/3}}(x^3 + x^2 yw + z^n)t = \mathcal{O}_X[(xt^{1/3})^{\mathrm{Int}}, ywt^{1/3}, z^n t]$$

is generated by $(xt^{1/3})^{\mathrm{Int}}$ and the coefficients $ywt^{2/3}, z^n t$.

Note that the standard form of the coefficient ideal is more complicated, though still easily computable in this case:

$$C_{xt^{1/3}}(x^3 + x^2 yw + z^n)t = \mathcal{O}_X[(xt^{1/3})^{\mathrm{Int}}, ft, D_{xt^{1/3}}(ft), D_{xt^{1/3}}^2(ft)] =$$
$$= \mathcal{O}_X[(xt^{1/3})^{\mathrm{Int}}, (x^3 + x^2 yw + z^n)t, (3x^2 + 2xyw)t^{2/3}, ywt^{1/3}],$$

3.5.10. *Coefficient ideals and admissibility.*

Lemma 3.5.11. *Let* $R = \mathcal{O}_X[f_j t^{b_j}]$ *be the Rees algebra on* X. *The following conditions are equivalent in a neighborhood of* $p \in X$.

    (1) $R \subseteq \mathcal{A}^{\mathrm{Int}} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x_k} t^{1/a_k}]^{\mathrm{Int}}$.
    (2) $C_{\overline{x}_1 t^{1/a_1}}(R) \subseteq \mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x_k} t^{1/a_k}]^{\mathrm{Int}}$
    (3) $C_{\overline{x}_1 t^{1/a_1}}((R)_{|H} \subseteq \mathcal{A}_{|H} = \mathcal{O}_H[\overline{x}_{2|H} t^{1/a_2}, \ldots, \overline{x}_{k|H} t^{1/a_k}]^{\mathrm{Int}}$.

Proof. We can pass to the completion $\widehat{\mathcal{O}}_{X,p}$, and replace $\overline{x}_i$ with $\overline{x_{i|H}}$, so that $D_{\overline{x}_1}$ split and are compatible with $\mathcal{A}$. Then $C_{\overline{x}_1 t^{1/a_1}}(R)$ is obtained from R by applying $\mathcal{D}_{\overline{x}_1 t^{1/a_1}}$. Note that $D_{\overline{x}_1}$ splits, it is compatible with $\overline{x}_{i|H}$ in $\overline{\mathcal{O}}_{X,p}$.
    (1) $\Rightarrow$ (2) by Lemma 3.4.3, the operator $\mathcal{D}_{\overline{x}_1 t^{1/a_1}}$ preserves both sides of the inclusion in (1) giving (2).
    (2) $\Rightarrow$ (3) The assertion in (3) follows from (2) simply by taking the restriction.
    (3) $\Rightarrow$ (1) By Lemma 3.5.6, $C_{\overline{x}_1 t^{1/a_1}}(R) \supset R$ can be written in the split form

$$C_{\overline{x}_1 t^{1/a_1}}(R) = \mathcal{O}_X[(\overline{x}t^{1/a_1})^{\mathrm{Int}}, C_{\overline{x}_1 t^{1/a_1}}((R)_{|H}] \subset \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x_k} t^{1/a_k}]^{\mathrm{Int}}.$$

♣

3.5.12. *Nested Rees algebras.* The following definition is motivated by the construction of the coefficient ideal.

Definition 3.5.13. Given a partial system $\overline{x}_1, \ldots, \overline{x}_n$ of local parameters compatible with E. Let

$$R_{|H_k} = \bigoplus R_{a|H_k} t^a$$

be the restriction of R to $H_k := V(\overline{x}_1, \ldots, \overline{x}_k)$, for $k \leq n$, and let

$$0 < a_1 < a_2 \ldots < a_k$$

be a sequence of rational numbers.

We say that the Rees algebra R is *strictly nested at H_k* if $H_k \subset X$ splits and

$$R = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k})^{\mathrm{Int}}, R_{|H_k}].$$

We say that the Rees algebra R is *nested at H_k* if it contains subalgebra

$$\mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{Int}}$$

and such that $\widehat{\mathcal{O}}_{X,p} \cdot R$ is strictly nested at H_k:

$$\widehat{\mathcal{O}_{X,p}} \cdot R = \widehat{\mathcal{O}_{X,p}}[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k})^{\mathrm{Int}}, R_{|H_k}],$$

The order $\mathrm{ord}_p(R_{|H_k})$ will be called the *nested order* of R at H_k and p.

Using this terminology one can rephrase the Lemma 3.5.6 we get

Lemma 3.5.14. *The coefficient ideal $C_{\overline{x}t^{1/a_1}}(R)$ is nested at $H = V(\overline{x})$.*
Moreover, if $(H, D_{\overline{x}})$ splits in $\mathcal{O}_X$ then $C_{\overline{x}t^{1/a_1}}(R)$ is strictly nested at H:

$$C_{\overline{x}t^{1/a_1}}(R) = \mathcal{O}_X[(\overline{x}t^{1/a_1})^{\mathrm{Int}}, C_{\overline{x}t^{1/a_1}}(R)_{|H}]$$

♣

In particular we can always write

$$\widehat{\mathcal{O}}_{X,p} \cdot C_{\overline{x}t^{1/a_1}}(R) = \widehat{\mathcal{O}}_{X,p}[(\overline{x}t^{1/a_1})^{\mathrm{Int}}, C_{\overline{x}t^{1/a_1}}(R)_{|H}]$$

3.5.15. *Nested coefficient ideals.* The definition of the coefficient ideal can be extended to the setting of nested Rees algebras.

Definition 3.5.16. Let $\overline{x}_1, \ldots, \overline{x}_n$ be a local system of local coordinates. Let

$$R = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k})^{\mathrm{Int}}, f_j t^{b_j}]_{j=1,\ldots,s}$$

be the Rees algebra nested at $H_k = V(\overline{x}_1, \ldots, \overline{x}_k)$. Assume $a_{k+1} \geq a_k \geq \ldots \geq a_1$. Then we shall call the coefficient ideal $C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R)$ at $\overline{x}_{k+1} t^{1/a_{k+1}}$ *nested* for R at $H_k = V(\overline{x}_1, \ldots, \overline{x}_1)$.

Lemma 3.5.17. *With the above notation the nested coefficient ideal of R at $\overline{x}_{k+1} t^{1/a_{k+1}}$, with respect to H_k can be written as*

$$C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R) :=$$
$$:= \mathcal{O}_X[(x_1 t^{1/a_1}, \ldots, x_{k+1} t^{1/a_{k+1}})^{\mathrm{Int}}, D^{\alpha}_{\overline{x}_{k+1} t^{1/a_{k+1}}}(f_j t^{b_j}), |\alpha|/a_{k+1} \leq b_j]$$

The subsequent statement is a straightforward extension of Lemma 3.5.7:

Lemma 3.5.18. *Let R be a Rees algebra nested at $H_k = V(\overline{x}_1, \ldots, \overline{x}_{k+1})$. The nested coefficient ideal $C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R)$ at $\overline{x}_{k+1} t^{1/a_{k+1}}$ is a nested Rees algebra at $H_{k+1} = V(\overline{x}_1, \ldots, \overline{x}_{k+1})$.*

Proof. By the assumption

$$\widehat{\mathcal{O}}_{X,p} \cdot R = \widehat{\mathcal{O}}_{X,p}[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k})^{\text{Int}}, R_{|H_k}]$$

Then by the above

$$\widehat{\mathcal{O}}_{X,p} \cdot C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R) = \widehat{\mathcal{O}}_{X,p}[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k})^{\text{Int}}, C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R_{|H_k})]$$

is also nested in H_k. On the other hand, by Lemma 3.5.7, $C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R_{|H_k})$ is nested in H_{k+1} on H_k. Consequently $C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R)$ is nested on X at H_{k+1}. ♣

3.5.19. *Nested coefficient ideals in the split form.* We can adapt Lemma 3.5.6 to strictly nested Rees algebras: Let $\overline{x}_1, \ldots, \overline{x}_n$ be a system of local coordinates compatible with E at p. Let

$$R = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k})^{\text{Int}}, f_j t^{b_j}]_{j=1,\ldots,s}$$

be a Rees algebra nested at $H_k = V(\overline{x}_1, \ldots, \overline{x}_k)$, for $k+1 \leq n$. Then

$$\widehat{\mathcal{O}}_{X,p} \cdot R = \widehat{\mathcal{O}}_{X,p}[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k})^{\text{Int}}, f_{j|H_k} t^{b_j}]_{j=1,\ldots,s}$$

with $f_{j|H_k} \in \mathcal{O}_{H_k}$. Moreover $(x_{k+1}, D_{\overline{x}_{k+1}})$ splits in $\text{Spec}(\widehat{\mathcal{O}}_{X,p})$ and we can write each generator $f_{j|H_k} t^{b_j}$ as

$$f_{j|H_k} t^{b_j} \equiv \sum_{|\alpha| < a a_{k+1}} c_{j\alpha} \overline{x}_{k+1}^{\alpha} t^{b_j} \equiv \sum_{|\alpha| < a a_{k+1}} (c_{j\alpha} t^{b_j - |\alpha|/a_{k+1}}) \overline{x}_{k+1}^{\alpha} t^{|\alpha|/a_{k+1}}$$

modulo the ideal

$$\widehat{\mathcal{O}}_{X,p}((\overline{x}_{k+1})^{\alpha} t^{b_j} \mid |\alpha| \geq b_j a_{k+1}) \subset \widehat{\mathcal{O}}_{X,p}[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\text{Int}},$$

where $c_{j\alpha} = c_{j\alpha}(\overline{x}_{k+2}, \ldots, \overline{x}_n) \in \mathcal{O}_{H_{k+1}} \subset \widehat{\mathcal{O}}_{X,p}$. Thus we obtain the result:

Lemma 3.5.20. *With the above assumptions and notations the completion of the nested coefficient ideal with respect to $\overline{x}_{k+1} t^{1/a_{k+1}}$ is equal to:*

$$\widehat{\mathcal{O}}_{X,p} \cdot C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R) = \widehat{\mathcal{O}}_{X,p}[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k})^{\text{Int}}, C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R)_{|H}] =$$

$$= \widehat{\mathcal{O}}_{X,p}[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_{k+1} t^{1/a_{k+1}})^{\text{Int}}, c_{j\alpha} t^{b_j - |\alpha|/a_{k+1}}, |\alpha| < b_j a_{k+1}]$$

with $c_{j\alpha} = c_{j\alpha}(\overline{x}_{k+2}, \ldots, \overline{x}_n) \in \mathcal{O}_{H_{k+1}} \subset \widehat{\mathcal{O}}_{X,p}$.

♣

3.5.21. *Strictly nested coefficient ideals in the split form.* The result can be stated for the strictly nested Rees algebras in view of Lemma 3.3.4.

Lemma 3.5.22. *Let $\overline{x}_1, \ldots, \overline{x}_n$ be a complete coordinate system. Let*

$$R = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k})^{\text{Int}}, f_j t^{b_j}]_{j=1,\ldots,s}$$

be a Rees algebra nested at $H_k = V(\overline{x}_1, \ldots, \overline{x}_k)$, for $k \leq n-1$, where $f_j \in \mathcal{O}_{H_k} \subset \mathcal{O}_X$. Assume that $H_{k+1} \subset X$ splits. Then upon the coordinate change $\overline{x}_1, \ldots, \overline{x}_k, \overline{x}_{k+1|H_k}, \ldots, \overline{x}_{n|H_k}$, $D_{\overline{x}_{k+1}} = \partial_{\overline{x}_{k+1}}$ splits in $\mathcal{O}_X$ and the coefficient ideal $C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R)$ is strictly nested at H_{k+1} and admits the split form

$$C_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R) = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_{k+1} t^{1/a_{k+1}})^{\text{Int}}, c_{j\alpha} t^{b_j - |\alpha|/a_{k+1}}, |\alpha| < b_j a_{k+1}]$$

with $c_{j\alpha} = c_{j\alpha}(\overline{x}_{k+2}, \ldots, \overline{x}_n) \in \mathcal{O}_{H_{k+1}} \subset \widehat{\mathcal{O}}_X$.

♣

Example 3.5.23. Let

$$R = \mathcal{O}_X[(xt^{1/3})^{\mathrm{int}}, (y_1^5 + y_1 y_2^6 + y_3^7)t]$$

be a Rees algebra strictly nested at $H_1 = V(x)$. The subvariety $H_2 = V(x, y_1)$ splits in $X = \mathrm{Spec}(K[x, y_1, y_2, y_3])$ and D_{y_1} splits on H_1 (so D_{y_1} vanishes on $\mathcal{O}_{H_2} = K[y_1, y_3] \subset K[x, y_1, y_2, y_3]$). Then the generator $(y_1^5 + y_1 y_2^6 + y_3^7)$ can be written in the coefficient form with respect $y_2 t^{1/5}$:

$$(y_1^5 + y_1 y_2^6 + y_3^7) = (y_1 t^{1/5})^5 + (y_1 t^{1/5}) y_2^6 t^{4/5} + y_3^7 t],$$

with the coefficients $c_{y^1} = y_2^6 t^{4/5}, c_0 = y_3^7 t$. Consequently the nested coefficient ideal with respect to $yt^{1/5}$ is equal to:

$$C_{yt^{1/5}} \mathcal{O}_X[(xt^{1/3})^{\mathrm{Int}}, (y_1^5 + y_1 y_2^6 + y_3^7)t] = \mathcal{O}_X[(xt^{1/3}, yt^{1/5})^{\mathrm{Int}}, y_2^6 t^{4/5}, y_3^7 t]$$

is generated by $(xt^{1/3}, yt^{1/5})^{\mathrm{Int}}$, and , $y_2^6 t^{4/5}$ and $y_3^7 t$.

As an immediate corollary from Lemma 3.5.11, by considering the restriction $R_{|H_k}$ we obtain its extension:

Lemma 3.5.24. *Let R be the Rees algebra on X nested at H_k, so that*

$$\widehat{\mathcal{O}}_{X,p} R = \widehat{\mathcal{O}}_{X,p}[(x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k})^{\mathrm{Int}}, R_{|H_k}].$$

The following conditions are equivalent.

(1) $R \subset \mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_n t^{1/a_n}]^{\mathrm{Int}}$.
(2) $\mathcal{C}_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R) \subset \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_n t^{1/a_n}]^{\mathrm{Int}}$
(3) $\mathcal{C}_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R)_{|H_{k+1}} \subset \mathcal{O}_{H_{k+1}}[\overline{x}_{k+2|H_{k+1}} t^{1/a_{k+2}}, \ldots, \overline{x}_{n|H_{k+1}} t^{1/a_n}]^{\mathrm{Int}}$.

Proof. Passing to the completion $\widehat{\mathcal{O}}_{X,p}$ we may assume that R is strictly nested. Moreover, by replacing $\overline{x}_{k+2}$ by $\overline{x}_{k+2|H_{k+1}}$ we can assume that $D_{\overline{x}_{k+1}}$ is compatible with $\mathcal{A}$. Let $R' := R_{|H_k}$. Observe that

$$\mathcal{C}_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R') = (\mathcal{C}_{\overline{x}_{k+1} t^{1/a_{k+1}}}(R))_{|H_k}.$$

By Lemma 3.5.11, we have the equivalent conditions for R'. On the other hand, since R is strictly nested the condition

$$R \subset \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_n t^{1/a_n}]^{\mathrm{Int}}$$

is equivalent to

$$R' = R_{|H_k} \subset \mathcal{O}_X[\overline{x}_{k+1|H_k} t^{1/a_{k+1}}, \ldots, \overline{x}_{n|H_k} t^{1/a_n}]^{\mathrm{Int}}.$$

Thus the result from Lemma 3.5.11.   ♣

3.6. **Maximal contact.**

3.6.1. *Cotangent ideal of Rees algebra.* From now on until the end of Chapter 3.7 we shall assume that the characteristic of the base field K is zero.

First, recall that given an ideal $\mathcal{I}$ on smooth variety X, and a positive integer a, the set of points where $\mathrm{ord}_p(\mathcal{I}) \geq a$ is described as $V(\mathcal{D}_{\overline{X}}^{\leq a-1}(\mathcal{I}))$, where $\mathcal{D}_{\overline{X}}^{\leq a-1}(\mathcal{I})$ is the ideal generated by all the derivatives $D_{\overline{x}^\alpha}(f)$, where $f \in \mathcal{I}$ and $0 \leq |\alpha| \leq a - 1$. If the order of $\mathcal{I}$ is a at a point then the order $\mathcal{D}_{\overline{X}}^{\leq a-1}(\mathcal{I})$ is equal to 1 and thus it contains a local parameter u, called *a maximal contact*, such that $V(u)$ contains $V(\mathcal{D}_{\overline{X}}^{\leq a-1}(\mathcal{I}))$. In such a case we shall call the ideal $T^{1/a}(\mathcal{I}) := \mathcal{D}_{\overline{X}}^{\leq a-1}(\mathcal{I})$ *cotangent.* The maximal contact plays a critical role in the classical Hironaka

resolution algorithm. It allows for the inductive reduction of the process to the hypersurface of maximal contact.

In our context, the meaning of the maximal contact is different. It is simply the coordinate in the gradation t^{1/a_1} occurring a some presentation of a maximal $\mathcal{R}$-admissible center $\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]$ for any Rees algebra R

Consider the admissibility condition

$$R \subset \mathcal{A}^{\mathrm{Int}} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{Int}}.$$

The idea of the cotangent ideal $T^{1/a_1}(R)$ is to gather the elements occurring in gradation t^{1/a_1}, obtained by applying differential operators $\mathcal{D}_{\overline{x} t^{1/a_1}}^{\alpha}$ to the generators of R, for $\overline{x}$ being a complete system of coordinates. The differential operators $\mathcal{D}_{\overline{x} t^{1/a_1}}^{\alpha}$ preserve the right side of the admissibility condition (See Lemma 3.4.3). Then the ideal $T^{1/a_1}(R)$ is necessarily contained in the gradation $\mathcal{A}_{1/a_1}^{\mathrm{Int}}$. Conversely by the maximal admissibility condition $T^{1/a_1}(R)t^{1/a_1}$ contains all the coordinates in $\overline{x}_1 t^{1/a_1}$ up to the higher order terms of the gradation $\mathcal{A}_{1/a_1}^{\mathrm{Int}}$. So, by the Replacement Lemma 3.1.29, we can assume that the $T^{1/a_1}(R)$ contains $\overline{x}_1 t^{1/a_1}$. This way we will determine $\overline{x}_1 t^{1/a_1}$- part of the center $\mathcal{A}$ on the right side of the inclusion with $\overline{x}_1 \in T^{1/a_1}$, which would be called a *maximal contact at a point*. A similar idea was used in the first version of [ATW19] to interpret the algebra of the center.

Consequently $T^{1/a_1}(R)$ generalizes $\mathcal{D}_X^{\leq a_1 - 1}(\mathcal{I})$ for the ideal $\mathcal{I}$ of order a at p. In particular, if $\mathcal{I}t \subset \mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{int}}$, then by Lemma 3.4.3,

$$t^{-(a_1-1/a_1)}\mathcal{D}_{\overline{X}}^{\leq a_1-1}(\mathcal{I}) = \mathcal{D}_{\overline{X}}^{\leq a_1-1}(\mathcal{I})t^{1/a_1} \subset \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{int}}.$$

Thus $\mathcal{D}_{\overline{X}}^{\leq a_1-1}(\mathcal{I}) \subset \mathcal{A}_{1/a_1}^{\mathrm{int}}$.

Definition 3.6.2. Given a Rees algebra $R = \mathcal{O}_X[f_j t^{b_j}]_{j=1,\ldots,s} = \bigoplus R_a t^a$ on an open affine subset U of X, and a rational number $a_1 > 0$.

$$T^{1/a_1}(R) := \sum_{|\alpha|=b_j a_1-1} \mathcal{O}_X \mathcal{D}_{\overline{x}}^{\alpha}(f_j) \quad \subseteq \quad \mathcal{T}^{\leq 1/a_1}(R) := \sum_{|\alpha|<b_j a_1} \mathcal{O}_X \mathcal{D}_{\overline{x}}^{\alpha}(f_j) \subseteq$$

$$\subseteq \mathcal{T}^{<1/a_1}(R) := \sum_{|\alpha| \leq b_j a_1} \mathcal{O}_X \mathcal{D}_{\overline{x}}^{\alpha}(f_j).$$

Remark 3.6.3. As the ideal $T^{1/a_1}(R)$ is in the gradation t^{1/a_1}, it can be written in the graded form

$$T^{1/a_1}(R)t^{1/a_1} := \sum_{|\alpha|=b_j a_1-1} \mathcal{D}_{\overline{x} t^{1/a_1}}^{\alpha}(f_j t^{b_j}).$$

Remark 3.6.4. The definition intentionally uses a specific representation of Rees algebras for the purpose of streamlining computations. The uniqueness of the center obtained during the resolution process in Section 3.7.1 is automatic and unaffected by any choices made.

The set of generators for R can be chosen arbitrarily, and we have the option to select the entire set of homogeneous elements of R as generators instead.

In the following examples, when we refer to $D_X^i(\mathcal{I}) \subset D_{\overline{X}}^{\leq i}(\mathcal{I})$, as the ideal generated by all the derivatives $D_{\overline{x}}^{\alpha}(f_i)$, where $f_i \in \mathcal{I}$ are generators of $\mathcal{I}$, and $|\alpha| = i$. This ideal is consistent with the definition of $T^{1/a_1}(R)$.

Example 3.6.5. Let $a_1 = 3 = \mathrm{ord}_0(R)$, where $R = \mathcal{O}_X[(x^3 + y^4 + z^5)t]$.
$$T^{1/3}(R)t^{1/3} = \mathcal{O}_X[\mathcal{D}_X^2(x^3 + y^4 + z^5)t^{1/3}] = \mathcal{O}_X[(x, y^2, z^3)t^{1/3}]$$

Example 3.6.6. $a_1 = 3 = \mathrm{ord}_0(R)$, where
$$R = \mathcal{O}_X([(x^3 + y^4 + z^5)t, (w^2x^4 + z^7)t^2, (y^2 + xz^2)t^{2/3}, w_1^3 t^{3/4}]$$

Then
$$T^{1/3}(R)t^{1/3} = (D_X^2(x^3 + y^4 + z^5), D_X^5(w^2x^4 + v^7), D_X(v^2 + xz^2))t^{1/3} =$$
$$= (x, w, v, y^2, z^2)t^{1/3},$$

with local paremeters x, w, v in gradation $t^{1/3}$. The element $w_1^3 t^{3/4}$ is of higher order $4 > 3$. It is thus ignored as $ba_1 = 3/4 \cdot 3 = 9/4$ is not integral. It will not contribute to the set of local parameters of gradation $t^{1/3}$. Corresponding local parameter is $w_1 t^{1/4}$ is of order 4.

On the other hand the ideal $T^{\leq 1/3}(R) = (x, w, v, y^2, z^2, w_1)$ is generated by $T^{1/3}(R)$, and $D^2(w_1^3) \sim w_1$, for the derivation order $2 < ba_1 = 9/4$. The element w_1 is however, in gradation $t^{1/4}$. The ideal $T^{\leq 1/3}(R)$ is used only in the context of Lemma 3.6.9.

In a more general context, we can state the following:

Lemma 3.6.7. *Let us assume that* $\mathrm{ord}_p(R) = a_1$. *Then*
$$T^{1/a_1}(R) + m_p^2 = \mathbf{C}_{1/a_1} + m_p^2$$
where $\mathbf{C}_{1/a_1} t^{1/a_1}$ *is the gradation of the algebra*
$$\mathbf{C}_{\overline{x} t^{1/a_1}} = \bigoplus \mathbf{C}_a t^a,$$
generated by all the elements $D_{\overline{x} t^{1/a_1}}^\alpha(ft^a)$, *where* $f \in R_a$, *and* $|\alpha| < aa_1$. *In particular,* $T^{1/a_1}(R) + m_p^2$ *is independent of the choice of generators.* ♣

3.6.8. *Singular locus, and the cotangent ideal.* The following lemma extends the analogous result for the ideals.

Lemma 3.6.9.   (1) $\mathrm{suppord}(R, \geq a_1) := \{q \in U \mid \mathrm{ord}_q(R) \geq a_1\} = V(\overline{T}^{\leq 1/a_1}(R))$.
  (2) $\mathrm{suppord}(R, > a_1) := \{q \in U \mid \mathrm{ord}_q(R) > a_1\} = V(\overline{T}^{< 1/a_1}(R))$.

Proof. (1)$\mathrm{ord}_q(R) \geq a_1$ if and only if $\mathrm{ord}_q(f_j) \geq b_j a_1$ for any j iff all $\mathcal{D}_{\overline{x}^\alpha}(f_j)$ vanish at q for $|\alpha| < b_j a_1$. (2) The reasoning is similar. ♣

3.6.10. *Maximal contact of Rees algebra.* The concept of *hypersurfaces of maximal contact* was initially introduced by Hironaka, Abhyankhar, and Giraud, and further developed in the works of Bierstone-Milman, Villamayor, and others. In the case where $\mathcal{I}$ is an ideal of order a_1, and no divisors are present a *maximal contact* is defined as a local parameter $u \in T^{1/a}(\mathcal{I}) = \mathcal{D}_X^{a-1}(\mathcal{I}) \subset \mathcal{D}_X^{\leq a-1}(\mathcal{I})$. Its zero locus $V(u)$ includes the set $V(\mathcal{D}_X^{\leq a-1}(\mathcal{I}))$, which consists of the *equimultiple locus* of all points where $\mathrm{ord}_p(\mathcal{I}) = a$.

Our definition of maximal contact is intended for Rees algebras. From our perspective, a maximal contact refers to a partial coordinate system $\overline{x}_1$ found in t^{1/a_1}-gradation of a maximal R-admissible center
$$\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{int}}$$

at a point p. It can be characterized using Corollary 3.6.20. This definition a priori does not need the property that $V(\overline{x}_1)$ contains the equimultiple locus and thus can be used in a nonzero characteristic. (See for instance Example 4.5.4)

The definition is straightforward when no divisors are present. In such cases, "maximal contact" refers to a maximal partial set of coordinates contained in $T^{1/a_1}(\mathcal{R})$. However, in general, maximal contact at a point p is a partial set of coordinates that is compatible with an SNC divisor, consisting of free coordinates in $T^{1/a_1}(\mathcal{R})$ along with the divisorial coordinates contained in $T^{1/a_1}(\mathcal{R}) + m_p^2$, thus tangential to $T^{1/a_1}(\mathcal{R})$.

The concept is specifically designed to enhance the efficiency of computations by eliminating unnecessary steps. Unlike in the standard approach, the notion incorporates multiple maximal contact coordinates simultaneously, which simplifies the representation of the main invariant and significantly aids in its computation. The idea of utilizing multiple maximal contacts is well-known among experts and is commonly employed in practical implementations for its effectiveness.

Definition 3.6.11. Given a Rees algebra R on X and a rational number $a_1 > 0$. By a *partial maximal contact of* (R, a_1) *on an open affine subset* U we mean a partial system of coordinates on U compatible with E:

$$\overline{x} = (x_1, \ldots, x_s, x_{s+1}, \ldots, x_r),$$

such that

    (1) If $i \leq s$ then $x_i \in T^{1/a_1}(R)(U)$ and x_i is free on U. In particular $\frac{\partial}{\partial x_i}(T^{1/a_1}R) = \mathcal{O}_U$.

    (2) If $s + 1 \leq i \leq r$, then $V(x_i) \in E$, and $\frac{\partial}{\partial x_i}(T^{1/a_1}R) = \mathcal{O}_U$,

If $p \in V(\overline{x})$, and $\mathrm{ord}_p(R) = a_1$, and the conditions (1) and (2) are satisfied in a neighborhood of a point p then $\overline{x}$ is a *partial maximal contact* of R at p.

If additionally $\frac{\partial}{\partial x_j}(T^{1/a_1}R) \neq \mathcal{O}_X$ at p for $j > r$, where $(x_1, \ldots, x_n)$ is a local system of parameters extending $(x_1, \ldots, x_r)$ then we say that $\overline{x}$ is a *maximal contact* of R at p.

We shall associate partial maximal contact with respect to (R, a) gradation t^{1/a_1} and write it in the graded form $\overline{x}t^{1/a_1}$.

Lemma 3.6.12. *Let $\overline{x}_1 t^{1/a_1}$ be a maximal contact for R at p, with $\mathrm{ord}_p(R) = a_1$. Then it is a partial maximal contact for (R, a_1) in a certain neighborhood U of p.* ♣

Proof. Let us consider an open neighborhood where the conditions (1) and (2) of Definition 3.6.11 are met. ♣

Remark 3.6.13. Note that a partial maximal contact on U exist if a_1 is an integer such that $\mathrm{ord}_p(R) \leq a_1$ for all $p \in U$. More specifically we have:

Lemma 3.6.14. *If $\overline{x}$ is a partial maximal contact of (R, a_1) on U, then $\mathrm{ord}_q(R) \leq a_1$ for any $q \in U$.*

Proof. It follows from the definition of the partial maximal contact that $\mathcal{D}_X(T^{1/a_1}R) = \mathcal{O}_U$. As a consequence, for every point q in U, there exists $\mathcal{D}_{\overline{x}^\alpha}(f_j)$ that is invertible at q, where $f_j t^{b_j}$ is a generator of R and $|\alpha| = b_j a_1$. ♣

Example 3.6.15. Let

$$R = \mathcal{O}_X[(x^3 + y^5 + z^7)t, (v^4 w^2)t^2]$$

where $a_1 = 3 = \mathrm{ord}_0(R)$, as in Example 3.6.6.

Then $T^{1/3}(R)t^{1/3} = (x, y^3, z^4, v, w)t^{1/3}$, and the maximal contact at 0 is given by $(x, v, w)t^{1/3}$ in gradation $t^{1/3}$. It is a partial maximal contact for $(R, 3)$ on X.

Lemma 3.6.16.   *(1) A maximal contact of R at $p \in X$ exists.*
  (2) Any partial maximal contact of R at p can be extended to a maximal contact of R at p.
  (3) The image of a maximal contact $\overline{x}$ at p in m_p/m_p^2 is determined uniquely. It is the smallest subspace compatible with E_p containing the image of $T_p^{1/a_1}(R)$.
  (4) The divisorial coordinates in the maximal contact $\overline{x}$ at p are uniquely determined and do not depend upon the maximal contact.

Proof. According to the definition, for a given a, we have $\mathrm{ord}_p(R_a) = aa_1$. Consequently, $D^{aa_1-1}(R_a) \subset T^{1/a_1}R$, and $\mathrm{ord}_p(D^{aa_1-1}(R_a)) = \mathrm{ord}_p(T^{1/a_1}R) = 1$.

Let $\mathcal{I}_{E,p}$ denote the ideal generated by the divisorial coordinates in E_p (i.e., in E passing through p). Consider the image

$$\overline{T}^{a_1}R := (T^{1/a_1}R + \mathcal{I}_{E,p} + m_p^2)/(\mathcal{I}_{E,p} + m_p^2)$$

of $T^{1/a_1}R$ in $m_p/(\mathcal{I}_{E,p}+m_p^2)$. Its basis is determined by a partial system of local free parameters $x_1, \ldots, x_s \in T^{1/a_1}(R)$. Then we can extend it to a partial coordinate system

$$\overline{x} := (x_1, \ldots, x_s, x_{s+1}, \ldots, x_r),$$

where x_i, for $i \geq s + 1$ are divisorial at p with the smallest image in m_p/m_p^2 containing the image of $T^{1/a_1}(R)$. Consequently, for any divisorial x_j, there exists an element $v = \sum c_i x_i \in T^{1/a_1} + m_p^2$ with $c_j \neq 0$ at p, and all x_i are divisorial.

Furthermore, $\overline{x}$ precisely represents a maximal contact of R at p. Conversely, any maximal contact of R at p can be constructed in this manner.

If $\overline{x}$ is any partial maximal contact, its image in $\overline{T}^{1/a_1}R$ can be extended to form a basis for $\overline{T}^{1/a_1}R$. This allows us to extend the free part of $\overline{x}$ to a maximal contact $\overline{x'}$. We need to show that $\overline{x'}$ contains the entirety of $\overline{x}$.

If $x_j \in \overline{x}$ is a divisorial coordinate, according to the previous discussion, $D_{x_j}(v) \neq 0$, where $v \in T^{1/a_1}(R)$ and $v = \sum a_i x_i, (\mathrm{mod}\, m_p^2)$, with x_i all being divisorial coordinates that include x_j. As the image of the maximal contact $\overline{x'}$ in m_p/m_p^2 necessarily includes the image of v, it also contains the images of all x_i with $a_i \neq 0$, including the coordinate x_j. Hence, $\overline{x} \subseteq \overline{x'}$.

In conclusion, the divisorial coordinates are precisely those that appear in the presentation of some $v \in T^{1/a_1}(R)$, where $v = \sum a_i x_i, (\mathrm{mod}\, m_p^2)$, with all x_i being divisorial. Therefore, the divisorial coordinates are uniquely determined.

♣

Lemma 3.6.17. *Let $R \subset \mathcal{A}^{\mathrm{Int}} = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{Int}}$ at a point $p \in X$ such that $a_1 = \mathrm{ord}_p(R)$. Then upon the change of the coordinate representation of $\mathcal{A}$, $\overline{x}_1$ contains a maximal contact $\overline{x}$ of R at p.*

Proof. According to Lemma 3.4.3, the operators $t^{-|\alpha|/a_1}\mathcal{D}_{x^\alpha}$ in the definition of $T^{1/a_1}(R)$ preserve $\mathcal{A}^{\text{Int}}$. Therefore, we have the following inclusion:

$$R \subset \mathcal{A}^{\text{Int}} = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\text{Int}}$$

implies that $T^{1/a_1}(R)t^{1/a_1} \subset \mathcal{A}^{\text{Int}}$. Consider the free part $(x_1, \ldots, x_r)$ of the maximal contact $\overline{x}$ in $T^{1/a_1}(R)$. According to Lemma 3.1.29, we can assume that

$$(x_1, \ldots, x_r)t^{1/a_1} \subseteq \overline{x}_1 t^{1/a_1} \subseteq \mathcal{A}^{\text{Int}}_{1/a_1},$$

after a coordinate change. Since $a_i > a_1$ for $i \geq 2$, the image of $\overline{x}_1$ in m_p/m_p^2 is the same as the image of $\mathcal{A}^{\text{Int}}_{1/a_1}$ in m_p/m_p^2 and includes the image of $T^{1/a_1}(R)$. By Lemma 3.6.16, it also includes the image of the maximal contact $\overline{x}$ of R. Moreover, $\overline{x}_1$ is compatible with the divisorial structure, indicating that it must also contain the divisorial part of the maximal contact itself. Therefore, $\overline{x}_1 \supseteq \overline{x}$.

♣

Definition 3.6.18. To a partial maximal contact $\overline{x} = (x_1, \ldots, x_r)$ of R on U, we can associate the invariant

$$\text{inv}^1(\overline{x}) = (1, \ldots, 1, 1_+, \ldots 1_+),$$

where the 1's correspond to the free maximal coordinates in $\overline{x}$, and the 1_+'s correspond to the coordinates in $\overline{x}$ that define the divisors on U.

Lemma 3.6.19. *Given a Rees algebra R of order a_1 at $p \in X$ with a maximal contact $\overline{x}$ of R at p. Then*

(1) $\text{inv}^1_p(R) = a_1\text{inv}(\overline{x})$.
(2) $R \subset \mathcal{O}_X[\overline{x}t^{1/a_1}, \overline{y}t^{1/a_2}]^{\text{Int}}$, *for some $a_2 > a_1$, and a coordinate system $(\overline{x}, \overline{y})$ at p compatible with E.*
(3) *If $\overline{x}$ is a partial maximal contact for (R, a_1) on U then $\text{inv}^1_p(R) \leq a_1\text{inv}(\overline{x})$.*

Proof. (1) and (2). If $R \subset \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}])$ at p, where $a_1 = \text{ord}_p(R)$, then according to Lemma 3.6.17, $\overline{x}_1$ contains a maximal contact of R after a coordinate change. Therefore, based on Definition 3.6.18, we have $\text{inv}^1_p(R) \leq a_1\text{inv}(\overline{x})$.

Moreover, note that since $\text{ord}_p(R) \geq a_1$, we have $\text{ord}_p(C_{\overline{x}t^{1/a_1}}R) \geq a_1$ and $\text{ord}_p(C_{\overline{x}t^{1/a_1}}R)_{|H} \geq a_1$. Suppose the latter order is equal to a_1. Then, since $C_{\overline{x}t^{1/a_1}}(R)_{|H}$ is generated by the coefficients as shown in Lemma 3.5.5, there exists a generator

$$ft^a = \sum c_\alpha \overline{x}^\alpha t^a \in R_a t^a$$

of R, with $c_\alpha = c_\alpha(\overline{y})$, for the corresponding system of local parameters $(\overline{x}, \overline{y})$, such that $\text{ord}_p(c_\alpha) = a_1 a - |\alpha|$ for some α. Moreover, for some β with $|\beta| = \text{ord}_p(c_\alpha) = a_1 a - |\alpha|$, the derivative $D_{\overline{y}^\beta}(c_\alpha) = D_{y_1^{b_1}, \ldots y_k^{b_k}}(c_\alpha)$ is invertible. Consequently, $D_{\overline{y}^\beta}D_{x^\alpha}(f)$ is invertible, and we have $D_{y_1^{b_1-1}, \ldots y_k^{b_k}}D_{x^\alpha}(f) \in T^{1/a_1}(R)$. Thus, $D_{y_1}(T^{1/a_1}(R))$ is invertible, contradicting the condition of maximal contact at a point as defined in Definition 3.6.11.

This implies that $\text{ord}_p(C_{\overline{x}t^{1/a_1}}R)|H = a_2 > a_1$. By Lemma 3.1.24, this further implies that $C_{\overline{x}t^{1/a_1}|H}R \in \mathcal{O}_H[\overline{y}t^{1/a_2}]^{\text{Int}}$. Hence, based on Lemma 3.5.11,

$$R \subseteq \mathcal{O}_X[\overline{x}t^{1/a_1}, \overline{y}t^{1/a_2}]^{\text{Int}},$$

which yields, by Lemma 3.1.21, that $\text{inv}^1_p(R) \geq a_1\text{inv}(\overline{x})$, resulting in equality.

(3) If $\overline{x}$ represents a partial maximal contact on U, then for any point $p \in U$, either $\mathrm{ord}_p(\mathcal{A}) < a_1$ or $\mathrm{ord}_p(\mathcal{A}) = a_1$. According to Lemma 3.6.17, in the latter case, $\overline{x}$ can be extended to a maximal contact $\overline{x'}$. Thus, we obtain the following:

$$\mathrm{inv}_p^1(R) = a_1\mathrm{inv}(\overline{x'}) \le a_1\mathrm{inv}(\overline{x}).$$

♣

Corollary 3.6.20. *Let $\overline{x}$ be a maximal contact of R at p. Let*

$$\mathcal{A} = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{int}}$$

be a maximal R-admissible center at a point $p \in X$. Then upon the change of the coordinate representation of $\mathcal{A}$, $\overline{x}_1 = \overline{x}$ is the maximal contact of R at p.

Proof. According to Lemma 3.6.17, $\overline{x}_1$ contains $\overline{x}$ after the coordinate change. Furthermore, based on Definition 3.6.18 and Lemma 3.6.19(1), we have

$$\mathrm{inv}_p^1(R) = a_1(\mathrm{inv}(\overline{x}_1)) = a_1(\mathrm{inv}(\overline{x})).$$

This implies the equality $\overline{x}_1 = \overline{x}$.   ♣

3.6.21. *Support of the invariant.* Let

$$\mathrm{suppinv}^1(R, \overline{b}_1) := \{p \in X \mid \mathrm{inv}_p^1(R) = \overline{b}_1\}$$

The following lemma highlights a fundamental property of maximal contact within our context:

Lemma 3.6.22. *Let $\overline{x}$ be a partial maximal contact of (R, a_1) on U, and set $b_1 := a_1\mathrm{inv}(\overline{x})$. Then we have*

$$\mathrm{suppinv}^1(R, \overline{b}_1) \subseteq V(\overline{x}).$$

Proof. Let's denote $\overline{x} = (x_1, \ldots, x_r, x_{r+1}, \ldots, x_s)$, where x_i are free on U for $1 \le i \le r$ and divisorial for $r + 1 \le i \le s$. Consider $q \in U \setminus V(x_i)$, where $x_i \in \overline{x}$ is a free coordinate.

In this case, we have $x_i \in T^{1/a_1}(R) = \mathcal{O}_X$, and $\mathcal{D}^{a_1 b_j - 1}(f_j)$ is invertible for some generator $f_j t^{b_j} \in R_{b_j}$. It follows that

$$\mathrm{ord}_q(f_j t^{b_j}) \le (a_1 b_j - 1)/b_j < a_1,$$

which implies $\mathrm{ord}_q(R) < a_1$. Therefore, we have $\mathrm{inv}_q^1(R) < \overline{b}_1$.

Consider $q \in V(x_1, \ldots, x_r) \cap U \setminus V(x_i)$, where $x_1, \ldots, x_r$ are free coordinates, and x_i with $i > r$ is a divisorial coordinate on U. Additionally, we have $D_{x_i}(T^{1/a_1}(R)) = \mathcal{O}_X$, which implies that x_i is a free coordinate at q since $q \notin V(x_i)$. Furthermore, for a certain local parameter $u \in T^{1/a_1}(R)$, we have $D_{x_i}(u)$ invertible. Consequently, u is linearly independent from $x_1, \ldots, x_r$ as well as from the other divisorial coordinates at q.

As a result, in a partial maximal contact $(x_1, \ldots, x_r, u)$ of R at q, there are at least $r + 1$ free coordinates. According to Lemma 3.6.19(3), this implies that

$$\mathrm{inv}_q^1(R) \le a_1\mathrm{inv}_1(x_1, \ldots, x_r, u) < \overline{b}_1 = a_1\mathrm{inv}(\overline{x}).$$

In this context, it is worth noting that the $(r + 1)$-th component of $a_1\mathrm{inv}_1(x_1, \ldots, x_r, u)$ is a_1, while for $\overline{b}_1 = a_1\mathrm{inv}(\overline{x})$, it is a_{1+}.

♣

Example 3.6.23. Consider the ideal $\mathcal{I} = (x_1^2 + x_2) \subset K[x_1, x_2]$, where x_1 is a free coordinate and x_2 is a divisorial coordinate. Then

$$T^1(\mathcal{I}t) = (x_1^2 + x_2)t$$

At the point 0, (x_2) represents a maximal contact, while on the scheme $X = \mathrm{Spec}(K[x_1, x_2])$, (x_2) corresponds to a partial maximal contact. Thus, we obtain the following:

$$\mathrm{inv}_0^1(\mathcal{I}t) = \mathrm{inv}^1(x_2) = (1_+) \quad \text{and}$$

$$\mathrm{suppinv}^1(\mathcal{I}t, (1_+)) \subseteq V(x_2).$$

When calculating the invariant $\mathrm{inv}_p^1(\mathcal{I}t)$ away from the origin, such as at $p \in V(\mathcal{I}) \setminus 0$, the coordinate $x_1' := x_2 + x_1^2$ is free. Therefore, $(x_1', \in T^2(\mathcal{I}t))$ represents a maximal contact at p, which yields the following:

$$\mathrm{inv}_p^1(\mathcal{I}t) = \mathrm{inv}^1(x_1') = (1) < (1_+).$$

Consequently, the invariant drops in a neighborhood of 0.

This example illustrates the concept of assigning heavier weights: $1_+ > 1$, to the divisorial coordinates.

Lemma 3.6.24. *Let $\overline{x} = (x_1, \ldots, x_r)$ be a partial maximal contact for (R, a_1) on an open subset U. Then*

(1) $\mathrm{suppinv}^1(R, \overline{b}_1) \subseteq V(T^{\leq 1/a_1}(R)) \cap V(\overline{x})$.
(2) $\overline{x}$ is a partial maximal contact at all points $q \in V(T^{\leq 1/a_1}(R)) \cap V(\overline{x})$ for which $\mathrm{ord}_q(R) = a_1$.
(3) $\mathrm{suppinv}^1(R, \overline{b}_1) = V(T^{\leq 1/a_1}(R)) \cap V(\overline{x}) \cap V(D_{x_{r+1}, \ldots, x_k}(T^{1/a_1}R)))$ is closed on U.
(4) $\overline{x}$ is a maximal contact for the points in $\mathrm{suppinv}^1(R, \overline{b}_1)$.

Proof. (1) and (2) Based on Lemmas 3.6.22 and 3.6.9, we have

$$\mathrm{suppinv}^1(R, \overline{b}_1) \subset V(T^{\leq 1/a_1}(R)) \cap V(\overline{x}).$$

Furthermore, according to Lemma 3.6.14, for points $q \in V(T^{\leq 1/a_1}(R)) \cap V(\overline{x})$, we have $\mathrm{ord}_q(R) \leq a_1$, and by Definition 3.6.11, $\overline{x}$ represents a partial maximal contact at q as long as $\mathrm{ord}_q(R) = a_1$.

(3) and (4). Let's assume that $D_{x_j}(T^{1/a_1}(R)) = \mathcal{O}_X$ at $q \in V(T^{\leq 1/a_1}(R)) \cap V(\overline{x})$ for some $j > r$.

If x_j is a divisorial coordinate, then $(\overline{x}, x_j)$ represents a partial maximal contact at q, and according to Lemma 3.6.19, we have $\mathrm{inv}_q(R) \leq a_1 \mathrm{inv}(\overline{x}, x_j) < \overline{b}_1$.

If x_j is a free coordinate at q, there exists a free local parameter $u \in T^{1/a_1}(R)$ such that $D_{x_j}(u)$ is invertible. In this case, $(x_1, \ldots, x_r, u)$ represents a partial maximal contact at q, and we have $\mathrm{inv}_q(R) \leq a_1 \mathrm{inv}(\overline{x}, x_j) < \overline{b}_1$.

Conversely, if $q \in V(T^{\leq 1/a_1}(R)) \cap V(\overline{x}) \cap V(D_{x_{r+1}, \ldots, x_k}(T^{1/a_1}R))$, then the condition (3) of Definition 3.6.11 is satisfied, and $\overline{x}$ represents a maximal contact at q. According to Lemma 3.6.19, we have $\mathrm{inv}_q^1(R) = a_1(\mathrm{inv}(\overline{x}))$.

♣

3.6.25. *Maximal contact for nested ideals.* We can directly extend the definition of maximal contact to nested Rees algebras. Consider a coordinate system $\overline{x}_1, \ldots, \overline{x}_n$ of local parameters that is compatible with E. Let

$$R = \mathcal{O}_X[(x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k})^{\mathrm{Int}}, f_j t^{b_j}]_{j=1,\ldots,s}$$

be a Rees algebra nested at $H_k := V(\overline{x}_1, \ldots, \overline{x}_k)$, with the nested order $\mathrm{ord}_p(R_{|H_k}) = a_{k+1}$. The *nested cotangent ideal* $T^{1/a_{k+1}}(R)$ of R can be defined as follows:

$$T_{H_K}^{1/a_{k+1}}(R) := \sum_{|\alpha| = b_j a_{k+1} - 1} \mathcal{O}_X \mathcal{D}_{\overline{x}_{k+1}, \ldots, \overline{x}_n}^{\alpha}(f_j)$$

in $t^{1/a_{k+1}}$- gradation. Its restriction $T_{H_K}^{1/a_{k+1}}(R)_{|H_k}$ defines the cotangent ideal of $R_{|H_k}$ on H_k. Consequently we define a *nested maximal contact* for R at H_i given by a partial system of coordinates

$$\overline{x}_{k+1} = (x_{k+1,1}, \ldots, x_{k+1,s}, x_{k+1,s+1}, \ldots, x_{k+1,r}))$$

on X, with free coordinates $x_{k+1,1}, \ldots, x_{k+1,s} \in T_{H_K}^{a_{k+1}}(R)$, and divisorial $x_{k+1,s+1}, \ldots, x_{k+1,r}$ such that its restriction $\overline{x}_{k+1|H_k}$ is a maximal contact for $R_{|H_k}$.

Corollary 3.6.26. *Let*

$$R = \mathcal{O}_X[(x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k})^{\mathrm{Int}}, f_j t^{b_j}]_{j=1,\ldots,s}$$

be a Rees algebra nested at $H_k := V(\overline{x}_1, \ldots, \overline{x}_k)$, and assume that

$$\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_n t^{1/a_n}]^{\mathrm{int}},$$

with $n \geq k$, is a maximal R- admissible center at $p \in X$ so that

$$R \subset \mathcal{A} = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{Int}}$$

Then the nested order $\mathrm{ord}_p(R_{|H_k}) = a_{k+1}$. Let $\overline{x}$ be a nested maximal contact of R at p and H_k. Then upon the change of the coordinate representation of $\mathcal{A}$, $\overline{x}_{k+1} = \overline{x}$ is the maximal contact of R at p.

Proof. Given that R is nested at H_k, it can be expressed in the split form in $\widehat{\mathcal{O}}_{X,p}$ as:

$$\widehat{\mathcal{O}}_{X,p} \cdot R = \widehat{\mathcal{O}}_{X,p}[(x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k})^{\mathrm{Int}}, R_{|H_k}]$$

Consequently,

$$\mathcal{A}_{|H_k} = \mathcal{O}_H[\overline{x}_{k+1|H_k} t^{1/a_{k+1}}, \ldots, \overline{x}_{n|H_k} t^{1/a_n}]^{\mathrm{int}}$$

serves as a maximal admissible center for $R_{|H_k}$. By applying Lemma 3.1.24, we can observe that $\mathrm{ord}_p(R_{|H_k}) = a_k$. According to Lemma 3.1.29, the free coordinates in $\overline{x}$ are a part of $\overline{x}_{k+1}$ after a coordinate change. Moreover, since $\overline{x}_{|H_k}$ represents a maximal contact for $R_{|H_k}$, we can deduce from Lemma 3.6.16(4) that $\overline{x}_{k+1|H_k}$ and $\overline{x}_{|H_k}$ share the same divisorial coordinates. Consequently, we obtain the equality $\overline{x}_{k+1} = \overline{x}$.

♣

3.7. Effective algorithm.

3.7.1. *The algorithm.* Consider an arbitrary Rees algebra R on a smooth variety X. In particular case, if an ideal $\mathcal{I}$ is given we consider the Rees algebra $R = \mathcal{O}_X[\mathcal{I}t]$. We shall construct a maximal $\mathcal{A}$-admissible center $\mathcal{R}$ at a given point $p \in X$. Assume, at first that the Rees center

$$\mathcal{A} = \mathcal{O}_X[(x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}},$$

which is maximal admissible at the point $p \in X$ for $\mathcal{R}$ on X exists. We shall use $\mathcal{A}$ only as the reference for our construction. All the operations are done for R without using or modifying the algebra $\mathcal{A}$. We will however modify the presentation of $\mathcal{A}$ so it will be compatible with the presentations of modifications of R.

The procedure is recursive and we put $R = R_1$. Assume that the center $\mathcal{A}$ is maximal admissible R, so that

$$R = R_1 \subseteq \mathcal{A} := \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{Int}}.$$

Using Lemma 3.1.24, we see that $\mathrm{ord}_p(R_1) = a_1$. Let $\overline{x}'_1 \subseteq T^{1/a_1}(R_1)$ be a maximal contact at $p \in X$, and put $H_1 := V(\overline{x}'_1)$.

By Lemma 3.6.20, one can change the presentation of the algebra $\mathcal{A}$ so that $\overline{x}_1 = \overline{x}'_1$ is a maximal contact. This modification does not affect the algebra $\mathcal{A}$ on the right, but only changes its presentation.

Let $R_2 := C_{\overline{x}_1 t^{1/a_1}}(R_1)$ be the coefficient ideal with respect to the maximal contact $\overline{x}_1$. By Lemma 3.5.7, R_2 is nested at $H_1 := V(\overline{x}_1)$, and can be written in $\widehat{\mathcal{O}}_{X,p}$ in the split form

$$\widehat{\mathcal{O}}_{X,p} R_2 = \widehat{\mathcal{O}}_{X,p}[(\overline{x}_1 t^{1/a_1})^{\mathrm{Int}}, R_{2|H_1}] = \widehat{\mathcal{O}}_{X,p}[(\overline{x}_1 t^{1/a_1})^{\mathrm{Int}}, C_{\overline{x}_1 t^{1/a_1}}(R_1)_{|H_1}]$$

Furthermore, according to Lemma 3.5.24, the same center $\mathcal{A}$ remains maximal admissible for $R_2 = C_{\overline{x}_1 t^{1/a_1}}(R_1)$.

This recursive procedure can be summarized as follows. Let R_i, for $i \geq 2$, be the algebra nested at $H_{i-1} = V(\overline{x}_1, \ldots, \overline{x}_{i-1})$ such that

$$\widehat{\mathcal{O}}_{X,p} R_i = \widehat{\mathcal{O}}_{X,p}[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_{i-1} t^{1/a_{i-1}})^{\mathrm{Int}}, R_{i|H_{i-1}}] \subseteq$$
$$\subseteq \widehat{\mathcal{O}}_{X,p} \mathcal{A} = \widehat{\mathcal{O}}_{X,p}[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{Int}},$$

where $\mathcal{A}$ is a maximal admissible center for R_i. According to Lemma 3.6.26, we have $\mathrm{ord}_p(R_i) = a_i$. We find a nested maximal contact $\overline{x}'_i$ for R_i at H_{i-1} and p. Using Lemma 3.6.26 we can modify the coordinates $\overline{x}_i, \ldots, \overline{x}_k$ in the presentation of $\mathcal{A}$ and assume that the maximal contact $\overline{x'}_i = \overline{x}_i$ occurs in the presentation of $\mathcal{A}$. Consequently, by Lemma 3.5.24,

$$R_{i+1} := C_{\overline{x}_i t^{1/a_i}}(R_i) \subseteq \mathcal{A}^{\mathrm{Int}} = \mathcal{O}_X[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{Int}},$$

where $\mathcal{A}$ on the right represents a maximal admissible center for R_{i+1}. Furthermore, by Lemma 3.5.18, R_{i+1} is nested at $H_i = V(\overline{x}_1, \ldots, \overline{x}_i)$, and can be written in $\widehat{\mathcal{O}}_{X,p}$ as

$$\widehat{\mathcal{O}}_{X,p} \cdot R_{i+1} = \widehat{\mathcal{O}}_{X,p}[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_i t^{1/a_i})^{\mathrm{Int}}, R_{i+1|H_i}] =$$
$$= \widehat{\mathcal{O}}_{X,p}[(\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_i t^{1/a_i})^{\mathrm{Int}}, C_{\overline{x}_{i+1}}(R_i)_{|H_i}],$$

We continue the procedure until we reach R_{k+1} which is nested at $H_k = V(\overline{x}_1, \ldots, \overline{x}_k)$, such that $R_{k+1|H_k} = 0$. Finally

$$R_{k+1} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{Int}} = \mathcal{A}^{\mathrm{Int}},$$

with the extended Rees algebra

$$\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w_{R,A}}, \overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}],$$

where $w_{R,A} = \mathrm{lcm}(a_1, \ldots, a_r, w_R)$.

3.7.2. *Uniqueness.* Throughout this process, we do not alter the algebra $\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^1]^{\mathrm{int}}$. However, we do modify the coordinates within $\mathcal{A}$, which gradually transform into maximal contacts for the nested Rees algebra on the left.

This demonstrates that the procedure, which is independent of $\mathcal{A}$, results in a unique, predetermined center $\mathcal{A}$ that is maximal admissible for R at p.

3.7.3. *Existence.*

Proposition 3.7.4. *(see also* [ATW17, Theorem 5.3.1] *(in the language of $\mathbb{Q}$-ideals)),). For any Rees algebra R on a smooth variety X over K with a SNC divisor E, and for any point $p \in X$ there exists a uniquely determined Rees center*

$$\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^1]^{\mathrm{int}}$$

which is a maximal admissible for R at p.

Moreover $\mathcal{A}^{\mathrm{Int}} = R_{k+1}$ is obtained by the recursive procedure $R_1 = R$, and $R_{i+1} = C_{\overline{x}_i t^{1/a_1}}(R_i)$ independent of $\mathcal{A}$, where

  (1) R_i is nested at $H_{i-1} := V(\overline{x}_1, \ldots, \overline{x}_{i-1})$, for $i-1 = 1, \ldots, k$, and $H_0 := X$.
  (2) $a_i = \mathrm{ord}_p(R_{i|H_{i-1}})$
  (3) $\overline{x}_i$ is a maximal contact for $R_{i|H_{i-1}}$.
  (4) $\mathrm{inv}_p(R) = (a_1 \mathrm{inv}(\overline{x}_1), \ldots, a_k \mathrm{inv}(\overline{x}_k))$.

Proof. The construction does not rely on the Rees center $\mathcal{A}$ on the right side of the admissibility condition at any step. Moreover, the inductive process leads to a Rees center $\mathcal{A}'$ that is admissible for R at p. We obtain a sequence of nested Rees algebras R_i at $H_{i-1} = V(\overline{x}_1, \ldots, \overline{x}_{i-1})$ such that $\mathrm{ord}_p(R_{i|H_{i-1}}) = a_i$ and $\overline{x}_i$ is a maximal contact for $R_{i|H_{i-1}}$. The final Rees center $\mathcal{A}' = R_{k+1}^{\mathrm{int}}$ is an admissible center for R.

Now, consider another admissible center

$$\mathcal{A}'' = \mathcal{O}_X[\overline{x}_1' t^{1/a_1'}, \ldots, \overline{x}_k' t^{1/a_k'}]^{\mathrm{int}},$$

for R at p. Suppose $\mathrm{inv}(\mathcal{A}'') \geq \mathrm{inv}(\mathcal{A}')$. We run the recursive procedure using $\mathcal{A}''$ on the right side. By Lemma 3.1.24, we conclude that $a_1 = a_1''$. Furthermore, according to Lemma 3.6.17(2), after a coordinate change, $\overline{x}_1''$ contains a maximal contact $\overline{x}_1$ such that $\mathrm{inv}^1(\overline{x}_1) \geq \mathrm{inv}^1(\overline{x}_1'')$. Since $\mathrm{inv}(\mathcal{A}'') \geq \mathrm{inv}(\mathcal{A}')$, it follows that $\mathrm{inv}^1(\overline{x}_1'') \geq \mathrm{inv}^1(\overline{x}_1)$. Therefore, we have $\overline{x}_1'' = \overline{x}_1$. We continue this process step by step, demonstrating that $a_i = a_i''$ and $\overline{x}_i'' = \overline{x}_i$ after a possible change in the coordinate representation of $\mathcal{A}'$. This shows that $\mathcal{A}'' = \mathcal{A}' = R_{k+1}^{\mathrm{int}}$ is the maximal admissible center for $\mathcal{I}$ at p.   ♣

3.7.5. *The inductive principle.* [ATW19],

Let R be a Rees algebra of order a_1 at p with a maximal contact $\overline{x}_1$, and let $H_1 := V(\overline{x}_1)$. By Lemma 3.5.11, a center $\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]^{\mathrm{int}}$ is maximal admissible for R if and only if $\mathcal{A}_{|H_1} = \mathcal{O}_{H_1}[\overline{x}_{2|H_1} t^{1/a_2}, \ldots, \overline{x}_{k|H_1} t^{1/a_k}]^{\mathrm{int}}$ is maximal admissible for $C_{\overline{x}_1(R)|H_1}$.

This leads to the following inductive formula, as stated in [ATW19]:

$$\mathrm{inv}_p(R) = (\mathrm{inv}_p^1(R), \mathrm{inv}_p(C_{\overline{x}_1}(R)_{|H_1})),$$

3.7.6. *Semicontinuity of canonical invariant. Local admissibility.* The inductive principle above ensures the semicontinuity of the invariant inv_p, as in [ATW19]. We proceed with an induction on $n = \dim(X)$ and assume that the function $p \mapsto \mathrm{inv}_p(R)$ is upper semicontinuous for any Rees algebra R' on any smooth variety H of dimension $n - 1$. Let R be a Rees algebra on a smooth X of dimension n.

The condition $\mathrm{inv}_p(R) \geq (\overline{b}_1, \ldots, \overline{b}_k)$ implies either $\mathrm{inv}_p^1(R) > \overline{b}_1$, or $\mathrm{inv}_p^1(R) = \overline{b}_1$ and $\mathrm{inv}_p(C_{\overline{x}_1|H_1}(R)) \geq (\overline{b}_2, \ldots, \overline{b}_k)$. This description corresponds to a closed subset, which follows from the semicontinuity of inv_p^1 as established in Lemma 3.6.24, and with the inductive assumption.

3.7.7. *Duality of Rees centers.* The Rees centers $\mathcal{A}$ possess a dual interpretation: they serve as admissible centers with a dummy variable t, and they also represent the extended algebras of the full cobordant blow-ups, where t^{-1} serves as the introduced coordinate on B. These two concepts are intimately connected and can be described by identical formulas, up to rescaling. In order to prevent any confusion, we will introduce a distinct variable t_B specifically for the algebras on B, where the differentiation between t and t_B becomes necessary.

3.7.8. *Controlled transforms of ideals.* [ATW19], Let $\mathcal{I}$ be an ideal on regular X. One writes $\mathcal{I}$-admissibility condition as:

$$\mathcal{I}t \subset \mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w_A}, \overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]$$

By employing the variable t_B and applying rescaling, we can rewrite this inclusion as follows:

$$\mathcal{I} \cdot t_B^{w_A} \subset \mathcal{O}_B = \mathcal{O}_X[t_B^{-1}, \overline{x}_1 t_B^{w_1}, \ldots, \overline{x}_k t_B^{w_k}] = \mathcal{O}_X[t_B^{-1}, \overline{x}_1', \ldots, \overline{x}_k'],$$

where $x_i' := x_i t^{w_i}$.

According to Lemma 2.5.1, the exceptional divisor on B_+ can be expressed as t_B^{-1}, which serves as a local parameter on B. Using this information, we can deduce that the full transform $\mathcal{O}_B \cdot I$ is divisible by $t_B^{-w_A}$ since $t^{w_A} \cdot \mathcal{O}_B \cdot \mathcal{I} = \mathcal{O}_B \cdot t_B^{w_A} \cdot \mathcal{I} \subset \mathcal{O}_B$.

Now, we introduce the concept of the *controlled transform* of the ideal $\mathcal{I}$, which is defined as:

$$\sigma^c(\mathcal{I}) := \mathcal{O}_B \cdot t_B^{w_A} \cdot \mathcal{I} \subset B.$$

3.7.9. *Strict transform of ideals.* Recall that

$$B_- = B \smallsetminus V(t^{-1}) = X \times \mathbb{G}_m \to X$$

is trivial over X. By *the strict transform* $\sigma^s(\mathcal{I})$ of an ideal $\mathcal{I}$ on X under a full cobordant blow-up $\sigma : B \to X$ of $\mathcal{A}$ on X we mean the schematic closure of $(\mathcal{O}_B \cdot \mathcal{I}_X)_{|B_-} = \mathcal{O}_{B_-} \cdot \mathcal{I}_X$. Thus

$$\sigma^s(\mathcal{I}) := \{t^a f \in \mathcal{O}_B \mid f \in \mathcal{O}_B \cdot \mathcal{I}_Y, \quad a \geq 0\}.$$

Consequently, the strict transform of a closed subscheme Y of X is the schematic closure Y^s of $Y \times G_m \subset B_- = X \times G_m$ in B. It is defined by $\mathcal{I}_{Y^s} = \sigma^s(\mathcal{I}_Y)$. This implies that $\sigma^c(\mathcal{I}) \subseteq \sigma^s(\mathcal{I})$, and thus $\mathrm{inv}(\sigma^c(\mathcal{I})) \geq \mathrm{inv}(\sigma^s(\mathcal{I}))$ for any $p \in B$.

Remark 3.7.10. Observe that for the cobordant blow-up $\sigma_+ : B_+ \to X$ the induced strict transform coincides with the standard definition $\sigma_+^s(\mathcal{I})$ as the schematic closure on B_+ of

$$\mathcal{O}_{\sigma_+^{-1}(X \smallsetminus V(\mathcal{J}))} \cdot \mathcal{I}_{|X \smallsetminus V(\mathcal{J})} = (\mathcal{O}_B \cdot \mathcal{I})_{|B_+ \cap B_-}.$$

3.7.11. *Controlled transforms of Rees algebras and double gradation.* The relationship between admissibility and the controlled transforms of ideals can be described using the concept of double gradation. Let $\mathcal{I}$ correspond to the gradation $\mathcal{I}t$ in the Rees algebra $\mathcal{A}_{\mathcal{I}} = \mathcal{O}_X[\mathcal{I}t]$. We can assign a double gradation $\mathcal{O}_B \cdot \mathcal{I} \cdot t_B^{w_A} t$ in the Rees algebra $\mathcal{O}_B[\mathcal{I} \cdot t_B^{w_A} t]$ on B, which can be interpreted as the controlled transform denoted by

$$\sigma^c(\mathcal{I}t) := (\mathcal{O}_B \cdot t_B^w \cdot \mathcal{I})t \subset \mathcal{O}_B[\mathcal{I} \cdot t_B^{w_A} t]$$

under the gradation t. One can simply write it as

$$\sigma^c(\mathcal{I}) := (\mathcal{O}_B \cdot t_B^w \cdot \mathcal{I}) \subset \mathcal{O}_B$$

In a more general setting, suppose we have a Rees algebra $R = \bigoplus R_a t^a$, and $\mathcal{A}$ is an R-admissible center. Then, we have the inclusion

$$R \subset \mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, \overline{x}_1 t^{1/a_1}, \dots, \overline{x}_k t^{1/a_k}],$$

where w is a multiple of $w_{R,A}$. By rescaling $\mathcal{A}$ with $t_B \mapsto t_B^w$ and defining $w_i := w/a_i \in \mathbb{Z}_{\geq 0}$, we obtain the cobordant blow-up of $\mathcal{A}^{\mathrm{ext}}$:

$$B = \mathrm{Spec}([t_B^{-1}, \overline{x}_1 t_B^{w_1}, \dots, \overline{x}_k t_B^{w_k}] = \mathrm{Spec}\,\mathcal{O}_B[t_B^{-1}, \overline{x}'_1, \dots, \overline{x}'_k],$$

where $x'_i := x_i t_B^{w_i}$. By combining the gradations on both t and t_B^w, we can express the admissibility of Rees algebras on B as follows:

$$\bigoplus R_a t_B^{a \cdot w} t^a \ \subset \ \mathcal{O}_B[t_B^{-1} t^{-1/w}, \overline{x}_1 t_B^{w_1} t^{1/a_1}, \dots, \overline{x}_k t_B^{w_k} t^{1/a_k}] \ \subset$$
$$\subset \ \mathcal{O}_B[t^{-1/w}, \overline{x}'_1 t^{1/a_1}, \dots, \overline{x}'_k t^{1/a_k}].$$

Hence, the Rees algebra on B can be denoted as:

$$\sigma^c(R) := \bigoplus (\mathcal{O}_B \cdot R_a \cdot t_B^{a \cdot w}) t^a,$$

and it will be referred to as the *controlled transform of R*.

3.7.12. *Cobordant blow-ups and admissibility.* Consequently, based on the previous discussion, we can conclude that:

Lemma 3.7.13. *If $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, \overline{x}_1 t^{1/a_1}, \dots, \overline{x}_k t^{1/a_k}]^{\mathrm{int}}$ is admissible for R, and $\sigma : B = \mathrm{Spec}_X(\mathcal{O}_B[t_B^{-1}, \overline{x}_1 t_B^{w_1}, \dots, \overline{x}_k t_B^{w_k}])$ is the full cobordant blow-up of $\mathcal{A}^{\mathrm{ext}}$ then the Rees center on B:*

$$\mathcal{A}_B^{\mathrm{ext}} := \sigma^c(\mathcal{A}^{\mathrm{ext}}) := \mathcal{O}_B[t^{-1/w}, \overline{x}_1 t_B^{w_1} t^{1/a_1}, \dots, \overline{x}_k t_B^{w_k} t^{1/a_k}]^{\mathrm{int}} =$$
$$\mathcal{O}_B[t^{-1/w}, \overline{x}'_1 t^{1/a_1}, \dots, \overline{x}'_k t^{1/a_k}]$$

is admissible for $\sigma^c(R) = \bigoplus \mathcal{O}_B R_a t_B^{wa} t^a$.   ♣

3.7.14. *Derivations on cobordant blow-up.* [ATW19],
    Consider be the full cobordant blow-up

$$B = \mathrm{Spec}_X(\mathcal{O}_X[t_B^{-1}, x_1 t_B^{w_1}, \dots, x_k t_B^{w_k}] \to X$$

of a center $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \dots, x_k t^{1/a_k}]$. The sheaf $\mathcal{D}_X$ of derivations on X is a coherent $\mathcal{O}_X$-module that is locally generated by the derivations D_{x_i}. Using chain rule we can write derivations $D_{x'_i}$ on B as $D_{x'_i} = t_B^{-w_i} D_{x_i}$, and $D_{x'_j} = D_{x_j}$.

This above formula can be extended using a double gradation principle: To obtain the controlled transform, we associate the derivations $t^{-1/a_i} D_{x_i}$ in the graded form with a double gradation with respect to t and t_B^w.

$$\sigma^c(t^{-1/a_i} D_{x_i}) = t^{-1/a_i} t_B^{-w_i} D_{x_i} = t^{-1/a_i} D_{x_j'}$$

In a similar manner, we define the "controlled transform" of $t^{-1/a_1} \mathcal{D}_X$ to be the subsheaf

$$\sigma^c(t^{-1/a_1} \mathcal{D}_X) := t^{-1/a_1} \mathcal{O}_B t_B^{-w_1} \mathcal{D}_X \subseteq t^{-1/a_1} \mathcal{D}_B$$

of the sheaf $\mathcal{D}_B$ of the derivations on B in gradation t^{-1/a_1}. Note that the sheaf $\mathcal{O}_B t_B^{-w_1} \cdot \mathcal{D}_X$ is generated by $t_B^{-(w_1-w_i)} t_B^{-w_i} D_{x_i} = t_B^{-(w_1-w_i)} D_{x_i'}$ for $i = 1, \ldots, k$ and $t_B^{-w_1} D_{x_i} = t_B^{-w_1} D_{x_i'}$ for $i = k+1, \ldots, n$.

3.7.15. *The order of the controlled transforms.*

Lemma 3.7.16. [ATW19], *Let $\sigma : B \to X$ be a cobordant blow-up of R-admissible center $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{1/w}, \overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]$, where $\mathrm{ord}_p(R) \leq a_1$ for $p \in X$. Then $\mathrm{ord}_{p'}(\sigma^c(R)) \leq a_1$ for $p' \in B$.*

Proof. Write $R = \mathcal{O}_X[f_j t^{b_j}]_{j=1,\ldots,s}$. If $\mathrm{ord}_p(R) = a_1$ then $\mathrm{ord}_p(f_j t^{b_j}) = b_j a_1$ for some j. Thus there exists α, with $|\alpha| = b_j a_1$, such that $\mathcal{D}_{x^\alpha} f_j$ is invertible. Consequently

$$1 \sim \mathcal{D}_{x^\alpha} f_j = (t_B^{-b_j w} t^{-b_j} \mathcal{D}_{x^\alpha})(f_j t_B^{b_j w} t^{b_j}) \in \sigma^c(\mathcal{D}_X^{b_j a_1} t^{-b_j})(\sigma^c(f_j t^{b_j}))$$
$$\subseteq \mathcal{D}_B^{b_j a_1} t^{-b_j}(\sigma^c(R_{b_j}) t^{b_j}) = \mathcal{D}_B^{b_j a_1}(\sigma^c(R_{b_j})) = \mathcal{O}_B,$$

which shows that $\mathrm{ord}_p \sigma^c(R) \leq a_1$. ♣

3.7.17. *Controlled transforms of cotangent ideal.* Similarly we have

Lemma 3.7.18. [ATW19], *Let $\sigma : B \to X$ be a cobordant blow-up of $R = \mathcal{O}_X[f_j t^{b_j}]$-admissible center $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{1/w}, \overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]$. If $T^{1/a_1}(R)$ is the cotangent ideal for R then*

$$\sigma^c(T^{1/a_1}(R) t^{1/a_1}) \subseteq (T^{1/a_1}(\sigma^c(R))) t^{1/a_1}.$$

Proof. Let $f_j t^{b_j} \in R_{b_j}$, and $|\alpha| = b_j a_1 - 1$. Then

$$(\mathcal{D}_{x^\alpha} f_j) t^{1/a_1} t_B^{w_1} = \sigma^c((\mathcal{D}_{x^\alpha} f_j) t^{1/a_1}) = \sigma^c(\mathcal{D}_{\overline{x} t^{1/a_1}}^\alpha f_j t^{b_j}) =$$
$$= \sigma^c(\mathcal{D}_{\overline{x} t^{1/a_1}}^\alpha)(\sigma^c(f_j t^{b_j}) \in \mathcal{D}_B^{|\alpha|} t^{-|\alpha|/a_1}(\sigma^c(R_{b_j}) t^{b_j}) =$$
$$= \mathcal{D}_B^{|\alpha|}(\sigma^c(R_{b_j})) t^{1/a_1} \subseteq (T^{1/a_1}(\sigma^c(R))) t^{1/a_1}.$$

♣

3.7.19. *Controlled transforms of a partial maximal contact.*

Lemma 3.7.20. [ATW19], *Let $\sigma : B \to X$ be a cobordant blow-up of $R = \mathcal{O}_X[f_j t^{b_j}]$-admissible center $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{1/w}, \overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]$. Assume that $\overline{x}_1 = (x_1, \ldots, x_r)$ is a partial maximal contact for (R, a_1) on an open affine subset U then $\sigma^c(\overline{x}_1)$ is a partial maximal contact for $\sigma^c(R)$ on $\sigma^{-1}(U)$.*

Proof. $\sigma^c(\overline{x}_1) = (x'_1, \ldots, x'_r)$ is a local system of coordinates. So if $x_i \in T^{1/a_1}(R) \cap \overline{x}_1$ is free then $x'_i = \sigma^c(x_i) \in T^{1/a_1}(\sigma^c(R)) \cap \sigma^c(\overline{x}_1)$ is free. Thus condition (1) of Definition 3.6.11 is satisfied.

If $x_i \in \overline{x}_1$ is divisorial, then using Lemma 3.7.18, we have

$$\mathcal{O}_B = \mathcal{O}_B \cdot D_{x_i}(T^{1/a_1}(R)) = \sigma^c(D_{x_i t^{1/a_1}}(T^{1/a_1}(R)t^{1/a_1}) =$$
$$= \sigma^c(D_{x_i t^{1/a_1}})\sigma^c(T^{1/a_1}(R)t^{1/a_1}) =$$
$$= t^{-1/a_1}D_{x'_i}\sigma^c((T^{1/a_1}(R)t^{1/a_1}) \subset D_{x'_i}(T^{1/a_1}(\sigma^c(R))$$

and thus condition (2) of Lemma 3.6.11 is also satisfied.

♣

3.7.21. *Controlled transform of the coefficient ideal.*

Lemma 3.7.22. *(see also* [ATW19]*) Let* $\sigma : B \to X$ *be a cobordant blow-up of* $R = \mathcal{O}_X[f_j t^{b_j}]$*-admissible algebra* $\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]$ *then* $\sigma^c(R) = \mathcal{O}_B[\sigma^c(f_j t^{b_j})]$*, and we have commutativity:*

$$\sigma^c(C_{\overline{x}_1 t^{1/a_1}}(R)) = C_{\overline{x}'_1 t^{1/a_1}}(\sigma^c(R)).$$

Proof. The assertion follows from the of the commutativity of the controlled transforms with derivations:

$$\sigma^c(D^\alpha_{\overline{x}_1 t^{1/a_1}}(f_j t^{b_j})) = \sigma^c(D^\alpha_{\overline{x}_1 t^{1/a_1}})(\sigma^c(f_j t^{b_j})) = D^\alpha_{\overline{x}_1 t^{1/a_1}}(\sigma^c(f_j t^{b_j}))$$

in $t^{|\alpha|}$-gradation for $|\alpha| < b_j a_1$.   ♣

3.7.23. *Restriction of cobordant blow-up to a maximal contact.*

Lemma 3.7.24. [ATW19], *If* $\mathcal{A}^{\text{ext}}$ *is admissible for* R *then* $\mathcal{A}^{\text{ext}}_{|H}$ *is admissible for* $R_{|H}$ *and* $\sigma^c(R)_{|H} = \sigma^c_H(R_{|H})$*, where* $H := V(\overline{x}_1)$*.*

The restriction of the blow-up $\sigma_X : B \to X$ *of* $\mathcal{A}^{\text{ext}}$ *to the strict transform* $H_B = V(\overline{x}'_1)$ *of* $H = V(\overline{x}_1)$ *is the cobordant blow-up* $\sigma_H : H_B \to H$ *of the restriction* $\mathcal{A}^{\text{ext}}_{|H}$*.*   ♣

3.7.25. *The centers with maximal invariant.*

Lemma 3.7.26. *(see also* [ATW19]*) Let* R *be a Rees algebra, and* $\mathcal{A}$ *be a maximal admissible center for* R *at a point* $p \in X$*. Then there exists an open neighborhood* U *of* p *such that*

$$\text{maxinv}_U(R) = \text{inv}_p(R) = \text{inv}(\mathcal{A})$$

on U *is attained at* $V(\mathcal{A})$*.*

Proof. Let $\overline{x}_1$ be a maximal contact at p for R. Then, by Lemma 3.6.12, it is a partial maximal contact on an open subset U of p. Thus by Lemma 3.6.19(3),

$$\text{inv}^1(R) \leq \overline{b}_1 := a_1 \text{inv}(\overline{x}_1).$$

In addition, by Lemma 3.6.24, the maximum $\max \text{inv}^1(R) = \overline{b}_1$ of the invariant $\text{inv}^1_q(R)$, where $q \in X$, is attained at the closed subset $\text{suppinv}^1(R, \overline{b}_1) \subset H_1 = V(\overline{x}_1)$, and $\overline{x}_1$ is a maximal contact along $\text{suppinv}^1(R, \overline{b}_1)$. Furthermore for the points in $\text{suppinv}^1(R, \overline{b}_1)$, we have

$$\text{inv}_p(R) = (\text{inv}^1_p(R), \text{inv}_p(C_{\overline{x}_1 t^{1/a_1}}(R)_{|H_1})),$$

Moreover $\mathcal{A}_{|H_1}$ is maximal admissible for R at p. It suffices to use induction on dimension to find an open neighborhood U of p such that $\mathrm{inv}_p(C_{\overline{x}_1 t^{1/a_1}}(R)_{|H_1})$ attains its maximal on $V(\mathcal{A}_{|H_1}) = V(\mathcal{A}) \subset H$.

♣

Proposition 3.7.27. , *Let $R = \bigoplus R_a$ be a Rees algebra on a smooth variety X over a field K such that $R_a \neq \mathcal{O}_X$ for any $a \in A$. There exists a unique center $\mathcal{A}(R) = \mathcal{A}$ such that*

(1) *The maximum $\mathrm{maxinv}(R) = (\overline{b}_1, \ldots, \overline{b}_k)$ of the invariant $\mathrm{inv}_p(R)$, where $p \in X$, is attained at $V(\mathcal{A})$.*
(2) *$\mathcal{A} = \mathcal{O}_X[\overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]$ is a maximal admissible center for R, with $\mathrm{inv}(\mathcal{A}) = (\overline{b}_1, \ldots, \overline{b}_k)$, where $\overline{b}_i = a_i \mathrm{inv}(\overline{x}_i)$.*

Proof. Lemma 3.7.26 implies that $\mathrm{inv}_p(R)$ is upper semicontinuous, and admits finitely many values. Consider the closed point set

$$S := \mathrm{suppinv}(R, (\overline{b}_1, \ldots, \overline{b}_k)) := \{p \in X \mid \mathrm{inv}(R) = (\overline{b}_1, \ldots, \overline{b}_k)\},$$

where $\mathrm{inv}_p(R)$, where $p \in X$, attains its maximal value $\mathrm{maxinv}(R) = (\overline{b}_1, \ldots, \overline{b}_k)$. Then for any point $p \in S$ let $\mathcal{A}$ be a maximal admissible center for R at p. Then

$$V(\mathcal{A}) = \mathrm{suppinv}(R, (\overline{b}_1, \ldots, \overline{b}_k))$$

locally around p. Moreover, $\mathcal{A}$ is a unique maximal admissible center for R at all points of $V(\mathcal{A})$ in a neighborhood of p. Thus $\mathcal{A}$ glues to a unique maximal admissible center for R along $V(\mathcal{A}) = \mathrm{suppinv}(R, (\overline{b}_1, \ldots, \overline{b}_k))$ as desired.

♣

3.7.28. *Cobordant blow-ups of the centers with maximal invariant.*

Proposition 3.7.29. *(see also* [ATW19]*) Let $R = \bigoplus R_a$ be a Rees algebra on a smooth variety X over a field K such that $R_a \neq \mathcal{O}_X$ for any $a \in A$. Let $\mathcal{A} = \mathcal{A}(R)$ be maximal admissible center for R at $V(\mathcal{A})$, and $B \to X$ be a full cobordant blow-up of $\mathcal{A}^{\mathrm{ext}}$. Then in a neighborhood of $V(\mathcal{A})$ such that $\mathrm{inv}_p(R)$ attains its maximum $\mathrm{maxinv}(\sigma^c(R)) = (\overline{b}_1, \ldots, \overline{b}_k)$ at $V(\mathcal{A})$ we have*

(1) *The maximum $\mathrm{maxinv}(\sigma^c(R)) = (\overline{b}_1, \ldots, \overline{b}_k)$ is attained at $V(\sigma^c(A))$ with the invariant $\mathrm{inv}(\sigma^c(R))$, and such that*

$$\sigma^c(A^{\mathrm{ext}}) = \mathcal{O}_B[t^{-1/w}, x_1' t^{1/a_1}, \ldots, x_k' t^{1/a_k}]$$

is maximal admissible center for $\sigma^c(R)$
(2) *$\mathrm{maxinv}(\sigma^c(R)) < (\overline{b}_1, \ldots, \overline{b}_k)$ on $B_+ = B \smallsetminus V(\sigma^c(A))$.*

Proof. We shall use induction on dimension of X. If $\dim(X) = 0$ then $\mathcal{A} = R = \mathcal{O}_X = \mathcal{O}_X[0]$, $p = X$, and $\mathrm{inv}_p(R) = ()$ has no entries and thus corresponds to the infinite sequence of ∞. Let $p \in V(\mathcal{A})$, and set $\mathrm{ord}_p(\mathcal{A}) = a_1$.

Consider a neighborhood U of p such that $\overline{x}_1$ is a maximal contact on U at p, and a partial maximal contact on U. Then, by Lemma 3.6.24, inv^1 attains its maximum $\overline{b}_1$ on $\mathrm{suppinv}^1(R, \overline{b}_1) \subset V(\overline{x}_1)$. Moreover $\overline{x}_1$ is a maximal contact along $\mathrm{suppinv}^1(R, \overline{b}_1)$.

Consequently for the points $p \in \mathrm{suppinv}^1(R, \overline{b}_1)$ we have

$$\mathrm{inv}_p(R) = (\mathrm{inv}_p^1(R), \mathrm{inv}_p(C_{\overline{x}_1 t^{1/a_1}}(R)_{|H_1})).$$

By Lemma 3.7.20, the controlled transform $\sigma^c(\overline{x}_1) = \overline{x}'_1$ is a partial maximal contact on $B_U = \sigma^{-1}(U)$ for $\sigma^c(R)$. Then, by Lemma 3.6.24, $\mathrm{maxinv}^1(\sigma^c(R)) \leq \overline{b}_1$, and it attains its value $\overline{b}_1$ on

$$\mathrm{suppinv}^1(\sigma^c(R), \overline{b}_1) \subseteq H'_1 = V(\overline{x}'_1),$$

and for the points p' in $\mathrm{suppinv}^1(R, \overline{b}_1)$ we have that $\overline{x}'_1$ is a maximal contact. Moreover, by Lemma 3.7.22, and Section 3.7.5 we get

$$\mathrm{inv}_{p'}(\sigma^c(R)) = (\mathrm{inv}^1_{p'}(\sigma^c(R)), \mathrm{inv}_{p'}(C_{\overline{x}'_1 t^{1/a_1}}(\sigma^c(R))_{|H'_1}).$$

By Lemma 3.7.24, the restriction of $B \to X$ to $H'_1 = V_B(\overline{x}'_1)$ is the cobordant blow-up $\sigma_{H'_1} : H'_1 \to H_1$ of $\mathcal{A}^{\mathrm{ext}}_{|H_1}$. By the inductive assumption on dimension the conditions (1), (2) of the Proposition are satisfied on H_1 for $C_{\overline{x}_1 t^{1/a_1}}(R)_{|H_1}$ and for the cobordant blow-up $\sigma_{H'_1} : H'_1 \to H_1$. Moreover, by Lemma 3.5.11, and the assumption, $\mathcal{A}^{\mathrm{ext}}_{|H_1}$ is maximal admissible for $C_{\overline{x}_1 t^{1/a_1}}(R)_{|H_1}$ on H_1, associated with $\mathrm{maxinv}(C_{\overline{x}_1 t^{1/a_1}}(R)_{|H_1})$.

Consequently, by the inductive argument $\sigma^c(\mathcal{A}^{\mathrm{ext}})_{|H_1} = \mathcal{A}^{\mathrm{ext}}_{B|H'_1}$ is maximal admissible for

$$\sigma^c_{H'_1}(C_{\overline{x}_1 t^{1/a_1}}(R)_{|H_1}) = C_{\overline{x}'_1 t^{1/a_1}}(\sigma^c(R))_{|H'_1}.$$

By Lemma 3.5.11 and the fact that $\overline{x}'_1$ is a partial maximal contact associated with $\mathrm{maxinv}^1(\sigma^c(R)) = \overline{b}_1$ we conclude that $\mathcal{A}^{\mathrm{ext}}_B$ is maximal admissible for $\sigma^c(R)$. By the inductive assumption, the maximum

$$\mathrm{maxinv}(\sigma^c(C_{\overline{x}'_1 t^{1/a_1}}(R)_{|H_1})) = (\overline{b}_2, \ldots, \overline{b}_k)$$

on B is attained at $V(\sigma^c_{H_1}(\mathcal{A}_{|H_1})$. Thus

$$\mathrm{maxinv}(\sigma^c(R)_{|H'_1}) = \max(\overline{b}_1, \mathrm{inv}(C_{\overline{x}'_1 t^{1/a_1}}(R)_{|H_1}) = (\overline{b}_1, \ldots, \overline{b}_k)$$

is attained at

$$V(\sigma^c_{H_1}(\mathcal{A}_{|H_1}) = V(\sigma^c(\mathcal{A})_{|H'_1}) = V(\sigma^c(\mathcal{A})) \subseteq H'_1,$$

and $\sigma^c(\mathcal{A})$ is a maximal admissible center for $\sigma^c(R)$ associated with $\mathrm{maxinv}\sigma^c(R)$.

♣

3.7.30. *Resolution principle.* (see also [ATW19] for (1), (2) and the second part of (4)) Summarizing the above, the resolution process consists of the following:

Given a rational Rees algebra on a smooth X with SNC divisor E over a field K of characteristic zero.

(1) The invariant $\mathrm{inv}_p(R)$ on X is semicontinuous. (Section 3.7.6)
(2) It attains its maximum $\mathrm{maxinv}_X(R)$ at a certain unique center $\mathcal{A}^{\mathrm{ext}}(R)$ on X. (Proposition 3.7.27)
(3) The invariant $\mathrm{inv}_p(\sigma^c(R))$ on the full cobordant blow-up $B \to X$ at $\mathcal{A}^{\mathrm{ext}}(R)$, attains its maximum

$$\mathrm{maxinv}_X(R) = \mathrm{maxinv}_B(\sigma^c(R))$$

exactly at the center $\sigma^c(\mathcal{A}^{\mathrm{ext}}(R))$ associated with the vertex $\mathrm{Vert}(B) = V(\sigma^c(\mathcal{A}^{\mathrm{ext}}))$ of the full cobordant blow-up $B \to X$.(Proposition 3.7.29(1))
(4) The invariant $\mathrm{inv}_p(.)$ drops on the cobordant blow-up

$$B_+ = B \smallsetminus V(\sigma^c(\mathcal{A}^{\mathrm{ext}}(R))) = B \smallsetminus \mathrm{Vert}(B)$$

after removing the vertex $\mathrm{Vert}(B)$, so that

$$\mathrm{maxinv}_X(R) > \mathrm{maxinv}_{B_+}(\sigma^c(R)).$$

(Proposition 3.7.29(2))

3.8. Properties of the invariant.

3.8.1. *The invariant* inv *at the smooth points.* Assume that Y is a smooth subvariety of codimension k on a smooth variety X and is described at $p \in Y$ by a partial set of free local parameters $Y = V(u_1, \ldots, u_k)$ compatible with an SNC divisor E. Then

$$\mathcal{A} := \mathcal{O}_X[(u_1, \ldots, u_k)t]$$

is a maximal $\mathcal{I}_Y$- admissible center at p, with

$$\mathrm{inv}_p(\mathcal{I}) = (1, \ldots, 1),$$

with k entries equal 1. Conversely, if $\mathrm{inv}_p(\mathcal{I}_Y) = (1, \ldots, 1)$ is as above then there exists a partial system of free local parameters $u_1, \ldots, u_k \in \mathcal{I}$ compatible with E, such that

$$\mathcal{O}_X[\mathcal{I}_Y t] \subseteq \mathcal{O}_X[(u_1, \ldots, u_k)t]^{\mathrm{int}} = \mathcal{O}_X[(u_1, \ldots, u_k)t].$$

So $\mathcal{I}_Y = (u_1, \ldots, u_k)$ is smooth generated by free coordinates and compatible with E for Y having SNC with E at $p \in X$.

3.8.2. *Torus action.* If X admits a torus action and R is T-stable, then the maximal admissible centers are canonical and unique and thus T-stable. One can run the algorithm in Section 3.7.1 using seminvariant maximal contacts and seminvariant derivations, with all intermediate Rees algebras R_i being T-stable. Thus one can choose inductively semiinvariant coordinates of the maximal admissible centers. These are the nested maximal contacts in the T-stable cotangent ideals $T^{1/a_i}(R_i)$.

Moreover, if additionally X admits a geometric quotient X/T with all orbits of dimension $\dim(T)$, then the number of semiinvariant coordinates in a T-stable center at a given point cannot exceed $\dim(X/T) = \dim(X) - \dim(T)$, as it describes a smooth T-stable subvariety of X of dimension at least $\dim(T)$. Consequently the set of values of $\mathrm{inv}_p(\mathcal{I})$ in the resolution process

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots$$

is contained in $(\mathbb{Q}_+)^k_{\geq 0}$, where

$$k = \dim(X_0) = \ldots = \dim(X_i) - \dim(\mathbb{G}^i_m).$$

3.8.3. *The descending chain condition.*

Lemma 3.8.4. *The set $\Gamma \subset (\mathbb{Q}_+)^k_{\geq 0}$, of the possible values of $\mathrm{inv}_p(R)$ of the T-stable Rees algebras on a smooth varieties X with action of torus T, and having a geometric quotient X/T of dimension k, which are generated by the gradations $R_{a^i}t^{a^i}$, for $i = 1, \ldots, s$ satisfies dcc.*

Proof. Donote by $a := \mathrm{lcm}(a^1, \ldots, a^s))$. We use induction on k. For $k = 1$, the possible values of $\mathrm{inv}_p(R) = \mathrm{inv}^1_p(R)$ are contained in $(1/a) \cdot \mathbb{N}_+$.

In general, we note that $\mathcal{A}$ is R-admissible if and only if $R_a t^a \subset \mathcal{A}$. Thus we have

$$\mathrm{inv}_p(R) = 1/a \cdot (\mathrm{inv}_p(R_a),$$

and the problem reduces to the ideals $\mathcal{I} = R_a$ on X.

Now, for any ideal $\mathcal{I}$ on X we have

$$\mathrm{inv}_p(\mathcal{I}) = (\mathrm{inv}_p^1(\mathcal{I}), \mathrm{inv}_p(\sigma_{H_1}^c (\mathcal{C}_{\overline{x}_1 t^{1/a_1}} (\mathcal{O}_X[(\mathcal{I}t)]|_{H_1})),$$

where the restricted coefficient ideal

$$C := \mathcal{C}_{\overline{x}_1 t^{1/a_1}} \mathcal{O}_X[(\mathcal{I}t)]|_{H_1}) = \bigoplus C_{a/a_1} \cdot t^{a/a_1}$$

is generated at gradations

$$C_{1/a_1} t^{1/a_1}, \ldots, C_{(a_1-1)/a_1} t^{(a_1-1)/a_1}$$

on H_1 with $\dim(H_1/T) < k$. By the inductive assumption the set of values of $\mathrm{inv}_p(C)$ satisfies dcc. Now since the set of values $\mathrm{inv}_p^1(\mathcal{I})$ satisfies dcc condition we conclude that the set of values $\mathrm{inv}_p(\mathcal{I})$ on X satisfies dcc.

Consequently the set of values of $\mathrm{inv}_p(R)$ in a resolution sequence also satisfies dcc. ♣

3.8.5. *Functoriality of the invariant.* [ATW19]

Lemma 3.8.6. *The invariant* $\mathrm{inv}_p(R)$ *and the maximal admissible center* $\mathcal{A}$ *are functorial under smooth morphisms, field extensions, and group actions.*

The functoriality of $\mathrm{inv}_p(R)$ and the maximal admissible centers $\mathcal{A}$ at p is a consequence of the functoriality of the resolution algorithm in Section 3.7.1 and functoriality of derivations, Rees algebras, coefficient ideals, and the maximal contacts.

3.9. **Final Conclusions.**

3.9.1. *Functorial Principalization.* [ATW19] (in a non-SNC setting),
    Let $\mathcal{I}$ be an ideal on X. We initiate the SNC resolution algorithm using cobordant blow-ups $\sigma_i : X_{i+1} \to X_i$ at the maximal $\mathcal{I}_i$-admissible centers $\mathcal{A}(\mathcal{I}_i)$ on X_i, associated with the maximum $\mathrm{maxinv}_{X_i}(\mathcal{I}_i)$ of $\mathrm{inv}_p(\mathcal{I}_i)$ on X_i. They form a sequence

$$X = X_0 \overset{\sigma_0}{\leftarrow} X_1 \overset{\sigma_1}{\leftarrow} \ldots \overset{\sigma_{k-1}}{\leftarrow} X_k = X',$$

Here the ideal $\mathcal{I}_0 := \mathcal{I}$, and $\mathcal{I}_{i+1} := \sigma_i^c(\mathcal{I}_i)$ is the controlled transform of $\mathcal{I}_i$. Each such blow-up reduces $\mathrm{maxinv}_{X_i}(\mathcal{I}_i)$ so that

$$\mathrm{maxinv}_{X_i}(\mathcal{I}_i) > \mathrm{maxinv}_{X_{i+1}}(\mathcal{I}_{i+1}),$$

where X_{i+1} is the coborandant blow-ups X_i at the center $\mathcal{A}(\mathcal{I}_i)$. By dcc property this is a finite process which continues until the maximum value

$$\mathrm{maxinv}_{X_k}(\sigma^c(\mathcal{I}_k)) = 0.$$

In this situation, the controlled transform

$$\sigma^c(\mathcal{I}) := \sigma_{k-1}^c(\mathcal{I}_i) \circ \ldots \circ \sigma_1^c(\mathcal{I}) = \mathcal{O}_X$$

of $\mathcal{I}$ becomes $\mathcal{O}_X$, and the full transform $\mathcal{O}_{X'} \cdot \mathcal{I}$ is locally the product of divisorial components. Note that each cobordant blow-up $X_{i+1} \to X_i$ inductively creates a scheme X_{i+1} with an action of $T_{i+1} = T_i \times G_m$. Since all constructions are canonical and functorial for smooth morphisms, the centers, the ideals $\mathcal{I}_i$ and the exceptional divisors are automatically T_i-stable. This proves Theorem 1.11.2.

3.9.2. *Embedded Desingularization.* [ATW19] In the process of embedded resolution of an irreducible subvariety $Y \subset X$ of codimension k, we consider the principalization of $\mathcal{I} = \mathcal{I}_Y$ of Y on X. We execute the algorithm applying the cobordant blow-ups at maximal $\mathcal{I}$-admissible centers $\mathcal{A} = \mathcal{A}(\mathcal{I})$. In this process, we use the strict transforms of the ideal $\mathcal{I} = \mathcal{I}_Y$ instead of the controlled transforms. We stop the algorithm when the invariant

$$\mathrm{maxinv}_X(\mathcal{I}) = (1, \ldots, 1),$$

with k entries equal 1. In such a case, the associated maximal admissible center with the invariant $\mathrm{inv}(\mathcal{A}) = (1, \ldots, 1)$ for the ideal $\mathcal{I}_{Y'}$ of the strict transform Y' of Y coincides with $\mathcal{O}_X[\mathcal{I}_{Y'}t]$, and thus, by Section 3.8.1, the strict transform Y' of Y has SNC with the exceptional divisor, which is also SNC.

Alternatively one can use the controlled transforms of ideals instead of the strict transforms in the process without changing the strategy. The process is slower and some centers need not to be contained in the strict transform of Y.

Both procedures show the existence of the functorial embedded desingularization by the smooth cobordant blow-ups as in Theorem 1.11.4. As a result, we obtain a smooth subvariety Y' having SNC with an SNC exceptional divisor E' on X', and with the torus action.

Each consecutive cobordant blow-up $X_{i+1} \to X_i$ will create inductively a variety X_{i+1} with an action of $T_{i+1} = T_i \times G_m$. Since all the constructions are canonical and functorial for smooth morphisms, the centers and subvariety $Y_i \subset X_i$ are automatically T_i-stable. On the level of the geometric quotients, we obtain the sequence of weighted blow-ups

$$X = X_0 \overset{\sigma_0}{\leftarrow} X_1/T_1 \overset{\sigma_1/T_1}{\leftarrow} \ldots \overset{\sigma_{k-1}/T_{k-1}}{\leftarrow} X_k/T_k = X'/T,$$

of varieties with quotient singularities such that the subvariety Y'/T of X'/T admits quotient singularities. When considering stack-theoretic quotients $[X_i/T_i]$, we obtain smooth stacks with a smooth substack $[Y'/T] \subset [X'/T]$ having SNC with the exceptional divisors. A similar process without exceptional SNC divisor was considered in [ATW19].

3.9.3. *Nonembedded SNC Resolution.* The nonembedded resolution is a consequnce of the embedded resolution using local embeddings and the functoriality properties. It will have SNC exceptional divisor.

We associate with a variety Y over K, initially without any divisors, a modified invariant $\widetilde{\mathrm{inv}}_p(Y)$ in the following way. We locally embed Y into a smooth variety X. Any two such embeddings $Y \subset X_1$ and $Y \subset X_2$ into smooth varieties of the same dimension are étale equivalent. If $\dim(X_1) + m = \dim(X_2)$, and $m \geq 0$ then the induced embeddings $Y \subseteq X_1 \subseteq \mathbf{A}_{X_1}^m$ and $Y \subset X_2$ are 'etale equivalent. Here the embedding

$$X_1 \subseteq \mathbf{A}_{X_1}^m = \mathrm{Spec}(\mathcal{O}_X[x_1, \ldots, x_m])$$

is defined by $V(x_1, \ldots, x_m)$.

For any local embeddings $Y \subset X_1 = V(x_1, \ldots, x_m) \subset \mathbf{A}_{X_1}^m$ and $Y \subset X_2$ with $\dim(X_1) + m = \dim(X_2)$ we get that $(x_1, \ldots, x_m)$ is partial maximal contact in gradation t. Then passing to the completion $\widehat{\mathcal{O}}_{X,p}$ the maximal admissibility condition

$$\widehat{\mathcal{O}}_{X,p} \cdot R_Y = \widehat{\mathcal{O}}_{X,p}[(x_1, \ldots, x_m)t, R_{Y|X_1}] \subset \widehat{\mathcal{O}}_{X,p} \cdot \mathcal{A} := \widehat{\mathcal{O}}_{X,p}[\overline{x}_1 t, \ldots \overline{x}_k t^{1/a_k}],$$

with $(x_1, \ldots, x_m) \subset \overline{x}_1$ is equivalent to the maximal admissibility $R_{Y|X_1} \subset \mathcal{A}_{Y|X_1}$, and thus

$$\mathrm{inv}_p(R_{Y,\mathbf{A}_{X_1}^m}) = (1, \ldots, 1, \mathrm{inv}_p(R_{Y|X_1}) =$$
$$= (1, \ldots, 1, \mathrm{inv}_p(\mathcal{I}_{Y,X_1})).$$

Moreover, by functoriality, we have

$$\mathrm{inv}_p(\mathcal{I}_{Y,X_2}) = \mathrm{inv}_p(\mathcal{I}_{Y,\mathbf{A}_{X_1}^m}) = (1, \ldots, 1, \mathrm{inv}_p(\mathcal{I}_{Y,X_1})),$$

For a closed embedding $Y \subset X$ into a smooth X of dimension n, and $p \in Y$, let $\mathrm{inv}_p(\mathcal{I}_Y) = \mathrm{inv}(b_1, \ldots, b_k)$. We define the invariant $\widetilde{\mathrm{inv}}_p(Y)$ on Y to be the equivalence class of the sequences $(b_1, \ldots, b_k)_n$, marked by $n = \dim(X)$, where $Y \subset X$ and $\mathrm{inv}_p(\mathcal{I}_Y) = (b_1, \ldots, b_k)$, with the equivalence relation

$$(b_1, \ldots, b_k)_n \simeq (1, \ldots, 1, b_1, \ldots, b_k)_{n+m},$$

where m is the number of ones in front of b_1. By the above the invariant is independent of embeddings and functorial for smooth morphisms.

One can compare two equivalence classes lexicographically, as before, by fixing their representatives with the same marking n.

We shall use the following

Lemma 3.9.4. *Let $Y \subset X_1$, and $Y \subset X_2$ be two different embeddings. For a point $p \in Y$ consider two maximal admissible centers at p: $\mathcal{A}_1$ for $\mathcal{I}_{Y,X_1}$ on X_1 and $\mathcal{A}_2$ for $\mathcal{I}_{Y,X_1}$. Denote by $B_{1+} \to X_1$, and $B_{2+} \to X_2$ the cobordant blow-ups respectively at $\mathcal{A}_1$ and $\mathcal{A}_2$, Then*

$$\mathcal{A}_{1|Y} = \mathcal{A}_{2|Y},$$

and consequently the restrictions of the cobordant blow-ups $B_{1+} \to X_1$, and $B_{2+} \to X_2$ to the strict transforms of Y coincide with the cobordant blow-up $B_{Y+} \to Y$ of $\mathcal{A}_{1|Y} = \mathcal{A}_{2|Y}$ on Y.

Proof. We can embed further X_1 and X_2 into an affine space $\mathbb{A}^N$ in such a way that the induced embeddings of Y to $\mathbb{A}^N$ coincide (see for intance [Wło05, Lemma 2.5.3]). This reduces the situation to the embedding $X_1 \subset X_2 = \mathbb{A}^N$.

Now $\mathcal{I}_{Y,X_1} = \mathcal{I}_{Y,X_2|X_1}$, and similarly $\mathcal{A}_1 = \mathcal{A}_{2|X_1}$ implying $\mathcal{A}_{1|Y} = \mathcal{A}_{2|Y}$ on Y. Consequently the morphism on the strict transforms $\sigma^c(Y) \to Y$ on X_1 or X_2 coincides, by Lemma 5.1.5, with the cobordant blow-up of $\mathcal{A}_{1|Y} = \mathcal{A}_{2|Y}$.
$$\clubsuit$$

We shall run the embedded algorithm using the centers associated with $\max \widetilde{\mathrm{inv}}$ on Y. It locally corresponds to the center $\mathcal{A}^{\mathrm{ext}}$ associated with $\mathrm{maxinv}(\mathcal{I}_Y)$ for a certain embedding $Y \subset X$.

Consider an open affine cover of Y by Y^j admitting closed embeddings $Y^j \subset X^j$ into smooth varieties X^j. Let $\mathcal{I}_{Y^j}$ be the corresponding ideal on X^j. We can assume that all X^j are of the same dimension. Let

$$\overline{Y} := \coprod Y_j \hookrightarrow \overline{X} := \coprod X_j$$

be the induced embedding of disjoint unions. Consider the natural étale projection $\overline{Y} \to Y$.

Now we shall run the embedded desingularization with the strict transforms for $\overline{Y} \hookrightarrow \overline{X}$, that the sequence of maximal $\mathcal{I}_{\overline{Y}_i}$-admissible centers $\overline{\mathcal{A}}_i$:

$$\overline{X} = \overline{X}_0 \overset{\sigma_0}{\leftarrow} \overline{X}_1 \overset{\sigma_1}{\leftarrow} \ldots \overset{\sigma_{k-1}}{\leftarrow} \overline{X}_k = \overline{X}',$$

with ideals $\mathcal{I}_{\overline{Y}_i}$ being the strict transform of $\mathcal{I}_{\overline{Y}_{i-1}}$, which induces the sequence of the strict transforms

$$\overline{Y} = \overline{Y}_0 \overset{\sigma_{1\overline{Y}}}{\leftarrow} \overline{Y}_1 \overset{\sigma_{1\overline{Y}}}{\leftarrow} \ldots \overset{\sigma_{k-1\overline{Y}}}{\leftarrow} \overline{Y}_k = \overline{Y}',$$

The latter is the sequence of the cobordant blow-ups with the induced centers $\overline{\mathcal{A}}_{i|\overline{Y}_1}$. By functoriality and Lemma 3.9.4, it descends to the sequence of the induced cobordant blow-ups

$$Y = Y_0 \overset{\sigma_{0Y}}{\leftarrow} Y_1 \overset{\sigma_{1Y}}{\leftarrow} \ldots \overset{\sigma_{k-1Y}}{\leftarrow} Y_k = Y',$$

on Y. Then, $\max \widetilde{\mathrm{inv}}$ drops to the minimal value $(1, 1, \ldots, 1)_n$ on $\overline{Y}'$ and on Y', where n is the dimension of the ambient variety $\overline{X}$, and the number of ones is equal to the codimension of the smooth subvariety $\overline{Y} \subset \overline{X}$.

The exceptional SNC divisor on a local ambient space $\overline{X}'$ is compatible with a smooth subvariety $\overline{Y}'$. Consequently, its restriction to $\overline{Y}'$ defines an SNC-exceptional divisor on $\overline{Y}'$ and determines a functorial nonembedded SNC resolution of $\overline{Y}$, while its descent determines a functorial nonembedded SNC resolution of Y. This completes the proof of Theorem 1.11.7. A similar non-SNC resolution was considered in [ATW19] ,

4. Free characteristic resolution of almost homogenous singularities

4.1. The weighted normal bundles to the centers.

4.1.1. *Weighted normal bundle.* Let X be a regular scheme.

Let $\mathcal{J}$ be a $\mathbb{Q}$-ideal locally written as $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$, and let $\mathcal{A}_{\mathcal{J}} = \mathcal{O}_X[t^{-1}, u_1 t^{w_1}, \ldots, u_k t^{w_k}]^{\mathrm{int}}$ be the associated Rees algebra. Once can consider the filtration

$$\{(\mathcal{J}^a)_X\}_{a \in \mathbb{Z}_{\geq 0}} = \{\mathcal{A}_{\mathcal{J}, a}\}_{a \in \mathbb{Z}_{\geq 0}}$$

of $\mathcal{O}_X$. Then $\mathrm{gr}_{\mathcal{J}}(\mathcal{O}_X)$ is a sheaf of graded $\mathcal{O}_X/\mathcal{J}_X = \mathcal{O}_{V(\mathcal{J})}$-modules on $V(\mathcal{J})$, which is locally on X, isomorphic to the graded ring

$$\begin{aligned}
\mathrm{gr}_{\mathcal{J}}(\mathcal{O}_X) &= \mathcal{O}_X/\mathcal{J}_X \oplus (\mathcal{J}_X/\mathcal{J}_X^2)t \oplus \ldots \oplus (\mathcal{J}_X^i/\mathcal{J}^{i+1})_X t^i \oplus \ldots = \\
&= (\mathcal{O}_X \oplus (\mathcal{J}_X)t \oplus \ldots \oplus (\mathcal{J}_X^i/t^i \oplus \ldots) / (\mathcal{J}_X \oplus \ldots \oplus (\mathcal{J}_X^i)t^{i-1} \oplus \ldots = \\
&= \mathcal{A}_{\mathcal{J}}[t^{-1}]/(t^{-1}\mathcal{A}_{\mathcal{J}}[t^{-1}]) = \mathcal{A}_{\mathcal{J}}^{\mathrm{ext}}/(t^{-1}\mathcal{A}_{\mathcal{J}}^{\mathrm{ext}}) = \mathcal{O}_B/(t^{-1} \cdot \mathcal{O}_B) = \\
&= \mathcal{O}_X[t^{-1}, u_1 t^{w_1}, \ldots, u_k t^{w_k}]/(t^{-1}) = \mathcal{O}_{V(\mathcal{J})}[u_1 t^{w_1}, \ldots, u_k t^{w_k}],
\end{aligned}$$

where u_i is in w_i gradation. We shall call the corresponding scheme the *weighted normal bundle* at the center $\mathcal{J}$, and denote it by

$$N_{\mathcal{J}}(X) := \mathrm{Spec}(\mathrm{gr}_{\mathcal{J}}(\mathcal{O}_X)).$$

Remark 4.1.2. A similar idea was considered independently in [QR22].

4.1.3. *The ideal of the initial forms.* With any function $f \in \mathcal{O}_{X,p}$, regular at $p \in V(\mathcal{J})$, such that $f \in ((\mathcal{J}^a)_X \smallsetminus (\mathcal{J}^{a+1})_X)$, for a certain $a \in \mathbb{N}$ one can associate the unique homogenous element- the initial form

$$\mathrm{in}(f) \in ((\mathcal{J}^a)_X/(\mathcal{J}^{a+1})_X)t^a \subset \mathrm{gr}_{\mathcal{J}}(\mathcal{O}_X).$$

Similarly, we associate with an ideal sheaf $\mathcal{I}$, the filtration $\mathcal{I}_a := \mathcal{I} \cap (\mathcal{J}^a)_X$, where $a \in \mathbb{N}$, and its ideal of the initial forms on $N_{\mathcal{J}}(X)$:

$$\mathrm{in}(\mathcal{I}) = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} (\mathcal{I}_a/(\mathcal{I}_{a+1}))t^a = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} (\mathcal{I}_a + (\mathcal{J}^{a+1})_X)/(\mathcal{J}^{a+1})_X t^a \subset \mathrm{gr}_{\mathcal{J}}(\mathcal{O}_X).$$

4.1.4. *The weighted normal bundle and the exceptional divisor.* The computations above imply the following extension of a classical result of Huneke-Swanson on extended Rees algebras and smooth blow-ups [HS06, Definition 5.1.5].

Lemma 4.1.5. *Let X be a regular scheme and $\sigma : B \to X$ be the full cobordant blow-up of the center $\mathcal{J}$. Then the exceptional divisor $V_B(t^{-1})$ is isomorphic to $N_{\mathcal{J}}(X)$:*

$$V_B(t^{-1}) = \mathrm{Spec}_X(\mathcal{O}_B/(t^{-1})) \simeq N_{\mathcal{J}}(X)$$

♣

4.1.6. *The strict transform and the ideal of the initial form.* We will need the above identification in the context of the strict transforms of the ideals.

Lemma 4.1.7. *With the previous assumptions and notation. Let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf on X. Let $\sigma^s(\mathcal{I}) \subset \mathcal{O}_B$ be the strict transform of $\mathcal{I}$. Then the natural isomorphism $\mathcal{O}_B/(t^{-1}) \to \mathrm{gr}_{\mathcal{J}}(\mathcal{O}_X)$ takes $\sigma^s(\mathcal{I})_{|V(t^{-1})}$ to $\mathrm{in}(\mathcal{I})$.*

Proof. Let $f \in \mathcal{I}$ such that $f \in (\mathcal{J}^a)_X \smallsetminus (\mathcal{J}^{a+1})_X$. Then the strict transform $\sigma^s(f) = t^a f \in (\mathcal{J}^a)_X t^a \subset \mathcal{O}_B$, and its reduction modulo t^{-1} is in $\mathcal{O}_B/(t^{-1} \cdot \mathcal{O}_B) = \mathrm{gr}_{\mathcal{J}}(\mathcal{O}_X)$ in the gradation

$$(\mathcal{J}^a)_X t^a/(t^{-1}\mathcal{O}_B \cap \mathcal{O}_X t^a) = ((\mathcal{J}^a)_X/(\mathcal{J}^{a+1})_X) \cdot t^a,$$

and naturally and bijectively corresponds to $\mathrm{in}(f) \in (\mathcal{J}^a)_X/(\mathcal{J}^{a+1})_X t^a \subset \mathrm{gr}_{\mathcal{J}}(\mathcal{O}_X)$.

♣

4.1.8. *Weighted normal cone.* One can extend the construction of the weighted normal bundle of a center $\mathcal{J}$ on a regular scheme to the weighted normal cone at any Rees algebra $R = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} R_a t^a \subset \mathcal{O}_X[t]$ on any noetherian scheme X. Consider the associated gradation

$$\mathrm{gr}_R(\mathcal{O}_X) = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} ((R_a)_X/(R_{a+1})_X)t^a = R/(R \cap t^{-1}R).$$

Then by *the weighted normal cone of X at R* we shall mean

$$C_R(X) := \mathrm{Spec}_{V(R)}(\mathrm{gr}_R(\mathcal{O}_X))$$

Definition 4.1.9. Let X be a regular scheme, and $Y \subset X$ be an integral, closed subscheme with an ideal sheaf $\mathcal{I}_Y$. Let $\mathcal{J}$ be a center on X with $V(\mathcal{J}) \subset Y$. Denote by

$$R_{\mathcal{J},Y} := \mathcal{O}_Y \cdot \mathcal{A}_{\mathcal{J}} = \mathcal{O}_Y(\mathcal{O}_X[\mathcal{J}t]_X)$$

the induced Rees algebra on Y. Then by the *weighted normal cone of Y at $V(\mathcal{J})$* we mean

$$C_{\mathcal{J}}(Y) := \mathrm{Spec}_{V(\mathcal{J})}(C_{R_{\mathcal{J},Y}}(Y)) = \mathrm{Spec}_{V(R)}(\mathrm{gr}_{R_{\mathcal{J},Y}}(\mathcal{O}_Y)).$$

Lemma 4.1.10. *With the above notation and assumptions*

$$C_{\mathcal{J}}(Y) = \operatorname{Spec}_{V(R)}(\operatorname{gr}_{R_{\mathcal{J},Y}}(\mathcal{O}_Y))$$

is the closed subscheme of the weighted normal bundle $\mathcal{N}_{\mathcal{J}}(X)$ which is defined by $\operatorname{in}(\mathcal{I}_Y) \subset \operatorname{gr}_{\mathcal{J}}(\mathcal{O}_X)$.

Proof. Consider the surjective morphism of sheaves $\phi : \mathcal{A}_{\mathcal{J}} \to R_{\mathcal{J},Y}$. Its kernel is generated by $(\mathcal{I}_Y \cdot \mathcal{O}_X[t]) \cap \mathcal{A}_{\mathcal{J}}$. So

$$R_{\mathcal{J},Y} \simeq \bigoplus \frac{(\mathcal{J}^a)_X}{\mathcal{I}_Y \cap (\mathcal{J}^a)_X} \cdot t^a.$$

The morphism ϕ defines the surjective morphism

$$\operatorname{gr}_{\mathcal{J}}(\mathcal{O}_X) = \bigoplus (\mathcal{J}^a)_X / (\mathcal{J}^{a+1})_X t^a \to \operatorname{gr}_R(\mathcal{O}_Y) = \bigoplus \frac{(\mathcal{J}^a)_X}{(\mathcal{J}^{a+1})_X + (\mathcal{I}_Y \cap (\mathcal{J}^a)_X)} \cdot t^a.$$

Its kernel is exactly

$$\operatorname{in}(\mathcal{I}_Y) = \bigoplus \frac{((\mathcal{I}_Y \cap (\mathcal{J}^a)_X) + (\mathcal{J}^{a+1})_X}{(\mathcal{J}^{a+1})_X} \cdot t^a.$$

♣

Remark 4.1.11. Assume that X is a variety over an algebraically closed field K and B is the cobordant blow-up of a center $\mathcal{J}$. It follows from Lemmas 4.1.5, 4.1.7, that when considering the natural projection $\pi : B \to \mathbb{A}^1 = \operatorname{Spec}(K[t^{-1}]$ we obtain the *deformation of the $X = \pi^{-1}(a)$, where $a \neq 0$ to the the normal weighted cone $N_{\mathcal{J}}(X) = \pi^{-1}(0) = V_B(t^{-1})$. Its restriction to $\pi_C : C := V(\sigma^s(\mathcal{I})) \to \mathbb{A}^1$ determines the *deformation of Y to the normal weighted cone $C_{\mathcal{J}}(Y)$*, with $Y = \pi_C^{-1}(a)$ for $a \neq 0$ and $C_{\mathcal{J}}(Y) = \pi_C^{-1}(0) = V_C(t^{-1})$. This observation was made by Quek and Rydh in [QR22].

4.2. **Almost homogenous singularities and their resolution.** For any ideal $\mathcal{I}$ on a regular scheme X let

$$\operatorname{Sing}(V(\mathcal{I})) := \operatorname{Sing}(\operatorname{Spec}_X(\mathcal{O}_X / \mathcal{I}_Y))$$

denote the singular locus of the scheme $\operatorname{Spec}_X(\mathcal{O}_X / \mathcal{I}_Y)$.

Definition 4.2.1. Let X be a regular scheme. Let $Y \subset X$ be an integral closed subscheme with an ideal sheaf $\mathcal{I}_Y$. We say that a regular subscheme $Z \subset Y$ is an *almost homogenous singularity* of Y if

(1) The singular locus of Y is $\operatorname{Sing}(Y) = Z$.
(2) There is a center $\mathcal{J}$ on X, such that $V(\mathcal{J}) = Z$, and

$$\operatorname{Sing}(C_{\mathcal{J}}(Y)) = V(\operatorname{in}(\mathcal{J})) = Z \subseteq N_{\mathcal{J}}(X).$$

Theorem 4.2.2. *Let X be a regular scheme. Let $Y \subset X$ be an integral closed subscheme. Let $Z \subset Y$ be an almost homogenous singularity of Y for a center $\mathcal{J}$. Let $N_{\mathcal{J}}(X)$ be the weighted normal bundle of X at $\mathcal{J}$, and $C_{\mathcal{J}}(Y) \subset N_{\mathcal{J}}(X)$ be the weighted normal cone of Y at $\mathcal{J}$. Assume that either X is universally catenary or the codimension of each component of $C_{\mathcal{J}}(Y) \smallsetminus Z$ in $N_{\mathcal{J}}(X)$ is equal to the codimension of Y in X.*

Then the cobordant blow-up $B_+ \to X$ defines a cobordant resolution of Y. That is, the strict transform Y' of Y is a regular subscheme of B_+ of the codimension equal to the codimension of Y in X.

Proof. The problem is local on X. Thus we can assume that $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$ and the full cobordant blow-up of $\mathcal{J}$ is given by

$$\sigma : B = \mathrm{Spec}(\mathcal{O}_X[t^{-1}, t_1^w u_1, \ldots, t^{w_k} u_k] \to X.$$

Assume that X is universally catenary, and thus B is catenary. Let d be the codimension of Y in X. Then for the morphism $\sigma_- : B_- = X \times \mathbb{G}_m \to X$, the inverse image $\sigma_-^{-1}(Y)$ is irreducible of codimension d. So it is its closure $Y' := \overline{\sigma^{-1}(Y)}$, which is the strict transform of Y:

$$\mathrm{codim}_B(Y') = d$$

Note that by the definition of the strict transform, t^{-1} is not a zero divisor of $\mathcal{O}_B/\mathcal{I}_{Y'}$. Then, by the Krull Hauptidealsatz, we have that each component of $Y' \cap V(t^{-1})$ is of codimension 1 in Y', and the codimension $d + 1$ in B. We conclude that each component of $Y' \cap V(t^{-1})$ is of codimension d in $V(t^{-1})$.

Note that $\mathrm{Sing}(Y') \smallsetminus V(t^{-1}) = \mathrm{Sing}(Y') \cap B_-$ is contained in

$$V_{B_-}(u_1, \ldots, u_k) = V_{B_-}(u_1 t^{w_1}, \ldots, u_k t^{w_k}) = V_{B_-}(u_1', \ldots, u_k'),$$

where $u_i' = u_i t^{w_i}$. On the other hand, by the assumption,

$$\mathrm{Sing}(\mathrm{in}(C_{\mathcal{J}}(Y)) \subset V(u_1 t^{w_1}, \ldots, u_n t^{w_n})$$

so via isomorphism from and Lemma 4.1.7 we obtain

$$\mathrm{Sing}(Y' \cap V_B(t^{-1})) \subset V_B(u_1', \ldots, u_n') \cap V_B(t^{-1}).$$

Then, by the assumption on codimension for any point $p \in (Y' \cap V(t^{-1})) \smallsetminus V(u_1', \ldots, u_n')$, we can find parameters $v_1, \ldots v_d \in (\mathcal{O}_B/t^{-1}) \cdot \mathcal{I}_{Y'}$ at p which vanish on $Y' \cap V(t^{-1})$. But these parameters come from local parameters in $\mathcal{I}_{Y'}$ on B at p. So they define a regular subscheme Y'' of B of codimension d, containing locally Y'. Thus Y'' locally coincides with Y' which must be regular at $p \in V(t^{-1}) \smallsetminus V_B(u_1', \ldots, u_n')$. Consequently $\mathrm{Sing}(Y')$ is contained in $V_B(u_1', \ldots, u_n')$, and Y' is a regular subscheme of $B_+ = B \smallsetminus V_B(u_1', \ldots, u_n')$ of codimension d.

♣

As a corollary from Theorem 4.2.2, we obtain the following:

Theorem 4.2.3. *Let X be a smooth scheme over a field K of any characteristic. Let $Y \subset X$ be a closed integral subscheme of X admitting an almost homogenous singularity $Z \subset Y$. There is a resolution of Y at Z, that is, a projective birational morphism $\phi : Y^{res} \to Y$ from a smooth variety Y^{res} with the exceptional locus $Z \subset Y$, such that $\phi^{-1}(Z)$ is an SNC divisor on Y'.*

Proof. Take the cobordant resolution $B_+ \to X$ from Theorem 4.2.2. By Corollary 5.2.9, the geometric quotient $B_+/\mathbb{G}_m$ is locally toric, and thus it can be canonically resolved by the combinatorial method of [Wło20, Theorem 7.17.1]. This produces the projective birational resolution $Y' \to Y$ of Y such that the inverse image of the singular point is an SNC divisor.

♣

A typical situation where the theorem can be applied is an isolated singularity.

Example 4.2.4. In Example 3.2.2 with the singularity

$$f = x_1^{c_1} + \ldots + x_n^{c_n}$$

we can assume that the characteristic of the field is nonzero and the singular locus is the origin $V(x_1, \ldots, x_n)$. Then the the cobordant blow-up of $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_n^{1/w_n})$ with $w_1 c_1 = \ldots = w_n c_n$ resolves the singularity. The same is still valid if

$$\mathrm{in}_{\mathcal{J}}(f) = x_1^{c_1} + \ldots + x_n^{c_n},$$

and we assume that both $\mathrm{Sing}(V(\mathrm{in}_{\mathcal{J}}(f)))$ and $\mathrm{Sing}(V(f))$ is the origin.

Example 4.2.5. Consider the mixed characteristic hypersurface Y in $X = \mathbb{A}_{\mathbb{Z}}^2 \smallsetminus V(k)$, where $k \in \mathbb{Z}$, defined by

$$f = x^p + p^p + y^k \in \mathbb{Z}[1/k][x, y],$$

with $p \nmid k$. Taking the derivative of the ideal (f) over $\mathbb{Z}[1/k]$ we get

$$\mathcal{D}(f) = (x^p + p^p + y^k, px^{p-1}, ky^{k-1})$$

we see that $\mathrm{Sing}(f) \subseteq V(x^p + p^p + y^k, px^{p-1}, ky^{k-1}) = V(x, y, p)$. Thus $Y := V(f)$ has an isolated singularity $V(x, y, p)$. After the coordinate change $x' := x + p$ we get the equation:

$$f = x'^p - p \cdot px'^{p-1} + \ldots + p \cdot p^{p-1}x' + y^k$$

Consider the center

$$\mathcal{J} := ((x')^{1/w_1}, p^{1/w_2}, y^{1/w_3})$$

with the integral weights satisfying

$$pw_1 = pw_2 + w_1 = kw_3.$$

We obtain the initial form

$$\mathrm{in}\, f = x'^p + p^p x' + y^k \in \mathbb{Z}_p[x't^{w_1}, pt^{w_2}, zt^{w_3}].$$

Here x', p, y correspond to the graded elements $x't^{w_1}, pt^{w_2}, zt^{w_3}$. Taking derivatives over $\mathbb{Z}_p$ with respect to the free variables x', p, y on $Z_p[x', p, z]$ we obtain

$$\mathcal{D}(\mathrm{in}\, f) = (x'^p + p^p x' + y^k, p^p, ky^{k-1}).$$

This implies that

$$\mathrm{Sing}(\mathrm{in}(f)) \subset V(x'^p + p^p x' + y^k, p^p, ky^{k-1}) = V(x', y, p) = V(\mathcal{J}).$$

Thus f defines an almost homogenous singularity for $\mathcal{J}$ and can be resolved by a single cobordant blow-up of $\mathcal{J}$.

Observe that the seemingly natural center $\mathcal{J}_1 = (x^{1/k}, p^{1/k}, y^{1/p})$ does not define an almost homogenous singularity (for $\mathcal{J}_1$), and the corresponding cobordant blow-up does not resolve singularity. In such a case

$$\mathrm{in}(f) = x^p + p^p + y^k = (x + p)^p + z^k \in \mathbb{Z}_p[x, y, p],$$

whence $\mathrm{Sing}(\mathrm{in}(f)) = V(x + p, z) \supsetneq V(\mathcal{J}_1)$. This also motivates the coordinate change above.

4.3. **Partial resolution by the invariants.** The method can be linked to different invariants and is related to the characteristic zero method.

For any ideal $\mathcal{I}$ on a regular scheme X, and an integer d let

$$\mathrm{supp}(\mathcal{I}, d) := \{p \in X \mid \mathrm{ord}_p(\mathcal{I}) \leq d\}.$$

Theorem 4.3.1. *Let $\mathcal{I}$ be an ideal on a regular scheme X. Let d be any natural integer. Assume that there exists a center $\mathcal{J}$ such that*

(1) $\mathrm{supp}(\mathcal{I}, d) = V(\mathcal{J}) \subset X$, *and*
(2) $\mathrm{supp}(\mathrm{in}(\mathcal{I}), d) = V(\mathcal{J}) \subset C_J(X)$.

Then for the cobordant blow-up $\sigma_+ : B_+ \to X$ of $\mathcal{J}$, we have:

$$\mathrm{ord}_{B_+}(\sigma^s(\mathcal{I})) < d$$

Proof. Let $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ be the center. If $q \in V(t^{-1}) \smallsetminus V(x_1', \ldots, x_k')$, then

$$\mathrm{ord}_q(\sigma^s(\mathcal{I}) \leq \mathrm{ord}_q(\sigma^s(\mathcal{I})_{|V(t^{-1})}) = \mathrm{ord}_q(\mathrm{in}(\mathcal{I})) < d.$$

If $q \in B \smallsetminus V(t^{-1}) \smallsetminus V(x_1', \ldots, x_k') = B_- \smallsetminus V(x_1', \ldots, x_k') = (X \smallsetminus V(\mathcal{J})) \times \mathbb{G}_m$, then since $\sigma(q) \in X \smallsetminus V(\mathcal{J})$ we conclude that

$$\mathrm{ord}_q(\sigma^s(\mathcal{I})) = \mathrm{ord}_{\sigma(q)}(\mathcal{I}) < d.$$

♣

4.4. **Resolution principle in any characteristic.** In practice, one can replace "ord" with a different type of invariants satisfying the properties used in the proof of the above lemma, like some modifications of the Hilbert-Samuel function or inv and inv_E for smooth schemes in characteristic zero or certain schemes in positive or mixed characteristic. One can link it to a more general principle:

Let $\sigma : B \to X$ be a full cobordant blow-up of a center $\mathcal{J}$. Let $Y \subset X$ be a closed subscheme. Let $D = V_B(t^{-1})$ be the exceptional divisor of π.

- The behavior of the strict transform $\sigma^s(Y)$ at the part of the vertex $\mathrm{Vert}(B) \smallsetminus D$ of B is controlled by the behavior of Y at $V(\mathcal{J})$.
- The behavior of the strict transform $\sigma^s(Y)$ at the part of the vertex $\mathrm{Vert}(B) \cap D$ is to a great extent controlled by the weighted normal cone $C_{\mathcal{J}}(Y)$ to the center $V(\mathcal{J})$.

Example 4.4.1. Let k be a field of characteristic $p > 2$. Let $Y \subset X = \mathrm{Spec}\, k[x, y, z]$ be a closed subscheme defined by

$$f = x^p + y^p z + z^k + x^{p-1} y^2 + x^{p+1} yz,$$

of order $p > 2$ with $p \nmid k(k-1)$, $k \geq 2p + 1$. Computing $\mathcal{D}^2(f)$ we obtain that

$$\mathrm{supp}(f, 3) = V(\mathcal{D}^2(f)) = V(x, y, z).$$

Then by solving the equations for the weights

$$p w_1 = p w_2 + w_3 = k w_3$$

we obtain that the initial form with respect to $\mathcal{J} = (x^{1/w_1}, y^{1/w_3}, z^{1/w_3})$ is given by

$$\mathrm{in}(f) = x^p + y^p z + z^k.$$

Then by the similar argument as before

$$\mathrm{supp}(\mathrm{in}(f), 3) = V(x, y, z) = V(\mathcal{J}) \subset C_{\mathcal{J}}(X)$$

The cobordant blow-up at $\mathcal{J}$ decreases the maximal order to 2, and the equation can be solved by further cobordant blow-ups.

Observe that if one considers "characteristic zero" center $\mathcal{J}_1 := (x^{\frac{1}{p+1}}, y^{\frac{1}{p}}, z^{\frac{1}{p}})$ associated with maximal admissible Rees center $\mathcal{O}_X[xt^{1/p}, yt^{1/(p+1)}, z^{1/(p+1)})]$ then $\mathrm{in}(f) = x^p + y^p z$, and

$$\mathrm{supp}(\mathrm{in}(f), 3) = V(x, y) \neq V(\mathcal{J}_1)$$

In fact the full cobordant blow-up $B'_+ \to X$ of $\mathcal{J}_1$ transforms f into

$$x^p + y^p z + t^{(p+1)-k} z^k + t^{1-p} x^{p-1} y^2 + t^{-3p-1} x^{p+1} yz$$

On $B_+ = B \smallsetminus V(x, y, z)$ in the chart $B \smallsetminus V(z)$, the maximal order does not decrease:

$$\mathrm{maxord}_{B_+}(\sigma^s(f)) = p.$$

4.5. **Homogenous subschemes.** Let V be a regular scheme and

$$X := \mathbb{A}^n_V = \mathrm{Spec}_V(\mathcal{O}_V[x_1, \ldots, x_n]))$$

be an affine space over V. Let $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ be a center on X. The center $\mathcal{J}$ defines the gradation on $\mathcal{O}_V[x_1, \ldots, x_n]$ via the isomorphism $x_i \mapsto t^{w_i} x_i$ for $i \leq k$, $x_i \mapsto x_i$:

$$\phi : \mathcal{O}_V[x_1, \ldots, x_n] \to \mathcal{O}_V[t^{w_1} x_1, \ldots, t^{w_k} x_k, x_{k+1}, \ldots, x_n]$$

It takes the monomial $x^\alpha = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \in \mathcal{J}_X^{a_1 w_1 + \ldots + a_k w_k}$ to $t^{a_1 w_1 + \ldots + a_k w_k} \cdot x^\alpha$.

Let $f \in \mathcal{J}_X^a \smallsetminus \mathcal{J}_X^{a+1}$ be a homogenous element in $\mathcal{O}_X = \mathcal{O}_V[x_1, \ldots, x_n]$ of grading a. Then

$$\phi(f) = t^a f \in \mathcal{O}_V[t^{w_1} x_1, \ldots, t^{w_k} x_k, x_{k+1}, \ldots, x_n].$$

The latter can be interpreted as the strict transform of f inside of the cobordant algebra $\mathcal{O}_B = \mathcal{O}_X[t^{-1}, t^{w_1} x_1, \ldots, t^{w_k} x_k]$.

Definition 4.5.1. With the above notation and assumption any subscheme of X defined by a homogenous ideal $\mathcal{I}$ will be called a *homogenous subscheme* with respect to the center $\mathcal{J}$.

This leads to a simple but useful result.

Lemma 4.5.2. *With the assumption and notation above. Let $\mathcal{I} \subset \mathcal{O}_V[x_1, \ldots, x_n]$ be a homogenous ideal with respect to $\mathcal{J}$. Let $\sigma : B \to X$ be the cobordant blow-up of $\mathcal{J}$, and $\pi : Y := X \times \mathbb{A}^1 \to X$ be the standard projection. Then*

$$B = \mathrm{Spec}(\mathcal{O}_X[t^{-1}, t^{w_1} x_1, \ldots, t^{w_k} x_k] = \mathrm{Spec}(\mathcal{O}_V[t^{-1}, t^{w_1} x_1, \ldots, t^{w_k} x_k, x_{k+1}, \ldots, x_n]$$

Thus there exists an isomorphism

$$B \simeq Y = X \times \mathbb{A}^1$$

over V, defined by

$$B = \mathrm{Spec}(\mathcal{O}_V[t^{-1}, t^{w_1} x_1, \ldots, t^{w_k} x_k, x_{k+1}, \ldots, x_n] \to Y = \mathrm{Spec}(\mathcal{O}_V[x_1, \ldots, x_n, y])$$

which takes

- $x_i \mapsto t^{w_i} x_i$ *for $i \leq k$,*
- $x_i \mapsto x_i$, $i > k$
- $y \mapsto t^{-1}$,
- $\mathcal{O}_Y \cdot \mathcal{I}$ *to the strict transform ideal $\sigma_B^s(\mathcal{I})$*
- $V(x_1, \ldots, x_n)$ *to the vertex V of B*

♣

Example 4.5.3. The lemma explains the transformation of the singularity

$$f = x_1^{c_1} + \ldots + x_k^{c_k},$$

from Example 3.2.2 which is taken to

$$\sigma^c(f) = \sigma^s(f) = (x_1')^{c_1} + \ldots + (x_k')^{c_k}.$$

Example 4.5.4. The following example is due to Narasimhman [Nar83]. Let

$$f = x^2 + yz^3 + zw^3 + y^7w \in k[x, y, z, w],$$

where k is a field of characteristic 2. Its singular locus $\mathrm{Sing}(f)$ coincides with

$$\mathrm{supp}(f, 2) = V(D(f)) = V(f, Y, Z, W),$$

where

$$Y := D_y(f) = z^3 + y^6w, \quad Z := D_z(f) = yz^2 + w^3, \quad W := D_w(f) = zw^2 + y^7.$$

Note that $F := f + yY + zZ = x^2 + yz^3$ is again binomial. Hence $\mathrm{Sing}(f) = V(F, Y, Z, W)$ is binomial, so it is a (non-normal) toric subvariety of $\mathbb{A}^4$ of dimension 1. One can compute that it coincides with the image $t \mapsto (t^{32}, t^7, t^{19}, t^{15})$.

Observe that the subscheme $V(f)$ is homogenous with respect to the center

$$\mathcal{J} := (x^{1/32}, y^{1/7}, z^{1/19}, t^{1/15}).$$

Thus, by Lemma 4.5.2, the cobordant blow-up of $\mathcal{J}$ transforms X into $B \simeq X \times \mathbb{A}^1_k$ with the same equation f describing $\sigma^s(f)$. Then $B_+ = B \smallsetminus V(x, y, z, w)$. The singular locus of $\sigma^s(f)$ on B_+ is simply the product

$$\{(t^{32}, t^7, t^{19}, t^{15}) \mid t \neq 0\} \times \mathbb{A}^1.$$

The functions Y, Z, and W are semiinvariant on B. Moreover $D_z(Y) = z^2$, $D_w(Z) = w^2$, and $D_y(W) = y^6$ are invertible on an open subset $B_{yzw} = B \smallsetminus V(yzw)$. On the other hand $D_y(Y) = D_z(Z) = D_w(W) \equiv 0$. Thus any two of Y, Z, W form a partial system of local parameters on B_{yzw}. Furthermore the set B_{yzw} contains

$$\mathrm{Sing}(\sigma^s(f)) \cap B_+ = V(\sigma^s(f), Y, Z, W) \cap B_+ = V(\sigma^s(f), Y, Z, W) \smallsetminus V(x, y, z, w),$$

as yzw is invertible on $\{(t^{32}, t^7, t^{19}, t^{15}) \mid t \neq 0\} \times \mathbb{A}^1$. In addition we can write on B_{yzw}:

$$\sigma^s(f) = x^2 + \frac{1}{z^2}YZ + \frac{y^6w^4}{z^2} = X^2 + \frac{1}{z^2}YZ,$$

where $X := x + \frac{y^3w^2}{z}$, z is a unit, and X, Y, Z form a partial system of local parameters. The subscheme $V_{B_+}(\sigma^s(f))$ is almost homogenous on $B_{xyz} \subset B_+$ with respect to the $\mathbb{G}_m$-stable smooth center described as $\mathcal{J}_1 := (X, Y, Z)$ on $B_{xyz} \subset B_+$. In fact, $\mathcal{J}_1$ is an ordinary ideal defining a smooth locus $\mathrm{Sing}(\sigma^s(f))$. The cobordant (or the ordinary) blow-up of $\mathcal{J}_1$ determines a cobordant resolution. Further applying locally toric resolution of [Wło20, Theorem 7.17.1] resolves the singularity functorially with SNC exceptional divisor.

Alternatively, the singularity can be resolved using the characteristic zero invariant "inv," but the process is longer and requires three cobordant blow-ups.

Remark 4.5.5. The Narasinhman example shows that the maximal contact, or even any regular hypersurface containing equimultiple locus, that is the subset of the points with the equal multiplicity of a function does not exist. In characteristic zero such property follows immediately from the definition of maximal contact $u \in \mathcal{D}^{a-1}(\mathcal{I})$ at a point p, where $\mathrm{ord}_p(\mathcal{I}) = a$. In our setting it is a consequence of Lemma 3.6.24(1). The equimultiple locus $\mathrm{supp}(f, 2) = \mathrm{Sing}(f)$ has embedding dimension 4 and cannot be embedded into a smooth hypersurface. In fact one can show that the invariant "inv" in positive characteristic is not upper semicontinuous, and its locus does not have good properties.

4.5.6. *Cobordant blow-ups vs blow-ups at smooth centers.* In the cobordant resolution of the Narasinhman example, the second cobordant blow-up was at the smooth center (X, Y, Z) with weights equal to 1. In general, we have:

Lemma 4.5.7. *Let X be a regular scheme. Let $B_+ \to X$ be a cobordant blow-up of $\mathcal{J}$, locally given by $\mathcal{J} = (u_1, \ldots, u_k)$ with weights 1. Let $Bl_{\mathcal{J}}(X) \to X$ be the ordinary blow-ups of the smooth center $\mathcal{J}$. Then the induced quotient morphsim*

$$B_+ \to B_+/\mathbb{G}_m \simeq Bl_{\mathcal{J}}(X)$$

is a locally trivial $\mathbb{G}_m$-bundle.

Proof. The problem is local on X. Then $B = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, u_1 t, \ldots, u_k t])$, and B_+ is covered by $B_{u_i t^w} = B \setminus V(u_i t) = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, u_1 t, \ldots, u_k t, (u_i t)^{-1}])$, and

$$B_{u_i t}/\mathbb{G}_m = \mathrm{Spec}_X(\mathcal{O}_X[\frac{u_1 t}{u_i t}, \ldots, \frac{u_k t}{u_i t}]) = \mathrm{Spec}_X(\mathcal{O}_X[\frac{u_1}{u_i}, \ldots, \frac{u_k}{u_i}]).$$

Thus the induced morphism $B_+/\mathbb{G}_m \to X$ is a weighted blow-up which coincides with the standard blow-up of $\mathcal{J}$. Since we have that $t^{-1} = (u_i t)^{-1} u_i$, we can write

$$\mathcal{O}_{B_{u_i t}} = \mathcal{O}_X[\frac{u_1}{u_i}, \ldots, \frac{u_k}{u_i}])[t^{-1}, u_i t, (u_i t)^{-1}] = \mathcal{O}_X[\frac{u_1}{u_i}, \ldots, \frac{u_k}{u_i}])[u_i t, (u_i t)^{-1}].$$

Thus locally $B_+ \simeq (Bl_{\mathcal{J}}) \times \mathbb{G}_m$. ♣

5. APPENDIX

5.1. **Cobordant blow-ups of Rees algebras and generalized weighted centers.**

5.1.1. *Generalized cobordant blow-ups.*

Definition 5.1.2. Let $R = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} R_a t^a \subset \mathcal{O}_X[t]$ be any Rees algebra on a Noetherian scheme Y, and let $R[t^{-1}]$ be an extended algebra of R. Then *the full cobordant blow-up* of R is given by $B := Spec_X(R[t^{-1}]) \to X$. The *cobordant blow-up* is simply $B_+ := B \setminus V(R)$.

5.1.3. *Generalized centers.* We generalize the centers accordingly.

Definition 5.1.4. A Rees center on a Noetherian scheme X is an extended Rees algebra $\mathcal{A}^{\mathrm{ext}}$ it can be locally presented as $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-w}, t^{1/a_1} f_1, \ldots, t^{1/a_k} f_k]$, where $V(f_1, \ldots, f_k)$ is a regular subscheme of X.

The following is a simple consequence of the definition:

Lemma 5.1.5. *If $f : X \to Y$ is a morphism of Noetherian schemes, and R_Y is a Rees algebra on Y then $R_X = \mathcal{O}_X \cdot R_Y$ is the induced Rees algebra on X. Let $B_Y \to Y$ and $B_X \to X$ denote the full cobordant blow-ups of Y at R_Y and X at R_X respectively. Then there are induced morphisms $f_B : B_X \to B_Y$, $f_{B_+} : B_{X+} \to B_{Y+}$ commuting with cobordant blow-ups to X and Y. Furtheremore, if $X \to Y$ is a closed immersion, then $B_X \to B_Y$ and $f_{B_+} : B_{X+} \to B_{Y+}$ are so.*

Proof. The morphism $f : X \to Y$ determines the homomorphism of graded sheaves $R_Y \to f_* R_X$, so that for any open subset $U \subset X$ and $V \subset Y$ such that $f(U) \subseteq V$, we have the induced graded homomorphisms $R_Y(V) \to R_X(U)$ which gives rise to $B_X \to B_Y$ taking the vertex V_X to the vertex V_Y. If f is a closed immersion then X can be thought as a closed subscheme of Y and we get the induced surjective homomorphism of graded $\mathcal{O}_Y$-modules $R_Y \to R_X$ which defines the closed immersions $f_B : B_X \to B_Y$, $f_{B_+} : B_{X+} \to B_{Y+}$.   ♣

5.1.6. *The maximal admissible centers on the subscheme are regular.*

Lemma 5.1.7. *Let X be a smooth variety over a field k of characteristic zero, and E be an SNC divisor on X. Let $Y \subset X$ be a reduced closed subscheme of a X which is not locally contained in the divisor E, and let $\mathcal{I}_Y$ be its ideal on X. Let $\mathcal{A}^{\mathrm{ext}}$ be a maximal $\mathcal{I}_Y$-admissible center locally at a point $p \in Y$ of the form $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-w}, t^{1/a_1} u_1 t^{1/a_k}, \ldots, t^{1/a_k} u_k]$. Then*

$$\mathcal{O}_Y \cdot \mathcal{A}^{\mathrm{ext}} = \mathcal{O}_Y[t^{-w}, t^{1/a_1} f_1 t^{1/a_k}, \ldots, t^{1/a_k} f_k],$$

for $f_i := u_{i|Y}$, is a Rees center on Y.

Proof. By definition $\mathcal{I}_Y t \subset \mathcal{A}^{\mathrm{ext}}$. So

$$Y = V(\mathcal{I}_Y) = V(\mathcal{O}_X[\mathcal{I}_Y t]) \supseteq V(\mathcal{A}^{\mathrm{ext}}) = V(u_1, \ldots, u_k)$$

Consequently

$$V(f_1, \ldots, f_k) = V(u_1, \ldots, u_k)_{|Y} = V(u_1, \ldots, u_k)$$

is regular on Y.

                                                                      ♣

5.2. **Geometric quotient and cobordant blow-ups.** In this section, we review some standard properties of the torus actions, usually stated over algebraically closed fields, and extended here to noetherian schemes.

Definition 5.2.1. Let a torus $T = \mathbb{G}_m^l$, where $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$ act on a scheme X. We say that an action of a torus T on scheme X is *relatively affine* if for any point $p \in X$ admits an open T-stable affine neighborhood U of p.

Let $\pi : X \to Y$ be an invariant morphism of schemes. We say that an action of a torus T on scheme X is *relatively affine over Y* if for any point $p \in X$ there is an open T-stable affine neighborhood U of p, mapping to an open affine neighborhood of V of $q = \pi(q) \in Y$, and a T-equivariant closed immersion

$$\phi_U : U \hookrightarrow V \times \mathbf{A}^n \times \mathbb{G}_m^r,$$

with the diagonal action of T on $\mathbf{A}^n \times \mathbb{G}_m^r$, and commuting with the invariant morphisms

If additionally, X admits geometric quotient, and U, and V can be chosen so that $V \times \mathbf{A}^n \times \mathbb{G}_m^r$ admits geometric quotient then we say that T admits *relatively affine geometric quotient* over Y.

We shall need a well known fact for the action of the torus on any affine schemes.

Lemma 5.2.2. *Let X be an scheme with a relatively affine action of torus T, with the group of characters $\chi(T)$. Then $\mathcal{O}_X$ admits a natural grading and*

$$\mathcal{O}_X = \bigoplus_{\alpha \in \chi(T)} (\mathcal{O}_X)_\alpha,$$

where T acts on $\mathcal{O}(X)_\alpha$ by the character t^α.

Proof. We can reduce the situation to the case of affine X. The natural grading on $\mathcal{O}(X)$, and thus on the sheaf $\mathcal{O}_X$ is given by the coaction

$$\rho : \mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathbb{Z}[T],$$

where $\mathbb{Z}[T] = \mathbb{Z}[t_1^\pm, \ldots, t_m^\pm]$. For any $f \in \mathcal{O}(X)$, $\rho(f) = \sum c_\alpha t^\alpha$, where $c_\alpha \in \mathcal{O}(X)$. Moreover it follows from the associativity

$$(\mathrm{id}_{\mathcal{O}(X)} \otimes \mu) \circ \rho = (\rho \times \mathrm{id}_{\mathbb{Z}[T]}) \circ \rho$$

where

$$\mu : \mathbb{Z}[T] \to \mathbb{Z}[T] \otimes \mathbb{Z}[T], \quad \mu(t) = tt'$$

is defined by the group multiplication, such that

$$\sum c_\alpha (tt')^\alpha = \sum \rho(c_\alpha)(t')^\alpha,$$

and hence $\rho(c_\alpha) = c_\alpha t^\alpha$. From existence of the identity map $1_T : \mathbb{Z}[t_1^\pm, \ldots, t_m^\pm] \to \mathbb{Z}$, such that

$$(1_T \otimes \mathrm{id}_{\mathcal{O}(X)}) \circ \rho = \mathrm{id}_{\mathcal{O}(X)}$$

we deduce that

$$f = (1_T \otimes \mathrm{id}_{\mathcal{O}(X)}) \circ \rho(f) = (1_T \otimes \mathrm{id}_{\mathcal{O}(X)})(\sum c_\alpha t^\alpha) = \sum c_\alpha.$$

Thus any element decomposes into a finite sum of the homogenous elements c_α of grading α, and

$$\mathcal{O}(X) = \bigoplus \mathcal{O}(X)_\alpha,$$

where T acts on $\mathcal{O}(X)_\alpha$ by the character t^α. ♣

Definition 5.2.3. Let X be a scheme with with a relatively affine action of torus T. By a T *-stable ideal* we mean a homogenous ideal $\mathcal{I}$ of $\mathcal{O}_X = \bigoplus_\alpha (\mathcal{O}_X)_\alpha$.

Definition 5.2.4. Let X be a normal scheme with a relatively affine action of torus T. A Q-ideal $\mathcal{J}$ on a normal scheme is said to be T *-stable* if is of the form $\mathcal{J} = \mathcal{I}^{1/n}$, where $\mathcal{I}$ is a T-stable ideal.

Lemma 5.2.5. *Let $\mathcal{J}$ be a T-stable center on a regular scheme X with a relatively affine action of T, and a T-stable SNC divisor E. Then one can write locally the center $\mathcal{J}$ as $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$, where $x_1, \ldots, x_k$ is a partial system of semiinvariant local parameters.*

Proof. We can assume that $\mathcal{J}$ has the form $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ at a point $p \in X$. Write $x_1 \in (\mathcal{J}^{w_1})_X$ as the sum of homogenous components $x_1 = \sum c_\alpha$. We conclude that $\mathrm{ord}_p(c_\alpha) = 1$ for some $x_1' := c_\alpha \in (\mathcal{J}^{w_1})_X$. So, by Lemma 3.1.29, one can replace x_1 with a semiinvariant x_1' in the presentation of $\mathcal{J}$. By the induction on dimension, we assume that $\mathcal{J}_{|V(x_1')} = ((x_2')^{1/w_2}, \ldots, (x_k')^{1/w_k})$ is represented by semiinvariant local parameters on $V(x_1')$. Moreover, there is a surjection of

graded rings $\mathcal{O}_X \to \mathcal{O}_{V(x_1')} = \mathcal{O}_X/V(x_1')$. Then x_i' can be lifted to the homogenous elements in $\mathcal{O}_X$. Set

$$\mathcal{J}' := ((x_1')^{1/w_1}, \ldots, (x_k')^{1/w_k}) = (x_1')^{1/w_1} + ((x_2')^{1/w_2}, \ldots, (x_k')^{1/w_k}).$$

Note that such a presentation is independent of the liftings of the elements x_i', $i \geq 2$ on $V(x_1')$ to U. Then $\mathcal{J}_{|V(x_1')} = \mathcal{J}'_{|V(x_1')}$, and their saturated powers $((\mathcal{J}_{|V(x_1')})^a)^{\mathrm{sat}} = ((\mathcal{J}'_{|V(x_1')})^a)^{\mathrm{sat}}$ for sufficiently divisible a are equal as ideals. Then in the completion

$$\widehat{\mathcal{O}_{X,p}} = x_1' \cdot \widehat{\mathcal{O}_{X,p}} \oplus \widehat{\mathcal{O}_{X,p}}/(x_1')$$

both ideals are equal so

$$\widehat{\mathcal{J}^a} := (x_1')^{a/w_1} + ((\mathcal{J}_{|V(x_1')})^a))^{\mathrm{sat}} = (x_1')^{a/w_1} + ((\mathcal{J}'_{|V(x_1')})^a))^{\mathrm{sat}} = \widehat{\mathcal{J}'^a}.$$

This implies that $\mathcal{J}^a = \mathcal{J}'^a$ as ideals and $\mathcal{J} = \mathcal{J}'$ as $\mathbb{Q}$-ideals. ♣

Lemma 5.2.6. *Let $\mathcal{I}_0$ be an ideal on a regular scheme X_0, and $X \to X_0$ be a scheme obtained from X_0 by a sequence of l cobordant blow-ups of regular (possibly reducible) weighted centers T-stable for the induced torus actions, and $\mathcal{I} = \sigma^s(\mathcal{I})$ be its controlled transform. Then X is a regular scheme with the relatively affine action of torus $T = \mathbb{G}_m^l$ over X_0 and X admits a relatively affine geometric quotient $X \to X/T$ of T over X_0. Moreover $\sigma^s(\mathcal{I})$ is a T-stable ideal on X.*

Proof. We use the induction on l. Let $X \to X_0$ be a sequence of $l-1$ equivariant cobordant blow-ups, and let $B_+ \to X$ be the cobordant blow-up of a T-stable center $\mathcal{J}$. Let $\pi : X \to X/T^{l-1}$ be a T-invariant morphism defined by the geometric quotient. Then B and B_+ admit an action of $T^l := T^{l-1} \times \mathbb{G}_m$, and $\pi\sigma_+ : B_+ \to X/T$ is T^l-invariant. In order to show that there exists a geometric quotient B_+/T^l it suffices to show that it exists for $\sigma_+^{-1}\pi^{-1}(V_\alpha)$ for open affine cover V_α of X/T. By the induction we can assume that each open affine $U := \pi^{-1}(V_\alpha)$ admits a T^{l-1}-equivariant embedding.

$$\phi : U \overset{\phi}{\hookrightarrow} W := U_0 \times \mathbb{A}^n \times \mathbb{G}_m^r$$

Furthermore the closed image $\phi(U)$ is generated by the seminvariant parameters $v_1, \ldots, v_r \in \Gamma(W, \mathcal{O}_W)$.

Also we can assume that the center on U is described as $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Consider the local presentation of B as in Section 2.5.2. Let us introduce the free variables denoted by $t^{-1}, x_1', \ldots, x_n'$ with the action

$$\mathrm{Spec}(\mathcal{O}_U[t^{-1}, x_1', \ldots, x_k']) \smallsetminus V(x_1', \ldots, x_k') = U \times (\mathbb{A}^{k+1} \smallsetminus V(x_1', \ldots, x_k')),$$

with the action of $\mathbb{G}_m$ on $\mathbb{A}^{k+1} = \mathrm{Spec}(\mathbb{Z}[t^{-1}, x_1', \ldots, x_k']))$, given by

$$(t^{-1} \cdot t^{-1}, t^{w_1} x_1', \ldots, t^{w_k} x_1'),$$

Then $B_U := \sigma^{-1}(U)$ is a closed subscheme of $U \times (\mathbb{A}^{k+1} \smallsetminus V(x_1', \ldots, x_k'))$:

$$(B_U)_+ = \mathrm{Spec}(\mathcal{O}_U[t^{-1}, x_1', \ldots, x_k']) \smallsetminus V(x_1', \ldots, x_k'))/(t^{-w_i} x_i' - x_i).$$

We consider the action of T^{l-1} on $(t^{-1}, x_1', \ldots, x_k')$ as on the corresponding elements of B, so the action of T^{l-1} on t^{-1} is trivial and the action of T^{l-1} on x_i' is the same as on $t^{w_i} x_i$.

Then for $(B_U)_{x_i} = D(x_i) = W \smallsetminus V(x_i) \subset (B_U)_+$ we obtain the T^l-equivariant closed embedding:

$$(B_U)_i \subset \mathrm{Spec}(\mathcal{O}_U[t^{-1}, x_1', \ldots, x_k', (x_i')^{-1}]) = U \times \mathbb{A}^k \times \mathbb{G}_m \hookrightarrow U_0 \times \mathbb{A}^{n+k} \times \mathbb{G}_m^{r+1},$$

such that $U_0 \times \mathbb{A}^{n+k} \times \mathbb{G}_m^{r+1}$ admits geometric quotient of the group action of T^l.
♣

Proposition 5.2.7. *Let Y be a regular affine scheme. Let T act on $\mathbf{A}^n \times \mathbb{G}_m^l$ diagonally so that the geometric quotient exists. Let Z be a closed T-stable regular subscheme of $W := Y \times \mathbf{A}^n \times \mathbb{G}_m^l$ defined by a partial system of semiinvariant local parameters.*

Then for any point $p \in Z$ there is an open affine T-stable neigbourhood $U \subset Z$ of p, over an open affine $V \subset Y$, and a T-equivariant closed immersion

$$U \overset{\phi}{\hookrightarrow} W_U := W = V \times \mathbb{G}_m^a \times (\mathbb{A}_k^s \times \mathbb{G}_m^b)$$

for some a, b, s, and the induced fiber square diagram :

$$
\begin{array}{ccccc}
U_Z & \overset{\phi}{\hookrightarrow} & W = V \times \mathbb{G}_m^a \times (\mathbb{A}_k^s \times \mathbb{G}_m^b) & \overset{\pi}{\to} & \mathbb{A}_k^s \times \mathbb{G}_m^b \\
\pi_X \downarrow & & \downarrow & & \pi_A \downarrow \\
U_Z/T & \overset{\phi/T}{\hookrightarrow} & W/Z = (V \times \mathbb{G}_m^a) \times X_\sigma & \overset{\pi/T}{\to} & X_\sigma = (\mathbb{A}_k^s \times \mathbb{G}_m^b)/T
\end{array}.
$$

(1) *T acts trivially on the factor $\mathbb{G}_m^a$.*
(2) *T acts diagonally on the factor $\mathbf{A}^s \times \mathbb{G}_m^b$, admits a geometric quotient of a diagonal action of T, and the action has finite stabilizers.*
(3) *ϕ, and ϕ/T are closed immersions, with their closed images $\phi(U)$, and $\phi_{U/T}(U/T)$ is defined by the system of invariant parameters $v_1, \ldots, v_k \in (\Gamma(W, \mathcal{O}_W)^T)$, whose restrictions to*

$$(\pi/T)^{-1}(q_0) = (V \times \mathbb{G}_m^a) \times \mathrm{Spec}(k(q_0))$$

also define regular parameters.

Proof. Let $x_1, \ldots, x_s$, for $s \leq n$ be the coordinates on $\mathbb{A}^s$ which are noinvertible at p on

$$W_0 = Y \times \mathbb{A}^n \times \mathbb{G}_m^l = Y \times \mathbb{A}^s \times \mathbb{A}^{n-s} \times \mathbb{G}_m^l,$$

and $y_{s+1}, \ldots, y_{n+l-s}$ are the coordinates on $\mathbb{A}^{n-s} \times \mathbb{G}_m^l$ which are invertible at p.

We can replace W with the open subset $W_0 = Y \times \mathbb{A}^s \times \mathbb{G}_m^c$, where $c := n-s+l$, and Z with $U := Z \cap W_0$

First, we show:

Lemma 5.2.8. *There is an open T-stable neighborhood U of any point p and a fiber square:*

$$
\begin{array}{ccc}
U & \overset{\phi}{\hookrightarrow} & W_U := (V \times \mathbf{A}^r \times \mathbb{G}_m^d) \\
\pi_U \downarrow & & \pi_W \downarrow \\
U/T = \mathrm{Spec}(\mathcal{O}(U)^T) & \overset{\phi/T}{\hookrightarrow} & W_U/T := V \times (\mathbf{A}^r \times \mathbb{G}_m^s)/T
\end{array},
$$

for some $d, r \in \mathbb{N}$, where ϕ, and ϕ/T are closed immersions, with their closed images $\phi(U)$, and $\phi_{U/T}(U/T)$ is defined by the system of invariant parameters $v_1, \ldots, v_k \in (\Gamma(W, \mathcal{O}_W)^T)$, whose restrictions to $V \times \{0\} \times \mathbb{G}_m^d$ also define regular parameters.

Proof. By the assumptions the image $U \subset W_0$, is described locally by a partial system of semiinvariant parameters $v_1, \ldots, v_t$.

Write any semiinvariant parameter v_j as

$$v_j = \sum c_\beta^j y^\beta, + \sum_{\alpha \neq 0} c_{\alpha\beta}^j x^\alpha y^\beta = v_0^j + w_0^j$$

where $c_{\alpha\beta}^j \in \mathcal{O}(Y)$, $v_0^j := \sum c_\beta^j y^\beta$, and $w_0^j := \sum_{\alpha \neq 0} c_{\alpha\beta}^j x^\alpha y^\beta$, with all monomials semiinvariant of the same weight. We will modify the sets of local parameters using the induction on j.

Assume that $v_1, \ldots, v_{j-1}$ is the set of first $j-1$ local parameters, which are invariant and such that their images in $\frac{m_p}{(x_1, \ldots, x_s) + m_p^2}$ are linearly independent. Consider two cases.

Case 1. $v_0^j \notin (v_1, \ldots, v_{j-1}) + (x_1, \ldots, x_s) + m_p^2$.

Then multiplying v_j by the inverse $y^{-\beta}$ for some β occuring in v_0^j we reduce the situation to the invariant parameter $v_j' := y^{-\beta} v_j$ on $U_0 \times \mathbb{A}^s \times \mathbb{G}_m^d$, such that $v_1, \ldots, v_j$ are invariant with their images in $\frac{m_p}{(x_1, \ldots, x_s) + m_p^2}$ are linearly independent.

Case 2. $v_0^j \in (v_1, \ldots, v_{j-1}) + (x_1, \ldots, x_s) + m_p^2$.

Then at least one $c_{\alpha\beta}^j x^\alpha y^\beta \in (x_1, \ldots, x_s)$ is a parameter at p. Thus $x^\alpha = x_i$ and $u_{\alpha\beta}^j := c_{\alpha\beta}^j y^\beta$ is invertible at p, and $c_{\alpha\beta}^j$ is such.

So, after replacing U with $U_{c_{\alpha\beta}^j}$, and clearing the unit $v_j' = (u_{\alpha\beta}^j)^{-1} v_j$ we can assume that the semiinvariant parameter v_j' is of the form $v_j' = x_i + h_j$, where $h_i := \sum a_{\alpha\beta} x^\alpha y^\beta$. We can write

$$v_j' = x_i + h_j = x_i (1 + \sum b_{\alpha\beta} x^\alpha y^\beta) + h_j' = g x_i + h_j',$$

where $h_j' := \sum d_{\alpha\beta} x^\alpha y^\beta$, where all x^α do not depend upon x_i, and

$$g := 1 + \sum b_{\alpha\beta} x^\alpha y^\beta$$

is invariant invertible at p. Further shrinking U to U_g and introducing new invertible variable y_{r+1} on G_m we obtain the induced T-equivariant embedding

$$\phi : U_g \hookrightarrow W' = W \times G_m = V \times \mathbb{A}^s \times \mathbb{G}_m^{r+1}$$

such that $y_{r+1} \mapsto g$. This also introduces the additional invariant local parameter $v_j'' := y_{r+1} - g$, such that

- $v_1, \ldots, v_{j-1}, v_j'', v_j', v_{j+1}, \ldots, v_r$ is a system of local parameters describing $\phi(U)$ on W'.
- $v_1, \ldots, v_{j-1}, v_j''$ are invariant with their images in $m_p/(x_1, \ldots, x_s) + m_p^2$ are linearly independent.

Then v_j' can be written modulo v_j'' as

$$v_j' = y_{r+1} x_i + h_i \sim x_i + h_j' \cdot y_{r+1}^{-1}$$

Consider a T^l-equivariant automorphism of W' taking $x_i + h_j' \cdot y_{r+1}^{-1} \mapsto x_i$, and $x_j \mapsto x_j$. Thus, after the coordinate change one can assume that the parameter $v_j' = y_{r+1} x_j$. This induces the embedding: into

$$\phi' : U_g \hookrightarrow W'' := V(x_i) = U_0 \times \mathbb{A}^{s-1} \times \mathbb{G}_m^{r+1},$$

and proves Lemma 5.2.8.   ♣

Let $\phi : U \hookrightarrow W_U := (V \times \mathbf{A}^r \times \mathbb{G}_m^d)$ be the closed T-equivariant embedding as in Lemma 5.2.8, whose image is defined by the invariant parameters $v_1, \ldots, v_k$ such that their restrictions to closed subscheme $V \times \{0\} \times \mathbb{G}_m^d$ also define the parameters.

Consider the splitting $\mathbb{G}_m^d = \mathbb{G}_m^a \times \mathbb{G}_m^b$, where $\mathbb{G}_m^a = (\mathbb{G}_m^d)^T$ is the maximal subtorus, such that T acts trivially on $\mathbb{G}_m^a$, and transitively on $\mathbb{G}_m^b$. Denote by $y_1, \ldots, y_a$ the coordinates in $\mathbb{G}_m^a$, and by $z_1, \ldots, z_b$ the coordinates in $\mathbb{G}_m^b$.

Let

$$\pi : W := V \times \mathbb{A}_k^s \times \mathbb{G}_m^d = V \times \mathbb{G}_m^a \times (\mathbb{A}_k^s \times \mathbb{G}_m^b) \to \mathbb{A}_k^s \times \mathbb{G}_m^b.$$

be the natural projection. Denote by $q \in \mathbb{A}_k^s \times \mathbb{G}_m^b$ the image $q = \pi(p)$, and by $p_0 \in W/T$ the image of p under the quotient $W \to W/T$.

Any invariant parameter v_j can be written as

$$v_j = \sum c_{\alpha\beta\gamma}^j x^\alpha y^\beta z^\gamma,$$

where $c_{\alpha\beta\gamma}^j \in \mathcal{O}(V)$. Then for any $\gamma \neq 0$ the action of T on z^γ is not trivial. Hence $\alpha \neq 0$ as the action on $x^\alpha z^\gamma$ is trivial .

This shows that

$$v_j = \sum c_\beta^j y^\beta + \sum_{\alpha \neq 0} c_{\alpha\beta\gamma}^j x^\alpha y^\beta z^\gamma$$

Thus the restriction of v_j to the smooth fiber

$$F_q = \pi^{-1}(q) \simeq V \times \mathbb{G}_m^b \times \mathrm{Spec}(k(q)) \subset V \times \{0\} \times \mathbb{G}_m^d$$

determines the parameters on F_q defined by $\sum c_\beta^j y^\beta$.

Let $q_0 \in \mathbb{A}_k^s \times \mathbb{G}_m^b/T$ be the image of q. The geometric quotient of W by T can be written as

$$W/T = U_0 \times \mathbb{G}_m^b \times (\mathbb{A}_k^s \times \mathbb{G}_m^b)/T = U_0 \times \mathbb{G}_m^a \times X_\sigma,$$

where

$$X_\sigma := (\mathbb{A}_k^s \times \mathbb{G}_m^b)/T$$

is a toric variety with the abelian quotient singularities.

The composition of morphisms

$$\psi : U_Z \xrightarrow{\phi} W = U_0 \times \mathbb{G}_m^a \times (\mathbb{A}_k^s \times \mathbb{G}_m^b) \xrightarrow{\pi} \mathbb{A}_k^s \times \mathbb{G}_m^b$$

induces

$$\psi/T : U_Z/T \xrightarrow{\phi/T} W/T = U_0 \times \mathbb{G}_m^a \times X_\sigma \xrightarrow{\pi/T} X_\sigma = (\mathbb{A}_k^s \times \mathbb{G}_m^b)/T,$$

for which the restrictions of $v_1, \ldots, v_k \in \mathcal{O}(U_Z/T) = \mathcal{O}(U_Z)^T$ at p_0 to the fiber

$$(\pi/T)^{-1}(q_0) = (V \times \mathbb{G}_m^a) \times \mathrm{Spec}(k(q_0))$$

also define regular parameters.

♣

Corollary 5.2.9. *Let X_0 be a smooth variety over a field k. Let X be obtained by a sequence of m cobordant blow-ups at smooth centers from Y, equivariant for the induced torus actions. Let Z be a regular T-stable closed subscheme of X, where $\mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}])$, and $T = \mathbb{G}^m$. Then the geometric quotient Z/T exists. Moreover, locally there exists a smooth morphism*

$$U_Z/T \to X_\sigma := (\mathbb{A}_k^s \times \mathbb{G}_m^b)/T,$$

where $U_Z \subset Z$ is an open affine T-stable and X_σ is a toric variety over k with abelian quotient singularities.

Proof. By Lemma 5.2.6, and Proposition 5.2.7, for any point $p_0 \in Z/T$ there is a smooth morphism

$$\pi/T : W/T = (V \times \mathbb{G}_m^b) \times X_\sigma \to X_\sigma,$$

and the closed immersion $U_Z/T \hookrightarrow W/T$, with the image defined by the functions $v_1, \ldots, v_r \in \mathcal{O}(W)^T = \mathcal{O}(W/T)$, and its restriction to the smooth fiber

$$(\pi/T)^{-1}(q_0) \simeq U_0 \times \mathbb{G}_m^b \times \mathrm{Spec}(k(q_0))$$

determines a partial system of local parameters. Thus, by [Sta, Lemma 10.99.3.] the morphism

$$\psi/T : U_Z/T \to X_\sigma := (\mathbb{A}_k^s \times \mathbb{G}_m^b)/T$$

is flat at p_0, and, by Lemma [Sta, Lemma 29.34.14(2)], it is smooth at p_0.

♣

Remark 5.2.10. The above result is not valid in the mixed characteristic. The morphism π/T, which is a projection along regular scheme is not flat, in general, and one needs to work with the presentation from Proposition 5.2.7 directly:

$$U/T \stackrel{\phi/T}{\hookrightarrow} V \times X_\sigma \stackrel{\pi/T}{\to} X_\sigma,$$

where V is a regular scheme, ϕ/T is a closed embedding, and π/T is the projection. Here the image $\phi/T(U/T)$ is defined by local parameters at $p_0 \in \phi/T(U/T)$ whose restriction to the regular fiber $F_{q_0} = (\pi/T)^{-1}(q_0)$, where $q_0 = (\pi/T)(p_0)$, are also parameters.

5.3. **Cobordant blow-ups in the logarithmic category.** Let X be a logarithmically regular scheme, with Zariski logarithmic structure (or a strict toroidal variety) and assume that $\mathcal{J}$ has a more general form

$$\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k}, m_1^{1/w_{k+1}}, \ldots, m_r^{1/w_{k+r}}),$$

where u_i are free parameters and m_i are monomials for the logarithmic structure on X then the full cobordant blow-up $B \to X$ at $\mathcal{J}$ can be described as

$$B = \mathrm{Spec}(\mathcal{O}_X[t^{-1}, t^{w_1}x_1, \ldots, t^{w_k}x_k, t^{w_{k+1}}m_1, \ldots, t^{w_{k+r}}m_r])^{int},$$

which is again a logarithmically regular scheme,(respectively strict toroidal variety). By functoriality, the construction extends to any logarithmically regular schemes (or toroidal varieties).

Initially, in the papers [ATW17], [ATW20], we considered the stack-theoretic blow-ups of the centers of the form $\mathcal{J} = (u_1, \ldots, u_k, m_1^{1/w_{k+1}}, \ldots, m_r^{1/w_{k+r}})$, in the context of Kummer étale topology on the logarithmic stacks. Then in [ATW19], we developed the formalism of the stack-theoretic blow-ups of the weighted centers of the form $(u_1^{1/w_1}, \ldots, u_k^{1/w_k})$.

Soon after Quek in [Que20, Section 3.2] studied the stack-theoretic blow-ups of the more general form

$$\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k}, m_1^{1/w_{k+1}}, \ldots, m_r^{1/w_{k+r}}) = (u_1^{1/w_1}, \ldots, u_k^{1/w_k}, Q^{1/w})$$

in the context of logarithmic stacks and Kummer étale topology. Here Q is generated by m_i^{w/w_i} for the corresponding w.

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