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FUNDAMENTAL DECAY MODE AND ASYMPTOTIC BEHAVIOUR OF POSITIVE SEMIGROUPS[1])

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1. INTRODUCTION

In 1957 J. LEHNER and M. WING published a paper, [8], in which a complete description of the spectrum of the linearized one-velocity Boltzmann operator in an infinite slab was given. They later in [9] applied their results to the neutron transport equation. In these papers it was shown that there is a continuous spectrum for the Boltzmann operator and that there may also be eigen-values in the half-plane $\operatorname{Re} \lambda > -\lambda^*$, where $\lambda^* \geqq 0$. Lehner showed in [7] that the situation is completely different if one considers a bounded domain G in place of an infinite slab and he actually gave some important results concerning the one-velocity Boltzmann operator in a sphere.

Further progress has been made by K. JÖRGENS [3] who has analysed the linearized energy-dependent Boltzmann equation under an additional assumption that both the space domain G and the velocity domain V are bounded and that V is bounded away from the origin.

A very important result is due to S. ALBERTONI and B. MONTAGNINI who have shown in [1] that if G is a convex bounded body whose volume is sufficiently small, then the point spectrum of the linearized velocity-dependent Boltzmann operator is empty.

A series of papers devoted to a detailed study of the spectrum of the linearized velocity-dependent Boltzmann operator has been published by I. VIDAV, see e.g. [19], [20].

Further progress can be traced in recent work by S. UKAI [18] and Y. SHIZUTA [16]. In [16] the theory for linear problems is extended and applied to some non-linear problems.

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The aim of this report is to show some asymptotic properties of certain semigroups of cone preserving operators of class (C_0). These properties are based on a lattice order in the space Y and consequently on the induced lattice order in $\mathscr{L}(Y)$ (see Section 2). The results obtained have rather important applications in the transport theory of particles. We note that in particular our theory covers the detailed description of the peripheral part of the spectrum of the linearized velocity-dependent Boltzmann operator. This gives a solution to Problem 10 of Kaper's Collection of Problems in [4].

2. DEFINITIONS AND NOTATION

Let Y be a real Banach space and let $X = Y \oplus \mathrm{i}Y$ be its complexification. By Y' and X' the corresponding dual spaces are denoted and by $\mathscr{L}(Y)$ and $\mathscr{L}(X)$ the spaces of bounded linear operators mapping Y and X into Y and X respectively.

It is assumed that $K \subset Y$ is a *closed, generating* and *normal cone* ([5]); that is K has the following properties:

(a) $x, y \in K \Rightarrow x + y \in K$, or else $K + K \subset K$,
(b) $x \in K$, $\alpha \geqq 0 \Rightarrow \alpha x \in K$, or else $\alpha K \subset K$,
(c) $x \in K$, $(-x) \in K \Rightarrow x = 0$, or else $K \cap (-K) = \{0\}$,
(d) $y \in Y \Rightarrow$ there are $y^{\pm} \in K$ such that $y = y^+ - y^-$, or else $Y = K - K$,
(e) there is $\delta > 0$ such that $x, y \in K \Rightarrow \|x + y\| \geqq \delta \max(\|x\|, \|y\|)$,
(f) $x_n \in K$, $\lim \|x_n - x\| = 0 \Rightarrow x \in K$.

A partial ordering is introduced into Y by setting $x \leqq y$ (or equivalently $y \geqq x$) whenever $(y - x) \in K$.

An operator $T \in \mathscr{L}(Y)$ is called *positive* (more precisely K-positive) if $TK \subset K$. A partial order is introduced into $\mathscr{L}(Y)$ by setting $S \leqq T$ (or $T \geqq S$) whenever $(T - S)K \subset K$.

An operator $T \in \mathscr{L}(Y)$ is called *K-irreducible*, if for every pair $x \in K$, $x \neq 0$, $x' \in K'$, $x' \neq 0$, where $K' = \{x' \in Y' : x'(x) \geqq 0 \text{ for all } x \in K\}$, there is a positive integer $p = p(x, x')$ such that $x'(T^p x) > 0$. Let us note that in [13] the concept semi-non-support operator was used originally.

Remark. The set K' also forms a closed normal generating cone in Y' and is called a *dual cone* with respect to K, [5].

An operator $T \in \mathscr{L}(Y)$ is said to be *K-irreducibly primitive*, if T^k is K-irreducible for every positive integer $k \geqq 1$.

If for every x and y in Y the sup $\{x, y\}$ and inf $\{x, y\}$ exist in the sense the ordering given by K, the space Y is called a *Riesz space*, or a *Banach lattice*, [14, Ch. II, pp. 46–153, in particular p. 81].

We briefly summarize some fundamental results concerning K-positive operators.

For a closed densely defined linear operator A mapping $\mathscr{D}(A) \subset X$ into X the symbol $\sigma(A)$ denotes its spectrum. The complement of $\sigma(A)$ in the complex plane is called the *resolvent set* and is denoted as $\varrho(A)$. For $T \in \mathscr{L}(X)$ the quantity $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ is called the *spectral radius of* T.

Let A be a linear operator mapping $\mathscr{D}(A) \subset Y$ into Y. Then the operator $\tilde{A}$ defined by $\tilde{A}z = Ax + iAy$ for $z = x + iy$, $x, y \in Y$, is called the *complex extension of* A. By definition we let $\sigma(A) = \sigma(\tilde{A})$ and $r(T) = r(\tilde{T})$ for $T \in \mathscr{L}(Y)$.

Proposition 1. *Let $T \in \mathscr{L}(Y)$ be K-positive. Then $r(T) \in \sigma(T)$.*

Proposition 2. *Let $T \in \mathscr{L}(Y)$ be K-positive and $\lambda \in \sigma(T)$, $|\lambda| = r(T)$, imply that λ is a pole of the resolvent operator. Then in the Laurent expansion*

$$(\mu I - T)^{-1} \equiv R(\mu, T) = \sum_{k=0}^{\infty} A_k(\lambda)(\mu - \lambda)^k + \sum_{k=1}^{q(\lambda)} B_k(\lambda)(\mu - \lambda)^{-k},$$

where $A_k(\lambda)$ and $B_k(\lambda)$ are in $\mathscr{L}(X)$, $B_{q(r)}(r)$ is K-positive; here $r = r(T)$ and $q(\lambda)$ is the multiplicity of λ as a pole of $R(\mu, T)$. In addition, $q(r) \geqq q(\lambda)$.

Corollary. *There exist eigen-vectors $x_0 \in K$ and $x'_0 \in K'$ such that $Tx_0 = rx_0$ and $T'x'_0 = rx'_0$.*

Proposition 3. *Let $T \in \mathscr{L}(Y)$ be K-irreducible and assume the assumptions of Proposition 2 hold. Then the sets $\mathscr{N} = \{y \in Y : (T - rI)^k y = 0$ for some $k = 0, 1, \ldots\}$ and $\mathscr{N}' = \{y' \in Y' : [(T - rI)']^k y' = 0$ for some $k = 0, 1, \ldots\}$ are one-dimensional. Moreover, $Tu = vu$, $u \in K$, $u \neq 0$, implies that $v = r$ and similarly, $T'u' = vu'$, $u' \in K'$, $u' \neq 0$, implies that $v = r$.*

Corollary. *$B_k(r) = \theta$ for $k > 1$ and $B_1 = B_1(r)$ is K-irreducible.*

Proposition 4. *Let $T \in \mathscr{L}(Y)$ be K-irreducibly primitive and let the assumptions of Proposition 2 hold. Then $\lambda \in \sigma(T)$, $|\lambda| = r(T)$, implies that $\lambda = r(T)$.*

Remark. It can be shown that Propositions 1–4 are valid in general Banach spaces whose ordering is given by a normal generating cone, not necessarily a lattice generating cone.

The validity of the next results depends essentially upon the lattice structure of Y.

Proposition 5. *Let $T \in \mathscr{L}(Y)$ be K-positive and such that its spectral radius is a pole of the resolvent operator $R(\mu, T)$ of order q with the residue $B_1(r)$, where $r = r(T)$, $q = q(r)$, having a finite dimensional range. Then all points $\lambda \in \sigma(T)$ for which $|\lambda| = r(T)$ are poles of $R(\mu, T)$ with multiplicites at most q. Moreover, $(1/r)^k \in \sigma(1/r)$, $k = 1, 2, \ldots$ whenever $\lambda \in \sigma(T)$, $|\lambda| = r(T)$.*

Proposition 6. *Let $T \in \mathscr{L}(Y)$ be K-irreducible and $r(T)$ be a pole of the resolvent operator. Then all other points $\lambda \in \sigma(T)$, $|\lambda| = r(T)$, are simple poles of the resolvent operator.*

The properties of K-positive operators shown above in Propositions 1–4 and the Corollaries can be found in the literature, e.g. [14, Chap. V], [13], [10], while Proposition 6 is shown first in [12]. A complete theory has been set up in [14, Chap. V, pp. 322–337].

4. MAIN RESULTS

Should the reader require any information regarding semigroups of operators, he is referred to [2].

It is always assumed here that the cone K producing the order in Y is a lattice generating cone. We note that our investigation is based on the following hypotheses, the fulfilment of which is assumed appropriately.

(i) A is the infinitesimal generator of a semigroup of operators $T(t; A)$ of class (C_0).

(ii) There is a $\hat{t} > 0$ such that $r(T(\hat{t}; A))$ is a pole of $R(\mu, T(\hat{t}; A))$ with a finite dimensional generalised eigenspace $\mathscr{N}_0$ (a Fredholm eigenvalue).

(iii) Let λ_0 be such that

$$\lambda \in \sigma(A) \Rightarrow \operatorname{Re} \lambda \leqq \operatorname{Re} \lambda_0 = \lambda_0 \tag{4.1}$$

and there exists a $\tilde{\lambda} \in \sigma(A)$ with $\operatorname{Re} \tilde{\lambda} = \lambda_0$.

(iv) The semigroup $T(t; A)$ is K-positive for $t > 0$.

(v) The semigroup $T(t; A)$ is K-irreducible for $t > 0$.

Theorem 1. *Let the hypotheses* (i)–(iv) *hold. Then*

$$\sigma(A) \cap \{\lambda : \operatorname{Re} \lambda = \lambda_0\} = \{\lambda_0\} \tag{4.2}$$

and

$$T(t; A) = e^{\lambda_0 t}[B_0 + Z(t)] + W(t), \tag{4.3}$$

where

$$B_0 = B_1 = (r(T(\hat{t}; A))), \quad B_0 Z(t) = Z(t) B_0 = Z(t), \tag{4.4}$$

$$B_0 W(t) = W(t) B_0 = \theta, \tag{4.5}$$

$$B_0 T(t; A) = T(t; A) B_0. \tag{4.6}$$

Moreover,

$$(4.7)\qquad \lim_{t\to+\infty} e^{-\lambda_0 t}\|W(t)\| = 0\,, \quad \lim_{t\to+\infty}\left\|t^{-q+1}Z(t) - \frac{1}{(q-1)!}B_{0,t}\right\| = 0\,,$$

where q is the multiplicity of λ_0 as a pole of $R(\mu, A)$.

In addition,

$$B_{0,q} = \lim_{\varrho\to\lambda_0,\varrho>\lambda_0} (\varrho - \lambda_0)^q R(\varrho, A)$$

is K-positive and, consequently, every $v_0 = B_{0,q}u$, where $u \in K$, $B_{0,q}u \neq 0$, is an eigen-vector of $A : Av_0 = \lambda_0 v_0$.

Proof. For every $t > 0$ the spectral radius $r(t) = r(T(t; A))$ is by Proposition 1 in the spectrum $\sigma(T(t; A))$. It is obvious that $r(t) = e^{\lambda_0 t}$. According to assumption (ii) $r(\hat{t})$ is a pole of $R(\mu, T(\hat{t}; A))$ and by the Niiro-Sawashima-Lotz-Schaefer theorem (Proposition 5) the whole peripheral spectrum of $T(\hat{t}; A)$ consists of a finite set of poles $\mu_0, \ldots, \mu_p$, $\mu_0 = e^{\lambda_0 \hat{t}} = r(t)$, $\mu_j = e^{\lambda_j \hat{t}}$, $\operatorname{Re}\lambda_j = \lambda_0$, $j = 0, \ldots, p$. The existence of a $\tilde{\lambda}$ in the spectrum of A such that $\operatorname{Re}\tilde{\lambda} = \lambda_0$, $\tilde{\lambda} \neq \lambda_0$, implies the existence of an infinite set of eigenvalues $\{\lambda_j\}$ of A with $\operatorname{Re}\lambda_j = \lambda_0$ and consequently contradicts the finiteness of the rank of B_0. Hence $p = 0$ and (4.2) follows.

Let

$$R(\lambda, A) = \sum_{k=0}^{\infty} A_k(\lambda - \lambda_0)^k + \sum_{k=1}^{q} B_k(\lambda - \lambda_0)^{-k}\,,$$

$1 \leqq q < +\infty$, be the Laurent expansion of the resolvent operator about the point λ_0 (see [17, p. 305]) and let $B = A - H$, where $H = [\lambda_0 B_1 + B_2]$.

It is easy to see that B is the infinitesimal generator of a semigroup of operators $T(t; B)$ of class (C_0) and that

$$T(t; A) = \exp\{tH\}\, T(t; B)\,, \quad t \geqq 0\,.$$

Since $BB_k = \theta$, $k = 1, \ldots, q$, we derive that

$$T(t; A) = \exp\{tH\} + T(t; B) - I$$

and consequently,

$$T(t; A) = P\exp\{tH\} + T(t; B) - P = P\exp\{tH\} + (I - P)\,T(t; B)\,,$$

where $P = B_1$.

The point λ_0 is isolated and so we conclude that there is an $\varepsilon > 0$ such that

$$\|T(t; B)\| \leqq e^{(\lambda_0-\varepsilon)t}\,, \quad t \geqq 0\,.$$

If we let

$$W(t) = (I - P)\,T(t; B)$$

and evaluate the exponential $\exp\{tH\}$, [17, p. 319] and use the fact that $q(j) = q(0)$ (Proposition 5), we obtain (4.3)–(4.7) immediately. The positivity of $B_{0,q(0)}$ is guaranteed by Proposition 2 and this completes the proof of Theorem 1.

Theorem 2. *Let the hypotheses* (i)–(iii) *and* (v) *hold. Then the relations* (4.2) *to* (4.7) *specify as follows*

$$\sigma(A) \cap \{\lambda : \operatorname{Re} \lambda = \lambda_0\} = \{\lambda_0\}, \tag{4.8}$$

$$T(t; A) = e^{\lambda_0 t} B_0 + W(t), \tag{4.9}$$

where

$$T(t; A)\, B_0 = B_0\, T(t; A), \quad B_0\, W(t) = W(t)\, B_0 = \theta, \tag{4.10}$$

and

$$\lim_{t \to +\infty} e^{-\lambda_0 t} \| W(t) \| = 0. \tag{4.11}$$

Moreover, $q = 1$, *where* q *is the multiplicity of* λ_0 *as a pole of* $R(\lambda, A)$.

Proof. K-irreducibility of $T(t; A)$ implies that $q = 1$ so that (4.8)–(4.11) follows as a consequence of (4.2)–(4.7). The proof is complete.

5. DECAY MODE AND ASYMPTOTIC PROPERTIES

The following Cauchy problem

$$\frac{d}{dt} u(t) = A\, u(t), \quad u(0) = u_0 \tag{5.1}$$

is considered, where A is a given linear operator. It is known that if A is the infinitesimal generator of a semigroup of operators of class (C_0) then, [2, p. 308]

$$u(t) = T(t; A)\, u_0 .$$

In various areas of applied mathematics and mathematical physics one needs to know the asymptotic behaviour of the solution $u(t)$ as t approaches $+\infty$.

Let A be the infinitesimal generator of a semigroup of operators $T(t; A)$ of class (C_0) and let μ_0 be its eigenvalue and $Av_0 = \mu_0 v_0$, $v_0 \neq 0$. Then $w(t) = e^{\mu_0 t} v_0$ is a solution of (5.1), where $u_0 = v_0$. Such a solution of (5.1) is called *decay mode* of the semigroup $T(t; A)$. A decay mode $w_0(t)$ is called *K-fundamental* if $w_0(0) \in K$, where K is a closed generating normal cone in the space Y.

Theorems 1 and 2 offer the following result.

Theorem 3. *Let the hypotheses* (i)–(iv) *hold. Then there exists at least one K-fundamental decay mode.*

If moreover, the assumption (v) *holds, then up to a multiplicative factor there exists exactly one K-fundamental decay mode $v(t)$ and we have that*

$$\lim_{t \to +\infty} e^{-\lambda_0 t} u(t) = c v_0 \tag{5.2}$$

for any solution $u(t)$ of (5.1), *where $v_0 = B_0 v$, $v \in K$, $v \neq 0$. Here*

$$c = v_0'(u(0)), \tag{5.3}$$

where v_0' is such that $v_0'(v) = u'(B_0 v)$ for all $v \in K$ with some $u' \in K'$, $u' \neq 0$ and

$$v_0'(v_0) = 1. \tag{5.4}$$

Proof. First, let the hypotheses (i)–(iv) hold. Then according to Theorem 1 $B_{0,q} \geqq \theta$ and there is a $u \in K$ such that $v_0 = B_{0,q} u \neq 0$. It follows that $v(t) = e^{\lambda_0 t} v_0$ is the desired decay mode, where $e^{\lambda_0 t} = r(t) = r(T(t; A))$.

If in addition $T(t; A)$ is K-irreducible for $t > t_0 \geqq 0$, there is by Proposition 3 exactly one normalized eigenvector of A in K, say $v_0 = B_0 u$, $u \in K$, $u \neq 0$. Let $u' \in K'$, $u' \neq 0$. Then by setting $v_0'(v) = u'(B_0 v)$ for all $v \in Y$ we have that $T(t; A)\, v_0 = e^{\lambda_0 t} v_0$ and $v_0'(T(t; A)\, v) = e^{\lambda_0 t}\, v_0'(v)$ for all $v \in Y$. It follows from (4.9) and (4.11) that (5.2) holds with (5.3) and (5.4). This concludes the proof.

Remark. As we may see from (5.2) the asymptotic behaviour of any solution of (5.1) with u_0 such that $B_0 u_0 \neq 0$ is nonoscillatory.

6. NEUTRON TRANSPORT EQUATION

In this section we give some applications of the previous theory to the linearized Boltzmann equation describing the transport of neutrons.

It is assumed here that G is a bounded convex domain with a lipschitzian boundary ∂G.

Let N denote the neutron density. The following equation

$$-LN = SN + FN + Q \tag{6.1}$$

is considered, where

$$LN \equiv -\mathbf{v} \operatorname{grad} N - v\, \Sigma(\mathbf{r}, \mathbf{v})\, N \quad \text{in} \quad G \tag{6.2}$$

and

$$N(\mathbf{r}, \mathbf{v}) = 0 \quad \text{for} \quad \mathbf{r} \in \partial\Omega \tag{6.3}$$

and $(\mathbf{v}\ \mathbf{n}) < 0$. where the normalized vector $\mathbf{n}$ has the direction of the outer normal; v is the length of $\mathbf{v}$

$$(6.4) \qquad SN \equiv \int_V v' \, \Sigma_s(\mathbf{v}' \to \mathbf{v}, \mathbf{r}) \, N(\mathbf{r}, v') \, d^3\mathbf{v}'$$

and

$$(6.5) \qquad FN \equiv \int_V v' \, \nu(\mathbf{v}') \, \Sigma_f(\mathbf{v}') \, \chi(\mathbf{v}' \to \mathbf{v}) \, N(\mathbf{r}, \mathbf{v}') \, d^3\mathbf{v}' .$$

In the above formulae Σ denotes the total macroscopic cross-section Σ_s the scattering cross-section kernel, Σ_f, the macroscopic fission cross-section, χ the fission spectrum, $\nu(\mathbf{v})$ the number of particles created by a particle with velocity $\mathbf{v}$ and Q denotes the external sources. The integration is carried out over the whole velocity space V, $V = [0, +\infty) \times \omega$, ω is the unit sphere.

In this paper it is assumed Q to be zero. We remark that $F = 0$ if the medium under consideration is non-multiplying (a moderator).

Let Y be any $L^p(G \times R^3, w)$ for $p \in (1, +\infty)$, where w is a suitable weight function.

It is known (e.g. [19]) that the Boltzmann operator $A = L + S + F$ is the infinitesimal generator of a strongly continuous semigroup of bounded operators. In fact L is the infinitesimal generator of a semigroup of operators $T(t; L)$ and the operators S and F are both bounded. The semigroup $T(t; L)$ can be evaluated explicitely by integrating the corresponding first-order differential equation and one can conclude that $T(t; L)$ is a semigroup of $\hat{K}$-positive operators; that is $T(t; L)$ leaves invariant the cone $\hat{K}$ of elements in Y with non-negative representatives. We also have that

$$\|T(t; L)\| \leqq e^{-\lambda^* t} ,$$

where

$$\lambda^* = \inf \{ v \, \Sigma(\mathbf{r}, \mathbf{v}) : \mathbf{r} \in R^3, \ \mathbf{v} \in V \} .$$

Since both operators S and F are defined by non-negative kernels, the semigroup $T(t; A)$ is also $\hat{K}$-positive [2, p. 403]. Moreover, since the kernel of S is almost everywhere positive, we may conclude that $T(t; A)$ is $\hat{K}$-irreducible for large $t > 0$.

The spectrum $\sigma(A)$ has the following structure: Every λ for which $\operatorname{Re} \lambda \leqq -\lambda^* \leqq 0$ belongs to the continuous spectrum ($\{\lambda : \operatorname{Re} \lambda \leqq -\lambda^*\} \subset C\,\sigma(A)$). On the other hand, λ, for which $\operatorname{Re} \lambda > -\lambda^* + r(S + F)$, belongs to the resolvent set $\varrho(A)$.

If the body G is sufficiently small, there are not additional points in $\sigma(A)$ except those in $\{\lambda : \operatorname{Re} \lambda \leqq -\lambda^*\}$, [1]. Therefore, we must assume that the strip $\{\lambda : -\lambda^* < \operatorname{Re} \lambda \leqq -\lambda^* + r(S + F)\}$ has a nonempty intersection with $\sigma(A)$. Let λ_0 be such that any λ for which $\operatorname{Re} \lambda > \lambda_0$ belongs to the resolvent set $\varrho(A)$, while there exists a $\lambda_1 \in \sigma(A)$ with $\operatorname{Re} \lambda_1 = \lambda_0$.

A standard procedure of investigating the behaviour of $N(t)$ as $t \to +\infty$ (see [15, pp. 201–213]) consists of estimating the semigroup operators by using the resolvent inversion formula [2, p. 349]

$$T(t; A) N_0 = \lim_{a \to \infty} \frac{1}{2\pi i} \int_{\nu - ia}^{\nu + ia} e^{\lambda t} R(\lambda, A) \, d\lambda ,$$

where $\nu > \max(0, \lambda_0)$ and $\lambda_0 = \operatorname{Re} \lambda_0$ is such that $\operatorname{Re} \lambda > \lambda_0$ implies that $\lambda \in \varrho(A)$. For such an approach complete information is needed about that part of the spectrum of A in the region $\operatorname{Re} \lambda > \lambda^*$. Contrary to this our theory developed in the previous sections is more straightforward and much simpler.

In order to apply our previous theory we have to show only that the point λ_0, the bound of the spectrum of $\sigma(A)$, is an isolated pole of the resolvent operator $R(\mu, T(\hat{t}; A))$. We emphasize this fact because a complete analysis of the existence of decay modes and the uniqueness of the fundamental decay mode can be carried through with no additional information about the spectrum of A. This makes our approach different from the analysis proposed by others. On the other hand, we describe only the peripheral part of the spectrum of the semigroup $T(t; A)$. If we make assumptions involving compactness about $ST(t; L) S$, [20], or other closely related assumptions, we can give a complete description of $\sigma(A)$. Actually, under certain assumptions concerning compactness of $T(t_1; L) S \ldots T(t_k; L) S$ it has been shown [19], [20], [15] that every $\mu \in \sigma(T(t; A))$, for which $|\mu| > e^{-\lambda^* t}$, has the form $\mu = e^{\lambda t}$, where λ is an isolated pole of $R(\nu, A)$ with finite-dimensional generated eigenspace $N(\lambda) = \{u \in Y : (A - \lambda I)^k = 0 \text{ for some } k = 1, 2, \ldots\}$. However, these assumptions are not fulfilled in general, e.g. for some models which allow also the case of inelastic scattering in the high-energy range [6].

The scattering operator S in (6.1) has the following structure

$$S = S_e + S_{in} ,$$

where the operator S_{in} is compact and both S_e and S_{in} are $\hat{K}$-positive. It can be shown easily that

$$R(a, A) = R(a, L + S_e) + R(a, A)(S_{in} + F) R(a, L + S_e) ,$$

where $a \in \varrho(A) \cap \varrho(L + S_e)$. It has already been mentioned that the semigroup $T(t; A)$ is $\hat{K}$-irreducible. It follows that $R(a, A)$ is also $\hat{K}$-irreducible if $a > \max(0, \lambda_0)$. This can be shown in a manner similar to Theorem 11.7.2. in [2, p. 353].

A crucial assumption for the applicability of our theory is the fulfilment of the strict inequality

$$r = r(R(a, A)) > r(R(a, L + S_e) = r_1 , \tag{6.6}$$

the relation $r \geqq r_1$ being trivial.

Actually we have

Lemma. *Under the hypothesis* (6.6) *for some* $a > \max(0, -\lambda^* + r(S + F))$ *the point* $e^{\lambda_0 t}$ *is a pole of* $R(\mu, T(t; A))$, $t > 0$, *with a finite dimensional generalised eigenspace, where* λ_0 *denotes the spectral bound of the infinitesimal generator* A.

Remark. We note that the validity of (6.6) is guaranteed by the compactness and $\hat{K}$-primitivity of $(S + F)\,T(t; L)\,(S + F)$ or by some other similar compactness and positivity assumptions as those mentioned above. The converse is obviously not necessarily true, as we have mentioned, in the case of inelastic scattering in the high-energy range.

Proof of the Lemma. From (6.6) we derive that

$$\sigma(S_e + L) \subset \{\lambda : \operatorname{Re} \lambda \leqq \nu_0\},$$

where

$$\frac{1}{a - \nu_0} = r(R(a, L + S_e))$$

and also

$$\frac{1}{a - \lambda_0} = r(R(a, A)).$$

The compactness of $C = S_{\text{in}} + F$ implies that C is $(L + S_e)$-smoothing and hence ([16]) every point $\varrho \in \sigma(T(t; A))$ with $|\varrho| > e^{(\nu_0 + \varepsilon)t}$, where $\nu_0 \in \varepsilon < \lambda_0$, is a pole of $R(\mu, T(t; A))$ with a finite dimensional generalised eigenspace. This in particular applies to $e^{\lambda_0 t}$ and completes the proof.

The conclusion of the Lemma implies that Theorems 1–3 apply to those cases of neutron transport theory where the assumption (6.6) holds. In our opinion, this is the case in most of the models used to date.

As a consequence we have the following final result.

Theorem. *If* (6.6) *holds then there exists* $\lambda_0 > \lambda^*$ *such that* $\operatorname{Re} \lambda_0 = \lambda_0 \in \sigma(A)$ *and* $\lambda \in \sigma(A)$ *implies that either* $\operatorname{Re} \lambda < \lambda_0$ *or* $\lambda = \lambda_0$. *The quantity* λ_0 *is a Fredholm eigenvalue of* A *with a one-dimensional generalized eigenspace* $\mathcal{N}_0$. *Therefore there exists exactly one normalized* $\hat{K}$*-fundamental decay mode* (λ_0, M_0) *and we have that for every solution* N *of*

$$\frac{d}{dt} N(t) = A\,N(t), \quad 0 \neq N(0) = N_0 \geqq 0,$$

that

$$\lim_{t \to +\infty} \|e^{-\lambda_0 t} N(t) - c M_0\| = 0,$$

where $c \geqq 0$ is a constant independent of t. More precisely, $cM_0 = PN_0$, where P is the residue of the Laurent expansion of $R(\mu, A)$ about the point λ_0.

We remark that this last theorem gives a solution to Problem 10 of Kaper's Collection of Problems in [4].

It should be noted that besides the already mentioned splitting $(L + S_e) + (S_{in} + F)$ some other splittings of the Boltzmann operator may exist such that $A = V + Q$, where Q is V-smoothing in the sense of [16]. Naturally, our wish is to have Q as possible large with respect to the magnitude of the spectral bound of V. This means that if $A = V_1 + Q_1 = V_2 + Q_2$, Q_j is V_j-smoothing, $j = 1, 2$, and

$$\frac{1}{a - v_1} = r(R(a, V_1)) < r(R(a, V_2)) = \frac{1}{a - v_2} < r(R(a,A))$$

where $a > \max(0, -\lambda^* + r(S + F))$, then the splitting $V_1 + Q_1$ gives more complete information than $V_2 + Q_2$. We have that every point $\lambda \in \sigma(A)$ for which $\operatorname{Re} \lambda > v_1$ is a Fredholm eigenvalue of A.

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