Open access • Journal Article • DOI:10.1007/S10240-004-0020-Z

Functoriality for the classical groups — Source link

James Cogdell, Henry H. Kim, Ilya Piatetski-Shapiro, Ilya Piatetski-Shapiro ...+1 more authors

Institutions: Oklahoma State University–Stillwater, University of Toronto, Tel Aviv University, Yale University ...+1 more institutions

Published on: 09 Sep 2004 - Publications Mathématiques de l'IHÉS (Springer-Verlag)

Topics: Converse theorem, Langlands dual group, Admissible representation, Automorphic form and Representation theory

FUNCTORIALITY FOR THE CLASSICAL GROUPS

by J. W. COGDELL, H. H. KIM, I. I. PIATETSKI-SHAPIRO, *and* F. SHAHIDI*

Functoriality is one of the most central questions in the theory of automorphic forms and representations [1, 2, 35, 36]. Locally and globally, it is a manifestation of Langlands' formulation of a non-abelian class field theory. Now known as the Langlands correspondence, this formulation of class field theory can be viewed as giving an arithmetic parameterization of local or automorphic representations in terms of admissible homomorphisms of (an appropriate analogue) of the Weil-Deligne group into the Langlands dual group or L-group. When this conjectural parameterization is combined with natural homomorphisms of the L-groups it predicts a transfer or lifting of local or automorphic representations of two reductive algebraic groups. As a purely automorphic expression of a global non-abelian class field theory, global functoriality is inherently an arithmetic process.

In this paper we establish global functoriality from the split classical groups $G_n = SO_{2n+1}$, SO_{2n}, or Sp_{2n} to an appropriate general linear group GL_N, associated to the natural embedding of L-groups, for globally generic cuspidal representations π of $G_n(\mathbf{A})$ over a number field k. We had previously presented functoriality for the case $G_n = SO_{2n+1}$ in [6], but were limited at that time by a lack of suitable local tools in the other cases. The present paper is by no means a simple generalization of [6]. There were serious local problems to be overcome in the development of the tools that now allow us to cover all three series of classical groups simultaneously and that will be applicable to other cases of functoriality in the future. In addition, we have completely determined the associated local images of functoriality and as a result are able to present several new applications of functoriality, including both global results concerning the Ramanujan conjecture for the classical groups and various applications to the local representation theory of the classical groups.

There are several approaches to the question of functoriality: the trace formula, the relative trace formula, and the Converse Theorem. In this work we use the Converse Theorem, which is an L-function method. The Converse Theorem itself states that if one has an irreducible admissible representation $\Pi \simeq \otimes'\Pi_v$ of $GL_N(\mathbf{A})$, then Π is in fact automorphic if sufficiently many of its twisted L-functions $L(s, \Pi \times \tau)$, with τ cuspidal automorphic representations of smaller $GL_m(\mathbf{A})$, are nice [7, 9]. As a vehicle for establishing functoriality from cuspidal representations $\pi = \otimes\pi'_v$ of some $G_n(\mathbf{A})$ to an automorphic representation of

* During the work on this project J. W. C. was partially supported by the NSA through grant MDA 904-03-1-0043, H. H. K. by a grant from NSERC and an AMS Centennial Fellowship, I. I. P-S. by the NSF through grant DMS-0072450, and F. S. by the NSF through grant DMS-0200325.

DOI 10.1007/s10240-004-0020-z

$\mathrm{GL}_N(\mathbf{A})$, there are three main steps. The first is to construct a candidate lift $\Pi = \otimes' \Pi_v$. This is done by locally lifting each local component representation π_v of $G_n(k_v)$ to Π_v of $\mathrm{GL}_N(k_v)$ in such a way that twisted local L- and ε-factors are matched. At the archimedean places and the finite places where π_v is unramified we may accomplish this local lift by using the local Langlands correspondence, i.e., the local arithmetic Langlands classification. At the remaining finite set of places where π_v is ramified we must finesse the lack of a local Langlands correspondence by using the stability of the local γ-factor under highly ramified twists. This highly ramified twist has the effect of "washing out" any subtle local properties of the representation and gives a matching of local representations π_v of $G_n(k_v)$ and Π_v of $\mathrm{GL}_N(k_v)$ for which the twisted local L- and ε-factors match after this highly ramified twist. We used this method in [6], however the key new ingredient here is a uniform method of expressing the local γ-factor as the Mellin transform of a Bessel function in fairly wide generality which is applicable in all of our cases as well as many more [55]. With this new general result in hand, the necessary stability result then follows from the asymptotic properties of the Bessel functions as in [8]. With this, we can finally lift π_v locally to Π_v at all places and form a candidate lift $\Pi = \otimes' \Pi_v$ such that $L(s, \Pi \times \tau) = L(s, \pi \times \tau)$ for all τ in a suitable twisting set. The second step is to then control the analytic properties of the twisted L-functions $L(s, \pi \times \tau)$ on the classical groups. As in our previous work, we control these L-functions through the Fourier coefficients of Eisenstein series – the Langlands-Shahidi method. Once we know that the $L(s, \pi \times \tau)$, and hence $L(s, \Pi \times \tau)$, are nice for a suitable twisting set of τ, we may move to the third main step, which is the application of the Converse Theorem for GL_N to the representation Π. This then gives global functoriality from any of the G_n to the appropriate GL_N (Theorem 1.1).

Assuming the existence of global functoriality, the result which we establish here, Ginzburg, Rallis, and Soudry had previously used their descent technique to characterize the image of global functoriality for globally generic representations of the split classical groups [56]. In particular, they show the image of global functoriality consists of isobaric sums of certain self-dual cuspidal representations of $\mathrm{GL}_d(\mathbf{A})$ satisfying an appropriate L-function criterion (Theorems 7.1 and 7.2). Using the rigidity of isobaric representations afforded by the strong multiplicity one theorem for isobaric representations of $\mathrm{GL}_N(\mathbf{A})$ [21], this implies that there is in fact no ambiguity in our global functorial lift coming from our use of the highly ramified twist and we are able to then compute explicitly the compatible local functorial lifts of the various series of generic representations of $G_n(k_v)$. For the case of $G_n = \mathrm{SO}_{2n+1}$ this was done in [30] and we follow that general method here, but again giving a uniform treatment for all split classical groups. For generic supercuspidal representations π_v we show that their lift is a local isobaric sum of certain self-dual supercuspidal representations of general linear groups, again satisfying the appropriate L-function condition (Theorem 7.3). With the local supercuspidal lift in hand, one can then use the classification of local generic discrete series representations for the classical groups [24, 25, 39, 40, 43] to compute the explicit form of their lifts (Proposition 7.3), and in turn work one's way up the classification to obtain explicit lifts of generic tempered representations (Proposition 7.4) and finally of arbitrary generic representations (Proposition 7.5). Finally we are able to refine these local results to compute the local factors of our global functorial lift Π (Theorem 7.4) with a second application of the Converse Theorem. This explicit knowledge of the local functorial lifts is crucial to our applications.

Let us note that in the case $G_n = SO_{2n+1}$ Jiang and Soudry [26, 27] were able to combine our global functoriality with the local descent to $\widetilde{Sp}_{2n}$ and then the theta correspondence to prove a Local Converse Theorem for SO_{2n+1} over a p-adic field. This allowed them to prove the injectivity of the local functorial lifts as we have defined them here and establish the local Langlands correspondence for SO_{2n+1}. Once the local descent is available in the other cases, we would expect similar results to follow. However, for the other classical groups the Local Converse Theorem will be more subtle since the torus does not act transitively on the set of generic characters. This will lead to more than one generic representation in each local L-packet, distinguished by their character of genericity. For a clean statement one may need to pass to similitude groups.

The global application we present is indeed of an arithmetic nature and concerns the Ramanujan conjecture for generic representations of the split classical groups. In the late 1970's, when the first counterexamples to the generalized Ramanujan conjecture for reductive groups were found for Sp_4 and U_3 [18], the Ramanujan conjecture for a general reductive group $G(\mathbf{A})$ was refined and conjectured to hold for generic cuspidal representations of quasi-split reductive groups [18, 45, 50]. On the other hand, Langlands, in Section 3 of [35], suggests that the Ramanujan conjecture should hold for cuspidal representations of quasi-split groups which functorially lift to isobaric representations of $GL_N(\mathbf{A})$ (cf. the Remark at the end of Section 10 here). This is the case for the globally generic representations of our classical groups $G_n(\mathbf{A})$ as we have noted above. Thus, with either formulation, we would expect that if $\pi \simeq \otimes' \pi_v$ is a generic cuspidal automorphic representation of $G_n(\mathbf{A})$ then each local component π_v should be tempered. This is widely believed to hold for GL_N. The best current general bounds towards Ramanujan for $GL_N(\mathbf{A})$ over a number field are those of Luo, Rudnick, and Sarnak [37]. Via functoriality we are able to link the Ramanujan conjecture for globally generic representations of the split classical groups to the Ramanujan conjecture for cuspidal representations of GL_N (Theorem 10.1). In particular, we show that the Ramanujan conjecture for these groups, in its strong form giving temperedness at all places, would follow from the Ramanujan conjecture for GL_N (Corollary 10.2), at least for globally generic cuspidal representations, and any bounds towards Ramanujan for GL_N, such as the Luo–Rudnick–Sarnak bounds, lead to similar bounds for the classical groups (Corollary 10.1). We note that once our results on functoriality are extended to the case of global function fields, which is primarily a matter of understanding the theory of L-functions for the classical groups over a global function field, then the Ramanujan conjecture for these groups over a global function field would become a theorem, thanks to Lafforgue's proof of the Ramanujan conjecture for GL_N over a global function field [33]. We hope to return to this extension in future papers.

Even though functoriality is inherently arithmetic, many of its applications are to the local representation theory of the groups G_n. These results seem difficult to establish locally on the classical groups themselves, but are rather straightforward applications of functoriality. The first local application presented in this paper is a proof of Mœglin's conjecture on the "dimension relation" for generic discrete series representations π_v of p-adic split classical groups of $G_n(k_v)$ [38]. This relation essentially states that the sum of the sizes of the Jordan blocks associated to π_v is equal to the dimension of the natural representation of the L-group of G_n, which is itself equal to the rank N of the general linear group GL_N to which π_v functorially lifts (Theorem 8.1). Our second application is to establishing of the basic properties of the conductor of a generic representation π_v of $G_n(k_v)$. The conductor is the exponent $f(\pi_v)$ occurring in the local ε-factor $\varepsilon(s, \pi_v, \psi_v)$. We show, as is known to be the case for general linear groups [19], that $f(\pi_v)$ is a non-negative integer and $f(\pi_v) = 0$ iff π_v is unramified (Theorem 9.1). E. Lapid has informed us that this should in turn have applications to the relative trace formula. Finally, we turn to one local application which in turn is expected to have global arithmetic applications. Using our bounds towards Ramanujan we show that the local normalized intertwining operators $N(s, \pi'_v \times \pi_v)$ with π_v a local component of a globally generic cuspidal representation π of $G_n(\mathbf{A})$ and π'_v a generic representation of $GL_m(k_v)$, are holomorphic and non-vanishing for $Re(s) \geq 0$ (Theorem 11.1). For $G_n = SO_{2n+1}$ this was done in [29]. This local result is necessary for the understanding of the global residual spectrum of the classical groups $G_n(\mathbf{A})$ [29].

While this project has been in the works for several years, the finalization of the proof of functoriality and the formulation of most of the applications took place while three of the authors were participants in the Thematic Program on Automorphic Forms held at the Fields Institute for Research in the Mathematical Sciences in the spring of 2003. We would like to thank the Fields Institute for providing us with a wonderful working environment. We would also like to thank D. Ban, C. Jantzen, G. Muić, and M. Tadić for helpful discussions on the classification of generic discrete series representations. Finally, we thank the referee for several pertinent comments and corrections.

CONTENTS

1. Functoriality for classical groups

Let k be a number field and let $\mathbf{A} = \mathbf{A}_k$ be its ring of adeles. We fix a non-trivial continuous additive character ψ of $\mathbf{A}$ which is trivial on the principal adeles k.

We will let G_n denote a split classical group of rank n defined over k. More specifically, we will consider the following three cases.

(i) *Odd orthogonal groups.* In this case $G_n = SO_{2n+1}$, the split special orthogonal group in $2n+1$ variables defined over k. For definiteness, we will take G_n as the connected component of the isometry group of the form $\Phi_{2n+1} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. The connected component of the Langlands dual group of G_n is ${}^LG_n^0 = Sp_{2n}(\mathbf{C})$.

(ii) *Even orthogonal groups.* In this case $G_n = SO_{2n}$, the split special orthogonal group in $2n$ variables defined over k. We will again take G_n as the connected component of the isometry group of the form $\Phi_{2n} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. The connected component of the Langlands dual group of G_n is ${}^LG_n^0 = SO_{2n}(\mathbf{C})$.

(iii) *Symplectic groups.* In this case $G_n = Sp_{2n}$, the symplectic group in $2n$ variables defined over k. For definiteness, we will take G_n as the isometry group of the alternating form $J_{2n} = \begin{pmatrix} 0 & \Phi_n \\ -\Phi_n & 0 \end{pmatrix}$. The connected component of the Langlands dual group of G_n is ${}^LG_n^0 = SO_{2n+1}(\mathbf{C})$.

In these realizations we can take the standard Borel subgroup of G_n to be represented by upper triangular matrices. We will denote this Borel subgroup by B_n and its unipotent radical by U_n. The abelianization of U_n is a direct sum of copies of k and we may use ψ to define a non-degenerate character of $U_n(\mathbf{A})$ which is trivial on $U_n(k)$. By abuse of notation we continue to call this character ψ.

In each of these cases there is a general linear group GL_N such that ${}^L G_n^0$ embeds naturally in $\mathrm{GL}_N(\mathbf{C}) = {}^L\mathrm{GL}_N^0$. Since both G_n and GL_N are split, this embedding completely determines an L-homomorphism $\iota : {}^L G_n \hookrightarrow {}^L\mathrm{GL}_N$ by extending ι to be the identity on the Weil group. By Langlands' principle of functoriality [1, 2, 5], associated to these L-homomorphisms there should be a *transfer* or *lift* of automorphic representations from $G_n(\mathbf{A})$ to $\mathrm{GL}_N(\mathbf{A})$ as in the following table.

G_n	${}^L G_n^0$	$\iota : {}^L G_n^0 \hookrightarrow {}^L\mathrm{GL}_N^0$	${}^L\mathrm{GL}_N^0$	GL_N
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbf{C})$	$\mathrm{Sp}_{2n}(\mathbf{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbf{C})$	$\mathrm{GL}_{2n}(\mathbf{C})$	GL_{2n}
SO_{2n}	$\mathrm{SO}_{2n}(\mathbf{C})$	$\mathrm{SO}_{2n}(\mathbf{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbf{C})$	$\mathrm{GL}_{2n}(\mathbf{C})$	GL_{2n}
Sp_{2n}	$\mathrm{SO}_{2n+1}(\mathbf{C})$	$\mathrm{SO}_{2n+1}(\mathbf{C}) \hookrightarrow \mathrm{GL}_{2n+1}(\mathbf{C})$	$\mathrm{GL}_{2n+1}(\mathbf{C})$	GL_{2n+1}

To be more precise, let $\pi = \otimes' \pi_v$ be an irreducible automorphic representation of $G_n(\mathbf{A})$.

For v a finite place of k where π_v is unramified the representation π_v of $G_n(k_v)$ is completely determined by its Satake parameter, a semi-simple conjugacy class $[t_v]$ in ${}^L G_n^0$ [2, 47]. $[t_v]$ then determines a semi-simple conjugacy class $[\iota(t_v)]$ in ${}^L\mathrm{GL}_N^0$. An unramified irreducible admissible representation Π_v of $\mathrm{GL}_N(k_v)$ is called the *local functorial lift* of π_v if its associated semi-simple conjugacy class in ${}^L\mathrm{GL}_N^0$ is $[\iota(t_v)]$, or equivalently, $L(s, \Pi_v) = \det(I - t_v q_v^{-s})^{-1} = L(s, \pi_v)$.

If v is an archimedean place, then by the arithmetic Langlands classification π_v is determined by an admissible homomorphism $\varphi_v : W_v \longrightarrow {}^L G_n^0$ where W_v is the local Weil group of k_v [2, 34]. The composition $\iota \circ \varphi_v$ is an admissible homomorphism of W_v into ${}^L\mathrm{GL}_N^0$ and hence determines a representation Π_v of $\mathrm{GL}_N(k_v)$ such that $L(s, \Pi_v) = L(s, \pi_v)$. This is again the *local functorial lift* of π_v.

An irreducible automorphic representation $\Pi = \otimes' \Pi_v$ of $\mathrm{GL}_N(\mathbf{A})$ is called a *functorial lift* of π if for every archimedean place v and for almost all non-archimedean places v for which π_v is unramified we have that Π_v is a local functorial lift of π_v. In particular this entails an equality of (partial) Langlands L-functions $L^S(s, \Pi) = \prod_{v \notin S} L(s, \Pi_v) = \prod_{v \notin S} L(s, \pi_v) = L^S(s, \pi)$. (We had previously referred to this lift as a weak lift, but there is nothing weak about it. This definition of a functorial lift is consistent with the formulations in [1, 5, 36].)

Let π be an irreducible cuspidal representation of $G_n(\mathbf{A})$. We say that π is *globally generic* if there is a cusp form $\varphi \in V_\pi$ such that φ has a non-vanishing ψ-Fourier coefficient along U_n, i.e., such that

$$\int_{U_n(k)\backslash U_n(\mathbf{A})} \varphi(ug)\psi^{-1}(u)\ du \neq 0.$$

Cuspidal automorphic representations of GL_n are always globally generic in this sense. For cuspidal automorphic representations of the classical groups this is a condition. In general the notion of being globally generic may depend on the choice of splitting of the group. However, as is shown in the Appendix to this paper, given a π which is globally generic with respect to some splitting there is always an "outer twist" which is globally generic with respect to a fixed splitting. This outer twist provides an abstract isomorphism between globally generic cuspidal representations and will not effect the L- or ε-factors nor the notion of the functorial lift. Hence we lose no generality in considering cuspidal representations that are globally generic with respect to our fixed splitting.

The principle result that we will prove in this paper is the following.

Theorem **1.1.** — *Let k be a number field and let π be an irreducible globally generic cuspidal automorphic representation of* $G_n(\mathbf{A})$. *Then π has a functorial lift to* $GL_N(\mathbf{A})$.

The low dimensional cases of this theorem, that is, when $n = 1$, are already well understood. As we will need them in the later sections of this paper, let us review them briefly here.

(i) *Odd orthogonal groups.* When $n = 1$ the split $SO_3 \simeq PGL_2$. The associated lifting from PGL_2 to GL_2 simply takes a representation π of $PGL_2(\mathbf{A})$ and views it as a representation Π of $GL_2(\mathbf{A})$ having trivial central character.

(ii) *Even orthogonal groups.* When $n = 1$ the split $SO_2 \simeq \mathbf{G}_m \simeq GL_1$. The natural embedding of L-groups then embeds GL_1 in GL_2 as a split rank one torus. The associated lifting then takes a character μ of $\mathbf{A}^\times \simeq GL_1(\mathbf{A})$ to the appropriate constituent of the induced representation $\mathrm{Ind}(\mu \otimes \mu^{-1})$, namely the isobaric sum $\Pi = \mu \boxplus \mu^{-1}$ which takes the local Langlands quotient at each place if there is reducibility [35]. Let us note that if we take a character μ of $\mathbf{A}^\times$ and let π_μ be the corresponding representation of $SO_2(\mathbf{A})$ then the standard L-function of π_μ is the degree two L-function associated to the standard embedding of L-groups discussed above, so indeed we have $L(s, \pi_\mu) = L(s, \mu)L(s, \mu^{-1}) = L(s, \mu \boxplus \mu^{-1})$, with similar equalities locally. In what follows, we will make recourse to the work of Mœglin and Tadić on local discrete series representations [39, 40]. In keeping with their conventions, we will adopt the convention that $SO_2(k_v)$, for a p-adic local field k_v, has no supercuspidal representations, nor discrete series representations.

(iii) *Symplectic groups.* When $n = 1$ then $Sp_2 \simeq SL_2$ and this functoriality is also well understood. The map on dual groups is then $PGL_2(\mathbf{C}) \simeq SO_3(\mathbf{C}) \hookrightarrow GL_3(\mathbf{C})$, which is the adjoint representation of $PGL_2(\mathbf{C})$. Thus if π is a generic cuspidal representation of $Sp_2(\mathbf{A}) \simeq SL_2(\mathbf{A})$ then its functorial lift Π to $GL_3(\mathbf{A})$ is the adjoint square lifting of Gelbart and Jacquet [11].

Thus we will concentrate primarily on the cases where $n \geq 2$.

The preparations for and proof of Theorem 1.1 when $n \geq 2$ will take place over the next five sections. Note that the case of $G_n = SO_{2n+1}$ is completely contained in our previous paper [6], but we include it here to provide a uniform treatment of all classical groups.

2. The Converse Theorem

In order to effect the functorial lifting from G_n to GL_N we will use the Converse Theorem for GL_N as we did in [6]. We give the formulation here.

Let us fix a finite set S of finite places of k. For each integer m, let

$$\mathscr{A}_0(m) = \{\tau \mid \tau \text{ is a cuspidal representation of } GL_m(\mathbf{A})\}$$
$$\mathscr{A}_0^S(m) = \{\tau \in \mathscr{A}_0(m) \mid \tau_v \text{ is unramified for all } v \in S\}.$$

We set

$$\mathscr{T}(N-1) = \coprod_{m=1}^{N-1} \mathscr{A}_0(m) \quad \text{and} \quad \mathscr{T}^S(N-1) = \coprod_{m=1}^{N-1} \mathscr{A}_0^S(m).$$

If η is a continuous character of $k^\times \backslash \mathbf{A}^\times$, let us set

$$\mathscr{T}(S;\eta) = \mathscr{T}^S(N-1) \otimes \eta = \{\tau = \tau' \otimes \eta \ : \ \tau' \in \mathscr{T}^S(N-1)\}.$$

Theorem **2.1** *(Converse Theorem). — Let* $\Pi = \otimes' \Pi_v$ *be an irreducible admissible representation of* $GL_N(\mathbf{A})$ *whose central character* ω_Π *is invariant under* $k^\times$ *and whose* L*-function* $L(s,\Pi) = \prod_v L(s,\Pi_v)$ *is absolutely convergent in some right half plane. Let* S *be a finite set of finite places of k and let* η *be a continuous character of* $k^\times \backslash \mathbf{A}^\times$*. Suppose that for every* $\tau \in \mathscr{T}(S;\eta)$ *the L-function* $L(s, \Pi \times \tau)$ *is* nice*, that is, satisfies*

1. $L(s,\Pi\times\tau)$ *and* $L(s,\widetilde{\Pi}\times\tilde{\tau})$ *extend to entire functions of* $s \in \mathbf{C}$,
2. $L(s,\Pi\times\tau)$ *and* $L(s,\widetilde{\Pi}\times\tilde{\tau})$ *are bounded in vertical strips, and*
3. $L(s,\Pi\times\tau)$ *satisfies the functional equation*

$$L(s,\Pi\times\tau) = \varepsilon(s,\Pi\times\tau)L(1-s,\widetilde{\Pi}\times\tilde{\tau}).$$

Then there exists an automorphic representation Π' *of* $GL_N(\mathbf{A})$ *such that* $\Pi_v \simeq \Pi'_v$ *for almost all v. More precisely,* $\Pi_v \simeq \Pi'_v$ *for all* $v \notin S$.

In the statement of the theorem, the twisted L- and ε-factors are defined by the products

$$L(s,\Pi\times\tau) = \prod_v L(s,\Pi_v\times\tau_v) \qquad \varepsilon(s,\Pi\times\tau) = \prod_v \varepsilon(s,\Pi_v\times\tau_v,\psi_v)$$

of local factors as in [7,6].

To motivate the next few sections, let us describe how we will apply this theorem to the problem of Langlands lifting from G_n to GL_N. We begin with our globally generic cuspidal automorphic representation $\pi = \otimes' \pi_v$ of $G_n(\mathbf{A})$. For each place v we need to associate to π_v an irreducible admissible representation Π_v of $GL_N(k_v)$ such that for every $\tau \in \mathcal{T}(S; \eta)$ we have

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v)$$
$$\varepsilon(s, \pi_v \times \tau_v, \psi_v) = \varepsilon(s, \Pi_v \times \tau_v, \psi_v).$$

For archimedean places v and those non-archimedean v where π_v is unramified, we take Π_v to be the local functorial lift of π_v described above. For those places v where π_v is ramified, we will take for Π_v an essentially arbitrary irreducible admissible generic representation of $GL_N(k_v)$ having trivial central character. However, we must choose our finite set of places S of k such that S contains the places where π_v is ramified and choose our character η of $k^\times \backslash \mathbf{A}^\times$ such that η_v is sufficiently highly ramified so that $L(s, \pi_v \times \eta_v)$, $L(s, \Pi_v \times \eta_v)$, $\varepsilon(s, \pi_v \times \eta_v, \psi_v)$, and $\varepsilon(s, \Pi_v \times \eta_v, \psi_v)$ are all standard. This will be possible by the result on the stability of the local γ-factors that we establish in Section 4.

Now consider the restricted tensor product $\Pi = \otimes' \Pi_v$. This is an irreducible representation of $GL_N(\mathbf{A})$. With the choices above we have

$$L(s, \pi \times \tau) = L(s, \Pi \times \tau)$$
$$\varepsilon(s, \pi \times \tau) = \varepsilon(s, \Pi \times \tau)$$

for $\mathrm{R}e(s) \gg 0$ and all $\tau \in \mathcal{T}(S; \eta)$. This is our candidate lift.

The theory of L-functions for $G_n \times GL_m$, which we address in the next section, will then guarantee that the twisted L-functions $L(s, \pi \times \tau)$ are nice for all $\tau \in \mathcal{T}(S; \eta)$. Then the $L(s, \Pi \times \tau)$ will also be nice and Π satisfies the hypotheses of the Converse Theorem. Hence there exists an irreducible automorphic representation Π' of $GL_N(\mathbf{A})$ such that $\Pi_v \simeq \Pi'_v$ for all archimedean v and almost all finite v where π_v is unramified. Hence Π' is a functorial lift of π.

3. L-functions for $\mathbf{G}_n \times \mathbf{GL}_m$

Let π be a globally generic cuspidal representation of $G_n(\mathbf{A})$. For τ a cuspidal representation of $GL_m(\mathbf{A})$ we will let $L(s, \pi \times \tau)$ be the completed L-function as defined in [51] via the theory of Eisenstein series. The local factors are then defined via the arithmetic Langlands classification for the archimedean places, through the Satake parameters for the finite unramified places, as given by the poles of the associated γ-factors (or local coefficients) if π_v and τ_v are tempered, by analytic extension if π_v

and τ_v are quasi-tempered, and via the representation theoretic Langlands classification otherwise.

The global theory of these twisted L-functions is now quite well understood.

Theorem **3.1.** — *Let* S *be a non-empty set of finite places of k. Let* η *be a character of* $k^\times\backslash\mathbf{A}^\times$ *such that, for some* $v \in$ S*, the square* η_v^2 *is ramified. Then for all* $\tau \in \mathscr{T}(\mathrm{S}; \eta)$ *the* L*-function* $\mathrm{L}(s, \pi \times \tau)$ *is* nice, *that is,*

1. $\mathrm{L}(s, \pi \times \tau)$ *is an entire function of s,*
2. $\mathrm{L}(s, \pi \times \tau)$ *is bounded in vertical strips of finite width, and*
3. we have the functional equation

$$\mathrm{L}(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau)\mathrm{L}(1 - s, \tilde{\pi} \times \tilde{\tau}).$$

Proof. — (1) In the case of $\mathrm{G}_n = \mathrm{SO}_{2n+1}$ we explicitly established this in [6]. In all cases this now follows from the more general Proposition 2.1 of [32]. Note that in view of the results of Muić [44] and of [4], the necessary result on normalized intertwining operators, Assumption 1.1 of [32], usually referred to as Assumption A [28], is valid in all cases as proved in [28, 31]. Note that this is the only part of the theorem where the twisting by η is needed.

(2) The boundedness in vertical strips of these L-functions is known in wide generality, which includes the cases of interest to us. It follows from Corollary 4.5 of [12] and is valid for all $\tau \in \mathscr{T}(\mathrm{N} - 1)$, provided one removes neighborhoods of the finite number of possible poles of the L-function.

(3) The functional equation is also known in wide generality and is a consequence of Theorem 7.7 of [51]. It is again valid for all $\tau \in \mathscr{T}(\mathrm{N} - 1)$. □

In order to mediate between the result as stated and the references for its proof, let us recall how these twisted L-functions are obtained from the theory of Eisenstein series.

Given our classical group G_n and a general linear group GL_m with $m \geq 1$ let G_{m+n} be the classical group of the same type as G_n, but of rank $m+n$. Then if we let $\mathrm{M} = \mathrm{GL}_m \times \mathrm{G}_n$ then M is a Levi subgroup of a standard maximal parabolic subgroup $\mathrm{P} = \mathrm{P}_{m,n} \subset \mathrm{G}_{m+n}$. Let $d = m + n$ and let $\mathrm{N} = \mathrm{N}_{m,n}$ be the unipotent radical of P.

Let $\mathbf{A}^{\times,1}$ denote the group of ideles of norm 1. Fix a subgroup $\mathrm{A}_+ \subset \mathbf{A}^\times$ such that $\mathrm{A}_+ \simeq \mathbf{R}_+^\times$ and $\mathbf{A}^\times = \mathbf{A}^{\times,1} \times \mathrm{A}_+$. It suffices to assume that τ is unitary and its central character is a character of $k^\times\backslash\mathbf{A}^\times$ which is trivial on A_+. Any cuspidal representation τ of $\mathrm{GL}_r(\mathbf{A})$ can be written as $\tau \simeq \tau' \otimes |\det|^{s'}$, where τ' is unitary with central character trivial on A_+, and then $\mathrm{L}(s, \pi \times \tau) = \mathrm{L}(s + s', \pi \times \tau')$. Note that if $\tau \in \mathscr{T}(\mathrm{S}; \eta)$, then so is τ'. Hence we may assume that τ is unitary.

With π and τ as in the theorem, then $\sigma = \tilde{\tau} \otimes \pi$ is a unitary globally generic representation of $\mathrm{M}(\mathbf{A})$. As such, we can form the induced representation

$$\mathrm{I}(s, \sigma) = \mathrm{Ind}_{\mathrm{P}(\mathbf{A})}^{\mathrm{G}_d(\mathbf{A})}(|\det|^s \tilde{\tau} \otimes \pi).$$

If α is the simple root associated to the maximal parabolic subgroup P and we let, as usual, $\tilde{\alpha} = \rho_P / \langle \rho_P, \alpha \rangle$ then as in [51]

$$I(s, \sigma) = \mathrm{Ind}_{P(\mathbf{A})}^{G_d(\mathbf{A})} (e^{\langle s\tilde{\alpha}, H_P \rangle} \sigma).$$

Since the adjoint action r of ${}^L M$ on the Lie algebra ${}^L\mathfrak{n}$ of ${}^L N$ has two irreducible constituents in general, that is, $r = r_1 \oplus r_2$, then the L-functions which naturally arise in the theory of intertwining operators and Eisenstein series for these representations will be a product $L(s, \sigma, r_1)L(2s, \sigma, r_2)$ where

$$L(s, \sigma, r_1) = L(s, \pi \times \tau),$$

the L-function of interest, and a second L-function, namely

$$L(2s, \sigma, r_2) = L(2s, \tau, Sym^2) \quad \text{if} \quad G_n = SO_{2n+1},$$

and if $m \geq 2$ and $G_n = Sp_{2n}$ or $G_n = SO_{2n}$, then

$$L(2s, \sigma, r_2) = L(2s, \tau, \wedge^2).$$

In these later two cases, if $m = 1$, then $r = r_1$ is irreducible.

4. Stability of γ-factors for $G_n \times GL_1$

This section is devoted to the formulation and proof of the stability of the local γ-factors for generic representations of the split classical groups. This result is necessary for defining a suitable local lift at the non-archimedean places where we do not have the local Langlands conjecture at our disposal. Following the ideas of [8] our method will be first to express the γ-factor as the Mellin transform of a certain partial Bessel function. This has been done in our cases as well as others in [55]. Then we will analyze the asymptotics of the Bessel functions as in [8] to obtain the stability. A more complete exposition and extensions to quasi-split groups will soon be available in [10].

For this section, let k denote a non-archimedean local field of characteristic zero. Let π be a generic irreducible admissible representation of $G_n(k)$ and let η be a continuous character of $GL_1(k) \simeq k^\times$. Let ψ be a fixed non-trivial additive character of k. Let $\gamma(s, \pi \times \eta, \psi)$ be the associated γ-factor as defined in Theorem 3.5 of [51]. These are defined inductively through the local coefficients of the local induced representations analogous to those given above. They are related to the local L- and ε-factors by

$$\gamma(s, \pi \times \eta, \psi) = \frac{\varepsilon(s, \pi \times \eta, \psi) L(1 - s, \tilde{\pi} \times \eta^{-1})}{L(s, \pi \times \eta)}.$$

The main result of this section is the following.

Theorem **4.1.** — *Let* π_1 *and* π_2 *be two irreducible admissible generic representations of* $G_n(k)$. *Then for every sufficiently highly ramified character* η *of* $k^\times$ *we have*

$$\gamma(s, \pi_1 \times \eta, \psi) = \gamma(s, \pi_2 \times \eta, \psi).$$

For the case of $G_n = SO_{2n+1}$ this is [8].

4.1. *Preliminaries on Bessel functions.* — Let us review the basic definitions from Section 3 of [8]. Note that, as their proofs show, the results in Section 3 of [8] are valid for any Chevalley group over k, not just SO_{2n+1}. In this paper we specialize them to the split classical groups.

Fix $G = G_n$ and recall that $B = B_n$ is the standard upper triangular Borel subgroup of G, $T = T_n$ the standard maximal split torus of B, i.e., the diagonal matrices in G_n, and $U = U_n$ is its unipotent radical. Let Φ^+ be the set of positive roots defining U and let Δ denote the associated simple roots. Let W be the Weyl group of G. Then $W \simeq N(T)/T$ and for each $w \in W$ we choose a representative in N(T), which by abuse of notation we will continue to call w. To be specific, for what follows it will be necessary to choose the representatives as in Section 2 of [55] (see Section 4.2 below). For $\alpha \in \Phi^+$ let U_α denote the one parameter root subgroup corresponding to α [57]. For any $w \in W$ let us set

$$U_w^- = \coprod_{\substack{\alpha>0 \\ w\alpha<0}} U_\alpha \quad \text{and} \quad U_w^+ = w^{-1}Bw \cap U$$

so that $U = U_w^+ U_w^-$.

Recall that we say that $w \in W$ *supports a Bessel function* if for every $\alpha \in \Delta$ such that $w\alpha > 0$ we have that $w\alpha \in \Delta$. If we let w_0 denote the long Weyl element of W then this is equivalent to $w_0 w$ being the long Weyl element of the Levi subgroup M_w of some standard parabolic subgroup $P_w \supset B$. In this case, U_w^- is the unipotent radical of P_w. Let A_w denote the center of M_w. Then

$$A_w = \{t \in T \mid w\alpha(t) = 1 \text{ for all } \alpha \in \Delta \text{ with } w\alpha > 0\}.$$

Suppose that $w \in W$ is such that w supports a Bessel function and the only $w' \in W$ with $w' \leq w$ in the Bruhat order which support a Bessel function are w itself and the identity e. This is equivalent to P_w being a maximal parabolic subgroup. Let $\alpha = \alpha_w$ be the simple root associated to P_w. There is an injection $\alpha^\vee$ from $k^\times$ into A_w such that $\alpha(\alpha^\vee(t)) = t$ for all $t \in k^\times$ and, setting $A_w^0 = \alpha^\vee(k^\times)$, we have the decomposition $A_w = ZA_w^0$, where $Z = Z_G$ is the (finite) center of G. (See, for example, the remarks after Assumption 5.1 in [55].)

Now let π be an irreducible admissible generic representation of $G(k)$. Let $v \in V_\pi$ be such that the associated Whittaker function $W_v \in \mathcal{W}(\pi, \psi)$ satisfies $W_v(e) = 1$. Then if $w \in W$ supports a Bessel function, and is a minimal non-trivial such with respect to the Bruhat order, we may formally define the associated Bessel function as the function on $A_w^0 \simeq k^\times$ defined by

$$J_{\pi,w}(\underline{a}) = \int_{U_w^-(k)} W_v(\underline{a}wu)\psi^{-1}(u)\ du.$$

Since the arguments of Section 4 of [8] again only depended on G being a Chevalley group, then by the Corollary to Proposition 4.2 we know that $J_{\pi,w}$ exists and is independent of the choice of $v \in V_\pi$ used to define it. This function is hard to work with. As a substitute, for every compact open subgroup $Y \subset U_w^-(k)$ we define the partial Bessel function $j_{\pi,w,v,Y}(\underline{a})$ by

$$j_{\pi,w,v,Y}(\underline{a}) = \int_{Y} W_v(\underline{a}wy)\psi^{-1}(y)\ dy.$$

In the case where π and w are fixed, we will simply write $j_{\pi,w,v,Y} = j_{v,Y}$.

4.2. *An integral representation for $\gamma(s, \pi \times \eta, \psi)$.* — Our proof of the stability of the γ-factor is based upon expressing the γ-factor as the Mellin transform of one of our Bessel functions.

Proposition **4.1.** — *Let π be a generic representation of $G_n(k)$ and η a non-trivial character of $k^\times$ such that η^2 is ramified. Then for each classical group G_n there exists a Weyl element w which supports a Bessel function and is minimal, non-trivial with this property, an elementary factor $g(s, \eta)$, and a rational number δ such that for every sufficiently large open compact subset $Y \subset U_w^-(k)$, setting $j_{v,Y} = j_{\pi,w,v,Y}$, we have*

$$\gamma(s, \pi \times \eta, \psi)^{-1} = g(s, \eta) \int_{k^\times} j_{v,Y}(\underline{a})\eta(a)|a|^{s-n+\delta}\ d^\times a.$$

The data for each classical group is as follows.

(i) If $G_n = SO_{2n+1}$, then the Weyl element is $w = \begin{pmatrix} & & 1 \\ & -I_{2n-1} & \\ 1 & & \end{pmatrix}$. The elementary factor is simply $g(s, \eta) = \eta(-1)^{n+1}$ and $\delta = 1/2$.

(ii) If $G_n = SO_{2n}$, then the Weyl element is $w = \begin{pmatrix} & & 1 \\ & -K_{2n-2} & \\ 1 & & \end{pmatrix}$ where

$$K_{2n-2} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

The elementary factor is $g(s, \eta) = \eta(-1)^{n+1}\gamma(2s, \eta^2, \psi)^{-1}$ and $\delta = 1$.

(iii) If $G_n = Sp_{2n}$, then the Weyl element is $w = \begin{pmatrix} & & -1 \\ & -I_{2n-2} & \\ 1 & & \end{pmatrix}$. The elementary factor is again $g(s, \eta) = \eta(-1)^{n+1}\gamma(2s, \eta^2, \psi)^{-1}$ and now $\delta = 0$.

In all cases, $\underline{a} = \mathrm{diag}(a, 1, \ldots, 1, a^{-1}) \in A_w^0$.

This proposition is essentially Corollary 1.2 of [55]. To obtain it in this form, we must relate the Bessel functions of [55] to the ones we have defined here. While this is essentially an exercise, it will be useful to have it written down.

4.2.1. *Corrections to [55].* — We begin with some minor corrections to [55]. In that paper the relevant Weyl elements w_0 were miscalculated. This results in the following corrections (in the notation of that paper).

(i) In the case $GL_1 \times SO_{2n+1} \subset SO_{2n+3}$ the relevant Weyl element w_0 given in (4.19) is replaced by

$$w_0 = \begin{pmatrix} & & (-1)^n \\ & -I_{2n+1} & \\ (-1)^n & & \end{pmatrix}.$$

This change only effects the elementary factor $g(s, \eta)$ in a minor way. It will change formula (7.12) to

$$C(s, \eta \otimes \sigma)^{-1} = \eta(-1)^{n+1}\gamma(2s, \eta^2, \psi)^{-1} \times$$
$$\times \int_{F^\times} j_{\tilde{v}, \overline{N}_0}\left(\begin{pmatrix} h & & \\ & I_{2n-1} & \\ & & h^{-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ & -I_{2n-1} & \\ 1 & & \end{pmatrix}\right) \eta(h)|h|^{s-n+1/2}\, d^\times h$$

with a similar change of $\eta(-1)$ to $\eta(-1)^{n+1}$ in formulas (1.5) and (1.6).

(ii) In the case $GL_1 \times SO_{2n} \subset SO_{2n+2}$ the relevant Weyl element w_0 given in (4.43) is replaced by

$$w_0 = \begin{pmatrix} & & (-1)^n \\ & -K_{2n} & \\ (-1)^n & & \end{pmatrix}.$$

The source of the error is an incorrect multiplication in (4.43). This change only effects the elementary factor $g(s, \eta)$ in a minor way. It will change formula (7.13) to

$$\begin{aligned} C(s, \eta \otimes \sigma)^{-1} &= \eta(-1)^{n+1}\gamma(2s, \eta^2, \psi)^{-1} \times \\ &\times \int_{F^\times} j_{\tilde{v},\overline{N}_0}\left(\begin{pmatrix} h & & \\ & I_{2n-2} & \\ & & h^{-1} \end{pmatrix}\begin{pmatrix} & & 1 \\ & -K_{2n-2} & \\ 1 & & \end{pmatrix}\right)\eta(h)|h|^{s-n+1}\, d^\times h \end{aligned}$$

with a similar change of $\eta(-1)$ to $\eta(-1)^{n+1}$ in formulas (1.5) and (1.7).

(iii) In the case $GL_1 \times Sp_{2n} \subset Sp_{2n+2}$ the relevant Weyl element w_0 given in (4.56) is replaced by

$$w_0 = \begin{pmatrix} & & (-1)^{n+1} \\ & -I_{2n} & \\ (-1)^n & & \end{pmatrix}.$$

The source of the error is an incorrect multiplication in (4.56). This change only effects the elementary factor $g(s, \eta)$ in a minor way. It will change formula (7.26) to

$$\begin{aligned} C(s, \eta \otimes \sigma)^{-1} &= \eta(-1)^{n+1}\gamma(2s, \eta^2, \psi)^{-1} \times \\ &\times \int_{F^\times} j_{\tilde{v},\overline{N}_0}\left(\begin{pmatrix} h & & \\ & I_{2n-2} & \\ & & h^{-1} \end{pmatrix}\begin{pmatrix} & & -1 \\ & -I_{2n-2} & \\ 1 & & \end{pmatrix}\right)\eta(h)|h|^{s-n}\, d^\times h \end{aligned}$$

with a similar change of $\eta(-1)$ to $\eta(-1)^{n+1}$ in formulas (1.5) and (1.7).

4.2.2. *A comparison of Bessel functions.* — For this section, let us use $\tilde{j}$ to denote any of the Bessel functions occurring in [55]. Our goal is to express the Bessel functions $\tilde{j}_{\tilde{v},\overline{N}_0}(\dot{m})$ occurring in Corollary 1.2 of [55] in terms of those we have defined here.

(i) Let $G_n = SO_m$ with $m = 2n+1$ or $2n$. In G_{n+1} consider the standard (upper triangular) maximal parabolic subgroup $P_{n+1} = M_{n+1}N_{n+1}$ with Levi subgroup $M_{n+1} = M \simeq GL_1 \times G_n$. In our geometric model, this would be the stabilizer of the isotropic line through $(0, ..., 0, 1)$. The unipotent radical then takes the form

$$N_{n+1}(k) = \left\{ n(t) = \begin{pmatrix} 1 & t & -\frac{1}{2}\langle t, t\rangle \\ & I_m & -t^* \\ & & 1 \end{pmatrix} \,\middle|\, t \in k^m \right\}$$

where $\langle t, t\rangle = t\Phi_m {}^t t$ and t^* is the adjoint of t with respect to this form.

Let π be our generic representation of $G_n(k)$. Then in the expression for $\tilde{j}_{\tilde{v},\overline{N}_0}(\dot{m})$ from Corollary 1.2 of [55] we have that

$$\dot{m} = \begin{pmatrix} h & & \\ & I_{m-2} & \\ & & h^{-1} \end{pmatrix} w = \underline{h} w \in G_n \subset M_{n+1}$$

where w is as in our integral representation and $\tilde{v} \in \mathscr{W}(\pi, \psi)$ with $W_{\tilde{v}}(e) = 1$. Here $\overline{N}_0 \subset \overline{N}(k)$ is a (suitable) open compact subgroup of the opposite unipotent subgroup $\overline{N}$ to $N = N_{n+1}$. In fact, the formulas hold for any such choice of $\tilde{v}$ and sufficiently large $\overline{N}_0$ (see Theorem 6.2 of [55]).

We now turn to the Bessel function itself as given in Theorem 6.2 of [55]. First $\tilde{j}_{\tilde{v},\overline{N}_0}(\dot{m}) = \tilde{j}_{\tilde{v},\overline{N}_0}(\dot{m}, y_0)$ with $y_0 \in k^\times$ satisfying $\mathrm{ord}_k(y_0) = -\mathrm{cond}(\psi) - \mathrm{cond}(\eta^2)$. Then the Bessel function is given by (6.26) of [55], which we can write as

$$\tilde{j}_{\tilde{v},\overline{N}_0}(\dot{m}, y_0) = \int_{U_{M,\dot{n}} \backslash U_M} W_{\tilde{v}}(\dot{m}u^{-1}) \varphi(u\alpha^\vee(y_0)^{-1}\alpha^\vee(\dot{x}_\alpha)\overline{\dot{n}}\alpha^\vee(\dot{x}_\alpha)^{-1}\alpha^\vee(y_0)u^{-1}) \psi(u)\ du.$$

Here $\alpha^\vee : k^\times \to Z_G \backslash Z_M$, $\dot{x}_\alpha \in k^\times$ a specified choice, and φ is the characteristic function of $\overline{N}_0$. Throughout, $\dot{n}$ is a specific $Z_M^0 U_M$-orbit representative in N and $\dot{m}$ and $\overline{\dot{n}}$ are related by $w_0^{-1}\dot{n} = \dot{m}\dot{n}'\overline{\dot{n}} \in M_{n+1}N_{n+1}\overline{N}_{n+1}$.

Let us first consider the domain of integration. By Proposition 4.4 or Proposition 4.8 of [55] we have that

$$U_{M,\dot{n}} = U'_{M,\dot{m}} = \{u \in U_M \mid \dot{m}u\dot{m}^{-1} \in U_M \text{ and } \psi(\dot{m}u\dot{m}^{-1}) = \psi(u)\}.$$

In our situation, $U_M = U_n \subset G_n$ and $\dot{m} = \underline{h}w$. Then $\dot{m}u\dot{m}^{-1} \in U_n$ iff $wuw^{-1} \in \underline{h}^{-1}U_n\underline{h} = U_n$, that is, $u \in U^+_{w^{-1}} = U^+_w$. Since $\underline{h}$ acts trivially on U^+_w we see that $U_{M,\dot{n}} = U^+_w$ so that we can take $U_{M,\dot{n}} \backslash U_M \simeq U^+_w \backslash U_n$ to be U^-_w, which we note depends only upon w.

Next we turn to the effect of the cutoff characteristic function φ. Taking $u \in U^-_w$ we see that the actual domain of integration is determined by the condition

$$u\alpha^\vee(y_0)^{-1}\alpha^\vee(\dot{x}_\alpha)\overline{\dot{n}}\alpha^\vee(\dot{x}_\alpha)^{-1}\alpha^\vee(y_0)u^{-1} \in \overline{N}_0.$$

A priori, this condition depends on $\overline{\dot{n}}$ which is related to $\dot{m}$ and hence $\underline{h}$. In fact, as we shall see, this is not the case. First note that this condition is equivalent to

$$u\alpha^\vee(\dot{x}_\alpha)\overline{\dot{n}}\alpha^\vee(\dot{x}_\alpha)^{-1}u^{-1} \in \alpha^\vee(y_0)\overline{N}_0\alpha^\vee(y_0)^{-1}.$$

But $\alpha^\vee(y_0)\overline{N}_0\alpha^\vee(y_0)^{-1}$ is another compact open subgroup of the same type, so we may ignore this in our situation. As in (7.1) of [55] we write

$$w_0^{-1}\dot{n}(t) = \dot{m}\dot{n}'\overline{\dot{n}}(y) \text{ where } \overline{\dot{n}}(y) = \begin{pmatrix} 1 & & \\ y & I_m & \\ -\frac{1}{2}y^*y & -y^* & 1 \end{pmatrix}$$

with $y \in k^m$ written as a column vector. Now according to section 7 of [55] in our situation we have $t = (1, 0, ..., 0, h)$ and $\dot{x}_\alpha = h^{-1}$. Then $y^* = 2\langle t, t\rangle^{-1} t = (h^{-1}, 0, ..., 0, 1)$ and

$$\alpha^\vee(\dot{x}_\alpha)\bar{\dot{n}}(y)\alpha^\vee(\dot{x}_\alpha)^{-1} = \alpha^\vee(h^{-1})\bar{\dot{n}}(y)\alpha^\vee(h) = \bar{\dot{n}}(y')$$

where $y' = {}^t(-h, 0, ..., 0, -1)$. So the condition on the cutoff of our domain of integration is that

$$u\bar{\dot{n}}({}^t(-h, 0, ..., 0, -1))u^{-1} \in \overline{\mathrm{N}}_0.$$

For certitude, let us take $\overline{\mathrm{N}}_0$ to be defined as

$$\overline{\mathrm{N}}_0 = \left\{ \bar{n}(y) = \begin{pmatrix} 1 & & \\ y & \mathrm{I}_m & \\ -\frac{1}{2}y^*y & -y^* & 1 \end{pmatrix} \Big| y_i \in \mathfrak{p}^{-\mathrm{M}_i} \right\}$$

for some sufficiently large integer vector $\mathrm{M} = (\mathrm{M}_1, ..., \mathrm{M}_m)$. As M increases, these exhaust $\overline{\mathrm{N}}$. Now recall that $u \in \mathrm{U}_w^-$ and this means that we can write

$$u = u(x) = \begin{pmatrix} 1 & x & -\frac{1}{2}x^*x \\ & \mathrm{I}_{m-2} & -x^* \\ & & 1 \end{pmatrix} \text{ with } x \in k^{m-2}$$

which we view as embedded in M via $u \in \mathrm{U}_n \simeq \mathrm{U}_\mathrm{M} \subset \mathrm{M}$. Then in general $u\bar{n}(y)u^{-1} = \bar{n}(uy)$ with $uy \in k^m$. In our situation $y = {}^t(-h, 0, ..., 0, -1)$ and so $u(x)y = {}^t\,(\frac{1}{2}x^*x - h, {}^tx^*, -1)$. Thus our domain of integration is over $\mathrm{Y} \subset \mathrm{U}_w^-(k)$ defined by the conditions

$$\mathrm{Y} = \left\{ u = u(x) \mid x_i \in \mathfrak{p}^{-\mathrm{M}_{m-i}} \text{ with } h \equiv \frac{1}{2}x^*x \pmod{\mathfrak{p}^{-\mathrm{M}_1}} \right\}.$$

To rid ourselves of the remaining dependence on h we enlarge $\overline{\mathrm{N}}_0$, which we are allowed to do. By choosing M_1 sufficiently large, which may depend on h and M_2, ..., M_{m-1}, we obtain a domain of integration

$$\mathrm{Y} = \left\{ u = u(x) \mid x_i \in \mathfrak{p}^{-\mathrm{M}_{m-i}},\ 1 \le i \le m-2 \right\}$$

which is now independent of h and with this choice of Y and M_1 we have

$$\tilde{j}_{\tilde{v},\overline{\mathrm{N}}_0}(\underline{h}w) = j_{\pi,\tilde{v},w,\mathrm{Y}}(\underline{h}) = j_{\tilde{v},\mathrm{Y}}(\underline{h}).$$

(ii) In the symplectic case $\mathrm{G}_n = \mathrm{Sp}_{2n}$ we must use the Bessel function $\tilde{j}'_{\tilde{v},\overline{\mathrm{N}}_0}(\dot{m}) = \tilde{j}'_{\tilde{v},\overline{\mathrm{N}}_0}(\dot{m}, y_0) = \tilde{j}_{\tilde{v},\overline{\mathrm{N}}_0}(\dot{m}\mathrm{H}, y_0)$ as in (7.24) and (7.25) of [55]. Here H is the matrix

$$\mathrm{H} = \begin{pmatrix} 1 & 0 & h \\ & \mathrm{I}_{2n-2} & 0 \\ & & 1 \end{pmatrix} \in \mathrm{U}_\mathrm{M} \simeq \mathrm{U}_n.$$

Its effect in computing the Bessel function is to replace $\bar{\bar{n}}$ by $\mathrm{H}\bar{\bar{n}}\mathrm{H}^{-1}$. But by (7.27) of [55] this matrix is represented by

$$\mathrm{H}\bar{\bar{n}}\mathrm{H}^{-1} = \begin{pmatrix} 1 & & \\ y_1 & \mathrm{I}_{2n} & \\ \mathrm{Y}_1 & y_1' & 1 \end{pmatrix}$$

with $y_1' = {}^t(-h^{-1}, 0, \ldots, 0, 1)$. Comparing this with our formula for $\bar{\bar{n}}(y)$ above in the orthogonal case we see that the same analysis will go through. So in this case we also have $\tilde{j}'_{\tilde{v},\overline{\mathrm{N}}_0}(\underline{h}w) = j_{\tilde{v},\mathrm{Y}}(\underline{h})$ for any sufficiently large open compact $\mathrm{Y} \subset \mathrm{U}_w^-(k)$.

4.2.3. *Proof of Proposition 4.1.* — With the identifications above, the fact that we have the integral representation of Proposition 4.1 is simply a restatement of Corollary 1.2 of [55]. To have the Proposition as stated, we must check that each Weyl element w that occurs both supports a Bessel function and is minimal non-trivial with this property. This is easy enough to check using the criterion in terms of parabolic subgroups from Proposition 3.2 of [8] mentioned above. □

Note that in the case of $\mathrm{G}_n = \mathrm{SO}_{2n+1}$ this integral representation is the same as that of Proposition 4.1 of [8] which was derived from Soudry's integral representation.

4.3. *Asymptotics of Bessel functions.* — In this section we investigate the asymptotics of the Bessel functions $j_{v,\mathrm{Y}}(\underline{a})$ defined above.

We will follow the development presented for SO_{2n+1} in [8]. The paper [8] was written for SO_{2n+1} because that was the only case in which there was an integral representation for the γ-factor in terms of Bessel functions. This integral representation was presented in Section 2 of [8] and that section is specific to SO_{2n+1}. Section 3 and the first parts of Section 4 of [8] rely only on results about Chevalley groups from, say, Steinberg's notes on Chevalley groups [57] and hence remain valid for any of our groups G_n. The remainder of Section 4 and Section 5 up through Proposition 5.1 of [8] are more or less formal and rely only on standard facts about the Bruhat decomposition, the Bruhat order, and the fact that the Weyl element w occurring in $j_{v,\mathrm{Y}}(\underline{a})$ has the property that w supports a Bessel function and that the only Weyl elements w' with $w' \leq w$ which support a Bessel function are w itself and the identity e. These facts remain true for our w as noted above, so the results of these sections of [8] remain valid in all our present cases. In particular, quoting Proposition 5.1 of [8] we have the following.

Proposition **4.2.** — *There exists a vector* $v'_\pi \in \mathrm{V}_\pi$ *and a compact neighborhood* BK_1 *of the identity* e *in* $\mathrm{B}\backslash\mathrm{G}_n$ *such that if* χ_1 *is the characteristic function of* BK_1*, we have that for all sufficiently large compact open* $\mathrm{Y} \subset \mathrm{U}_w^-(k)$

$$j_{v,\mathrm{Y}}(\underline{a}) = \int_{\mathrm{Y}} \mathrm{W}_v(\underline{a}wy)\chi_1(\underline{a}wy)\psi^{-1}(y)\ dy + \mathrm{W}_{v'_\pi}(\underline{a}).$$

From this point on the situation is slightly different from that in [8] because in the cases $G_n = SO_{2n}$ and $G_n = Sp_{2n}$ the groups have non-trivial finite centers. Still following [8], for each simple root α let $t \mapsto u_\alpha(t)$ be the associated one parameter subgroup of U. For any positive integer M let

$$U(M) = \langle u_\alpha(t) \mid \alpha \in \Delta;\ \ |t| \le q^M \rangle.$$

This is a compact open subgroup of $U(k)$ and as M grows these exhaust U. For any $v \in V_\pi$ let us set

$$v_M = \frac{1}{Vol(U(M))} \int_{U(M)} \psi^{-1}(u)\pi(u)v\ du.$$

Since (π, V_π) is smooth this is actually a finite sum and so $v_M \in V_\pi$.

Then as noted in [8] as long as Y is sufficiently large relative to M we may choose v'_π and K_1 in Proposition 4.2 such that $K_1 \subset Stab(v_M)$ and we have

$$\int_Y W_v(\underline{a}wy)\chi_1(\underline{a}wy)\psi^{-1}(y)\ dy = \int_Y W_{v_M}(\underline{a}wy)\chi_1(\underline{a}wy)\psi^{-1}(y)\ dy.$$

Consider this latter integral. If we write $\underline{a}wy = u\underline{t}k_1$ with $u \in U(k)$, $\underline{t} \in T(k)$, and $k_1 \in K_1$, so that $u\underline{t}k_1 \in BK_1$, then since $K_1 \subset Stab(v_M)$ we have $W_{v_M}(\underline{a}wy) = \psi(u)W_{v_M}(\underline{t})$. As shown in Lemma 4.1 of [8] the support of W_{v_M} on the torus T is contained in the set

$$T_M = \{\underline{t} \in T(k) \mid \alpha(\underline{t}) \in 1 + \mathfrak{p}^M \text{ for all simple } \alpha\}.$$

For M′ a positive integer, let us set $T^1_{M'} = \{\underline{t} \in T(k) \mid \underline{t} \equiv I \pmod{\mathfrak{p}^{M'}}\}$.

Lemma **4.1.** — *For* M *sufficiently large,* $T_M \subset Z \cdot T^1_{M'}$ *where* Z *is the center of* G_n *and* $M' = M - ord(2)$.

Proof. — Let us consider the case of $G_n = SO_{2n}$. The others are handled accordingly. With our basis, we can write an element $\underline{t}$ of the torus as $\underline{t} = diag(t_1, ..., t_n, t_n^{-1} ..., t_1^{-1})$. The simple roots are then $\alpha_i(\underline{t}) = t_i/t_{i+1}$ for $i = 1, ..., n-1$ and $\alpha_n(\underline{t}) = t_{n-1}t_n$. If $\underline{t} \in T_M$ then $\alpha_{n-1}(\underline{t}) \in 1 + \mathfrak{p}^M$ and $\alpha_n(\underline{t}) \in 1 + \mathfrak{p}^M$ implies their ratio lies in there as well, that is, $t_n^2 \in 1 + \mathfrak{p}^M$.

In general, if $t^2 \in 1+\mathfrak{p}^M$ then t is a unit satisfying $t^2 - 1 \equiv 0 \pmod{\mathfrak{p}^M}$. Letting $P(X) = X^2 - 1$ we have that $ord(P'(t)) = ord(2)$ and the roots of $P(X) = 0$ in $\mathcal{O}$ are ± 1. Thus, say by Corollary 1 of Theorem 2 in ch.III, §4, no.4 of [3], we know $t \equiv \pm 1 \pmod{\mathfrak{p}^{M'}}$ where $M' = M - ord(2)$.

Thus $t_n \in \pm 1 + \mathfrak{p}^{M'}$. Then since $\alpha_{n-1}(\underline{t}) = t_{n-1}/t_n \in 1 + \mathfrak{p}^M \subset 1 + \mathfrak{p}^{M'}$ we have that $t_{n-1} \in \pm 1 + \mathfrak{p}^{M'}$ and that the sign of t_n and t_{n-1} must be the same. Continuing with the rest of the roots in this manner, we find that $\pm\underline{t} \in T^1_{M'}$ and we are done since $Z = \{\pm 1\}$. □

Hence if $\underline{t} \in T_M$ we can further write $\underline{t} = z\underline{t}^1$ with $z \in Z$ and $\underline{t}^1 \in T^1_{M'}$. It is easy to check that for $\underline{t} \in T_M$ we have $W_v(\underline{t}) = W_{v_M}(\underline{t})$, so that if we choose M from the beginning so that $T^1_{M'} \subset T \cap \mathrm{Stab}(v)$ then we see that $W_{v_M}(\underline{t}) = W_v(\underline{t}) = W_v(z\underline{t}^1) = \omega_\pi(z)W_v(\underline{t}^1) = \omega_\pi(z)$.

So, in our integral, we see that $W_{v_M}(\underline{a}wy)\chi_1(\underline{a}wy) \neq 0$ iff $\underline{a}wy \in UT_MK_1$ or $y \in (\underline{a}w)^{-1}UT_MK_1$. If we write this decomposition as $\underline{a}wy = u\underline{t}k_1 = u(\underline{a}wy)z(\underline{a}wy)\underline{t}^1k_1$, then we find

$$\int_Y W_v(\underline{a}wy)\chi_1(\underline{a}wy)\psi^{-1}(y)\, dy = \int_{Y\cap(\underline{a}w)^{-1}UT_MK_1} \psi(u(\underline{a}wy))\psi^{-1}(y)\omega_\pi(z(\underline{a}wy))\, dy.$$

Then our previous proposition can now be written as follows.

Proposition **4.3.** — *Fix $v \in V_\pi$ such that $W_v(e) = 1$ and choose M sufficiently large so that $T^1_{M'} \subset T \cap \mathrm{Stab}(v)$. There exists a vector $v'_\pi \in V_\pi$ and a compact open subgroup K_1 such that for $Y \subset U^-_w(k)$ sufficiently large we have*

$$j_{v,Y}(\underline{a}) = \int_{Y\cap(\underline{a}w)^{-1}UT_MK_1} \psi(u(\underline{a}wy))\psi^{-1}(y)\omega_\pi(z(\underline{a}wy))\, dy + W_{v'_\pi}(\underline{a}).$$

This proposition gives us the asymptotics of $j_{v,Y}(\underline{a})$ in the following sense. The function $W_{v'_\pi}$ is a smooth Whittaker function and hence vanishes for a large and exhibits the standard asymptotics of the Whittaker function as a goes to zero. Thus the integral expression contains all asymptotics of the Bessel function as a gets large. Even though this integral is a complicated exponential sum, it only depends on π through its central character ω_π.

4.4. *Stability of γ-functions depending on the central character.* — As an immediate consequence of Proposition 4.3 we obtain the following stability result.

Proposition **4.4.** — *Let π_1 and π_2 be two irreducible admissible generic representations of $G_n(k)$ having the same central character. Then for every sufficiently highly ramified character η of $k^\times$ we have*

$$\gamma(s, \pi_1 \times \eta, \psi) = \gamma(s, \pi_2 \times \eta, \psi).$$

Proof. — Let $v_i \in V_{\pi_i}$ be chosen such that for each we have $W_{v_i}(e) = 1$. Choose a large integer M such that $T^1_{M'} \subset T \cap \mathrm{Stab}(v_i)$. Let K_0 be a compact open subgroup of G_n such that $K_0 \subset \mathrm{Stab}(v_1) \cap \mathrm{Stab}(v_2)$. Then in Proposition 4.3 we may take

$$K_1 = \bigcap_{u\in U(M)} u^{-1}K_0u$$

as in Section 6 of [8], that is, we can take K_1 to be the same for π_1 and π_2. Then by Proposition 4.3 there exist vectors $v'_{\pi_i} \in V_{\pi_i}$ such that

$$j_{v_i,Y}(\underline{a}) = \int_{Y\cap(\underline{a}w)^{-1}UT_MK_1} \psi(u(\underline{a}wy))\psi^{-1}(y)\omega_{\pi_i}(z(\underline{a}wy))\ dy + W_{v'_{\pi_i}}(\underline{a}).$$

Since the central characters of π_1 and π_2 agree, we have

$$j_{v_1,Y}(\underline{a}) - j_{v_2,Y}(\underline{a}) = W_{v'_{\pi_1}}(\underline{a}) - W_{v'_{\pi_2}}(\underline{a}).$$

If we now turn to Proposition 4.1 we find that as long as η^2 is ramified we have

$$\begin{aligned}\gamma(s, \pi_1 \times \eta, \psi)^{-1} - &\gamma(s, \pi_2 \times \eta, \psi)^{-1} \\ &= g(s, \eta) \int_{k^\times} (j_{v_1,Y}(\underline{a}) - j_{v_2,Y}(\underline{a}))\eta(a)|a|^{s-n+\delta}\ d^\times a \\ &= g(s, \eta) \int_{k^\times} (W_{v'_{\pi_1}}(\underline{a}) - W_{v'_{\pi_2}}(\underline{a}))\eta(a)|a|^{s-n+\delta}\ d^\times a.\end{aligned}$$

But the Whittaker functions are smooth. So for $Re(s) \gg 0$ and η sufficiently highly ramified we have

$$\int_{k^\times} W_{v'_{\pi_i}}(\underline{a})\eta(a)|a|^{s-n+\delta}\ d^\times a \equiv 0.$$

Thus for $Re(s) \gg 0$ we have

$$\gamma(s, \pi_1 \times \eta, \psi)^{-1} - \gamma(s, \pi_2 \times \eta, \psi)^{-1} \equiv 0$$

and then by the principle of analytic continuation this must be true for all s. Thus

$$\gamma(s, \pi_1 \times \eta, \psi) = \gamma(s, \pi_2 \times \eta, \psi)$$

and we are done. □

4.5. *Computation of the stable forms.* — To complete the proof of Theorem 4.1, as well as for application in the proof of Theorem 1.1, we will compute an explicit formula for the stable form of the γ-factor. In order to do this, let π_1 be any irreducible admissible generic representation of $G_n(k)$ with central character ω. Take $\mu_1, \ldots, \mu_n$ to be n characters of $k^\times$. Then $\mu_1 \otimes \cdots \otimes \mu_n$ defines a character of $T_n(k)$ and we assume that upon restriction to the center $Z_n \subset T_n(k)$ this character agrees with the central character ω of π_1. Then if we let $\pi_2 = \mathrm{Ind}_{B_n(k)}^{G_n(k)}(\mu_1 \otimes \cdots \otimes \mu_n)$ then for an appropriate choice of the μ_i (in "general position") this representation will be irreducible admissible generic and have central character ω. Thus for all sufficiently highly ramified η we have

$$\gamma(s, \pi_1 \times \eta, \psi) = \gamma(s, \pi_2 \times \eta, \psi).$$

We can explicitly compute the right hand side of this formula. By first using the multiplicativity of the γ-factor [52] we obtain

$$\gamma(s, \pi_2 \times \eta, \psi) = \prod_{j=1}^{n} \gamma(s, \mu_j\eta, \psi)\gamma\left(s, \mu_j^{-1}\eta, \psi\right)$$

if $G_n = SO_{2n+1}$ or $G_n = SO_{2n}$, while if $G_n = Sp_{2n}$ we obtain

$$\gamma(s, \pi_2 \times \eta, \psi) = \gamma(s, \eta, \psi) \prod_{j=1}^{n} \gamma(s, \mu_j\eta, \psi)\gamma\left(s, \mu_j^{-1}\eta, \psi\right).$$

This computes the stable form of γ-factor in terms of abelian γ-factors.

Proposition **4.5.** — *Let π be any irreducible admissible generic representation of $G_n(k)$ with central character ω and let $\mu_1, \ldots, \mu_n$ be any choice of characters of $k^\times$ in general position such that $\mu_1 \otimes \cdots \otimes \mu_n$ agrees with ω upon restriction to the center. Then for every sufficiently highly ramified character η we have*

$$\gamma(s,\pi \times \eta, \psi) = \begin{cases} \prod_{j=1}^{n} \gamma(s, \mu_j\eta, \psi)\gamma\left(s, \mu_j^{-1}\eta, \psi\right) & G_n = SO_{2n+1},\ SO_{2n} \\ \gamma(s, \eta, \psi)\prod_{j=1}^{n} \gamma(s, \mu_j\eta, \psi)\gamma\left(s, \mu_j^{-1}\eta, \psi\right) & G_n = Sp_{2n} \end{cases}.$$

4.6. *Proof of Theorem 4.1.* — To complete the proof of Theorem 4.1 it will suffice to show that the stable form of the γ-factor computed in Proposition 4.5 is actually independent of the central character ω. There is an elementary reason for this (see the comments at the end of this section), but a reason which is particularly adapted to our application is the following.

First take G_n to be SO_{2n+1} or SO_{2n}. Then in either case the standard embedding of the L-groups predicts a functoriality to GL_N with $N = 2n$. In either of these cases, let Π be the induced representation of $GL_N(k)$ induced from these same characters, that is,

$$\Pi = \mathrm{Ind}_{B_N(k)}^{GL_N(k)}\left(\mu_1 \otimes \cdots \otimes \mu_n \otimes \mu_n^{-1} \otimes \cdots \otimes \mu_1^{-1}\right).$$

Then Π is a generic representation of $GL_N(k)$ having trivial central character and by multiplicativity of the γ-factors for GL_N [20] we also have

$$\gamma(s, \Pi \times \eta, \psi) = \prod_{j=1}^{n} \gamma(s, \mu_j\eta, \psi)\gamma\left(s, \mu_j^{-1}\eta, \psi\right).$$

Thus

$$\gamma(s, \pi \times \eta, \psi) = \gamma(s, \Pi \times \eta, \psi).$$

On the other hand, by the stability of γ-factors for GL_N [23] we know that the stable form of the γ-factor on GL_N depends only on the central character. Since Π has trivial central character no matter the central character ω of π, we see that the stable form of the γ-factor for G_n is independent of the central character. This establishes Theorem 4.1 in these cases.

The case of $G_n = Sp_{2n}$ is similar. Take π an irreducible admissible generic representation of $Sp_{2n}(k)$ and take characters $\mu_1, \ldots \mu_n$ so that for sufficiently ramified η we have

$$\gamma(s, \pi \times \eta, \psi) = \gamma(s, \eta, \psi) \prod_{j=1}^{n} \gamma(s, \mu_j \eta, \psi) \gamma\left(s, \mu_j^{-1} \eta, \psi\right).$$

Now the functorial lift should be to GL_{2n+1}, so we take Π to be the generic representation of $GL_{2n+1}(k)$ with trivial central character given by

$$\Pi = \operatorname{Ind}_{B_{2n+1}(k)}^{GL_{2n+1}(k)} \left(\mu_1 \otimes \cdots \otimes \mu_n \otimes 1 \otimes \mu_n^{-1} \otimes \cdots \otimes \mu_1^{-1}\right).$$

Then multiplicativity of γ-factors for GL_N [20] gives

$$\gamma(s, \Pi \times \eta, \psi) = \gamma(s, \eta, \psi) \prod_{j=1}^{n} \gamma(s, \mu_j \eta, \psi) \gamma\left(s, \mu_j^{-1} \eta, \psi\right)$$

as well, so that

$$\gamma(s, \pi \times \eta, \psi) = \gamma(s, \Pi \times \eta, \psi)$$

for all sufficiently highly ramified η. But again the stable form of the γ factor for GL_N depends only on the central character of Π [23], which is trivial no matter what the central character of π. Thus the stable form of $\gamma(s, \pi \times \eta, \psi)$ is independent of the central character of π as well. This completes the proof of Theorem 4.1 in this case as well. □

We end this section with two corollaries of our stability results. The first is a corollary of Proposition 4.5 combined with Theorem 4.1 and the following observations. In the notation of Proposition 4.5, for η sufficiently highly ramified, each $\mu_i \eta$ will also be highly ramified, so that $L(s, \mu_j \eta) \equiv 1$. Then $\gamma(s, \mu_j \eta, \psi) = \varepsilon(s, \mu_j \eta, \psi)$. Similarly, by [54] as soon as η is sufficiently highly ramified we have $L(s, \pi \times \eta) \equiv 1$, so that $\gamma(s, \pi \times \eta, \psi) = \varepsilon(s, \pi \times \eta, \psi)$ as well. Thus we obtain the stability of local ε-factors as well as their stable form.

Corollary **4.1.** — *Let* π *be an irreducible admissible generic representation of* $G_n(k)$ *and let* $\mu_1, \ldots, \mu_n$ *be characters of* $k^\times$ *in general position. Then for every sufficiently ramified character* η *we have*

$$\varepsilon(s,\pi \times \eta, \psi) = \begin{cases} \prod_{j=1}^{n} \varepsilon(s, \mu_j\eta, \psi)\varepsilon\big(s, \mu_j^{-1}\eta, \psi\big) & G_n = SO_{2n+1},\ SO_{2n} \\ \varepsilon(s, \eta, \psi)\prod_{j=1}^{n} \varepsilon(s, \mu_j\eta, \psi)\varepsilon\big(s, \mu_j^{-1}\eta, \psi\big) & G_n = Sp_{2n} \end{cases}.$$

Our second corollary combines the proof of Theorem 4.1 with the above observations on the stability of the local L-factors, both for G_n and GL_N.

Corollary **4.2.** — *Let* π *be an irreducible admissible generic representation of* $G_n(k)$. *Let* Π *be any irreducible admissible representation of* $GL_N(k)$ *with trivial central character (*N *as in Theorem 1.1). Then for all sufficiently ramified characters* η *of* $k^\times$ *we have*

$$L(s, \pi \times \eta) \equiv 1 \equiv L(s, \Pi \times \eta) \quad \textit{and} \quad \varepsilon(s, \pi \times \eta, \psi) = \varepsilon(s, \Pi \times \eta, \psi).$$

As was pointed out by the referee, the formulas in Proposition 4.5 and Corollary 4.1 can be simplified as follows. As we noted above, for highly ramified characters, there is no difference in the γ-factors and the ε-factors. The ε-factors for characters of $k^\times$ can then be computed via Gauss sums. As long as η is sufficiently highly ramified with respect to μ we have that there exists c_η such that $\varepsilon(s, \mu\eta, \psi) = \mu(c_\eta)\varepsilon(s, \eta, \psi)$. Thus under these conditions we have

$$\varepsilon(s, \mu\eta, \psi)\varepsilon(s, \mu^{-1}\eta, \psi) = \varepsilon(s, \eta, \psi)^2$$

which then leads to

$$\gamma(s, \pi \times \eta, \psi) = \gamma(s, \eta, \psi)^N$$

in Proposition 4.5 in all cases and

$$\varepsilon(s, \pi \times \eta, \psi) = \varepsilon(s, \eta, \psi)^N$$

in Corollary 4.1 in all cases, where the natural functoriality is from G_n to GL_N. In the case of Proposition 4.5 this formula provides the elementary proof of the independence of the stable form from the central character of π alluded to above. We chose to leave our original proof since it then leads naturally to Corollary 4.2. These formulas for the stable form can then be obtained after the fact by using stability and then taking π to be induced from trivial characters (again as was pointed out by the referee).

5. The candidate lift

We now return to k denoting a number field. Let $\pi = \otimes' \pi_v$ be a globally generic cuspidal representation of $G_n(\mathbf{A})$. In this section we will construct our candidate $\Pi = \otimes' \Pi_v$ for the functorial lift of π as an irreducible admissible representation of $GL_N(\mathbf{A})$. We will construct Π by constructing each local component, or local lift, Π_v. There will be three cases: (i) the archimedean lift, (ii) the non-archimedean unramified lift, and finally (iii) the non-archimedean ramified lift.

5.1. *The archimedean lift.* — Let v be an archimedean place of k. By the arithmetic Langlands classification [34, 2], π_v is parameterized by an admissible homomorphism $\phi_v : W_{k_v} \to {}^L G_n^0$ where W_{k_v} is the Weil group of k_v. By composing with $\iota : {}^L G_n(\mathbf{C}) \hookrightarrow GL_N(\mathbf{C})$ we have an admissible homomorphism $\Phi_v = \iota \circ \phi_v : W_{k_v} \longrightarrow GL_N(\mathbf{C})$ and this defines an irreducible admissible representation Π_v of $GL_N(k_v)$.

$$\begin{array}{ccccccc} & & {}^L G_n & \xrightarrow{\iota} & {}^L GL_N & & \\ \pi_v & \longrightarrow & \phi_v \nwarrow & & \nearrow \Phi_v & \longrightarrow & \Pi_v. \\ & & & W_{k_v} & & & \end{array}$$

Then Π_v is the local functorial lift of π_v. We take Π_v as our local lift of π_v.

The local archimedean L- and ε-factors defined via the theory of Eisenstein series that we are using are the same as the Artin factors defined through the arithmetic Langlands classification [49]. Since the embedding $\iota : {}^L G_n(\mathbf{C}) \hookrightarrow GL_N(\mathbf{C})$ is the standard representation of the L-group of $G_n(k_v)$ then by the definition of the local L- and ε-factors given in [2] we have

$$L(s, \pi_v) = L(s, \iota \circ \phi_v) = L(s, \Pi_v)$$

and

$$\varepsilon(s, \pi_v, \psi_v) = \varepsilon(s, \iota \circ \phi_v, \psi_v) = \varepsilon(s, \Pi_v, \psi_v)$$

where in both instances the middle factor is the local Artin-Weil L- and ε-factor attached to representations of the Weil group as in [59].

If τ_v is an irreducible admissible representation of $GL_m(k_v)$ then it is in turn parameterized by an admissible homomorphism $\phi'_v : W_{k_v} \longrightarrow GL_m(\mathbf{C})$. Then the tensor product homomorphism $(\iota \circ \phi_v) \otimes \phi'_v : \mathrm{W}_{k_v} \longrightarrow \mathrm{GL}_{m\mathrm{N}}(\mathbf{C})$ is admissible and again we have by definition

$$\mathrm{L}(s, \pi_v \times \tau_v) = \mathrm{L}(s, (\iota \circ \phi_v) \otimes \phi'_v) = \mathrm{L}(s, \Pi_v \times \tau_v)$$

and

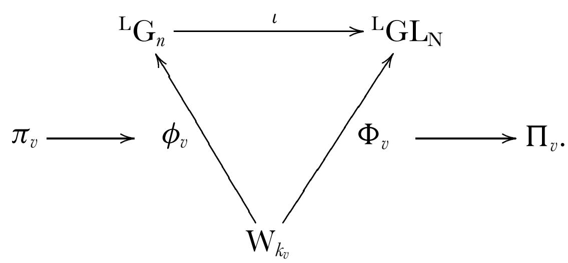
$$\varepsilon(s, \pi_v \times \tau_v, \psi_v) = \varepsilon(s, (\iota \circ \phi_v) \otimes \phi'_v, \psi_v) = \varepsilon(s, \Pi_v \times \tau_v, \psi_v).$$

This then gives the following matching of the twisted local L- and ε-factors.

Proposition **5.1.** — *Let v be an archimedean place of k and let π_v be an irreducible admissible generic representation of* $\mathrm{G}_n(k_v)$*,* Π_v *its local functorial lift to* $\mathrm{GL}_\mathrm{N}(k_v)$*, and* τ_v *an irreducible admissible generic representation of* $\mathrm{GL}_m(k_v)$*. Then*

$$\mathrm{L}(s, \pi_v \times \tau_v) = \mathrm{L}(s, \Pi_v \times \tau_v) \quad \textit{and} \quad \varepsilon(s, \pi_v \times \tau_v, \psi_v) = \varepsilon(s, \Pi_v \times \tau_v, \psi_v).$$

5.2. *The non-archimedean unramified lift.* — Now let v be an non-archimedean place of k and assume that π_v is an unramified representation. By the unramified arithmetic Langlands classification or the Satake classification [2,47], π_v is parameterized by an unramified admissible homomorphism $\phi_v : \mathrm{W}_{k_v} \to {}^{\mathrm{L}}\mathrm{G}_n^0$ where W_{k_v} is the Weil group of k_v. By composing with $\iota : {}^{\mathrm{L}}\mathrm{G}_n(\mathbf{C}) \hookrightarrow \mathrm{GL}_\mathrm{N}(\mathbf{C})$ we have an unramified admissible homomorphism $\Phi_v = \iota \circ \phi_v : \mathrm{W}_{k_v} \longrightarrow \mathrm{GL}_\mathrm{N}(\mathbf{C})$ and this defines an irreducible admissible representation Π_v of $\mathrm{GL}_\mathrm{N}(k_v)$ [15,17].



Then Π_v is again the local functorial lift of π_v and we take it as our local lift.

More specifically, any irreducible admissible generic unramified representation π_v of $\mathrm{G}_n(k_v)$ occurs as a subrepresentation of an induced representation from n unramified characters $\mu_{1,v}, \ldots, \mu_{n,v}$, that is

$$\pi_v \subset \mathrm{Ind}_{\mathrm{B}_n(k_v)}^{\mathrm{G}_n(k_v)}(\mu_{1,v} \otimes \cdots \otimes \mu_{n,v}).$$

If we normalize the local class field theory isomorphism so that a geometric Frobenius F_v corresponds to the uniformizer ϖ_v of k_v, then since π_v is unramified it is determined by and determines the semi-simple conjugacy class, its Satake class, associated to the diagonal matrix

$$\phi_v(\mathrm{F}_v) = \mathrm{diag}\big(\mu_{1,v}(\varpi), \ldots, \mu_{n,v}(\varpi_v), \mu_{n,v}(\varpi_v)^{-1}, \ldots, \mu_{1,v}(\varpi_v)^{-1}\big)$$

in the cases $\mathrm{G}_n = \mathrm{SO}_{2n+1},\ \mathrm{SO}_{2n}$ and to

$$\phi_v(\mathrm{F}_v) = \mathrm{diag}\big(\mu_{1,v}(\varpi), \ldots, \mu_{n,v}(\varpi_v), 1, \mu_{n,v}(\varpi_v)^{-1}, \ldots, \mu_{1,v}(\varpi_v)^{-1}\big)$$

in the case $\mathrm{G}_n = \mathrm{Sp}_{2n}$.

Then the semi-simple conjugacy class in $\mathrm{GL}_\mathrm{N}(\mathbf{C})$ determining Π_v is $\Phi_v(\mathrm{F}_v) = \iota \circ \phi_v(\mathrm{F}_v)$ whose Satake class is represented by the same diagonal matrix viewed as an element of ${}^\mathrm{L}\mathrm{GL}_\mathrm{N}$. Hence Π_v is the unique unramified constituent of the induced representation

$$\Xi_v = \mathrm{Ind}_{\mathrm{B}_{2n}(k)}^{\mathrm{GL}_{2n}(k)}\big(\mu_{1,v} \otimes \cdots \otimes \mu_{n,v} \otimes \mu_{n,v}^{-1} \otimes \cdots \otimes \mu_{1,v}^{-1}\big)$$

in the cases $\mathrm{G}_n = \mathrm{SO}_{2n+1},\ \mathrm{SO}_{2n}$ and of

$$\Xi_v = \mathrm{Ind}_{\mathrm{B}_{2n+1}(k)}^{\mathrm{GL}_{2n+1}(k)}\big(\mu_{1,v} \otimes \cdots \otimes \mu_{n,v} \otimes 1_v \otimes \mu_{n,v}^{-1} \otimes \cdots \otimes \mu_{1,v}^{-1}\big)$$

in the case $\mathrm{G}_n = \mathrm{Sp}_{2n}$. In terms of Langlands' local isobaric sums, we have

$$\Pi_v = \begin{cases} \mu_{1,v} \boxplus \cdots \boxplus \mu_{n,v} \boxplus \mu_{n,v}^{-1} \boxplus \cdots \boxplus \mu_{1,}^{-1} & \mathrm{G}_n = \mathrm{SO}_{2n+1},\ \mathrm{SO}_{2n} \\ \mu_{1,v} \boxplus \cdots \boxplus \mu_{n,v} \boxplus 1_v \boxplus \mu_{n,v}^{-1} \boxplus \cdots \boxplus \mu_{1,v}^{-1} & \mathrm{G}_n = \mathrm{Sp}_{2n} \end{cases}.$$

We will again need to know that the twisted L- and ε-factors agree for π_v and Π_v.

Proposition **5.2.** — *Let v be a non-archimedean place of k and let π_v be an irreducible admissible generic unramified representation of* $\mathrm{G}_n(k_v)$. *Let Π_v be its functorial local lift to* $\mathrm{GL}_\mathrm{N}(k_v)$, *and τ_v an irreducible admissible generic representation of* $\mathrm{GL}_m(k_v)$. *Then*

$$\mathrm{L}(s, \pi_v \times \tau_v) = \mathrm{L}(s, \Pi_v \times \tau_v) \quad \textit{and} \quad \varepsilon(s, \pi_v \times \tau_v, \psi_v) = \varepsilon(s, \Pi_v \times \tau_v, \psi_v).$$

Proof. — We will prove this by explicitly computing both sides in terms of the above data.

On the general linear group side, either utilizing the local Langlands correspondence for GL_N over a p-adic field [15, 17], as we did in the case of archimedean fields, or directly utilizing the results of [20], specifically Theorem 3.1 and Theorem 9.5, it is routine to compute that

$$\mathrm{L}(s, \Pi_v \times \tau_v) = \prod_{j=1}^{n} \mathrm{L}(s, \tau_v \times \mu_{j,v})\mathrm{L}\big(s, \tau_v \times \mu_{j,v}^{-1}\big)$$

$$\varepsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{j=1}^{n} \varepsilon(s, \tau_v \times \mu_{j,v}, \psi_v)\varepsilon\big(s, \tau_v \times \mu_{j,v}^{-1}, \psi_v\big)$$

in the cases $G_n = SO_{2n+1}, SO_{2n}$ while

$$L(s, \Pi_v \times \tau_v) = L(s, \tau_v)\prod_{j=1}^{n} L(s, \tau_v \times \mu_{j,v})L\left(s, \tau_v \times \mu_{j,v}^{-1}\right)$$

$$\varepsilon(s, \Pi_v \times \tau_v, \psi_v) = \varepsilon(s, \tau_v, \psi_v)\prod_{j=1}^{n} \varepsilon(s, \tau_v \times \mu_{j,v}, \psi_v)\varepsilon\left(s, \tau_v \times \mu_{j,v}^{-1}, \psi_v\right)$$

in the case of $G_n = Sp_{2n}$.

For the unramified representation π_v of the classical group $G_n(k_v)$ the argument is as in [6]. First, by the multiplicativity of γ-factors [51, 52] we have that

$$\gamma(s, \pi_v \times \tau_v, \psi_v) = \prod_{j=1}^{n} \gamma(s, \tau_v \times \mu_{j,v}, \psi_v)\gamma\left(s, \tau_v \times \mu_{j,v}^{-1}, \psi_v\right)$$

in the cases $G_n = SO_{2n+1}, SO_{2n}$ and that

$$\gamma(s, \pi_v \times \tau_v, \psi_v) = \gamma(s, \tau_v, \psi_v)\prod_{j=1}^{n} \gamma(s, \tau_v \times \mu_{j,v}, \psi_v)\gamma\left(s, \tau_v \times \mu_{j,v}^{-1}, \psi_v\right)$$

for $G_n = Sp_{2n}$. Hence to obtain the factorization of the ε-factors it suffices to combine this with the factorization of the L-factors.

To obtain the factorization of the L-functions we will use the definition of the L-functions as in [51]. Since π_v and τ_v are generic, then they are both full induced from generic tempered representations in Langlands order. For the classical groups this is Muić (see Theorem 5.1 of [43] or Theorem 1.1 of [44]) while for the linear groups it is Zelevinsky [61] or Jacquet and Shalika [22]. Thus we may write

$$\pi_v \simeq \mathrm{Ind}_{Q(k_v)}^{G_n(k_v)}\left(\pi'_{1,v}\nu^{a_1} \otimes \cdots \otimes \pi'_{r,v}\nu^{a_r} \otimes \pi''_v\right)$$

with each $\pi'_{j,v}$ tempered on some $GL_{n_j}(k_v)$, ν the character $\nu(g) = |\det(g)|_v$ for $g \in GL_{n_j}(k_v)$, π''_v tempered on $G_{n_0}(k_v)$, $a_1 > \ldots > a_r$, and Q the standard parabolic with Levi of the form $GL_{n_1} \times \cdots \times GL_{n_r} \times G_{n_0}$. Similarly

$$\tau_v \simeq \mathrm{Ind}_{Q'(k_v)}^{GL_m(k_v)}\left(\tau'_{1,v}\nu^{b_1} \otimes \cdots \otimes \tau'_{t,v}\nu^{b_t}\right)$$

with each $\tau'_{i,v}$ tempered on some $GL_{m_i}(k_v)$, $b_1 > \cdots > b_t$, and Q′ the standard parabolic with Levi $GL_{m_1} \times \cdots \times GL_{m_t}$. Note that under our assumptions, each $\pi'_{j,v}$ is full induced from unitary characters and π''_v is the unique irreducible generic unramified subrepresentation of such. Then by definition ([51], Section 7)

$$L(s, \pi_v \times \tau_v) = \prod_{i,j} L\left(s + a_j + b_i, \pi'_{j,v} \times \tau'_{i,v}\right)L\left(s - a_j + b_i, \tilde{\pi}'_{j,v} \times \tau'_{i,v}\right) \times$$
$$\times \prod_i L\left(s + b_i, \pi''_v \times \tau'_{i,v}\right).$$

Now consider the factors on the right hand side. Begin with the $GL_{n_j} \times GL_{m_i}$ factors. Since $\pi'_{j,v}$ is a full induced from unitary characters, say $\pi'_{j,v} \simeq \mathrm{Ind}(\mu^j_{1,v} \otimes \cdots \otimes \mu^j_{n_j,v})$, and the fact that $\tau'_{i,v}$ is tempered, then by either [20] or [52] we have

$$L(s, \pi'_{j,v} \times \tau'_{i,v}) = \prod_{\ell} L(s, \mu^j_{\ell,v} \times \tau'_{i,v}).$$

The results of [52] apply equally well to $G_{n_0} \times GL_{m_i}$ and if we write $\pi''_v \subset \mathrm{Ind}(\mu''_{1,v} \otimes \cdots \otimes \mu''_{\ell,v})$ with the $\mu''_{j,v}$ unitary, then by Theorem 5.2 of [52] we have

$$L(s, \pi''_v \times \tau'_{i,v}) = \prod_{j=1}^{\ell} L(s, \tau'_{i,v} \times \mu''_{j,v}) L(s, \tau'_{i,v} \times \mu''^{\,-1}_{j,v})$$

in the cases $G_n = SO_{2n+1}, SO_{2n}$ and

$$L(s, \pi''_v \times \tau'_{i,v}) = L(s, \tau'_{i,v}) \prod_{j=1}^{\ell} L(s, \tau'_{i,v} \times \mu''_{j,v}) L(s, \tau'_{i,v} \times \mu''^{\,-1}_{j,v})$$

for $G_n = Sp_{2n}$. Note that Conjecture 5.1 of [52], which is a hypothesis of Theorem 5.2 there, is known in our case by Theorem 4.1 of [4].

We have now factored the L-functions for π_v all the way down to the characters occurring in its Satake class $\phi_v(F_v)$. If we now reconstruct these decompositions we find

$$L(s, \pi_v \times \tau_v) = \prod_{j=1}^{n} L(s, \tau_v \times \mu_{j,v}) L(s, \tau_v \times \mu^{-1}_{j,v})$$

when $G_n = SO_{2n+1}, SO_{2n}$ and

$$L(s, \pi_v \times \tau_v) = L(s, \tau_v) \prod_{j=1}^{n} L(s, \tau_v \times \mu_{j,v}) L(s, \tau_v \times \mu^{-1}_{j,v})$$

for $G_n = Sp_{2n}$. If we combine this with our factorization of the γ-factor above we obtain

$$\varepsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{j=1}^{n} \varepsilon(s, \tau_v \times \mu_{j,v}, \psi_v) \varepsilon(s, \tau_v \times \mu^{-1}_{j,v}, \psi_v)$$

for $G_n = SO_{2n+1}, SO_{2n}$ and

$$\varepsilon(s, \pi_v \times \tau_v, \psi_v) = \varepsilon(s, \tau_v, \psi_v) \prod_{j=1}^{n} \varepsilon(s, \tau_v \times \mu_{j,v}, \psi_v) \varepsilon(s, \tau_v \times \mu^{-1}_{j,v}, \psi_v)$$

when $G_n = Sp_{2n}$.

Comparing our expressions for $L(s, \Pi_v \times \tau_v)$ and $L(s, \pi_v \times \tau_v)$ as well as those for the ε-factors, we obtain our result. □

5.3. *The non-archimedean ramified lift.* — We are left with the case of a non-archimedean place v of k where the local component π_v of π is ramified. Now we do not have the local Langlands correspondence to give us a natural local functorial lift. Instead we will use the results of Section 4.

In this case, that is when π_v is ramified, we take for our local lift any irreducible admissible representation Π_v of $\mathrm{GL}_N(k_v)$ having trivial central character. For simplicity we will take Π_v to be self-contragredient as well, but this is not essential. Given π_v and this Π_v then by the results of Section 4, particularly Corollary 4.2, we know that for every sufficiently highly ramified character η_v of $\mathrm{GL}_1(k_v)$ we have

$$\begin{aligned} &L(s, \pi_v \times \eta_v) \equiv 1 \equiv L(s, \Pi_v \times \eta_v) \quad \text{and} \\ &\varepsilon(s, \pi_v \times \eta_v, \psi_v) = \varepsilon(s, \Pi_v \times \eta_v, \psi_v). \end{aligned}$$

Thus the L- and ε-factors for π_v and Π_v agree when twisted by sufficiently ramified representations of $\mathrm{GL}_1(k_v)$. There is a natural extension of this to $\mathrm{GL}_m(k_v)$ given in the following proposition.

Proposition **5.3.** — *Let v be an non-archimedean place of k. Let π_v be an irreducible admissible generic representation of $G_n(k_v)$ and let Π_v be an irreducible admissible representation of $\mathrm{GL}_N(k_v)$ having trivial central character. Let τ_v be an irreducible admissible generic representation of $\mathrm{GL}_m(k_v)$ of the form $\tau_v \simeq \tau_{0,v} \otimes \eta_v$ with $\tau_{0,v}$ unramified and η_v sufficiently ramified as above. Then*

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v) \quad \textit{and} \quad \varepsilon(s, \pi_v \times \tau_v, \psi_v) = \varepsilon(s, \Pi_v \times \tau_v, \psi_v).$$

Proof. — The proof of this proposition is similar to that of Proposition 5.2. Since $\tau_{0,v}$ is unramified and generic we can write it as a full induced representation from characters [22]

$$\tau_{0,v} \simeq \mathrm{Ind}_{B'_m(k_v)}^{\mathrm{GL}_m(k_v)}(\chi_{1,v} \otimes \cdots \otimes \chi_{m,v})$$

with each $\chi_{i,v}$ unramified. If we let $\chi_{i,v}(x) = |x|_v^{b_i}$ and let $\nu(x) = |x|_v$, then we may write τ_v as

$$\tau_v \simeq \mathrm{Ind}_{B'_m(k_v)}^{\mathrm{GL}_m(k_v)}\left(\eta_v \nu^{b_1} \otimes \cdots \otimes \eta_v \nu^{b_m}\right).$$

Arguing as in the proof of Proposition 5.2, but now factoring τ_v according to its characters, we find

$$L(s, \pi_v \times \tau_v) = \prod_{i=1}^{m} L(s + b_i, \pi_v \times \eta_v)$$

and

$$\varepsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{m} \varepsilon(s + b_i, \pi_v \times \eta_v, \psi_v).$$

On the other hand, by the same results of [20] we also have

$$\mathrm{L}(s, \Pi_v \times \tau_v) = \prod_{i=1}^{m} \mathrm{L}(s + b_i, \Pi_v \times \eta_v)$$

and

$$\varepsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{m} \varepsilon(s + b_i, \Pi_v \times \eta_v, \psi_v).$$

By Corollary 4.2 of Section 4 we see that after factoring the L- and ε-factors for π_v and Π_v twisted by such τ_v the factors are term by term equal for η_v sufficiently highly ramified. This establishes the proposition. □

5.4. *The global candidate lift.* — Return now to the global situation. Let $\pi \simeq \otimes' \pi_v$ be a globally generic cuspidal representation of $\mathrm{G}_n(\mathbf{A})$. Let S be a finite set of finite places such that for all non-archimedean places $v \notin \mathrm{S}$ we have π_v is unramified. For each $v \notin \mathrm{S}$ let Π_v be the local functorial lift of π_v as in Section 5.1 or 5.2. For the places $v \in \mathrm{S}$ we take Π_v to be any irreducible admissible self-contragredient representation of $\mathrm{GL}_{\mathrm{N}}(k_v)$ having trivial central character as in Section 5.3. Then the restricted tensor product $\Pi \simeq \otimes' \Pi_v$ is an irreducible admissible self-contragredient representation of $\mathrm{GL}_{\mathrm{N}}(\mathbf{A})$ having trivial central character. This is our candidate lift.

For each place $v \in \mathrm{S}$ choose a sufficiently highly ramified character η_v so that Proposition 5.3 is valid. Let η be any idele class character of $\mathrm{GL}_1(\mathbf{A})$ which has local component η_v at those $v \in \mathrm{S}$. Then combining Propositions 5.1–5.3 we obtain the following result on our candidate lift.

Proposition **5.4.** — *Let π be a globally generic cuspidal representation of $\mathrm{G}_n(\mathbf{A})$ and let Π be the candidate lift constructed above as a representation of $\mathrm{GL}_{\mathrm{N}}(\mathbf{A})$. Then for every representation $\tau \in \mathscr{T}(\mathrm{S}; \eta) = \mathscr{T}^{\mathrm{S}}(\mathrm{N}-1) \otimes \eta$ we have*

$$\mathrm{L}(s, \pi \times \tau) = \mathrm{L}(s, \Pi \times \tau) \qquad \textit{and} \qquad \varepsilon(s, \pi \times \tau) = \varepsilon(s, \Pi \times \tau).$$

6. Global functoriality

Let us now prove Theorem 1.1. We begin with our globally generic cuspidal representation of $\mathrm{G}_n(\mathbf{A})$. Decompose $\pi \simeq \otimes' \pi_v$ into its local components and let S be a non-empty set of non-archimedean places such that for all non-archimedean places $v \notin S$ we have that π_v is unramified.

Let $\Pi \simeq \otimes' \Pi_v$ be the irreducible admissible representation of $GL_N(\mathbf{A})$ constructed in Section 5 as our candidate lift. By construction Π is self-contragredient, has trivial central character, and is the local functorial lift of π at all places $v \notin S$.

Choose η, an idele class character, such that its local components η_v are sufficiently highly ramified at those $v \in S$ so that Proposition 5.4 is valid. Furthermore, since we have taken S non-empty, we may choose η so that for at least one place $v_0 \in S$ we have that $\eta_{v_0}^2$ is also ramified. Then Theorem 3.1 is also valid. Fix this character.

We are now ready to apply the Converse Theorem to Π. Consider any representation $\tau \in \mathcal{T}(S; \eta)$. By Proposition 5.4 we have that

$$L(s, \pi \times \tau) = L(s, \Pi \times \tau) \qquad \text{and} \qquad \varepsilon(s, \pi \times \tau) = \varepsilon(s, \Pi \times \tau).$$

On the other hand, by Theorem 3.1 we know that each $L(s, \pi \times \tau)$ and hence $L(s, \Pi \times \tau)$ is nice. Thus Π satisfies the hypotheses of the Converse Theorem, Theorem 2.1. Hence there is an automorphic representation $\Pi' \simeq \otimes' \Pi'_v$ of $GL_N(\mathbf{A})$ such that $\Pi'_v \simeq \Pi_v$ for all $v \notin S$. But for $v \notin S$, by construction Π_v is the local functorial lift of π_v. Hence Π' is a functorial lift of π as required in the statement of Theorem 1.1. □

7. The image of functoriality

In this section we would like to investigate the image of functoriality. Assuming the existence of global functoriality, the global image has been analyzed in the papers of Ginzburg, Rallis, and Soudry using their method of descent [13, 56]. For completeness, we recall their global results below. Related results in the case $G_n = SO_{2n+1}$ can be found in [29, 30].

We then turn to what global functoriality implies about the local image of functoriality at the non-archimedean places, including those where the representation is ramified. In the case of $G_n = SO_{2n+1}$ this has been carried out by Jiang and Soudry using functoriality plus the local descent [26, 27], with related results obtained in [30] without using the descent. In this paper we will follow the development of [30] since the local descent has not been completed in the other cases. These local results are needed for our applications in Sections 8–11, particularly our results towards Ramanujan we present in Section 10.

7.1. *The global image of functoriality.* — From their method of descent of automorphic representations from $GL_N(\mathbf{A})$ to the classical groups $G_n(\mathbf{A})$ and its local analogues, Ginzburg, Rallis, and Soudry were able to characterize the image of functoriality from generic representations before this was known to exist, that is, before our result [13,56]. As the results are slightly different for $G_n = SO_{2n+1}$ and $G_n = SO_{2n}$, Sp_{2n} we will state them separately.

For the odd orthogonal group, the result takes the following form [13,56].

Theorem **7.1.** — *Let π be a globally generic cuspidal representation of $G_n(\mathbf{A}) = SO_{2n+1}(\mathbf{A})$. Then any functorial lift of π to an automorphic representation Π of $GL_{2n}(\mathbf{A})$ has trivial central character and is of the form*

$$\Pi = \mathrm{Ind}(\Pi_1 \otimes \cdots \otimes \Pi_d) = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

where each Π_i is a unitary self-dual cuspidal representation of $GL_{N_i}(\mathbf{A})$ such that the partial L-function $L^T(s, \Pi_i, \Lambda^2)$, for any sufficiently large finite set of places T containing all archimedean places, has a pole at $s = 1$ and $\Pi_i \not\simeq \Pi_j$ for $i \neq j$. Moreover, any such Π is the functorial lift of some π as above.

Note that the condition that $L^T(s, \Pi_i, \Lambda^2)$ has a pole at $s = 1$ implies that $N_i = 2n_i$ is even and each Π_i has trivial central character. In particular, the cuspidal image of functoriality consists of all self-dual cuspidal representations of $GL_{2n}(\mathbf{A})$ having trivial central character and whose (partial) exterior square L-function has a pole at $s = 1$; the non-cuspidal part of the image consists of all irreducible isobaric sums of such. As observed in [13], those π which do not lift to cuspidal Π are in fact cuspidal endoscopic lifts from products of smaller odd special orthogonal groups.

For the cases of SO_{2n} and Sp_{2n} the result is similar with the exterior square L-function replaced by the symmetric square L-function [56].

Theorem **7.2.** — *Let π be a globally generic cuspidal representation of $G_n(\mathbf{A}) = SO_{2n}(\mathbf{A})$, $n \geq 2$, or $G_n(\mathbf{A}) = Sp_{2n}(\mathbf{A})$. Then any functorial lift of π to an automorphic representation Π of $GL_N(\mathbf{A})$ has trivial central character and is of the form*

$$\Pi = \mathrm{Ind}(\Pi_1 \otimes \cdots \otimes \Pi_d) = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

where each Π_i is a unitary self-dual cuspidal representation of $GL_{N_i}(\mathbf{A})$ such that the partial L-function $L^T(s, \Pi_i, Sym^2)$, for any sufficiently large finite set of places T containing all archimedean places, has a pole at $s = 1$ and $\Pi_i \not\simeq \Pi_j$ for $i \neq j$. Moreover, any such Π is the functorial lift of some π as above.

There are two remarks to be made on this result. First, the cuspidal image of functoriality from $SO_{2n}(\mathbf{A})$ consists of all self-dual cuspidal representations of $GL_{2n}(\mathbf{A})$ having trivial central character and whose (partial) symmetric square L-function has a pole at $s = 1$ and the functorial image from $Sp_{2n}(\mathbf{A})$ consists of the same type of representations of $GL_{2n+1}(\mathbf{A})$. If the image is not cuspidal, then it consists of an isobaric sum of such representations which are then the functorial lifts from products of smaller symplectic groups or even special orthogonal groups. However, since we lose the condition of trivial central character (except for the representation Π itself) the Π_i could be functorial lifts from quasi-split even special orthogonal groups.

Let us remark for future use that in cases where one might not have the descent method it may still be possible to prove that the image of functoriality is an isobaric representation of GL_N by using facts about the local unitary dual and the Langlands-Shahidi method of analyzing L-functions. This type of argument can be found in [29, 30] where this method is used for the case of $G_n = SO_{2n+1}$. Similar arguments work for our other cases as well and have the potential of working in more general situations. Note that in all following applications, it is only the fact that the image is an isobaric sum of unitary cuspidal representations that is necessary, so these results do not rely on having a descent theory.

There are several facts about classical groups that can be deduced from the existence of the functorial lift to GL_N combined with the characterization of the image. One immediate consequence is that we have lost no information at the places where we did not have a local functorial lift. This is possible since we have a strong multiplicity one result for isobaric representations of $GL_N(\mathbf{A})$ [21].

Corollary **7.1.** — *Let π be a globally generic cuspidal representation of G_n and let Π be its functorial lift to $GL_N(\mathbf{A})$. Then Π is completely determined by requiring that Π_v be the local functorial lift of π_v at almost all places v of k, that is, no global information is lost from those local places where local functoriality is not known.*

7.2. *The local image of functoriality.* — One type of consequence of global functoriality combined with the fact that the image is the isobaric sum of unitary cuspidal representations is that we can fill in some facts about the local components of the lift of globally generic cusp forms on classical groups. Since the local functorial lifts are completely understood at the archimedean places, in this section we will always take v to be a non-archimedean place of k.

We begin with the unramified local lift.

Proposition **7.1.** — *Let $\pi \simeq \otimes' \pi_v$ be a globally generic cuspidal representation of $G_n(\mathbf{A})$. Let v be a non-archimedean place of k at which π_v is unramified. Then the unramified local functorial lift Π_v, as defined in Section 5.2, is generic. In particular the induced representation Ξ_v introduced there is irreducible and equal to Π_v.*

Proof. — Since the global functorial lift Π of π is either cuspidal or a full induced representation from cuspidals, Π is generic. Thus all of its local components are as well. □

For the case of $G_n = SO_{2n+1}$ this was proved by purely local methods in [6]. However for the other two cases this is not a purely local fact, but rather a consequence of π_v being a component of a globally generic cuspidal representation.

At a general non-archimedean place v we have the following result towards establishing generic local functoriality at all places. It encompasses the above proposition as well.

Proposition **7.2.** — *Let v be a non-archimedean place of k and let π_v be an irreducible admissible generic representation of $G_n(k_v)$ which appears as a local component of some globally generic cuspidal representation. Then there exists a unique generic representation Π_v of $GL_N(k_v)$ such that for every supercuspidal representation ρ_v of $GL_m(k_v)$ we have*

$$\gamma(s, \pi_v \times \rho_v, \psi_v) = \gamma(s, \Pi_v \times \rho_v, \psi_v).$$

In particular, this is true for any irreducible generic supercuspidal representation π_v. Moreover, if π_v is the component at v of a globally generic cuspidal representation π and Π the functorial lift of π then, as the notation suggests, this Π_v is the local component of Π at the place v.

Proof. — Take π_v as the local component at v of the globally generic cuspidal representation π. Let Π be the functorial lift of π to an automorphic representation of $GL_N(\mathbf{A})$. Then Π, and hence each of its local components Π_v, is generic.

We first show the existence of one such Π_v. If π_v is unramified, then the statement follows from Proposition 5.2. In general, let ρ_v be as in the statement of the proposition. Then by Proposition 5.1 of [51] there is a cuspidal representation ρ' of $GL_m(\mathbf{A})$ such that at the place v the local component of ρ' is the given ρ_v and at all other finite places $w \neq v$ we have ρ'_w is unramified. Let S be a finite set of finite places such that π_w is unramified for $w \notin S$ and let $S' = S - \{v\}$. Let η be an idele class character such that η_v is trivial and η_w is sufficiently highly ramified at $w \in S'$ so that

$$\gamma\big(s, \pi_w \times \big(\rho'_w \otimes \eta_w\big), \psi_w\big) = \gamma\big(s, \Pi_w \times \big(\rho'_w \otimes \eta_w\big), \psi_w\big) \tag{7.1}$$

as in the proof of Proposition 5.3.

Let $\rho = \rho' \otimes \eta$. Note that, since η_v is trivial, the local component of ρ at v is still our given ρ_v. We have the global functional equations

$$L(s, \pi \times \rho) = \varepsilon(s, \pi \times \rho) L(1 - s, \tilde{\pi} \times \tilde{\rho})$$

and

$$L(s, \Pi \times \rho) = \varepsilon(s, \Pi \times \rho) L(1 - s, \widetilde{\Pi} \times \tilde{\rho})$$

which we can write in the form

$$\gamma(s, \pi_v \times \rho_v, \psi_v) = \left(\prod_{w \in S'} \gamma(s, \pi_w \times \rho_w, \psi_w)^{-1}\right) \frac{L^S(s, \pi \times \rho)}{\varepsilon^S(s, \pi \times \rho, \psi) L^S(1 - s, \tilde{\pi} \times \tilde{\rho})}$$

and

$$\gamma(s, \Pi_v \times \rho_v, \psi_v) = \left(\prod_{w\in S'} \gamma(s, \Pi_w \times \rho_w, \psi_w)^{-1}\right) \frac{L^S(s, \Pi \times \rho)}{\varepsilon^S(s, \Pi \times \rho, \psi) L^S(1-s, \widetilde{\Pi} \times \tilde{\rho})}.$$

By Propositions 5.1 and 5.2 we have that

$$\frac{L^S(s, \pi \times \rho)}{\varepsilon^S(s, \pi \times \rho, \psi) L^S(1-s, \tilde{\pi} \times \tilde{\rho})} = \frac{L^S(s, \Pi \times \rho)}{\varepsilon^S(s, \Pi \times \rho, \psi) L^S(1-s, \widetilde{\Pi} \times \tilde{\rho})},$$

while for $w \in S'$ we have $\gamma(s, \pi_w \times \rho_w, \psi_w) = \gamma(s, \Pi_w \times \rho_w, \psi_w)$ by (7.1). Hence

$$\gamma(s, \pi_v \times \rho_v, \psi_v) = \gamma(s, \Pi_v \times \rho_v, \psi_v).$$

This shows the existence of such Π_v. The uniqueness follows from the "local converse theorem for GL_N", that is, a generic admissible irreducible representation of $GL_N(k_v)$ is uniquely determined by its γ-factor with twists by supercuspidal representations of all smaller rank general linear groups, as in the Remark after the Corollary of Theorem 1.1 of Henniart [16].

If π_v is a generic supercuspidal representation of $G_n(k_v)$ then by Proposition 5.1 of [51] it occurs as the local component of a globally generic cuspidal representation of $G_n(\mathbf{A})$, hence the above reasoning applies.

The final statement of the proposition has in fact been shown in the beginning part of the proof since we took for π an arbitrary global cuspidal representation of $G_n(\mathbf{A})$ with local component π_v and arrived at the uniquely defined local generic lift Π_v. □

We will refer to Π_v as the *local functorial lift* of π_v. This terminology agrees with the usual one at those places $v \notin S$. As was shown in [27] this is completely justifiable in the case of SO_{2n+1}.

This result for $G_n = SO_{2n+1}$ was one of the ingredients of Jiang and Soudry's proof of a "local converse theorem" for SO_{2n+1} which in turn was a key ingredient in their analysis of local functoriality and the local Langlands correspondence for generic representations of $SO_{2n+1}(k_v)$ for a p-adic place v [26,27]. Hopefully, once the details of the local descent theory are worked out for SO_{2n} and Sp_{2n} this proposition will play a similarly useful role.

However, even without the full strength of the descent, we can still say much about the local image of our functorial lift. We will follow the method of [30] where similar results were proved for $G_n = SO_{2n+1}$. We begin with the following lemma.

Lemma **7.1.** — *Let π_v be a local component of the globally generic cuspidal representation π. Assume that π_v is tempered. Then the local functorial lift Π_v is also tempered.*

Proof. — We are assuming that

$$\gamma(s, \pi_v \times \rho_v, \psi_v) = \gamma(s, \Pi_v \times \rho_v, \psi_v)$$

for every supercuspidal representation ρ_v of $\mathrm{GL}_m(k_v)$.

We first extend this to twisting by discrete series representations of $\mathrm{GL}_m(k_v)$. If σ_v is a discrete series, then σ_v can be realized as the irreducible quotient $\delta(\rho_v, t)$ of the induced representation

$$\Xi_v = \mathrm{Ind}\big(\rho_v \nu^{-\frac{t-1}{2}} \otimes \cdots \otimes \rho_v \nu^{\frac{t-1}{2}}\big)$$

associated to the segment

$$\Delta = \big[\rho_v \nu^{-\frac{t-1}{2}}, \rho_v \nu^{\frac{t-1}{2}}\big] = \big\{\rho_v \nu^{-\frac{t-1}{2}}, \rho_v \nu^{-\frac{t-1}{2}+1}, \ldots, \rho_v \nu^{\frac{t-1}{2}}\big\}$$

as in [61] where ρ_v is a supercuspidal representation of an appropriate general linear group and t is a positive integer. Then using the multiplicativity of γ-factors on both sides [52, 20] we have

$$\begin{aligned}
\gamma(s, \pi_v \times \sigma_v, \psi_v) &= \prod_{j=0}^{t-1} \gamma\left(s + \frac{t-1}{2} - j, \pi_v \times \rho_v, \psi_v\right)\\
&= \prod_{j=0}^{t-1} \gamma\left(s + \frac{t-1}{2} - j, \Pi_v \times \rho_v, \psi_v\right)\\
&= \gamma(s, \Pi_v \times \sigma_v, \psi_v).
\end{aligned}$$

We next claim that for any discrete series representation σ_v of $\mathrm{GL}_m(k_v)$ we have

$$\mathrm{L}(s, \pi_v \times \sigma_v) = \mathrm{L}(s, \Pi_v \times \sigma_v).$$

Since π_v and σ_v are both tempered, then by definition [51] $\mathrm{L}(s, \pi_v \times \sigma_v)^{-1}$ is the normalized polynomial part of the numerator of $\gamma(s, \pi_v \times \sigma_v, \psi_v)$. Since we have equality of the twisted γ-factors, our equality would follow from the similar statement for $\mathrm{L}(s, \Pi_v \times \sigma_v)$.

Since Π_v is generic and unitary then by the classification of unitary generic representations of $\mathrm{GL}_{\mathrm{N}}(k_v)$ [58] we can write

$$\begin{aligned}
\Pi_v = \mathrm{Ind}\big(\delta_{1,v}\nu^{r_1} \otimes \cdots \otimes \delta_{k,v}\nu^{r_k} \otimes \delta_{k+1,v} \otimes \cdots \otimes \delta_{k+\ell,v}\\
\otimes \delta_{k,v}\nu^{-r_k} \otimes \cdots \otimes \delta_{1,v}\nu^{-r_1}\big)
\end{aligned}$$

with each $\delta_{i,v}$ a discrete series representation and $0 < r_k \leq \cdots \leq r_1 < \frac{1}{2}$. Again using the multiplicativity of the γ-factors from [20] we have

$$\gamma(s, \Pi_v \times \sigma_v, \psi_v) = \prod_{j=1}^{k} \gamma(s + r_j, \delta_{j,v} \times \sigma_v, \psi_v)\gamma(s - r_j, \delta_{j,v} \times \sigma_v, \psi_v) \times \\ \times \prod_{i=1}^{\ell} \gamma(s, \delta_{k+i,v} \times \sigma_v, \psi_v).$$

By definition [20]

$$\gamma(s, \delta_{i,v} \times \sigma_v, \psi_v) = \frac{\varepsilon(s, \delta_{i,v} \times \sigma_v, \psi_v)\mathrm{L}(1-s, \tilde{\delta}_{i,v} \times \tilde{\sigma}_v)}{\mathrm{L}(s, \delta_{i,v} \times \sigma_v)}.$$

Hence we see that the numerator of $\gamma(s, \Pi_v \times \sigma_v, \psi_v)$ in the factorization is given, up to a monomial factor coming from the ε-factors, by

$$\left(\prod_{j=1}^{k} \mathrm{L}(s + r_j, \delta_{j,v} \times \sigma_v,)\mathrm{L}(s - r_j, \delta_{j,v} \times \sigma_v) \prod_{i=1}^{\ell} \mathrm{L}(s, \delta_{k+i,v} \times \sigma_v)\right)^{-1}.$$

Since $\delta_{i,v}$ and σ_v are both unitary discrete series, $\mathrm{L}(s, \delta_{i,v} \times \sigma_v)$ has no poles in $\mathrm{R}e(s) > 0$ [20] and so this numerator can have zeros only in $\mathrm{R}e(s) < \frac{1}{2}$ since $0 < r_j < \frac{1}{2}$.

Similarly the denominator of $\gamma(s, \Pi_v \times \sigma_v, \psi_v)$ in the factorization is the polynomial

$$\left(\prod_{j=1}^{k} \mathrm{L}(1 - s - r_j, \tilde{\delta}_{j,v} \times \tilde{\sigma}_v,)\mathrm{L}(1 - s + r_j, \tilde{\delta}_{j,v} \times \tilde{\sigma}_v) \prod_{i=1}^{\ell} \mathrm{L}(1-s, \tilde{\delta}_{k+i,v} \times \tilde{\sigma}_v)\right)^{-1}$$

and this can have zeros only in the region $\mathrm{R}e(1-s) < \frac{1}{2}$, that is, $\mathrm{R}e(s) > \frac{1}{2}$.

Hence the numerator and denominator coming from the factorization of the γ-factor are relatively prime. Consequently, from the equality of γ-factors we can conclude that

$$\mathrm{L}(s, \pi_v \times \sigma_v) \\ = \prod_{j=1}^{k} \mathrm{L}(s + r_j, \delta_{j,v} \times \sigma_v,)\mathrm{L}(s - r_j, \delta_{j,v} \times \sigma_v) \prod_{i=1}^{\ell} \mathrm{L}(s, \delta_{k+i,v} \times \sigma_v).$$

On the other hand, by [20] we can compute that

$$\mathrm{L}(s, \Pi_v \times \sigma_v) \\ = \prod_{j=1}^{k} \mathrm{L}(s + r_j, \delta_{j,v} \times \sigma_v,)\mathrm{L}(s - r_j, \delta_{j,v} \times \sigma_v) \prod_{i=1}^{\ell} \mathrm{L}(s, \delta_{k+i,v} \times \sigma_v)$$

and hence

$$L(s, \pi_v \times \sigma_v) = L(s, \Pi_v \times \sigma_v)$$

as desired.

We can now prove that Π_v is tempered. We write Π_v as above and consider the equality of twisted L-factors with $\sigma_v = \tilde{\delta}_{i,v}$ with $1 \leq i \leq k$. By Theorem 4.1 of [4], since $\delta_{i,v}$ and π_v are both tempered we know that $L(s, \pi_v \times \tilde{\delta}_{i,v})$ is holomorphic for $Re(s) > 0$. On the other hand, as noted above we have the factorization

$$\begin{aligned} &L(s, \Pi_v \times \tilde{\delta}_{i,v}) \\ &= \prod_{j=1}^{k} L(s + r_j, \delta_{j,v} \times \tilde{\delta}_{i,v},)L(s - r_j, \delta_{j,v} \times \tilde{\delta}_{i,v}) \prod_{j=1}^{\ell} L(s, \delta_{k+j,v} \times \tilde{\delta}_{i,v}). \end{aligned}$$

The term $L(s - r_i, \delta_{i,v} \times \tilde{\delta}_{i,v})$ produces a pole at $s = r_i$ and since the local L-factors are never zero, this persists to a pole of $L(s, \Pi_v \times \tilde{\delta}_{i,v})$ at $s = r_i > 0$. This is a contradiction unless no non-zero exponents occur in Π_v, that is, $k = 0$ and

$$\Pi_v = \mathrm{Ind}(\delta_{1,v} \otimes \cdots \otimes \delta_{\ell,v})$$

is a full induced representation from unitary discrete series, that is, is tempered. □

With this lemma in hand, it is easy to determine the structure of the local functorial lift of any supercuspidal representation of $G_n(k_v)$.

Theorem **7.3.** — *(a) Let π_v be a supercuspidal representation of the group $SO_{2n+1}(k_v)$ and let Π_v be its local functorial lift in the sense of Proposition 7.2. Then Π_v is of the form*

$$\Pi_v \simeq \mathrm{Ind}(\Pi_{1,v} \otimes \cdots \otimes \Pi_{d,v})$$

where each $\Pi_{i,v}$ is an irreducible supercuspidal self-dual representation of some $GL_{2n_i}(k_v)$ such that $L(s, \Pi_{i,v}, \wedge^2)$ has a pole at $s = 0$ and $\Pi_{i,v} \not\simeq \Pi_{j,v}$ for $i \neq j$.

(b) Let π_v be a supercuspidal representation of $SO_{2n}(k_v)$, $n \geq 2$, or $Sp_{2n}(k_v)$ and let Π_v be its local functorial lift in the sense of Proposition 7.2. Then Π_v is of the form

$$\Pi_v \simeq \mathrm{Ind}(\Pi_{1,v} \otimes \cdots \otimes \Pi_{d,v})$$

where each $\Pi_{i,v}$ is an irreducible supercuspidal self-dual representation of some $GL_{N_i}(k_v)$ such that $L(s, \Pi_{i,v}, Sym^2)$ has a pole at $s = 0$ and $\Pi_{i,v} \not\simeq \Pi_{j,v}$ for $i \neq j$.

Proof. — As part (a) of this theorem was established in both [26] and [30], by essentially the same method, we will restrict ourselves to part (b). The proof is essentially the same as that of part (a).

Recall from the proof of Lemma 7.1 that we know the local functorial lift Π_v is tempered and of the form

$$\Pi_v = \mathrm{Ind}(\delta_{1,v} \otimes \cdots \otimes \delta_{d,v})$$

with each $\delta_{i,v}$ discrete series. Furthermore, for any discrete series representation σ_v of $\mathrm{GL}_m(k_v)$ we know that

$$\mathrm{L}(s, \pi_v \times \sigma_v) = \mathrm{L}(s, \Pi_v \times \sigma_v).$$

We now claim that each $\delta_{i,v}$ is in fact supercuspidal. We can realize $\delta_{i,v}$ as the irreducible quotient $\delta(\rho_{i,v}, t_i)$ of the induced representation

$$\Xi_{i,v} = \mathrm{Ind}\big(\rho_{i,v}\nu^{-\frac{t_i-1}{2}} \otimes \cdots \otimes \rho_{i,v}\nu^{\frac{t_i-1}{2}}\big)$$

associated to the segment $[\rho_{i,v}\nu^{-\frac{t_i-1}{2}}, \rho_{i,v}\nu^{\frac{t_i-1}{2}}]$ as in [61] where $\rho_{i,v}$ is a supercuspidal representation of an appropriate general linear group and t_i is a positive integer. We can then apply our equality of twisted L-factors with $\sigma_v = \tilde{\delta}_{i,v}$ as follows.

From the general linear group side we know by [20] or [51] that

$$\mathrm{L}(s, \Pi_v \times \tilde{\delta}_{i,v}) = \prod_{j=1}^{d} \mathrm{L}(s, \delta_{j,v} \times \tilde{\delta}_{i,v})$$

and that

$$\mathrm{L}(s, \delta_{i,v} \times \tilde{\delta}_{i,v}) = \prod_{k=0}^{t_i-1} \mathrm{L}(s+k, \rho_{i,v} \times \tilde{\rho}_{i,v}).$$

Now $\mathrm{L}(s, \rho_{i,v} \times \tilde{\rho}_{i,v})$ has a pole at $s = 0$ so that $\mathrm{L}(s + t_i - 1, \rho_{i,v} \times \tilde{\rho}_{i,v})$ has a pole at $s = -(t_i - 1)$. Since local L-functions are never zero, this persists to give a pole of $\mathrm{L}(s, \Pi_v \times \tilde{\delta}_{i,v})$ at $s = 1 - t_i$.

On the classical group side, from either [51] or the explicit computations in [31] we have that

$$\mathrm{L}(s, \pi_v \times \tilde{\delta}_{i,v}) = \mathrm{L}\big(s + \tfrac{t_i-1}{2}, \pi_v \times \tilde{\rho}_{i,v}\big)$$

since π_v is supercuspidal. Since $\mathrm{L}(s, \pi_v \times \tilde{\rho}_{i,v})$ can have poles only for $\mathrm{R}e(s) = 0$, we see that $\mathrm{L}(s, \pi_v \times \tilde{\delta}_{i,v})$ can only have poles on the line $\mathrm{R}e(s) = -\frac{t_i-1}{2}$.

These locations of poles are inconsistent unless $t_i = 1$, that is $\delta_{i,v} = \rho_{i,v}$ is supercuspidal. So now let us write

$$\Pi_v = \mathrm{Ind}(\rho_{1,v} \otimes \cdots \otimes \rho_{d,v})$$

with each $\rho_{i,v}$ supercuspidal.

To see that each $\rho_{i,v}$ is self-dual, we consider the equality

$$L(s, \pi_v \times \tilde{\rho}_{i,v}) = L(s, \Pi_v \times \tilde{\rho}_{i,v}).$$

Then the right hand side has a pole at $s = 0$ as above. For the left hand side to have a pole at $s = 0$ we must have that $\rho_{i,v}$ is self-dual by [53]. Moreover, in this case, the order of the pole on the left hand side is one while the order of the pole on the right hand side is the number of j such that $\rho_{i,v} \simeq \rho_{j,v}$. Hence we see that each $\rho_{i,v}$ is self dual and $\rho_{i,v} \not\simeq \rho_{j,v}$ if $i \neq j$.

We finally come to the L-function condition. Recall we are in the case that $G_n = SO_{2n}$, $n \geq 2$, or Sp_{2n}. By the previous analysis, $L(s, \pi_v \times \rho_{i,v})$ has a pole at $s = 0$. On the other hand in these situations [51] implies that the product

$$L(s, \pi_v \times \rho_{i,v}) L(2s, \rho_{i,v}, \wedge^2)$$

has a simple pole at $s = 0$. Since this pole is accounted for by $L(s, \pi_v \times \rho_{i,v})$ we can conclude that $L(s, \rho_{i,v}, \wedge^2)$ has no pole at $s = 0$. On the other hand, from [53] we know that

$$L(s, \rho_{i,v} \times \rho_{i,v}) = L(s, \rho_{i,v}, Sym^2) L(s, \rho_{i,v}, \wedge^2).$$

Since the left hand side always has a pole at $s = 0$, in our cases this must come from the symmetric square term, that is, $L(s, \rho_{i,v}, Sym^2)$ has a pole at $s = 0$. This completes the proof of the theorem. □

In [26] Jiang and Soudry were able to then use the descent method to show that in part (a) of the theorem the local functoriality taking π_v to Π_v is bijective and that the description of Π_v given determines the image completely, that is, the lift is onto the set of Π_v with these properties. We expect a similar result in part (b) when the descent theory is completed.

To continue with our analysis of the local image of functoriality, we will need to deal with generic representations π_v of $G_n(k_v)$ which may or may not occur as components of globally generic cusp forms. To this end, we make the following definition independent of whether π_v occurs as a component of a cuspidal representation.

Definition **7.1.** — *Let* π_v *be an irreducible admissible generic representation of* $G_n(k_v)$. *We will say that an irreducible admissible representation* Π_v *of* $GL_N(k_v)$ *is* a local functorial lift *of* π_v *if for every supercuspidal representation* ρ_v *of* $GL_m(k_v)$ *we have*

$$L(s, \pi_v \times \rho_v) = L(s, \Pi_v \times \rho_v) \ \textit{and} \ \varepsilon(s, \pi_v \times \rho_v, \psi_v) = \varepsilon(s, \Pi_v \times \rho_v, \psi_v).$$

Note that given the interrelations among L, γ, and ε, this definition could equivalently be stated as

$$L(s, \pi_v \times \rho_v) = L(s, \Pi_v \times \rho_v) \text{ and } \gamma(s, \pi_v \times \rho_v, \psi_v) = \gamma(s, \Pi_v \times \rho_v, \psi_v)$$

and this is the formulation that is easiest to work with. This definition is consistent with the previous definitions given at the places where we can define a local functorial lift via the local Langlands correspondence and is consistent with that given by the image of global functoriality for components of globally generic cuspidal representations.

We would next like to compute the local functorial lift of a generic discrete series representation π_v of $G_n(k_v)$. We first recall some facts and notation from the representation theory of general linear groups [61], some of which we have used before. If ρ is a supercuspidal representation of some $GL_d(k_v)$ and a and b are in $\frac{1}{2}\mathbf{Z}$ with $a \geq b$ and $a - b \in \mathbf{Z}$, then $\Delta = [\nu^b\rho, \nu^a\rho] = \{\nu^b\rho, \nu^{b+1}\rho, ..., \nu^a\rho\}$ is referred to as a segment and $\delta(\Delta) = \delta([\nu^b\rho, \nu^a\rho])$ denotes the unique irreducible quotient of the induced representation

$$\mathrm{Ind}(\nu^b\rho \otimes \cdots \otimes \nu^a\rho).$$

Then $\delta([\nu^b\rho, \nu^a\rho])$ is an essentially square integrable representation of the group $GL_{d(a-b+1)}(k_v)$. If $a \in \mathbf{Z}$, $a \geq 1$, and ρ is unitary supercuspidal we will let $\delta(\rho, a) = \delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho])$. Then $\delta(\rho, a)$ is a unitary discrete series representation of $GL_{da}(k_v)$. If ρ is self-dual then so is $\delta(\rho, a)$.

Now recall from the classification of generic discrete series representations π_v of classical groups [24, 25, 39, 40, 42] that such π_v can be realized as a subrepresentation of an induced representation of the form

$$\xi_v = \mathrm{Ind}\big(\delta_1 \otimes \cdots \otimes \delta_r \otimes \delta'_1 \otimes \cdots \otimes \delta'_\ell \otimes \pi_{0,v}\big) \tag{7.2}$$

where $\pi_{0,v}$ is a generic supercuspidal representation of a smaller classical group $G_{n_0}(k_v)$ of the same type (possibly the trivial representation of $G_0(k_v)$),

$$\delta_i = \delta\Big(\big[\nu^{-\frac{(b_i-1)}{2}}\rho_i, \nu^{\frac{(a_i-1)}{2}}\rho_i\big]\Big)$$

with ρ_i a self-dual supercuspidal representation of an appropriate $GL_{d_i}(k_v)$ and $a_i > b_i > 0$ integers of the same parity and

$$\delta'_j = \delta\Big(\big[\nu^{\epsilon_j}\rho'_j, \nu^{\frac{(a'_j-1)}{2}}\rho'_j\big]\Big)$$

with ρ'_j a self-dual supercuspidal representation of an appropriate $GL_{d'_j}(k_v)$, $a'_j > 0$ an integer and $\epsilon_j = \frac{1}{2}$ if a'_j is even and $\epsilon_j = 1$ if a'_j is odd. The representations

$\rho'_1, \ldots, \rho'_\ell$ are all distinct and we have that ρ'_j can occur only if the induced representation $\mathrm{Ind}(\rho'_j\nu^s \otimes \pi_{0,v})$ is reducible at $s = \frac{1}{2}$ or $s = 1$ (but these conditions are not sufficient). These reducibilities are discussed in [51]. The integer a'_j determining the exponents will then be even if the reducibility point is $s = \frac{1}{2}$ and it will be odd if the reducibility point is $s = 1$. This last reducibility is equivalent to $\mathrm{L}(s, \rho'_j \times \pi_{0,v})$ having a pole at $s = 0$ [51].

Let us briefly indicate how we derive this from the work of Mœglin and Tadić [39,40]. More details can be found in Section 8 below. We will use freely the terminology from these papers. Note that while the body of these papers deal with the cases $\mathrm{G}_n = \mathrm{SO}_{2n+1}$ and $\mathrm{G}_n = \mathrm{Sp}_{2n}$, Section 16 of [40] discusses the extension of these results to $\mathrm{G}_n = \mathrm{SO}_{2n}$, with the convention that $\mathrm{SO}_2(k_v)$ does not have supercuspidal or discrete series representations. First we consider the Jordan blocks associated to a generic supercuspidal representation $\pi_{0,v}$ of $\mathrm{G}_n(k_v)$. We will let ρ denote a self dual supercuspidal representation of an appropriate $\mathrm{GL}_{d_\rho}(k_v)$. Combining Theorem 8.1 of [51] and the definition of $\mathit{Jord}(\pi_{0,v})$ [39,40] we can easily see that

$$\mathit{Jord}(\pi_{0,v}) = \left\{(\rho, 1) \middle| \mathrm{Ind}\big(\rho\nu^s \otimes \pi_{0,v}\big) \text{ is reducible at } s = 1\right\}$$

and the set of extended Jordan blocks $\mathit{Jord}'(\pi_{0,v})$ is then

$$\mathit{Jord}'(\pi_{0,v}) = \mathit{Jord}(\pi_{0,v}) \cup \left\{(\rho, 0) \middle| \mathrm{Ind}\big(\rho\nu^s \otimes \pi_{0,v}\big) \text{ is reducible at } s = \tfrac{1}{2}\right\}.$$

Note that once one assumes that $\mathrm{Ind}(\rho\nu^s \otimes \pi_{0,v})$ reduces somewhere, then reduction at $s = 1/2$ is equivalent to the L-function $\mathrm{L}(s, \rho, \mathrm{R})$ having a pole at $s = 0$, where we have let $\mathrm{R} = \mathrm{S}ym^2$ if $\mathrm{G}_n = \mathrm{SO}_{2n+1}$ and $\mathrm{R} = \wedge^2$ if $\mathrm{G}_n = \mathrm{SO}_{2n}$ or Sp_{2n} [51,53]. Let us write $\mathit{Jord}'(\pi_{0,v}) = \{(\rho'_j, a_j)\}$. If π_v^+ is a strongly positive generic discrete series representation [39,40] and $\pi_{0,v} = \pi^+_{cusp}$ is its partial cuspidal support, then $\pi_{0,v}$ must be generic. Then by Proposition 4.1 of [39] or Section 7 of [40] we know that for each $(\rho'_j, a_j) \in \mathit{Jord}'(\pi_{0,v})$ there exist integers $a'_j \geq a_j$ and of the same parity such that if we let $\delta'_j = \delta([\rho'_j\nu^{\frac{(a_j+1)}{2}}, \rho'_j\nu^{\frac{(a'_j-1)}{2}}])$, with δ'_j associated to empty segments omitted, then π_v^+ is the unique irreducible subrepresentation of

$$\xi_v^+ = \mathrm{Ind}\big(\delta'_1 \otimes \cdots \otimes \delta'_\ell \otimes \pi_{0,v}\big).$$

This is in agreement with our characterization. Our characterization of a general generic discrete series representation π_v then follows inductively from Lemma 3.1 and Section 4.2 of [39]. From there we see that there is a strongly positive discrete series representation π_v^+ of a smaller classical group of the same type and a sequence of self-dual supercuspidal representations ρ_i of $\mathrm{GL}_{d_{\rho_i}}(k_v)$ and integers $a_i > b_i > 0$ of the same parity such that if we let

$$\delta_i = \delta\Big(\Big[\nu^{-\frac{(b_i-1)}{2}}\rho_i, \nu^{\frac{(a_i-1)}{2}}\rho_i\Big]\Big)$$

then π_v will occur as a subrepresentation of

$$\xi'_v = \mathrm{Ind}\big(\delta_1 \otimes \cdots \otimes \delta_r \otimes \pi_v^+\big).$$

If π_v is generic, then so must π_v^+ be and if we combine this with the characterization above of generic strongly positive discrete series and use the transitivity of induction we obtain our characterization. For $G_n = SO_{2n+1}$ or Sp_{2n} the characterization can also be derived from Jantzen's work [24, 25].

Returning to our generic discrete series representation π_v realized as a subrepresentation of (7.2), let $\Pi_{0,v}$ be the local functorial lift of $\pi_{0,v}$ as constructed in Theorem 7.3. Then if we consider the induced representation of $GL_N(k_v)$ defined by

$$\begin{aligned}\Xi_v = \mathrm{Ind}\big(&\delta_1 \otimes \cdots \otimes \delta_r \otimes \delta'_1 \otimes \cdots \otimes \delta'_\ell \\ &\otimes \Pi_{0,v} \otimes \widetilde{\delta}'_\ell \otimes \cdots \otimes \widetilde{\delta}'_1 \otimes \widetilde{\delta}_r \otimes \cdots \otimes \widetilde{\delta}_1\big)\end{aligned} \tag{7.3}$$

then this induced representation has a unique generic constituent Π_v [61].

Proposition **7.3.** — *Let π_v be a generic discrete series representation of $G_n(k_v)$ realized as a subrepresentation of (7.2). Then π_v has a local functorial lift Π_v to $GL_N(k_v)$, given by the generic constituent of (7.3), which is self-dual, generic, and tempered.*

Proof. — For simplicity, let us rearrange the inducing data for ξ_v to write it in the form

$$\xi'_v = \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \pi_{0,v}\big)$$

where each $\tau_{i,v}$ is a self-dual discrete series representation of an appropriate $GL_{m_i}(k_v)$, $r_m \le \cdots \le r_1$, and $\pi_{0,v}$ is our generic supercuspidal representation of an appropriate smaller classical group $G_{n_0}(k_v)$ of the same type. Then if we consider the induced representation of $GL_N(k_v)$ defined by

$$\Xi'_v = \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \Pi_{0,v} \otimes \tau_{m,v}\nu^{-r_m} \otimes \cdots \otimes \tau_{1,v}\nu^{-r_1}\big),$$

which is a rearrangement of the inducing data for Ξ_v, then this induced representation has a unique generic subrepresentation which is Π_v [61].

We claim that Π_v is a local functorial lift of π_v, that is, we have

$$L(s, \pi_v \times \rho_v) = L(s, \Pi_v \times \rho_v) \ \text{ and } \ \gamma(s, \pi_v \times \rho_v, \psi_v) = \gamma(s, \Pi_v \times \rho_v, \psi_v).$$

As we have used several times, from the multiplicativity of γ-factors as in [52] for the classical group side and, for example [20], for the general linear group side, we have

$$\begin{aligned}\gamma(s, \pi_v \times \rho_v, \psi_v) &= \gamma(s, \Pi_v \times \rho_v, \psi_v) \\ &= \gamma(s, \Pi_{0,v} \times \rho_v, \psi_v) \prod_{i=1}^{m} \gamma(s \pm r_i, \tau_{i,v} \times \rho_v, \psi_v).\end{aligned}$$

To obtain the equality of L-functions, we will directly prove that Π_v is tempered. Once π_v and Π_v are both tempered, then the equality of the L-factors follows from the equality of γ-factors by [51].

If we now return to Ξ_v as given by (7.3),

$$\Xi_v = \mathrm{Ind}\big(\delta_1 \otimes \cdots \otimes \delta_r \otimes \delta'_1 \otimes \cdots \otimes \delta'_\ell \otimes \Pi_{0,v} \otimes \widetilde{\delta}'_\ell \otimes \cdots \otimes \widetilde{\delta}'_1 \otimes \widetilde{\delta}_r \otimes \cdots \otimes \widetilde{\delta}_1\big),$$

then Π_v is the unique generic constituent of Ξ_v and we can explicitly compute this constituent using induction in stages.

First, consider the contribution of $\delta_i \otimes \widetilde{\delta}_i$ for indices $1 \leq i \leq r$. Replacing δ_i by its inducing data and then rearranging, we find that the induced representation of $\mathrm{GL}_{d_i(a_i+b_i)}(k_v)$ given by $\mathrm{Ind}(\delta_i \otimes \widetilde{\delta}_i)$ is a quotient of the larger induced representation

$$\Xi_{i,v} = \mathrm{Ind}\Big(\big(\rho_i\nu^{\frac{-(b_i-1)}{2}} \otimes \cdots \otimes \rho_i\nu^{\frac{(a_i-1)}{2}}\big) \otimes \big(\rho_i\nu^{\frac{-(a_i-1)}{2}} \otimes \cdots \otimes \rho_i\nu^{\frac{(b_i-1)}{2}}\big)\Big)$$

and hence a constituent of

$$\mathrm{Ind}\Big(\big(\rho_i\nu^{\frac{-(a_i-1)}{2}} \otimes \cdots \otimes \rho_i\nu^{\frac{(a_i-1)}{2}}\big) \otimes \big(\rho_i\nu^{\frac{-(b_i-1)}{2}} \otimes \cdots \otimes \rho_i\nu^{\frac{(b_i-1)}{2}}\big)\Big).$$

The generic constituent of this induced representation is visibly the self-dual tempered representation $\mathrm{Ind}(\delta(\rho_i, a_i) \otimes \delta(\rho_i, b_i))$. Hence in computing the generic constituent of Ξ_v we may replace each $\delta_i \otimes \widetilde{\delta}_i$ by $\delta(\rho_i, a_i) \otimes \delta(\rho_i, b_i)$ in the inducing data.

Next consider the contribution of $\delta'_j \otimes \widetilde{\delta}'_j$ when the associated integer a'_j determining the exponents is even. In this case, replacing δ'_j by its inducing data we see that $\mathrm{Ind}(\delta'_j \otimes \widetilde{\delta}'_j)$ is a constituent of the larger induced representation

$$\Xi'_{j,v} = \mathrm{Ind}\big(\rho'_j\nu^{-\frac{(a'_j-1)}{2}} \otimes \cdots \otimes \rho'_j\nu^{-\frac{1}{2}} \otimes \rho'_j\nu^{\frac{1}{2}} \otimes \cdots \otimes \rho'_j\nu^{\frac{(a'_j-1)}{2}}\big)$$

which has as its unique generic constituent the self-dual discrete series representation given by $\delta(\rho'_j, a'_j)$. So for the purpose of computing the generic constituent of Ξ_v we may replace $\delta'_j \otimes \widetilde{\delta}'_j$ by $\delta(\rho'_j, a'_j)$ in the inducing data.

Finally let us consider the contribution of the δ'_j when the exponent a'_j is odd. Recall that this is possible only if $L(s, \pi_{0,v} \times \rho'_j) = L(s, \Pi_{0,v} \times \rho'_j)$ has a pole at $s = 0$. By Theorem 7.3 we know that we can write

$$\Pi_{0,v} \simeq \mathrm{Ind}(\Pi_{1,v} \otimes \cdots \otimes \Pi_{d,v})$$

with each $\Pi_{i,v}$ a self-dual supercuspidal representation of an appropriate general linear group $\mathrm{GL}_{d_i}(k_v)$ with the $\Pi_{i,v}$ distinct. But these $\Pi_{i,v}$ are then precisely the supercuspidal representations ρ'' for which $L(s, \Pi_{0,v} \times \rho'')$ has a pole at $s = 0$. Hence if we let

ρ''_j with $j = 1, \ldots, \ell''$ denote those ρ'_j with a'_j odd, then $\{\rho''_1, \ldots, \rho''_{\ell''}\} \subset \{\Pi_{1,v}, \ldots \Pi_{d,v}\}$ and we can write

$$\Pi_{0,v} = \mathrm{Ind}\big(\rho''_1 \otimes \cdots \otimes \rho''_{\ell''} \otimes \Pi'_{0,v}\big)$$

with $\Pi'_{0,v}$ the tensor product of the $\Pi_{i,v}$ which were not among the ρ''_j. Now, for each ρ''_j, consider the contribution $\delta''_j \otimes \rho''_j \otimes \widetilde{\delta}''_j$ to our generic constituent. If we replace δ''_j by its inducing data, we see that $\mathrm{Ind}(\delta''_j \otimes \rho''_j \otimes \widetilde{\delta}''_j)$ is a constituent of the larger induced representation

$$\Xi''_{j,v} = \mathrm{Ind}\Big(\rho''_j \nu^{-\frac{(a''_j-1)}{2}} \otimes \cdots \otimes \rho''_j \nu^{-1} \otimes \rho''_j \otimes \rho''_j \nu \otimes \cdots \otimes \rho''_j \nu^{\frac{(a''_j-1)}{2}}\Big)$$

and this representation has as its unique generic constituent the self dual discrete series representation $\delta(\rho''_j, a''_j)$ So in the inducing data for Ξ_v we may replace $\delta''_j \otimes \rho''_j \otimes \widetilde{\delta}''_j$ by $\delta(\rho''_j, a''_j)$ and not effect the generic constituent.

If we put these all back together, we find that our Π_v is now the unique generic constituent of the induced representation

$$\mathrm{Ind}\big(\delta(\rho_1, a_1) \otimes \delta(\rho_1, b_1) \otimes \cdots \otimes \delta(\rho_r, a_r) \otimes \delta(\rho_r, b_r) \otimes \delta\big(\rho'_1, a'_1\big) \otimes \cdots \otimes \delta\big(\rho'_\ell, a'_\ell\big) \otimes \Pi'_{0,v}\big).$$

But this representation is induced from self-dual unitary discrete series and is hence generic, tempered, self-dual, and irreducible. Thus this irreducible induced representation is precisely our local lift. It is a self-dual, generic, tempered representation of $\mathrm{GL}_N(k_v)$. □

We would like to point out that the proof given for Proposition 2.6 of [30] is incorrect and should be replaced by the preceding proof.

As a corollary, let us give the more precise form of the lift we obtained.

Corollary **7.2.** — *Let π_v be a generic discrete series representation of $\mathrm{G}_n(k_v)$ realized as a subrepresentation of (7.2). Then π_v has a local functorial lift Π_v to $\mathrm{GL}_N(k_v)$ given by the irreducible induced representation*

$$\mathrm{Ind}\big(\delta(\rho_1, a_1) \otimes \delta(\rho_1, b_1) \otimes \cdots \otimes \delta(\rho_r, a_r) \otimes \delta(\rho_r, b_r) \otimes \delta\big(\rho'_1, a'_1\big) \otimes \cdots \otimes \delta\big(\rho'_\ell, a'_\ell\big) \otimes \Pi'_{0,v}\big).$$

We next consider a general generic tempered representation π_v of the group $\mathrm{G}_n(k_v)$. Since π_v is generic and tempered it is a direct summand of an induced representation of the form

$$\mathrm{Ind}(\delta_{1,v} \otimes \cdots \otimes \delta_{m,v} \otimes \sigma_{0,v}) \tag{7.4}$$

where the $\delta_{i,v}$ are discrete series representations of appropriate $\mathrm{GL}_{n_i}(k_v)$ for $i = 1 \ldots, m$ and $\sigma_{0,v}$ is a generic discrete series of $\mathrm{G}_{n_0}(k_v)$ for a smaller classical group of the same type. Now set Π_v to be the induced representation of $\mathrm{GL}_\mathrm{N}(k_v)$ given by

$$\Pi_v = \mathrm{Ind}(\delta_{1,v} \otimes \cdots \otimes \delta_{m,v} \otimes \Pi_{0,v} \otimes \tilde{\delta}_{m,v} \otimes \cdots \otimes \tilde{\delta}_{1,v})$$

where $\Pi_{0,v}$ is the local functorial lift of $\sigma_{0,v}$ defined in Proposition 7.3. This representation is then irreducible, self-dual, generic, and tempered. Then arguing by the multiplicativity of the local γ- and L-factors as before we have that

$$\mathrm{L}(s, \pi_v \times \rho_v) = \mathrm{L}(s, \Pi_v \times \rho_v) \text{ and } \gamma(s, \pi_v \times \rho_v, \psi_v) = \gamma(s, \Pi_v \times \rho_v, \psi_v)$$

for all requisite supercuspidal ρ_v. Hence Π_v is a local functorial lift of π_v. Thus we have established the following proposition.

Proposition **7.4.** — *Let π_v be a generic tempered representation of the group $\mathrm{G}_n(k_v)$ given as a summand of (7.4). Then π_v has a local functorial lift to a representation Π_v of $\mathrm{GL}_n(k_v)$ given by*

$$\Pi_v = \mathrm{Ind}\big(\delta_{1,v} \otimes \cdots \otimes \delta_{m,v} \otimes \Pi_{0,v} \otimes \tilde{\delta}_{m,v} \otimes \cdots \otimes \tilde{\delta}_{1,v}\big),$$

where $\Pi_{0,v}$ is the generic local functorial lift of $\sigma_{0,v}$. The lift Π_v is self-dual, generic, and tempered.

Finally, let π_v be an arbitrary irreducible admissible generic representation of $\mathrm{G}_n(k_v)$. By the work of Muić [44] on the standard module conjecture we know that π_v is a full induced representation of the form

$$\pi_v \simeq \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \tau_{0,v}\big) \tag{7.5}$$

where each $\tau_{i,v}$ is a tempered representation of an appropriate $\mathrm{GL}_{n_i}(k_v)$, the exponents can be taken so that $0 < r_m < \cdots < r_1$, and $\tau_{0,v}$ is a generic tempered representation of a smaller classical group $\mathrm{G}_{n_0}(k_v)$ of the same type, except in the case where $\mathrm{G}_n = \mathrm{SO}_{2n}$, $\tau_{0,v}$ is the trivial representation of $\mathrm{G}_0(k_v)$ and $n_m = 1$, in which case we must allow

$$\pi_v \simeq \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m}\big) \tag{7.6}$$

with $0 \leq |r_m| < r_{m-1} < \cdots < r_1$. (In particular, see Section 4 of [44] for the elaboration of these cases.) Then on $\mathrm{GL}_\mathrm{N}(k_v)$ we can either form the induced representation

$$\Xi_v = \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \Pi_{0,v} \otimes \tilde{\tau}_{m,v}\nu^{-r_m} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-r_1}\big) \tag{7.7}$$

where $\Pi_{0,v}$ is the local functorial lift of $\tau_{0,v}$ as constructed in Proposition 7.4 if we are in the situation (7.5) or

$$\Xi_v = \begin{cases} \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \tilde{\tau}_{m,v}\nu^{-r_m} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-r_1}\big) & \text{if } r_m \geq 0 \\ \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tilde{\tau}_{m,v}\nu^{-r_m} \otimes \tau_{m,v}\nu^{r_m} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-r_1}\big) & \text{if } r_m < 0 \end{cases} \tag{7.8}$$

in case (7.6). There is no reason for Ξ_v to be irreducible any more. However, with the exponents arranged in the Langlands ordering, Ξ_v has a unique irreducible quotient, which we denote by Π_v.

Proposition **7.5.** — *If π_v is an irreducible admissible generic representation of $G_n(k_v)$ given by (7.5) (respectively (7.6)), then it has a local functorial lift Π_v given by the unique irreducible quotient of (7.7) (respectively (7.8)).*

Proof. — Once again, we must show the equality of the twisted γ- and L- factors. We will do this in the general case (7.5), the exceptional case (7.6) being handled in the same manner. Hence assume π_v is of the form (7.5).

By the multiplicativity of γ factors and L-factors for general linear groups (Theorem 3.1 and Proposition 9.4 of [20]) we know that even for any tempered representation τ'_v of $GL_m(k_v)$ we have

$$\begin{aligned}&\gamma\big(s, \Pi_v \times \tau'_v, \psi_v\big)\\ &= \gamma\big(s, \Pi_{0,v} \times \tau'_v, \psi_v\big) \prod_{j=1}^{m} \gamma\big(s + r_j, \tau_{j,v} \times \tau'_v, \psi_v\big)\gamma\big(s - r_j, \tilde{\tau}_{j,v} \times \tau'_v, \psi_v\big)\end{aligned}$$

and

$$L\big(s, \Pi_v \times \tau'_v\big) = L\big(s, \Pi_{0,v} \times \tau'_v\big) \prod_{j=1}^{m} L\big(s + r_j, \tau_{j,v} \times \tau'_v\big)L\big(s - r_j, \tilde{\tau}_{j,v} \times \tau'_v\big).$$

On the classical group side, we still retain multiplicativity by Theorem 5.2 of [52] (see also the discussion in Section 5 of [32] where the condition of being a subrepresentation is removed) so that

$$\begin{aligned}&\gamma\big(s, \pi_v \times \tau'_v, \psi_v\big)\\ &= \gamma\big(s, \tau_{0,v} \times \tau'_v, \psi_v\big) \prod_{j=1}^{m} \gamma\big(s + r_j, \tau_{j,v} \times \tau'_v, \psi_v\big)\gamma\big(s - r_j, \tilde{\tau}_{j,v} \times \tau'_v, \psi_v\big).\end{aligned}$$

Since the representation π_v is no longer tempered, its L-function is defined through the Langlands classification in [51] and so by definition

$$L\big(s, \pi_v \times \tau'_v\big) = L\big(s, \tau_{0,v} \times \tau'_v\big) \prod_{j=1}^{m} L\big(s + r_j, \tau_{j,v} \times \tau'_v\big)L\big(s - r_j, \tilde{\tau}_{j,v} \times \tau'_v\big).$$

If we take $\tau'_v = \rho_v$ to be supercuspidal, then by the previous proposition (or the definition of being a local functorial lift) we have

$$\begin{aligned}&L(s, \tau_{0,v} \times \rho_v) = L(s, \Pi_{0,v} \times \rho_v) \quad \text{and}\\ &\gamma(s, \tau_{0,v} \times \rho_v, \psi_v) = \gamma(s, \Pi_{0,v} \times \rho_v, \psi_v).\end{aligned}$$

Hence indeed Π_v is a local functorial lift of π_v. □

To proceed we will next show that control of supercuspidal twists as in the definition of local functorial lift is sufficient to control all generic twists.

Lemma **7.2.** — *Let π_v be an irreducible admissible generic representation of $G_n(k_v)$ and let Π_v be the local functorial lift constructed in the previous proposition. Then for any irreducible admissible generic representation π'_v of $GL_m(k_v)$ we have*

$$L(s, \pi_v \times \pi'_v) = L(s, \Pi_v \times \pi'_v) \text{ and } \gamma(s, \pi_v \times \pi'_v, \psi_v) = \gamma(s, \Pi_v \times \pi'_v, \psi_v).$$

Proof. — The argument is as before, now using multiplicativity in the other variable. Once again, we will present the argument in the general case where π_v is given by (7.5) and its lift Π_v by (7.7). The exceptional case of (7.6) is handled accordingly.

Since π'_v is generic, we can write π_v as a full induced representation from either tempered or discrete series [61]. We take

$$\pi'_v \simeq \mathrm{Ind}(\tau'_{1,v}\nu^{b_1} \otimes \cdots \otimes \tau'_{k,v}\nu^{b_k})$$

with each $\tau'_{j,v}$ tempered and $b_1 > \cdots > b_k$.

Again by Theorem 3.1 and Proposition 9.4 of [20] on the general linear side we have

$$\gamma(s, \Pi_v \times \pi'_v, \psi_v) = \prod_{i=1}^{k} \Big[\gamma(s + b_i, \Pi_{0,v} \times \tau'_{i,v}, \psi_v) \times \prod_{j=1}^{m} \gamma(s + b_i + r_j, \tau_{j,v} \times \tau'_{i,v}, \psi_v)\gamma(s + b_i - r_j, \tilde{\tau}_{j,v} \times \tau'_{i,v}, \psi_v)\Big]$$

and

$$L(s, \Pi_v \times \pi'_v) = \prod_{i=1}^{k} \Big[L(s + b_i, \Pi_{0,v} \times \tau'_{i,v}) \times \prod_{j=1}^{m} L(s + b_i + r_j, \tau_{j,v} \times \tau'_{i,v})L(s + b_i - r_j, \tilde{\tau}_{j,v} \times \tau'_{i,v})\Big].$$

On the classical group side we obtain the similar factorizations for the same reasons as in the previous proposition. This reduces us to showing that

$$\gamma(s, \tau_{0,v} \times \tau'_v, \psi_v) = \gamma(s, \Pi_{0,v} \times \tau'_v, \psi_v) \quad \text{and}$$
$$L(s, \tau_{0,v} \times \tau'_v) = L(s, \Pi_{0,v} \times \tau'_v)$$

for τ'_v a tempered representation of $GL_m(k_v)$ when $\Pi_{0,v}$ is the local functorial lift of $\tau_{0,v}$ as above. But now both $\tau_{0,v}$ and $\Pi_{0,v}$ are tempered and we know the equality of the twisted γ- and L-factors when $\tau'_v = \rho_v$ is supercuspidal. We then first write our general tempered τ'_v as

$$\tau'_v = \mathrm{Ind}(\delta_{1,v} \otimes \cdots \otimes \delta_{k,v})$$

with each $\delta_{i,v}$ discrete series and use multiplicativity once again to reduce to $\tau'_v = \delta_v$ discrete series. Then for the discrete series we realize δ_v as $\delta(\rho_v, t)$, the generic quotient of the induced representation

$$\mathrm{Ind}\big(\rho_v \nu^{-\frac{t-1}{2}} \otimes \cdots \otimes \rho_v \nu^{\frac{t-1}{2}}\big)$$

with ρ_v supercuspidal and t a positive integer as before. Using multiplicativity of γ-factors as always gives

$$\gamma(s, \tau_{0,v} \times \delta_v, \psi_v) = \gamma(s, \Pi_{0,v} \times \delta_v, \psi_v)$$

and by direct calculation as in [20] and [31] we have

$$\begin{aligned}
\mathrm{L}(s, \tau_{0,v} \times \delta_v) &= \prod_{j=0}^{t-1} \mathrm{L}\left(s + \frac{t-1}{2} - j, \tau_{0,v} \times \rho_v\right) \\
&= \prod_{j=0}^{t-1} \mathrm{L}\left(s + \frac{t-1}{2} - j, \Pi_{0,v} \times \rho_v\right) \\
&= \mathrm{L}(s, \Pi_{0,v} \times \delta_v).
\end{aligned}$$

This completes the proof of the lemma. □

We are now able to determine the image of local functoriality in general for components of globally generic cuspidal representations.

Theorem **7.4.** — *Let*

$$\pi_v \simeq \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \tau_{0,v}\big)$$

be an irreducible generic representation of $\mathrm{G}_n(k_v)$ *as in (7.5) or (7.6). Suppose that* π_v *is a local component of a globally generic cuspidal representation* π *of* $\mathrm{G}_n(\mathbf{A})$. *Then its local functorial lift* Π_v *is self-dual, generic and has the form*

$$\Pi_v = \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \Pi_{0,v} \otimes \tilde{\tau}_{m,v}\nu^{-r_m} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-r_1}\big)$$

with $\Pi_{0,v}$ *the local functorial lift of* $\tau_{0,v}$ *defined above if* π *is as in (7.5) and by*

$$\Pi_v = \begin{cases} \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \tilde{\tau}_{m,v}\nu^{-r_m} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-r_1}\big) & \text{if } r_m \geq 0 \\ \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tilde{\tau}_{m,v}\nu^{-r_m} \otimes \tau_{m,v}\nu^{r_m} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-r_1}\big) & \text{if } r_m < 0 \end{cases}$$

if π *is as in (7.6).*

Proof. — By definition, we know that Π_v is the Langlands quotient of this induced representation. Hence once we show that Π_v is generic the induced representation will be irreducible and thus equal to Π_v.

We will establish this by using the Converse Theorem once again. Let $\pi = \otimes'\pi_w$. Let S be the set of finite places where π is ramified. For the $w \notin S$ we have constructed a local functorial lift in Propositions 5.1 and 5.2. For the places $w \in S$, which include v, we will take Π_w to be the local functorial lift defined in Proposition 7.5. Then let $\Pi = \otimes'\Pi_w$. This is an irreducible admissible representation of $GL_N(\mathbf{A})$ with trivial central character and is our candidate lift. Moreover, by Propositions 5.1 and 5.2 combined with Lemma 7.2 for the places in S we have that for all cuspidal representations $\tau \in \mathcal{T}(N-1)$

$$L(s, \pi \times \tau) = L(s, \Pi \times \tau) \qquad \text{and} \qquad \varepsilon(s, \pi \times \tau) = \varepsilon(s, \Pi \times \tau).$$

Now let $T = \{w_0\}$ be a singleton set containing one non-archimedean place, say where π_{w_0} is unramified. In particular, $w_0 \neq v$ for our fixed place v. Let η any idele class character which is sufficiently ramified at w_0 so that Theorem 3.1 is true for all $\tau \in \mathcal{T}(T;\eta)$. Then $L(s, \Pi \times \tau)$ is also nice for all $\tau \in \mathcal{T}(T;\eta)$. Now applying the Converse Theorem we find a global functorial lift Π' of π such that $\Pi'_w \simeq \Pi_w$ for all $w \neq w_0$, so that in particular $\Pi_v = \Pi'_v$. By Theorem 7.1 or Theorem 7.2 we know that Π' and hence Π'_v is generic. Hence Π_v is generic. □

8. A conjecture of Mœglin

Let k_v be a non-archimedean local field of characteristic 0, which we take to be a local component of our number field k.

In the recent work on the characterization of discrete series representations of the p-adic classical groups by Mœglin and Tadić [39,40], to each discrete series representation π_v of $G_n(k_v)$ they have attached a triple

$$\pi_v \mapsto (\mathit{Jord}(\pi_v), \pi_{cusp}, \epsilon_{\pi_v})$$

where $\mathit{Jord}(\pi_v)$ is the set of Jordan blocks attached to π_v, π_{cusp} is the partial cuspidal support of π_v, a supercuspidal representation of a smaller classical group $G_{n_0}(k_v)$ of the same type, and ϵ_{π_v} is a partially defined function $\epsilon_{\pi_v} : \mathit{Jord}(\pi_v) \to \{\pm 1\}$. We will be most interested in the Jordan blocks. The set $\mathit{Jord}(\pi_v)$ consists of pairs (ρ, a) where ρ is a self dual supercuspidal representation of some $GL_{d_\rho}(k_v)$ and a is a natural number. By definition [39] a pair $(\rho, a) \in \mathit{Jord}(\pi_v)$ iff

1. ρ is self-dual,
2. the induced representation $\mathrm{Ind}(\delta(\rho, a) \otimes \pi_v)$ of the group $G_{n+ad_\rho}(k_v)$ is irreducible, and
3. a is even if $L(s, \rho, R)$ has a pole at $s = 0$ and odd otherwise.

As in [39, 40], we have let R denote Sym^2 if $G_n = SO_{2n+1}$ and $R = \wedge^2$ if $G_n = SO_{2n}$ or Sp_{2n} and the L-functions $L(s, \rho, R)$ are as in [53]. The partial cuspidal support π_{cusp} is the unique supercuspidal representation of a smaller classical group $G_{n_0}(k_v)$ such that π_v occurs as a subrepresentation of $\mathrm{Ind}(\tau_v \otimes \pi_{cusp})$ for some convenient irreducible representation τ_v of $GL_{n-n_0}(k_v)$. The function ϵ_{π_v} will play no role for us so we will not describe it.

Let N denote the rank of the general linear group to which the discrete series representation π_v of $G_n(k_v)$ should functorially lift. Motivated by the conjectural Langlands correspondence and conjectures of Arthur, Mœglin has conjectured [38–40] that one should have the *dimension relation*

$$\sum_{(\rho,a)\in \mathit{Jord}(\pi_v)} ad_\rho = N$$

relating the size of the Jordan blocks and the dimension of the natural representation of the dual group ${}^L G_n$, which is N. One can find a discussion of this relation and its motivation in the Introductions to [38] and [40], where it is noted that this equality would follow from Arthur's conjectures. In [38] Mœglin has established the inequality

$$\sum_{(\rho,a)\in \mathit{Jord}(\pi_v)} ad_\rho \leq N$$

in general.

Given its relation with the local Langlands correspondence, and hence functoriality, it should not be surprising that as a first local consequence of the existence of global functoriality for the classical groups G_n, particularly the construction of the local lift of a generic discrete series representation π_v in Proposition 7.3, we can establish this conjecture for the case of generic discrete series representations of split classical groups.

Theorem **8.1.** — *Let* π_v *be a generic discrete series representation of some* $G_n(k_v)$. *Let* N *be the rank of the general linear group to which* π_v *functorially lifts. Then*

$$\sum_{(\rho,a)\in \mathit{Jord}(\pi_v)} ad_\rho = N.$$

Proof. — Let us first suppose that $\pi_v = \pi_{0,v}$ is generic supercuspidal. Then, as we have noted in Section 7.2, in this case the Jordan blocks $\mathit{Jord}(\pi_{0,v})$ can be characterized as [39, 40]

$$\mathit{Jord}(\pi_{0,v}) = \{(\rho, 1) \mid \mathrm{Ind}(\rho\nu^s \otimes \pi_{0,v}) \text{ is reducible at } s = 1\}.$$

On the other hand, by Theorem 7.3 we know that $\pi_{0,v}$ lifts functorially to

$$\Pi_{0,v} \simeq \mathrm{Ind}(\Pi_{1,v} \otimes \cdots \otimes \Pi_{d,v})$$

where each $\Pi_{i,v}$ is an irreducible supercuspidal self-dual representation of some $\mathrm{GL}_{\mathrm{N}_i}(k_v)$ and $\Pi_{i,v} \not\simeq \Pi_{j,v}$ for $i \neq j$. By Theorem 8.1 of [51] we know that $\mathrm{Ind}(\rho\nu^s \otimes \pi_{0,v})$ is reducible at $s = 1$ iff $\mathrm{L}(s, \rho \times \pi_{0,v})$ has a pole at $s = 0$. But this last is equivalent to

$$\mathrm{L}(s, \rho \times \pi_{0,v}) = \mathrm{L}(s, \rho \times \Pi_{0,v}) = \prod_{i=1}^{d} \mathrm{L}(s, \rho \times \Pi_{i,v})$$

having a pole at $s = 0$. Since local L-functions are never zero, this is the case iff $\rho = \Pi_{i,v}$ for some $i = 1, \ldots, d$. Hence we see that

$$\mathit{Jord}(\pi_{0,v}) = \{(\Pi_{i,v}, 1) \mid i = 1, \ldots, d\}$$

so that

$$\sum_{(\rho,a)\in \mathit{Jord}(\pi_v)} ad_\rho = \sum_{i=1}^{d} \mathrm{N}_i = \mathrm{N}.$$

This establishes the theorem for generic supercuspidal representations.

Next we let $\pi_v = \pi_v^+$ be a generic strongly positive discrete series representation as in [39,40]. Then by Section 5.3 of [39] (see also Section 5 of [40]) we know that the associated triple $(\mathit{Jord}(\pi_v^+), \pi_{cusp}^+, \epsilon_{\pi_v^+})$ is admissible of alternated type. Let $\pi_{cusp}^+ = \pi_{0,v}$, which must be generic if π_v^+ is. As in [39,40] and Section 7.2 above, the set of extended Jordan blocks $\mathit{Jord}'(\pi_{0,v})$ is then

$$\mathit{Jord}'(\pi_{0,v}) = \mathit{Jord}(\pi_{0,v}) \cup \left\{(\rho, 0) \mid \mathrm{Ind}(\rho\nu^s \otimes \pi_{0,v}) \text{ is reducible at } s = \tfrac{1}{2}\right\}$$

and once one assumes that $\mathrm{Ind}(\rho\nu^s \otimes \pi_{0,v})$ reduces somewhere, then reduction at $s = 1/2$ is equivalent to the L-function $\mathrm{L}(s, \rho, \mathrm{R})$ having a pole at $s = 0$ [51]. Let us enumerate this set as $\mathit{Jord}'(\pi_{0,v}) = \{(\rho_j', a_j)\}$. Then by Section 2 of [40], particularly formula (2-7), we know that $\mathit{Jord}(\pi_v^+)$ is in bijection with $\mathit{Jord}'(\pi_{0,v})$, this bijection preserves the supercuspidal representations ρ_j' occurring, and if we enumerate $\mathit{Jord}(\pi_v^+)$ in accordance with $\mathit{Jord}'(\pi_{0,v})$ then $\mathit{Jord}(\pi_v^+) = \{(\rho_j', a_j')\}$ where $a_j' \geq a_j$ and of the same parity. Then, as we observed in Section 7.2, Proposition 4.1 of [39] or Section 7 of [40] give that π_v^+ is the unique subrepresentation of

$$\xi_v^+ = \mathrm{Ind}\left(\delta_1' \otimes \cdots \otimes \delta_\ell' \otimes \pi_{0,v}\right)$$

where $\delta_j' = \delta([\rho_j'\nu^{\frac{(a_j+1)}{2}}, \rho_j'\nu^{\frac{(a_j'-1)}{2}}])$, with δ_j' associated to empty segments omitted. Note that the only way a segment $[\rho_j'\nu^{\frac{(a_j+1)}{2}}, \rho_j'\nu^{\frac{(a_j'-1)}{2}}]$ can be empty is if $a_j' = a_j = 1$, since by definition each $a_j' \geq 1$ and $a_j \in \{0, 1\}$. The local functorial lift Π_v^+ of this representation is then given in Proposition 7.3 or Corollary 7.2. From the statement of Proposition 7.3 we know that Π_v^+ is the generic constituent of

$$\Xi_v^+ = \mathrm{Ind}\big(\delta_1' \otimes \cdots \otimes \delta_\ell' \otimes \Pi_{0,v} \otimes \widetilde{\delta}_\ell' \otimes \cdots \otimes \widetilde{\delta}_1'\big).$$

In the course of the proof of that proposition and the derivation of the form of Π_v^+ given in Corollary 7.2 we successively replaced parts of the induction data for Ξ_v by associated discrete series representations. We now interpret these replacements in terms of the Jordan blocks of π_v^+. For the δ_j' associated to (ρ_j', a_j') with a_j' even, so that part of $\mathcal{J}ord(\pi_v^+)$ corresponding to a pair $(\rho_j', 0)$ in $\mathcal{J}ord'(\pi_{0,v})$, the factor $\delta_j' \otimes \widetilde{\delta}_j'$ was replaced by $\delta(\rho_j', a_j')$ in the inducing data for Ξ_v. For the remaining δ_j', namely those associated to (ρ_j', a_j') with a_j' odd and greater than one, then $\rho_j' = \Pi_{i,v}$ for one of the factors $\Pi_{i,v}$ of the lift $\Pi_{0,v}$ of $\pi_{0,v}$ and then $\delta_j' \otimes \Pi_{i,v} \otimes \widetilde{\delta}_j'$ was replaced by $\delta(\rho_j', a_j')$ in the inducing data. Finally, we were left with those factors $\Pi_{i,v} = \rho_j'$ of $\Pi_{0,v}$ for which the associated $a_j' = 1$ and these remain. Then as in Corollary 7.2 we have

$$\Pi_v^+ = \mathrm{Ind}\big(\delta\big(\rho_1', a_1'\big) \otimes \cdots \otimes \delta(\rho_\ell', a_\ell') \otimes \Pi_{0,v}'\big)$$

where $\Pi_{0,v}'$ is the tensor product of the $\Pi_{i,v} = \rho_j'$ with $a_j' = 1$, that is, corresponding to the empty segments above. Thus

$$\begin{aligned}
\mathrm{N} &= \sum_{j=1}^{\ell} a_j' d_{\rho_j'} + \sum_{(\rho_j',1)\in\mathcal{J}ord(\pi_v)} d_{\rho_j'} \\
&= \sum_{\substack{(\rho_j',a_j')\in\mathcal{J}ord(\pi_v)\\ a_j=0}} a_j' d_{\rho_j'} + \sum_{\substack{(\rho_j',a_j')\in\mathcal{J}ord(\pi_v)\\ a_j=1,a_j'>1}} a_j' d_{\rho_j'} + \sum_{(\rho_j',1)\in\mathcal{J}ord(\pi_v)} d_{\rho_j'} \\
&= \sum_{(\rho_j',a_j')\in\mathcal{J}ord(\pi_v)} a_j' d_{\rho_j'}
\end{aligned}$$

and the theorem is true for strongly positive generic discrete series.

Finally, we take π_v to be an arbitrary generic discrete series representation of $\mathrm{G}_n(k_v)$. Then, as in Section 7.2, using inductively Lemma 3.1 and Section 4.2 of [39] we may realize π_v as a subrepresentation of

$$\xi_v = \mathrm{Ind}\big(\delta_1 \otimes \cdots \otimes \delta_r \otimes \pi_v^+\big)$$

with π_v^+ a generic strongly positive discrete series and

$$\delta_i = \delta\big(\big[\nu^{-\frac{(b_i-1)}{2}}\rho_i, \nu^{\frac{(a_i-1)}{2}}\rho_i\big]\big)$$

for self-dual supercuspidal representations ρ_i of $\mathrm{GL}_{d_{\rho_i}}(k_v)$ and integers $a_i > b_i > 0$ of the same parity. Then by Proposition 4.2 of [39] we know that $\mathit{Jord}(\pi_v)$ is the union of $\mathit{Jord}(\pi_v^+)$ and the set $\{(\rho_i, a_i), (\rho_i, b_i)\}$ and that these sets are disjoint. If we let Π_v^+ be the local functorial lift of π_v^+ discussed in the previous paragraph, then we can interpret Corollary 7.2 as saying that the functorial lift of π_v is given by

$$\Pi_v = \mathrm{Ind}\big(\delta(\rho_1, a_1) \otimes \delta(\rho_1, b_1) \otimes \cdots \otimes \delta(\rho_r, a_r) \otimes \delta(\rho_r, b_r) \otimes \Pi_v^+\big).$$

If Π_v^+ is a representation of $\mathrm{GL}_{\mathrm{N}^+}(k_v)$ then we see that

$$\mathrm{N} = \mathrm{N}^+ + \sum_{i=1}^{r}(a_i + b_i)d_{\rho_i} = \mathrm{N}^+ + \sum_{(\rho,a)\in \mathit{Jord}(\pi_v)-\mathit{Jord}(\pi_v^+)} a d_\rho$$

and if we combine this with the result for generic strongly positive discrete series above we obtain our statement in this case as well. □

9. The conductor of a generic representation

Let v be a non-archimedean place of k and let π_v be a generic representation of $\mathrm{G}_n(k_v)$ for one of our classical groups. Let q_v be the order of the residue field of k_v. We will assume that our local additive character ψ_v is normalized to have conductor zero, that is, ψ_v is trivial on the integers $\mathcal{O}_v$ and non-trivial on $\varpi_v^{-1}\mathcal{O}_v$.

Let us recall the basic structure of the local ε-factor of π_v. In Section 3 of [51] the basic local γ-factor

$$\gamma(s, \pi_v, \psi_v) = \gamma(s, \pi_v \times 1_v, \psi_v)$$

is defined (with 1_v the trivial representation of $\mathrm{GL}_1(k_v)$) and it is shown that

$$\gamma(s, \pi_v, \psi_v)\gamma\big(1-s, \tilde{\pi}_v, \psi_v^{-1}\big) = 1.$$

The γ-factor and ε-factor are related by

$$\gamma(s, \pi_v, \psi_v) = \frac{\varepsilon(s, \pi_v, \psi_v)\mathrm{L}(1-s, \tilde{\pi}_v)}{\mathrm{L}(s, \pi_v)}$$

with $\varepsilon(s, \pi_v, \psi_v)$ a monomial in q_v^s, as in Section 7 of [51], and we will also have that

$$\varepsilon(s, \pi_v, \psi_v)\varepsilon\big(1-s, \tilde{\pi}_v, \psi_v^{-1}\big) = 1.$$

Thus we may write

$$\varepsilon(s, \pi_v, \psi_v) = \varepsilon\left(\tfrac{1}{2}, \pi_v, \psi_v\right) q_v^{-f(\pi_v)\left(s-\frac{1}{2}\right)}$$

with $f(\pi_v) \in \mathbf{Z}$. The number $\varepsilon(\frac{1}{2}, \pi_v, \psi_v)$ is then called the *local root number* attached to π_v (and ψ_v) and either the exponent $f(\pi_v)$ or the exponential $q_v^{f(\pi_v)}$ is called the (arithmetic) *conductor* of π_v. For our purposes, we will take the exponent $f(\pi_v)$ as the conductor. If π_v is unitary, then $\tilde{\pi}_v = \overline{\pi}_v$ and we have that the local root number satisfies $|\varepsilon(\frac{1}{2}, \pi_v, \psi_v)| = 1$. We could make similar definitions for the ε-factors of pairs $\varepsilon(s, \pi_v \times \pi_v', \psi_v)$ with π_v' a generic representation of $\mathrm{GL}_m(k_v)$.

Of course, for representations of $\mathrm{GL}_{\mathrm{N}}(k_v)$ there is an analogous definition of root number and conductor [14, 19]. One of the principle results of [19] is the following (see Theorem 5.1 and Remark 5.4).

Theorem **9.1.** — *Let* Π_v *be an irreducible admissible representation of* $\mathrm{GL}_{\mathrm{N}}(k_v)$*. Then* $f(\Pi_v)$ *is a non-negative integer, i.e.,* $f(\Pi_v) \geq 0$*, and* $f(\Pi_v) = 0$ *iff* Π_v *is unramified.*

For the case of generic Π_v, in [19] they then go on to give a structural interpretation of the integer $f(\Pi_v)$ in terms of the existence of vectors stable under appropriate open compact subgroups of Hecke type. We will not pursue this finer result here, but we will establish the following analogue of the basic facts on the conductor for the classical groups G_n.

Theorem **9.2.** — *Let* π_v *be an irreducible admissible generic representation of* $\mathrm{G}_n(k_v)$*. Then* $f(\pi_v) \geq 0$ *and* $f(\pi_v) = 0$ *iff* π_v *is unramified.*

Proof. — In Section 7 we have attached to π_v an irreducible admissible representation Π_v of $\mathrm{GL}_{\mathrm{N}}(k_v)$ such that

$$\varepsilon(s, \pi_v \times \rho_v, \psi_v) = \varepsilon(s, \Pi_v \times \rho_v, \psi_v)$$

for all supercuspidal representations ρ_v of $\mathrm{GL}_m(k_v)$. In particular, for $m = 1$ and $\rho_v = 1_v$ we have

$$\varepsilon(s, \pi_v, \psi_v) = \varepsilon(s, \Pi_v, \psi_v).$$

Thus for our local functorial lift Π_v of π_v we have the matching of both the conductors $f(\pi_v) = f(\Pi_v)$ and the root numbers $\varepsilon(\frac{1}{2}, \pi_v, \psi_v) = \varepsilon(\frac{1}{2}, \Pi_v, \psi_v)$. In particular this implies that $f(\pi_v) \geq 0$. Furthermore, by construction, if π_v is unramified then so is Π_v, so that if π_v is unramified we have $f(\pi_v) = 0$.

We are left with showing that if π_v is irreducible, admissible, generic and $f(\pi_v) = 0$, then π_v is unramified. If $f(\pi_v) = 0$ and Π_v is the local functorial lift of π_v then $f(\Pi_v) = 0$ and Π_v must be unramified.

First, suppose that π_v is supercuspidal. In the low dimensional cases of $SO_3 \simeq PGL_2$ or $Sp_2 \simeq SL_2$ one can check directly using the description of the lift given in Section 1 that the local functorial lifts can never be unramified. Thus we may assume $n \geq 2$. Then, by Theorem 7.3, Π_v is of the form

$$\Pi_v \simeq \mathrm{Ind}(\Pi_{1,v} \otimes \cdots \otimes \Pi_{d,v})$$

with each $\Pi_{i,v}$ a supercuspidal representation of some GL_{N_i}. The only way this could be unramified is if $d = N$ and each $\Pi_{i,v}$ were an unramified self-dual character. But the $\pi_{i,v}$ are distinct and there are only two unramified self-dual characters. So this would be possible only if $N = 2$. But since we are taking $n \geq 2$ we always have $N \geq 4$. So the lift of a supercuspidal representation cannot be unramified.

Next, suppose that π_v is a generic discrete series. Again, the low dimensional cases can be handled individually given their description in Section 1, so we may assume $n \geq 2$. Then as in Proposition 7.3 we realize π_v as a subrepresentation of an induced representation of the form

$$\xi_v = \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \pi_{0,v}\big)$$

where each $\tau_{i,v}$ is a self-dual discrete series representation of an appropriate $GL_{n_i}(k_v)$, $r_m \leq \cdots \leq r_1$, and $\pi_{0,v}$ is our generic supercuspidal representation of an appropriate smaller classical group $G_{n_0}(k_v)$ of the same type. The local functorial lift Π_v is then the generic constituent of

$$\Xi_v = \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \Pi_{0,v} \otimes \tau_{m,v}\nu^{-r_m} \otimes \cdots \otimes \tau_{1,v}\nu^{-r_1}\big)$$

where $\Pi_{0,v}$ is the local functorial lift of $\pi_{0,v}$. For Π_v to be unramified, all of the inducing data in Ξ_v must be unramified. By the above, $\Pi_{0,v}$ is never unramified. Hence $\pi_{0,v}$ cannot be present and π_v is a subrepresentation of

$$\xi_v = \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m}\big).$$

Then by Corollary 7.2 we know that Π_v is a full induced of the form

$$\Pi_v = \mathrm{Ind}(\delta(\rho_1, t_1) \otimes \cdots \otimes \delta(\rho_m, t_m))$$

with each ρ_i a self-dual supercuspidal representations of appropriate $GL_{d_i}(k_v)$. Again, for this to be unramified, we must have each $\delta(\rho_i, t_i)$ unramified. But this is possible only if each ρ_i is a self-dual unramified character, that is $\rho_i = 1$ or $\rho_i = \nu^{i\pi/\log(q_v)}$, and $t_i = 1$. Irreducibility then forces Π_v to be a representation of $GL_1(k_v)$ or $GL_2(k_v)$, which as we have seen is impossible if $n \geq 2$. Hence the local functorial lift of a generic discrete series representation is never unramified.

Now suppose that π_v is a tempered generic representation of $G_n(k_v)$. Then as in Proposition 7.4 we have that π_v is the direct summand of an induced representation of the form

$$\mathrm{Ind}(\delta_{1,v} \otimes \cdots \otimes \delta_{m,v} \otimes \sigma_{0,v})$$

where the $\delta_{i,v}$ are discrete series representations of appropriate $GL_{n_i}(k_v)$ for $i = 1 \ldots, m$ and $\sigma_{0,v}$ is a generic discrete series of $G_{n_0}(k_v)$ for a smaller classical group of the same type. Then, as in that proposition, its local functorial lift is

$$\Pi_v = \mathrm{Ind}\big(\delta_{1,v} \otimes \cdots \otimes \delta_{m,v} \otimes \Pi_{0,v} \otimes \tilde{\delta}_{m,v} \otimes \cdots \otimes \tilde{\delta}_{1,v}\big)$$

where $\Pi_{0,v}$ is the local functorial lift of $\sigma_{0,v}$. If this is to be unramified, then all of its inducing data must be unramified. In particular, by the previous analysis $\Pi_{0,v}$ cannot be present since it is never unramified. Hence π_v is a direct summand of

$$\mathrm{Ind}(\delta_{1,v} \otimes \cdots \otimes \delta_{m,v})$$

with the $\delta_{i,v}$ unramified. But again, the only unramified discrete series representations of $GL_d(k_v)$ are the unramified unitary characters of $GL_1(k_v)$. Hence π_v is unramified and our theorem is true in this case.

In general, as in Proposition 7.5, we write an arbitrary irreducible admissible generic representation of $G_n(k_v)$ in the form

$$\pi_v \simeq \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \tau_{0,v}\big)$$

where each $\tau_{i,v}$ is a tempered representation of an appropriate $GL_{n_i}(k_v)$ and $\tau_{0,v}$ is a tempered representation of a smaller classical group $G_{n_0}(k_v)$ of the same type as in (7.5) or (7.6). Then Π_v is taken to be the unique irreducible quotient of

$$\Xi_v = \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \Pi_{0,v} \otimes \tilde{\tau}_{m,v}\nu^{-r_m} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-r_1}\big)$$

where $\Pi_{0,v}$ is the local functorial lift of $\tau_{0,v}$ if we are in the situation of (7.5) or

$$\Xi_v = \begin{cases} \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tau_{m,v}\nu^{r_m} \otimes \tilde{\tau}_{m,v}\nu^{-r_m} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-r_1}\big) & \text{if } r_m \geq 0 \\ \mathrm{Ind}\big(\tau_{1,v}\nu^{r_1} \otimes \cdots \otimes \tilde{\tau}_{m,v}\nu^{-r_m} \otimes \tau_{m,v}\nu^{r_m} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-r_1}\big) & \text{if } r_m < 0 \end{cases}$$

in case (7.6). Π_v can be unramified only if $\Pi_{0,v}$ is unramified and all the $\tau_{i,v}$ are unramified. But as we have shown above, if $\Pi_{0,v}$ is unramified, so is $\tau_{0,v}$, each tempered representation $\tau_{i,v}$ with $i \geq 1$ is a full induced from unitary discrete series, and for $\tau_{i,v}$ to be unramified, each discrete series must also be unramified. But the only unramified unitary discrete series are the unitary characters of $GL_1(k_v)$. So for Π_v to be unramified, each $\tau_{i,v}$ must be induced from unramified unitary characters of $GL_1(k_v)$.

Then π_v will a full induced representation from unramified representations, that is, π_v must be unramified.

Thus, in all cases, we have shown that if $f(\pi_v) = 0$, then π_v is unramified. This completes the proof of the theorem. □

This result is expected to have applications to the relative trace formula (communications with E. Lapid) among others.

10. The Ramanujan conjecture

We first recall the current formulation of the Ramanujan conjecture for generic cuspidal representations of quasi-split groups as in [18], [45], or [50]. This conjecture was made after the counter-examples to the more general conjecture were found [18]. We reiterate this conjecture formally here.

Conjecture **10.1.** — *Let* G *be a quasi-split reductive group over k. Then every globally generic cuspidal representation* $\pi = \otimes' \pi_v$ *of* $\mathrm{G}(\mathbf{A})$ *satisfies the Ramanujan conjecture, that is, each local component* π_v *is tempered.*

As a global consequence of functoriality, we obtain bounds towards Ramanujan for globally generic cuspidal representations of our classical groups G_n by pulling back the known bounds for GL_N.

Let us formulate estimates towards Ramanujan in the following terms [46]. Let $\Pi = \otimes' \Pi_v$ be a unitary cuspidal representation of $\mathrm{GL}_m(\mathbf{A})$. If v is any place of k then Π_v is a unitary generic representation of $\mathrm{GL}_m(k_v)$ and hence by [60, 61] can be written as a full induced

$$\Pi_v \simeq \mathrm{Ind}\big(\Pi_{1,v}\nu^{a_{1,v}} \otimes \cdots \otimes \Pi_{t,v}\nu^{a_{t,v}}\big)$$

with $a_{1,v} > \cdots > a_{t,v}$ and each $\Pi_{i,v}$ tempered. We will say that Π satisfies condition $\mathrm{H}(\theta_m)$ with $\theta_m \geq 0$ (allowing for the possibility that the bound is dependent on the rank of the group) if for all places v the exponents in Π_v satisfy

$$-\theta_m \leq a_{i,v} \leq \theta_m.$$

By the classification of the unitary generic dual for $\mathrm{GL}_m(k_v)$ we have that trivially every cuspidal Π satisfies $\mathrm{H}(\frac{1}{2})$. The best result known for a general number field is that of Luo, Rudnick, and Sarnak which states that any cuspidal representation Π of $\mathrm{GL}_m(\mathbf{A})$ satisfies $\mathrm{H}(\frac{1}{2} - \frac{1}{m^2+1})$. The Ramanujan conjecture is that all cuspidal Π satisfy condition $\mathrm{H}(0)$.

Similarly, if G_n is any of our classical groups and $\pi = \otimes' \pi_v$ is a generic cuspidal representation of $G_n(\mathbf{A})$ then by [44] or [60] we know that at every place we have that π_v is also a full induced

$$\pi_v \simeq \mathrm{Ind}\big(\tau_{1,v}\nu^{b_{1,v}} \otimes \cdots \otimes \tau_{t,v}\nu^{b_{t,v}} \otimes \tau_{0,v}\big)$$

where each $\tau_{i,v}$ is a tempered representation of an appropriate $GL_{n_i}(k_v)$, and $\tau_{0,v}$ is a generic tempered representation of a smaller classical group $G_{n_0}(k_v)$ of the same type as in (7.5) or (7.6). We will similarly say that π satisfies $H(\theta_n)$ if for all places we have

$$-\theta_n \leq b_{i,v} \leq \theta_n.$$

For these groups, the classification of the generic unitary dual gives the trivial estimate of $H(1)$.

Theorem **10.1.** — *Let π be a globally generic cuspidal representation of $G_n(\mathbf{A})$ and let N be the rank of the general linear group to which π functorially lifts. Suppose that for all $m \leq N$ we know that cuspidal representations of $GL_m(\mathbf{A})$ satisfy condition $H(\theta_m)$ with $\theta_r \geq \theta_m$ for $r > m$. Then π satisfies $H(\theta_N)$.*

Proof. — Let Π be the functorial lift of π to $GL_N(\mathbf{A})$.

At the archimedean places, this follows from local functoriality since that is completely understood in terms of the arithmetic Langlands parameterization.

Let v be a non-archimedean place of k at which π_v is unramified. Let us give the argument in terms of Satake parameters at these places since this is more elementary and does not depend on the bulk of the work in Section 7.2. In the notation of Section 5.2 the Satake class of π_v is represented by

$$\phi_v(F_v) = \mathrm{diag}\big(\mu_{1,v}(\varpi), \ldots, \mu_{n,v}(\varpi_v), \mu_{n,v}(\varpi_v)^{-1}, \ldots, \mu_{1,v}(\varpi_v)^{-1}\big)$$

in the cases $G_n = SO_{2n+1}, SO_{2n}$ or

$$\phi_v(F_v) = \mathrm{diag}\big(\mu_{1,v}(\varpi), \ldots, \mu_{n,v}(\varpi_v), 1, \mu_{n,v}(\varpi_v)^{-1}, \ldots, \mu_{1,v}(\varpi_v)^{-1}\big)$$

when $G_n = Sp_{2n}$. Its Satake parameters are then the complex numbers $\alpha_{j,v} = \mu_{i,v}(\varpi_v)^{\pm 1}$. As noted in Section 5.3, the local component Π_v of the functorial lift is represented by the same class, viewed as a diagonal matrix in $GL_N(\mathbf{C})$ and hence has the same Satake parameters.

If Π is unitary cuspidal, then by hypothesis the Satake parameters will satisfy the bounds

$$q_v^{-\theta_N} \leq |\alpha_{j,v}| \leq q_v^{\theta_N}.$$

If Π is not cuspidal, but rather induced from unitary cuspidal representations Π_i of $GL_{N_i}(\mathbf{A})$ with $N_i < N$ as in Theorems 7.1 or 7.2 then the Satake parameters of π_v will be distributed among those of the $\Pi_{i,v}$ and hence satisfy the possibly better estimates

$$q_v^{-\theta_N} \leq q_v^{-\theta_{N_i}} \leq |\alpha_{j,v}| \leq q_v^{\theta_{N_i}} \leq q_v^{\theta_N}.$$

Hence π_v satisfies $H(\theta_N)$.

In general, a local component π_v will be of the form

$$\pi_v \simeq \mathrm{Ind}\big(\tau_{1,v}\nu^{b_{1,v}} \otimes \cdots \otimes \tau_{t,v}\nu^{b_{t,v}} \otimes \tau_{0,v}\big)$$

where each $\tau_{i,v}$ is a tempered representation of an appropriate $\mathrm{GL}_{n_i}(k_v)$ and $\tau_{0,v}$ is a generic tempered representation of a smaller classical group $G_{n_0}(k_v)$ of the same type as in (7.5) or (7.6). Then as we have seen in Theorem 7.4

$$\Pi_v = \mathrm{Ind}\big(\tau_{1,v}\nu^{b_{1,v}} \otimes \cdots \otimes \tau_{t,v}\nu^{b_{t,v}} \otimes \Pi_{0,v} \otimes \tilde{\tau}_{t,v}\nu^{-b_{t,v}} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-b_{1,v}}\big)$$

with $\Pi_{0,v}$ the local functorial lift of $\tau_{0,v}$ if π is as in (7.5) and by

$$\Pi_v = \begin{cases} \mathrm{Ind}\big(\tau_{1,v}\nu^{b_{1,v}} \otimes \cdots \otimes \tau_{t,v}\nu^{b_{t,v}} \otimes \tilde{\tau}_{t,v}\nu^{-b_{t,v}} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-b_{1,v}}\big) & \text{if } b_{t,v} \geq 0 \\ \mathrm{Ind}\big(\tau_{1,v}\nu^{b_{1,v}} \otimes \cdots \otimes \tilde{\tau}_{t,v}\nu^{-b_{t,v}} \otimes \tau_{t,v}\nu^{b_{t,v}} \otimes \cdots \otimes \tilde{\tau}_{1,v}\nu^{-b_{1,v}}\big) & \text{if } b_{t,v} < 0 \end{cases}$$

if π is as in (7.6). By Proposition 7.4 we know $\Pi_{0,v}$ is tempered. If our global lift Π is cuspidal, then by condition $H(\theta_N)$ we have

$$-\theta_N \leq b_{i,v} \leq \theta_N$$

and hence π_v satisfies $H(\theta_N)$ at this place. If instead Π is of the form $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d$ as in Theorem 7.1 or 7.2 with Π_i a unitary cuspidal representation of $\mathrm{GL}_{N_i}(\mathbf{A})$ with $N_i < N$ then as the exponents distribute out they will each satisfy a possibly better bound

$$-\theta_N \leq -\theta_{N_i} \leq b_{j,v} \leq \theta_{N_i} \leq \theta_N.$$

Hence we have that π_v satisfies $H(\theta_N)$ at these places as well. □

If we combine this result with the Ramanujan bounds of Luo, Rudnick, and Sarnak for the general linear groups mentioned above, we obtain non-trivial Ramanujan bounds for generic cuspidal representations of the classical groups.

Corollary **10.1.** — *Let π be a globally generic cuspidal representation of* $G_n(\mathbf{A})$ *and let* N *be the rank of the general linear group to which π functorially lifts. Then π satisfies condition* $H(\frac{1}{2} - \frac{1}{N^2+1})$.

For the case of $G_n = SO_{2n+1}$ bounds just better than $H(\frac{1}{2})$, having exponents strictly less than $\frac{1}{2}$, were established in the course of proving Proposition 4.1 of [29].

Of a general nature, we can now state, for the first time, that the Ramanujan conjecture for generic cuspidal representations of the classical groups follows from the Ramanujan conjecture for general linear groups, which is widely held to be true.

Corollary **10.2.** — *If the Ramanujan conjecture holds for all cuspidal representations of* $GL_m(\mathbf{A})$ *for all m, that is, every cuspidal representation of* $GL_m(\mathbf{A})$ *satisfies condition* $H(0)$, *then it holds for all globally generic cuspidal representations of the classical groups* $G_n(\mathbf{A})$.

Of course, from our proof it is clear that to have Ramanujan for G_n it suffices to know it for GL_m with all $m \leq N$.

Remark. — Our work seems to shed light on what form a general Ramanujan conjecture for these groups should take in terms of functoriality. As suggested by Langlands [35], those cuspidal representations of $G_n(\mathbf{A})$ which defy Ramanujan should not functorially lift to any isobaric representation of $GL_N(\mathbf{A})$ and in particular not lift to any unitary isobaric one, by which is meant an isobaric sum of unitary cuspidal representations. If the lift is unitary isobaric, then by the Ramanujan conjecture for $GL_N(\mathbf{A})$ the lift would be tempered. Since conjecturally the tempered representations should be characterized, locally and globally, by the boundedness of the image of the associated arithmetic Langlands parameters, then the temperedness of the lift would imply the temperedness of the original representation. One could also give an argument of this type using Arthur's parameters and their connection with temperedness. This would then verify Langlands suggestion. On the other hand, assuming the conjecture on global genericity of tempered L-packets and the Ramanujan conjecture for GL_N, then, from the fact that generic cuspidal lifts are unitary isobaric (our Theorem 1.1 and Theorems 7.1 and 7.2), one can easily verify the converse. Note that in these arguments it is crucial that the map of L-groups under consideration is an embedding; more pathological L-homomorphisms could easily allow more pathological behavior under functoriality. Consequently, it seems reasonable to conjecture that a cuspidal representation of $G_n(\mathbf{A})$ is tempered if and only if its conjectural lift to $GL_N(\mathbf{A})$, associated to the natural embedding of the L-groups, is unitary isobaric. We would like to emphasize that the condition for cuspidal temperedness is that the lift is unitary isobaric, rather than tempered isobaric; so, for example, if one replaced G_n by GL_N and used the identity L-homomorphism then this would become the standard Ramanujan conjecture for GL_N itself. On the other hand, the residual representations of the classical groups $G_n(\mathbf{A})$ should lift to residual representations on $GL_N(\mathbf{A})$ and these should then account for those isobaric images that are not unitary isobaric.

11. Normalized local intertwining operators

We finish with one local result that follows from our bounds towards Ramanujan. We expect this result to have many applications, particularly in the study of the residual spectrum of classical groups.

Once again, let v be a non-archimedean place of k. Let π_v be an irreducible admissible unitary generic representation of $G_n(k_v)$ and let π'_v be an irreducible admissible unitary generic representation of $GL_m(k_v)$. As in Section 3, let G_{m+n} be the classical group of the same type as G_n but of rank $m+n$ and let P be the standard parabolic subgroup with Levi subgroup $M = GL_m \times G_n$. Then $\sigma_v = \pi'_v \otimes \pi_v$ is a unitary generic representation of $M(k_v)$ and we may form the induced representation

$$I(s, \sigma_v) = I(s, \pi'_v \otimes \pi_v) = \mathrm{Ind}_{P(k_v)}^{G_{m+n}(k_v)}\big(|\det|^s \pi'_v \otimes \pi_v\big).$$

Associated to this induced representation is a normalized intertwining operator $N(s, \sigma_v, w) = N(s, \pi'_v \otimes \pi_v, w)$ as in [51]. (For the case of $G_n = SO_{2n+1}$ see [6] or [29].)

Theorem **11.1.** — *Suppose that π_v is a local component of a globally generic cuspidal representation π of $G_n(\mathbf{A})$. Then for any irreducible admissible unitary generic representation π'_v of $GL_m(k_v)$ the normalized intertwining operator $N(s, \pi'_v \times \pi_v, w)$ is holomorphic and non-zero for $\mathrm{R}e(s) \geq 0$.*

For the case of $G_n = SO_{2n+1}$ this result is Proposition 4.1 of [29]. However, for the argument there to be complete, the lemma below is also needed. It should be pointed out that the lemma is independent of whether the representations involved occur as components of generic cuspidal representations or not.

Lemma **11.1.** — *Let τ'_v and τ_v be irreducible generic tempered representations of $GL_m(k_v)$ and $G_n(k_v)$, respectively. Then the normalized intertwining operator $N(s, \tau'_v \otimes \tau_v, w)$ is holomorphic and non-zero in the region $\mathrm{R}e(s) > -1/2$.*

Proof. — We follow the method of Lemma 4.3 of [31]. For simplicity we will drop the dependence of the normalized intertwining operators on the Weyl elements w since these elements play no role in the argument.

In general it is known that for tempered representations $N(s, \tau'_v \otimes \tau_v)$ is holomorphic and non-zero for $\mathrm{R}e(s) \geq 0$ in all our cases (see Lemma 4.2 in [31] for example). To extend this holomorphy and non-vanishing to $\mathrm{R}e(s) > -1/2$, we first reduce to discrete series representations by writing

$$\tau'_v = \mathrm{Ind}\big(\delta'_{1,v} \otimes \cdots \otimes \delta'_{k,v}\big)$$

with each $\delta'_{i,v}$ a unitary discrete series of appropriate smaller general linear groups and realizing τ_v as a direct summand of an induced representation of the form

$$\mathrm{Ind}(\delta_{1,v} \otimes \cdots \otimes \delta_{r,v} \otimes \sigma_{0,v})$$

with each $\delta_{i,v}$ a unitary discrete series representation of a general linear group for $i = 1, \ldots, r$ and $\sigma_{0,v}$ a generic unitary discrete series representation of a smaller classical group of the same type. Then the normalized intertwining operator $\mathrm{N}(s, \tau'_v \otimes \tau_v)$ will factor into a product of rank one normalized intertwining operators of the form $\mathrm{N}(s, \delta'_{i,v} \otimes \delta_{j,v})$ and $\mathrm{N}(s, \delta'_{i,v} \otimes \sigma_{0,v})$ [48]. Again, by [41] each $\mathrm{N}(s, \delta'_{i,v} \otimes \delta_{j,v})$ with $j \geq 1$ is holomorphic and non-vanishing for $\mathrm{R}e(s) > -1$. This reduces us to controlling normalized intertwining operators of the form $\mathrm{N}(s, \delta'_v \otimes \sigma_v)$ for δ'_v a unitary discrete series representation of some $\mathrm{GL}_m(k_v)$ and σ_v a unitary generic discrete series of a classical group $\mathrm{G}_n(k_v)$. Again, we know holomorphy and non-vanishing for $\mathrm{R}e(s) \geq 0$ and we are interested in pushing this to $\mathrm{R}e(s) > -1/2$.

For normalized intertwining operators associated to generic unitary discrete series we use the classification of these representations to reduce to supercuspidal representations. To this end, we again realize δ'_v as $\delta'_v = \delta(\rho'_v, t)$, now realized as the generic subrepresentation of the induced representation of the form

$$\Xi'_v = \mathrm{Ind}\big(\rho'_v \nu^{\frac{t-1}{2}} \otimes \cdots \otimes \rho'_v \nu^{-\frac{t-1}{2}}\big)$$

with ρ'_v a unitary supercuspidal representation of a smaller general linear group and t a positive integer [61]. Similarly, by [40] we can realize σ_v as a subrepresentation of

$$\Xi_v = \mathrm{Ind}\big(\rho_{1,v} \nu^{\frac{a_1}{2}} \otimes \cdots \otimes \rho_{r,v} \nu^{\frac{a_r}{2}} \otimes \rho_{0,v}\big)$$

where each $\rho_{i,v}$ with $i \geq 1$ is a supercuspidal representation of a general linear group, the a_i are positive integers, and $\rho_{0,v}$ is a generic supercuspidal representation of a smaller classical group of the same type. Then by transitivity of induction, the induced representation $\mathrm{I}(s, \delta'_v \otimes \sigma_v)$ is a subrepresentation of $\mathrm{I}(s, \Xi'_v \otimes \Xi_v)$ and $\mathrm{N}(s, \delta'_v \otimes \sigma_v)$ is obtained as the restriction of $\mathrm{N}(s, \Xi'_v \otimes \Xi_v)$ to $\mathrm{I}(s, \delta'_v \otimes \sigma_v)$. So it suffices to understand $\mathrm{N}(s, \Xi'_v \otimes \Xi_v)$. The normalized intertwining operator $\mathrm{N}(s, \Xi'_v \otimes \Xi_v)$ may have poles or zeros in $\mathrm{R}e(s) \geq 0$, but by the above result these will not occur when we restrict to $\mathrm{I}(s, \delta'_v \otimes \sigma_v)$. What we will be interested in is whether $\mathrm{N}(s, \Xi'_v \otimes \Xi_v)$ can have any poles or zeros in the region $-1/2 < \mathrm{R}e(s) < 0$.

This normalized intertwining operator once again factors into rank one normalized intertwining operators of the form $\mathrm{N}(2s + t - 1 - j, \rho'_v \otimes \rho'_v)$ with $j = 1, \ldots, 2t - 3$, of the form $\mathrm{N}(s + \frac{t-1}{2} \pm \frac{a_i}{2} - j, \rho'_v \otimes \rho_{i,v})$ with $0 \leq j \leq t - 1$ and $i = 1, \ldots, r$, or of the form $\mathrm{N}(s + \frac{t-1}{2} - j, \rho'_v \otimes \rho_{0,v})$ with $j = 0, \ldots, t - 1$. For the supercuspidal normalized intertwining operators we know that each $\mathrm{N}(s, \rho'_v \otimes \rho_{i,v})$, $0 \leq i \leq r$, is holomorphic except possibly on the lines $\mathrm{Re}(s) = -1$ and $\mathrm{Re}(s) = -1/2$ by Lemma 4.1 of [31]. Since all of our normalized intertwining operators are evaluated at either $s + \frac{b}{2}$ with integer b or $2s + c$ with integer c, we see that none of these has a pole in the region $-1/2 < \mathrm{Re}(s) < 0$.

Reconstructing our representations, we see that each $\mathrm{N}(s, \delta'_v \otimes \sigma_v)$ with δ'_v and σ_v unitary generic discrete series have no poles in the region $\mathrm{Re}(s) > -1/2$ and then the same is true for our $\mathrm{N}(s, \tau'_v \otimes \tau_v)$ with τ_v and τ'_v unitary tempered representations.

Once we have holomorphy, non-vanishing follows from Zhang's Lemma (Theorem 3 of [62], see also Lemma 4.7 of [31]). This then completes the lemma. □

We now turn to the proof of our theorem.

Proof. — Since π_v is a unitary generic representation then, as we have done several times, we can write it as

$$\pi_v \simeq \mathrm{Ind}\big(\tau_{1,v}\nu^{a_1} \otimes \cdots \otimes \tau_{m,v}\nu^{a_m} \otimes \tau_{0,v}\big)$$

where each $\tau_{i,v}$ is a tempered representation of an appropriate $\mathrm{GL}_{n_i}(k_v)$ and $\tau_{0,v}$ is a generic tempered representation of a smaller classical group $\mathrm{G}_{n_0}(k_v)$ of the same type as in (7.5) or (7.6). Since π_v is a local component of a globally generic cuspidal representation we know from Corollary 10.1 that the exponents satisfy the bounds

$$0 \leq |a_m| < a_{m-1} < \cdots < a_1 \leq \frac{1}{2} - \frac{1}{\mathrm{N}^2+1} < \frac{1}{2}.$$

Similarly for π'_v we have from the classification of unitary generic representation of $\mathrm{GL}_m(k_v)$ [58] that

$$\pi'_v = \mathrm{Ind}\big(\tau'_{1,v}\nu^{b_1} \otimes \cdots \otimes \tau'_{d,v}\nu^{b_d} \otimes \tau'_{0,v} \otimes \tau'_{d,v}\nu^{-b_d} \otimes \cdots \otimes \tau'_{1,v}\nu^{-b_1}\big)$$

with each $\tau'_{i,v}$ a tempered representation of an appropriate smaller general linear group and such that the exponents satisfy

$$0 < b_d < \cdots < b_1 < \frac{1}{2}.$$

The induced representation to which $\mathrm{N}(s, \pi'_v \otimes \pi_v, w)$ is associated is $\mathrm{I}(s, \pi'_v \otimes \pi_v)$ and if we replace π'_v and π_v by their realizations as induced representations and use transitivity of induction we see that the normalized intertwining operator $\mathrm{N}(s, \pi'_v \otimes \pi_v, w) = \mathrm{N}(s, \pi'_v \otimes \pi_v)$ will factor into a product of rank one normalized intertwining operators of one of the forms $\mathrm{N}(s \pm a_i \pm b_j, \tau'_{j,v} \otimes \tau'_{i,v})$, $\mathrm{N}(2s \pm b_i \pm b_j, \tau'_{i,v} \otimes \tau_{j,v})$ or $\mathrm{N}(s \pm b_j, \tau'_{j,v} \otimes \tau_{0,v})$ [48]. Again we have dropped the dependence on the Weyl elements since they do not effect the argument.

The rank one normalized intertwining operators of the form $\mathrm{N}(s, \tau'_{j,v} \otimes \pi_{i,v})$ with $i > 0$ are holomorphic for $\mathrm{R}e(s) > -1$ [41]. With our bounds on the exponents this implies that each operator $\mathrm{N}(s \pm a_i \pm b_j, \tau'_{j,v} \otimes \tau_{i,v})$ is holomorphic for $\mathrm{R}e(s) \geq 0$. Similarly, each operator $\mathrm{N}(2s \pm b_i \pm b_j, \tau'_{i,v} \otimes \tau'_{j,v})$ is holomorphic for $\mathrm{R}e(2s) \geq 0$, i.e., $\mathrm{R}e(s) \geq 0$. Since we now know from our lemma that each $\mathrm{N}(s, \tau'_{j,v} \otimes \tau_{0,v})$ is holomorphic for $\mathrm{R}e(s) > -1/2$ we see that each $\mathrm{N}(s \pm b_j, \tau'_{j,v} \otimes \tau_{0,v})$ is holomorphic for $\mathrm{R}e(s) \geq 0$ as desired.

Thus $\mathrm{N}(s, \pi' \otimes \pi, w)$ is holomorphic for $\mathrm{R}e(s) \geq 0$ and so by Zhang's Lemma again (Theorem 3 of [62]) it is non-vanishing there as well. □

A. Appendix

The following appendix addresses the issue of non-degeneracy of cuspidal representations with respect to different characters. This is relevant here since neither SO_{2n} nor Sp_{2n} is of adjoint type. For future use, we will present the argument in a more general context than the rest of the current paper.

We let k be a number field as before, $\mathbf{A}$ its ring of adeles, and $\psi = \otimes_v \psi_v$ be a non-trivial character of $k\backslash\mathbf{A}$. Let $\Gamma = \mathrm{Gal}(\bar{k}/k)$.

Let G be a quasisplit connected reductive algebraic group over k. We fix a k-Borel subgroup $\mathrm{B} = \mathrm{TU}$ with T a maximal torus and U its unipotent radical. Let $\mathrm{P} = \mathrm{MN}$ be a maximal parabolic subgroup of G with the Levi decomposition satisfying $\mathrm{N} \subset \mathrm{U}$ and $\mathrm{T} \subset \mathrm{M}$.

If Δ' denotes the set of (non–restricted) simple roots of T in U, let $\{\mathrm{X}_{\alpha'}\}_{\alpha' \in \Delta'}$ be a Γ-invariant set of root vectors, giving what we will call in short a k-splitting. Then $\{\mathrm{X}_{\alpha'}\}_{\alpha' \in \Delta'}$ is a k_v-splitting for each completion k_v of k. It then defines a character χ_v of $\mathrm{U}(k_v)$ by

$$\chi_v\Big(\prod_{\alpha \in \Delta'} \exp(x_{\alpha',v}\mathrm{X}_{\alpha'})\Big) = \psi_v\Big(\sum_{\alpha' \in \Delta'} x_{\alpha',v}\Big). \tag{A.1}$$

We understand that if $\mathrm{X}_{\beta'} = \sigma(\mathrm{X}_{\alpha'}), \alpha', \beta' \in \Delta$, then $x_{\beta',v} = \sigma(x_{\alpha',v})$, $\sigma \in \Gamma$. Let $\chi = \otimes_v \chi_v$ be the corresponding non–degenerate character of $\mathrm{U}(k)\backslash\mathrm{U}(\mathbf{A})$. We use χ to also denote its restriction to $\mathrm{U_M}(\mathbf{A}) = \mathrm{U}(\mathbf{A}) \cap \mathrm{M}(\mathbf{A})$.

Denote by $r = \bigoplus_{i=1}^{m} r_i$, as usual (cf. [51]), the adjoint action of ${}^{\mathrm{L}}\mathrm{M}$ on ${}^{\mathrm{L}}\mathfrak{n}$, the Lie algebra of ${}^{\mathrm{L}}\mathrm{N}$.

Let $\pi = \otimes_v \pi_v$ be a globally χ-generic cuspidal representation of $\mathrm{M}(\mathbf{A})$. The machinery of our method [51] then defines a global L-function $\mathrm{L}(s, \pi, r_i)$ and a global ε-factor $\varepsilon(s, \pi, r_i)$ for each i, $1 \leq i \leq m$, such that

$$\mathrm{L}(s, \pi, r_i) = \varepsilon(s, \pi, r_i)\mathrm{L}(1 - s, \tilde{\pi}, r_i). \tag{A.2}$$

The purpose of this appendix is to show that the choice of the k-splitting has no effect on $\varepsilon(s,\pi,r_i)$ and $L(s,\pi,r_i)$. More precisely, we will show that if one changes the splitting and accordingly π, the same $\varepsilon(s,\pi,r_i)$ and $L(s,\pi,r_i)$ are obtained.

We start with the following well–known lemma.

Lemma **A.1.** — *Let* Z_G *be the center of* G. *Assume* $H^1(Z_G) = \{1\}$. *Then* $T(k)$ *acts transitively on the set of generic characters of* $U(k)\backslash U(\mathbf{A})$.

Proof. — Assume χ is defined by

$$\chi\Big(\prod_{\alpha'\in\Delta'}\exp(x_{\alpha'}X_{\alpha'})\Big) = \psi\Big(\sum_{\alpha'\in\Delta'}\kappa_{\alpha'}x_{\alpha'}\Big), \tag{A.3}$$

where $\kappa_{\alpha'} = \kappa_{\sigma(\alpha')} \in k^\times$ for all $\alpha' \in \Delta'$ and $\sigma \in \Gamma$, since χ is a generic character of $U(k)\backslash U(\mathbf{A})$.

Choose $t \in T(\bar{k})$ such that $\alpha'(t) = \kappa_{\alpha'}$ for all $\alpha' \in \Delta'$. Then $\sigma(\alpha'(t)) = \alpha'(t)$. Moreover $\kappa_{\sigma^{-1}(\alpha')} = \sigma^{-1}(\alpha'(\sigma(t)))$ implies $\alpha'(\sigma(t)) = \alpha'(t)$. Thus

$$\alpha'(t^{-1}\sigma(t)) = \alpha'(t)^{-1}\alpha'(\sigma(t)) = 1$$

for all $\alpha' \in \Delta'$ and therefore $\sigma \mapsto t^{-1}\sigma(t)$ defines a class in $H^1(Z_G) = \{1\}$. Choose $z \in Z_G$ such that $t^{-1}\sigma(t) = z\sigma(z)^{-1}$. Then $\alpha'(tz) = \kappa_{\alpha'}$ for all $\alpha' \in \Delta'$ and $tz \in T(k)$. The lemma is now complete. □

By Proposition 5.4 of [55], we embed G into $\tilde{G}$ sharing the same derived group as G and satisfying $H^1(Z_{\tilde{G}}) = \{1\}$. Let $\tilde{B} = \tilde{T}U$ be a k-Borel subgroup of $\tilde{G}$ containing B and moreover assume $\tilde{T} \supset T$. Observe that $T = B \cap \tilde{T}$. Then $\tilde{T}(k)$ acts transitively on generic characters of $U(k)\backslash U(\mathbf{A})$. Observe that $\tilde{T}(k)$ normalizes M and $M(k)$ as well as $M(\mathbf{A})$, since as $\bar{k}$-groups, $\tilde{T} = TZ_{\tilde{G}}$.

Given a cusp form $\phi \in V_\pi$ in the space of π and $t \in \tilde{T}(k)$, define ϕ_t by $\phi_t(m) = \phi(t^{-1}mt)$, $m \in M(\mathbf{A})$. Then ϕ_t is a cusp form which is χ_t-generic (globally), where $\chi_t(u) = \chi(t^{-1}ut)$, $u \in U(\mathbf{A})$. Let $\pi_t(m) = \pi(t^{-1}mt)$. Then the representation π_t on the space $V_\pi = \{\phi\}$ of π is equivalent to the right regular action of $M(\mathbf{A})$ on the space

$$\{\phi_t | \phi \in V_\pi\}.$$

Moreover, if $\pi_t = \otimes_v \pi_{t,v}$, then $\pi_{t,v} = \pi_{v,t}$ for each v, where $\pi_{v,t} = \pi_v(t^{-1}mt)$.

Given f in the space of

$$I(s,\pi) = \mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}(\pi \otimes \exp\langle s\tilde{\alpha},\ H_M(\ )\rangle),$$

define $f_t(g) = f(t^{-1}gt)$. The set of all such f_t comprises the space of $\mathrm{I}(s, \pi_t)$. Finally

$$\mathrm{I}(s, \pi_t) = \otimes_v \mathrm{I}(s, \pi_{v,t}).$$

The general machinery of our method then leads to the functional equation

$$\textbf{(A.4)} \qquad \mathrm{L}(s, \pi_t, r_i) = \varepsilon(s, \pi_t, r_i)\mathrm{L}(1 - s, \tilde{\pi}_t, r_i)$$

for each i, $1 \leq i \leq m$.

The local L-functions $\mathrm{L}(s, \pi_{t,v}, r_{i,v})$ are defined by means of intertwining operators and local coefficients [51,49]. In fact, if σ is an irreducible supercuspidal χ_v-generic representation of $\mathrm{M}(k_v)$, then $\mathrm{L}(s, \sigma, r_{i,v})$ and $\mathrm{L}(s, \sigma_t, r_{i,v})$ are determined inductively precisely by poles of local standard intertwining operators such as $\mathrm{A}(s, \sigma)$ and $\mathrm{A}(s, \sigma_t)$ acting on $\mathrm{I}(s, \sigma)$ and $\mathrm{I}(s, \sigma_t)$, respectively. Moreover, if we use the definition

$$\textbf{(A.5)} \qquad \mathrm{A}(s, \sigma)f(g) = \int_{\overline{\mathrm{N}}(\mathrm{F}_v)} f(\overline{n}g)\,d\overline{n},$$

where $\overline{\mathrm{N}} = w_0^{-1}\mathrm{N}w_0$, then

$$\textbf{(A.6)} \qquad \mathrm{A}(s, \sigma_t)f_t = (d\overline{n}/d(t^{-1}\overline{n}t))(\mathrm{A}(s, \sigma)f)_t.$$

Thus

$$\textbf{(A.7)} \qquad \mathrm{L}(s, \sigma, r_{i,v}) = \mathrm{L}(s, \sigma_t, r_{i,v}).$$

The equality (A.7) of L-functions for a general σ follows from the inductive definition of L-functions by means of local coefficients and Langlands classification (cf. [51]).

Comparing functional equations (A.2) and (A.4) one gets

$$\textbf{(A.8)} \qquad \varepsilon(s, \pi_t, r_i) = \varepsilon(s, \pi, r_i)$$

for every i, $1 \leq i \leq m$.

From this discussion it now follows that to define and study global ε-factors, for example their stability, it is enough to take π which is generic with respect to the most convenient splitting.

REFERENCES

1. J. Arthur, The principle of functoriality, *Bull. Am. Math. Soc.*, **40** (2002), 39–53.
2. A. Borel, Automorphic L-functions, *Proc. Symp. Pure Math.*, **33**, part 2 (1979), 27–61.
3. N. Bourbaki, Algèbre Commutative, Ch. 3 et 4, *Actualités Scientifiques et Industrielles*, no. 1293. Hermann, Paris (1961).
4. W. Casselman and F. Shahidi, On reducibility of standard modules for generic representations, *Ann. Sci. Éc. Norm. Supér.*, **31** (1998), 561–589.
5. J. W. Cogdell, Dual groups and Langlands Functoriality, *An Introduction to the Langlands Program* (J. Bernstein and S. Gelbart, eds.). Birkhäuser, Boston (2003), 251–268.
6. J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi, On lifting from classical groups to GL_N, *Publ. Math., Inst. Hautes Étud. Sci.*, **93** (2001), 5–30.
7. J. W. Cogdell and I. I. Piatetski-Shapiro, Converse Theorems for GL_n, *Publ. Math., Inst. Hautes Étud. Sci.*, **79** (1994), 157–214.
8. J. W. Cogdell and I. I. Piatetski-Shapiro, Stability of gamma factors for $SO(2n+1)$, *Manuscr. Math.*, **95** (1998), 437–461.
9. J. W. Cogdell and I. I. Piatetski-Shapiro, Converse Theorems for GL_n, II, *J. Reine Angew. Math.*, **507** (1999), 165–188.
10. J. W. Cogdell, I. I. Piatetski-Shapiro, and F. Shahidi, On stability of local γ-factors, in preparation.
11. S. Gelbart and H. Jacquet, A relation between automorphic representations of GL(2) and GL(3), *Ann. Sci. Éc. Norm. Supér., IV. Sér.,* **11** (1978), 471–542.
12. S. Gelbart and F. Shahidi, Boundedness of automorphic L-functions in vertical strips, *J. Am. Math. Soc.*, **14** (2001), 79–107.
13. D. Ginzburg, S. Rallis, and D. Soudry, Generic automorphic forms on $SO(2n+1)$: functorial lift to $GL(2n)$, endoscopy, and base change, *Int. Math. Res. Not.*, **2001**, no. 14 (2001), 729–764.
14. R. Godement and H. Jacquet, Zeta Functions of Simple Algebras, *Lect. Notes Math.*, **260**. Springer-Verlag, Berlin (1972).
15. M. Harris and R. Taylor, The Geometry and Cohomology of Some Simple Shimura Varieties, *Ann. Math. Stud.*, **151**. Princeton University Press, Princeton (2001).
16. G. Henniart, Caractérisation de la correspondance de Langlands locale par les facteurs ϵ de paires, *Invent. Math.*, **113** (1993), 339–350.
17. G. Henniart, Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps p-adique, *Invent. Math.*, **139** (2000), 439–455.
18. R. Howe and I. I. Piatetski-Shapiro, A counter-example to the "generalized Ramanujan conjecture" for (quasi)-split groups, *Proc. Symp. Pure Math.*, **33**, part 1 (1979), 315–322.
19. H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, Conducteur des représentations du groupe linéaire., *Math. Ann.*, **256** (1981), 199–214.
20. H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, Rankin-Selberg convolutions, *Am. J. Math.*, **105** (1983), 367–464.
21. H. Jacquet and J. Shalika, On Euler products and the classification of automorphic representations, *Am. J. Math.*, **103** (1981), 499–558 and 777–815.
22. H. Jacquet and J. Shalika, The Whittaker models for induced representations, *Pac. J. Math.*, **109** (1983), 107–120.
23. H. Jacquet and J. Shalika, A lemma on highly ramified ϵ-factors, *Math. Ann.*, **271** (1985), 319–332.
24. C. Jantzen, On square integrable representations of classical p-adic groups, *Can. J. Math.*, **52** (2000), 539–581.
25. C. Jantzen, On square integrable representations of classical p-adic groups II, *Represent. Theory*, **4** (2000), 127–180.
26. D. Jiang and D. Soudry, The local converse theorem for $SO(2n+1)$ and applications, *Ann. Math.*, **157** (2003), 743–806.
27. D. Jiang and D. Soudry, Generic representations and local Langlands reciprocity law for p-adic SO_{2n+1}, *Contributions to Automorphic Forms, Geometry and Number Theory (Shalikafest 2002)* (H. Hida, D. Ramakrishnan, and F. Shahidi, eds.). Johns Hopkins University Press, Baltimore, to appear.
28. H. Kim, Langlands–Shahidi method and poles of automorphic L-functions, II, *Isr. J. Math.*, **117** (2000), 261–284.

29. H. KIM, Residual spectrum of odd orthogonal groups, *Int. Math. Res. Not.* **2001**, no. 17 (2001), 873–906.

30. H. KIM, Applications of Langlands' functoriality of odd orthogonal groups, *Trans. Am. Math. Soc.*, **354** (2002), 2775–2796.

31. H. KIM, On local L-functions and normalized intertwining operators, *Can. J. Math.*, to appear.

32. H. KIM and F. SHAHIDI, Functorial products for $GL_2 \times GL_3$ and the symmetric cube for GL_2, *Ann. Math.*, **155** (2002), 837–893.

33. L. LAFFORGUE, Chtoucas de Drinfeld at correspondance de Langlands, *Invent. Math.*, **147** (2002) 1–241.

34. R. P. LANGLANDS, On the classification of irreducible representations of real algebraic groups, *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, Math. Surv. Monogr., **31**. Am. Math. Soc., Providence, RI (1989), 101–170.

35. R. P. LANGLANDS, Automorphic representations, Shimura varieties, and motives. Ein Märchen, *Proc. Symp. Pure Math.*, **33**, part 2 (1979), 205–246.

36. R. P. LANGLANDS, Where stands functoriality today, *Proc. Symp. Pure Math.*, **61** (1997), 457–471.

37. W. LUO, Z. RUDNICK, and P. SARNAK, On the generalized Ramanujan conjecture for $GL(n)$, *Proc. Symp. Pure Math.*, **66**, part 2 (1999), 301–310.

38. C. MŒGLIN, Points de réducibilité pour les induits de cuspidals, preprint (2001).

39. C. MŒGLIN, Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité, *J. Eur. Math. Soc.*, **4** (2002), 141–200.

40. C. MŒGLIN and M. TADIĆ, Construction of discrete series for p-adic classical groups, *J. Am. Math. Soc.*, **15** (2002), 715–786.

41. C. MŒGLIN and J-L. WALDSPURGER, Le spectre résiduel de $GL(n)$, *Ann. Sci. Éc. Norm. Supér.*, **22** (1989), 605–674.

42. G. MUIĆ, On generic irreducible representations of $Sp(n, F)$ and $SO(2n+1, F)$, *Glas. Mat., III. Ser.*, **33 (53)** (1998), 19–31.

43. G. MUIĆ, Some results on square integrable representations; Irreducibility of standard representations, *Int. Math. Res. Not.*, **1998**, no. 14 (1998), 705–726.

44. G. MUIĆ, A proof of Casselman-Shahidi's conjecture for quasi-split classical groups, *Can. Math. Bull.*, **44** (2001), 298–312.

45. I. I. PIATETSKI-SHAPIRO, Multiplicity one theorems, *Proc. Symp. Pure Math.*, **33**, part 1 (1979), 209–212.

46. P. SARNAK, Estimates for Rankin-Selberg L-functions and quantum unique ergodicity, *J. Funct. Anal.*, **184** (2001), 419–453.

47. I. SATAKE, Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields, *Publ. Math., Inst. Hautes Étud. Sci.*, **18** (1963), 5–69.

48. F. SHAHIDI, On certain L-functions, *Am. J. Math.*, **103** (1981), 297–355.

49. F. SHAHIDI, Local coefficients as Artin factors for real groups, *Duke Math. J.*, **52** (1985), 973–1007.

50. F. SHAHIDI, On the Ramanujan conjecture and finiteness of poles for certain L-functions, *Ann. Math.*, **127** (1988), 547–584.

51. F. SHAHIDI, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups, *Ann. Math.*, **132** (1990), 273–330.

52. F. SHAHIDI, On multiplicativity of local factors, *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday*, part II (Ramat Aviv, 1989), Israel Math. Conf. Proc., **3**, Weizmann, Jerusalem (1990), 279–289.

53. F. SHAHIDI, Twisted endoscopy and reducibility of induced representations for p-adic groups, *Duke Math. J.*, **66** (1992), 1–41.

54. F. SHAHIDI, Twists of a general class of L-functions by highly ramified characters, *Can. Math. Bull.*, **43** (2000), 380–384.

55. F. SHAHIDI, Local coefficients as Mellin transforms of Bessel functions; Towards a general stability, *Int. Math. Res. Not.*, **2002**, no. 39 (2002), 2075–2119.

56. D. SOUDRY, On Langlands functoriality from classical groups to GL_n, *Astérisque*, to appear.

57. R. STEINBERG, *Lectures on Chevalley Groups*. Yale Lecture Notes, New Haven (1967).

58. M. TADIĆ, Classification of unitary representations in irreducible representations of general linear groups (non-Archimedean case), *Ann. Sci. Éc. Norm. Supér., IV. Sér.*, **19** (1986), 335–382.

59. J. TATE, Number theoretic background, *Proc. Symp. Pure Math.*, **33**, part 2 (1979), 3–26.

60. D. VOGAN, Gelfand-Kirillov dimension for Harish-Chandra modules, *Invent. Math.*, **48** (1978), 75–98.

61. A. Zelevinsky, Induced representations of reductive $\mathfrak{p}$–adic groups, II, *Ann. Sci. Éc. Norm. Supér.*, **13** (1980), 165–210.

62. Y. Zhang, The holomorphy and nonvanishing of normalized local intertwining operators, *Pac. J. Math.*, **180** (1997), 386–398.

J. W. C.
Department of Mathematics,
Oklahoma State University,
Stillwater, OK 74078, USA
cogdell@math.okstate.edu

H. H. K.
Department of Mathematics,
University of Toronto,
Toronto, ON M5S 3G3, Canada
henrykim@math.toronto.edu

I. I. P.-S.
Department of Mathematics,
Yale University,
New Haven, CT 06520, USA
and
School of Mathematics,
Tel Aviv University,
Tel Aviv 69978, Israel
ilya@math.yale.edu

F. S.
Department of Mathematics,
Purdue University,
West Lafayette, IN 47907, USA
shahidi@math.purdue.edu

Manuscrit reçu le 10 août 2003.