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Fundamental Groups of Rationally Connected Varieties

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Dedicated to William Fulton

1. Introduction

Let X be a smooth, projective, unirational variety, and let $U \subset X$ be an open set. The aim of this paper is to find a smooth rational curve $C \subset X$ such that the fundamental group of $C \cap U$ surjects onto the fundamental group of U. Following the methods of [K4] and [Co], a positive answer over $\mathbb{C}$ translates to a positive answer over any p-adic field. This gives a rather geometric proof of the theorem of [Hb] about the existence of Galois covers of the line over p-adic fields (1.4). We also obtain a slight generalization of the results of [Co] about the existence of certain torsors over open subsets of the line over p-adic fields (1.6).

If $U = X$ then $\pi_1(X)$ is trivial (cf. (2.3)), thus any rational curve C will do. If $X \setminus U$ is a divisor with normal crossings and if C intersects every irreducible component of $X \setminus U$ transversally, then the *normal* subgroup of $\pi_1(U)$ generated by the image of $\pi_1(C \cap U)$ equals $\pi_1(U)$ by a simple argument. (See e.g. the beginning of (4.2).) It is also not hard to produce rational curves C such that the image of $\pi_1(C \cap U)$ has finite index in $\pi_1(U)$ (cf. (3.3)). These results suggest that we are very close to a complete answer, but surjectivity is not obvious. Differences between surjectivity and finiteness of the index appear in many similar situation; see, for instance, [K1, Part I] or [NR].

The present proof relies on the machinery of rationally connected varieties developed in the papers [KMM1; KMM2; KMM3]. The relevant facts are recalled in Section 2.

The main geometric result is the following theorem.

THEOREM 1.1. *Let K be an algebraically closed field of characteristic* 0, *and let X be a smooth projective variety over K that is rationally connected* (2.1). *Let $U \subset X$ be an open subset and $x_0 \in U$ a point. Then there is an open subset* $0 \in V \subset \mathbb{A}^1$ *and a morphism $f : V \to U$ such that $f(0) = x_0$ and*

$$\pi_1(V, 0) \twoheadrightarrow \pi_1(U, x_0) \text{ is surjective.}$$

Moreover, we can assume that the following also hold:

(1) $H^1(\mathbb{P}^1, \bar f^* T_X(-2)) = 0$, *where $\bar f : \mathbb{P}^1 \to X$ is the unique extension of f*;
(2) *$\bar f$ is an embedding if* $\dim X \geq 3$ *and an immersion if* $\dim X = 2$.

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COROLLARY 1.2. *Let K be a p-adic field, and let X be a smooth projective variety over K that is rationally connected over $\bar{K}$. Let $U \subset X$ be an open subset and $x_0 \in U(K)$ a point. Then there is an open subset $0 \in V \subset \mathbb{A}^1$ and a morphism $f: V \to U$ (all defined over K) such that $f(0) = x_0$ and*

$$\pi_1(V, 0) \twoheadrightarrow \pi_1(U, x_0) \text{ is surjective,}$$

where π_1 here denotes the algebraic fundamental group.

REMARK 1.3. More generally, (1.2) holds for any field K of characteristic 0 such that every curve with a smooth K-point contains a Zariski dense set of K-points. Characterizations of this property are given in [P, 1.1]. The following are some interesting classes of such fields:

(1) fields complete with respect to a discrete valuation;
(2) quotient fields of local Henselian domains;
(3) $\mathbb{R}$ and all real closed fields;
(4) pseudo-algebraically closed fields (cf. [FJ, Ch. 10]).

COROLLARY 1.4 [Hb]. *Let G be a finite group, and let K be a field of characteristic 0 as in (1.3). Then there is a Galois cover $g: C \to \mathbb{P}^1_K$ with Galois group G such that C is geometrically irreducible and $g^{-1}(0:1) \cong G$.*

Proof. Let $G \subset \mathrm{GL}(n, K)$ be a faithful representation. Set $U = \mathrm{GL}(n)/G$ with quotient map $h: \mathrm{GL}(n) \to U$, and let x_0 be the image of the identity matrix. Then U is unirational; thus, by (1.2) there is a $0 \in V \subset \mathbb{A}^1$ and a morphism $f: V \to U$ such that $\pi_1(V) \twoheadrightarrow \pi_1(U)$ is onto. The map $h: \mathrm{GL}(n) \to U$ is étale and proper, and thus it corresponds to a quotient $\pi_1(U) \twoheadrightarrow G$. The fiber product $W := \mathrm{GL}(n) \times_U V \to V$ corresponds to the surjective homomorphism

$$\pi_1(V) \twoheadrightarrow \pi_1(U) \twoheadrightarrow G.$$

Thus W is connected and $W \to V$ is a Galois cover with Galois group G. Since W has a K-point, it is also geometrically connected. The preimage of $0 \in V$ is isomorphic to G (the disjoint union of $|G|$ copies of Spec K). The cover $W \to V$ can be extended to a (ramified) Galois cover of the whole $\mathbb{P}^1_K$. □

REMARK 1.5. The foregoing proof works in positive characteristic if we know that, for every subgroup $H < G$, the quotient $\mathrm{GL}(n)/H$ has a smooth compactification.

The following result was proved by [Co] for finite groups, which is probably the most important for applications.

COROLLARY 1.6. *Let K be a field of characteristic 0 as in (1.3), let G be a linear algebraic group scheme over K, and let A be a principal homogeneous G-space. Then there is an open set $0 \in V \subset \mathbb{A}^1_K$ and a geometrically irreducible G-torsor $g: W \to V$ such that $g^{-1}(0) \cong A$ (as a G-space).*

Proof. Assume that G acts on A from the left and choose an embedding $G \subset \mathrm{GL}(n)$ over K. $A \times \mathrm{GL}(n)$ has a diagonal left action by G and a right action by $\mathrm{GL}(n)$ acting only on $\mathrm{GL}(n)$.

The right G-action makes the morphism

$$h\colon G\backslash(A\times \mathrm{GL}(n)) \to G\backslash(A\times \mathrm{GL}(n))/G =: U$$

into a G-torsor. Let $x_0 \in U$ be the image of $G\backslash(A\times G)$. The fiber of h over x_0 is isomorphic to A. Let G^0 be the connected component of G. Then $G\backslash(A\times \mathrm{GL}(n)) \to G\backslash(A\times \mathrm{GL}(n))/G^0$ is smooth with connected fibers and $G\backslash(A\times \mathrm{GL}(n))/G^0 \to U$ is étale and proper. Let $0 \in V \subset \mathbb{A}^1$ and $f\colon V \to U$ be as in the proof of (1.4). Then $W := (G\backslash(A\times \mathrm{GL}(n))) \times_U V$ works. □

A geometric application is as follows.

COROLLARY 1.7. *For every* $2 \le g \le 13$ *there is an open set* $0 \in V_g \subset \mathbb{C}$ *and a smooth proper morphism with genus-g fibers* $S_g \to V_g$ *such that the image of the monodromy representation is the full Teichmüller group.*

Proof. The moduli of curves is unirational for $g \le 13$. Apply (1.1) to the open subset of curves without automorphisms $U_g \subset M_g$. □

2. Rationally Connected Varieties and Morphisms of Rational Curves

Rationally connected varieties were introduced in [KMM2] as a higher-dimensional generalization of rational and unirational varieties. A surface is rationally connected if and only if it is rational. In higher dimensions, rationality and unirationality are very hard to check. The notion of rational connectedness concentrates on rational curves on a variety. The following characterizations were developed in [KMM2; K2, IV.3; K3, 4.1.2].

DEFINITION–THEOREM 2.1. Let K be an algebraically closed field of characteristic 0. A smooth proper variety X over K is called *rationally connected* if it satisfies any of the following equivalent properties.

(1) There is an open subset $\emptyset \ne X^0 \subset X$ such that, for every $x_1, x_2 \in X^0$, there is a morphism $f\colon \mathbb{P}^1 \to X$ satisfying $x_1, x_2 \in f(\mathbb{P}^1)$.
(2) For every $x_1, \dots, x_n \in X$, there is a morphism $f\colon \mathbb{P}^1 \to X$ satisfying $x_1, \dots, x_n \in f(\mathbb{P}^1)$.
(3) There is a morphism $f\colon \mathbb{P}^1 \to X$ such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$. (This is equivalent to f^*T_X being ample.)
(4) There is a variety P and a dominant morphism $F\colon \mathbb{P}^1 \times P \to X$ such that $F((0:1)\times P)$ is a point. We can also assume that $H^1(\mathbb{P}^1, F_p^*T_X(-2)) = 0$ for every p, where $F_p := F\big|_{\mathbb{P}^1\times\{p\}}$.
(5) Let $z_1, \dots, z_n \in \mathbb{P}^1$ be distinct points and let $m_1, \dots, m_n$ be natural numbers. For each $i = 1, \dots, n$, let $f_i\colon \operatorname{Spec} K[t]/(t^{m_i}) \to X$ be a morphism. Then there is a variety P and a dominant morphism $F\colon \mathbb{P}^1 \times P \to X$ such that:
   (a) the Taylor series of F_p at z_i coincides with f_i up to order m_i for every i and $p \in P$; and
   (b) $H^1(\mathbb{P}^1, F_p^*T_X(-\sum m_i)) = 0$ for every i and p.

Another easy result that we need is the following.

LEMMA 2.2 (cf. [K2, II.3.5.4, II.3.10.1, II.3.11]). *Let X and P be smooth varieties, and let $F\colon \mathbb{P}^1 \times P \to X$ be a dominant morphism such that $F((0:1)\times P)$ is a point. Then there is a dense open set $P^0 \subset P$ such that $H^1(\mathbb{P}^1, F_p^* T_X(-2)) = 0$ for every $p \in P^0$, where $F_p := F|_{\mathbb{P}^1\times\{p\}}$.*

Conversely, let $f\colon \mathbb{P}^1 \to X$ be a morphism such that $H^1(\mathbb{P}^1, f^* T_X(-2)) = 0$. Then there is a pointed variety $p_0 \in P$ and a dominant morphism $F\colon \mathbb{P}^1 \times P \to X$ such that $F((0:1)\times P)$ is a point, F is smooth away from $(0:1)\times P$, and $F_{p_0} = f$.

The following result was proved by [S] for unirational varieties and by [C; KMM3] for the general case.

PROPOSITION 2.3. *A smooth, proper, rationally connected variety is simply connected.*

In the course of the proof we repeatedly encounter the following situation. We have morphisms $f^i\colon \mathbb{P}^1 \to X$ each passing through the same point $x_0 \in X$. We would like to have a family of morphisms $f_t\colon \mathbb{P}^1 \to X$ such that the union of the maps f^i can be considered as the limit of the maps f_t as $t \mapsto 0$. The following lemma is a technical formulation of this idea. Its statement is a bit complicated, since we also want to keep track of the field over which the f_t are defined.

LEMMA 2.4 (cf. [K4, 3.2]). *Let K be a field and let $x_0 \in X$ be a smooth, proper, pointed K-scheme. Let S be a 0-dimensional reduced K-scheme, and let $f_0\colon \mathbb{P}^1_S \to X$ be a morphism such that*

(1) *$H^1(\mathbb{P}^1_S, f_0^* T_X(-2)) = 0$ and*
(2) *$f_0(S \times \{(0:1)\}) = \{x_0\}$.*

Then there exist

(3) *a smooth pointed curve $0 \in D$ over K,*
(4) *a smooth surface Y with a proper morphism $h\colon Y \to D$ and a section $B \subset Y$ of h, and*
(5) *a morphism $F\colon Y \to X$*

such that:

(6) *$h^{-1}(0)$ is the union of $\mathbb{P}^1_S$ with a copy B_0 of $\mathbb{P}^1_K$ such that $B_0 \cap \mathbb{P}^1_S = S \times \{(0:1)\}$ and $B_0 \cap B$ is a single point;*
(7) *F restricted to $\mathbb{P}^1_S$ coincides with f_0 and $F(B_0 \cup B) = \{x_0\}$;*
(8) *$h^{-1}(D^0) \cong \mathbb{P}^1_K \times D^0$, where $D^0 := D \setminus \{0\}$; and*
(9) *$H^1(\mathbb{P}^1_d, F_d^* T_X(-2)) = 0$ for every $d \in D^0$, where $F_d := F|_{h^{-1}(d)}$.*

Proof. Let us start with any curve $0 \in D'$ and $Y' := \mathbb{P}^1 \times D'$. Let $S' \subset \mathbb{P}^1 \times \{0\}$ be a subscheme isomorphic to S, and let Y'' be the blow-up of $S' \subset D'$ with projection $h'\colon Y'' \to D'$. We can define a morphism $f'\colon (h')^{-1}(0) \to X$ by setting f' to be f_0 on the exceptional divisor of $Y'' \to Y'$ and the constant morphism to $\{x_0\}$ on the birational transform of $\mathbb{P}^1 \times \{0\}$. Fix any section $B' \subset Y'$ that does not pass through S'. We are done if f' can be extended to $F'\colon Y'' \to X$ as required. In general this is not possible, but such an extension exists after a suitable étale base change $(0 \in D) \to (0 \in D')$. This is proved in [KMM2, 1.2] and [K4, 2.2]. □

3. Fundamental Groups of Fibers of Morphisms

We need some easy results about the variation of fundamental groups for fibers of nonproper morphisms.

LEMMA 3.1. *Let K be an algebraically closed field of characteristic* 0, *let $\bar{Z}$ and D be irreducible K-varieties, and let $f\colon \bar{Z} \to D$ be a smooth and proper morphism with connected fibers. Let $Z \subset \bar{Z}$ be an open subset such that $(\bar{Z} \setminus Z) \to D$ is smooth. Let $z_0 \in Z(K)$ be a K-point, with $d_0 = f(z_0)$ and Z_0 the fiber of $Z \to D$ through z_0. Then there is an exact sequence*

$$\pi_1(Z_0, z_0) \to \pi_1(Z, z_0) \to \pi_1(D, d_0) \to 1. \qquad (*)$$

Proof. Over $\mathbb{C}$, the fibration $Z(\mathbb{C}) \to D(\mathbb{C})$ is a topological fiber bundle; thus we have the exact sequence $(*)$. To settle the algebraic case, let $Y \to Z$ be any connected finite degree étale cover and extend it to a finite morphism $\bar{Y} \to \bar{Z}$, where $\bar{Y}$ is normal. Since $\bar{Z} \setminus Z$ is smooth over D, the same holds for $\bar{Y} \to D$. (This is a special case of Abhyankar's lemma; cf. [G, XIII.5.2].) The generic fiber of $\bar{Y} \to D$ is irreducible. Since $\bar{Y} \to D$ is smooth and proper, every fiber is irreducible. This is equivalent to the exactness of $(*)$. □

The following technical lemma is an upper semicontinuity statement for the fundamental groups of fibers of nonproper morphisms.

LEMMA 3.2. *Let K be an algebraically closed field of characteristic* 0, *let W be a normal surface over K, and let $f\colon W \to D$ be a* (*not necessarily proper*) *morphism to a curve with connected fibers. Let $B \subset W$ be a connected subset, one of whose irreducible components is a section of f. Let $d_0 \in D$ be a K-point and C_0 an irreducible component of $f^{-1}(d_0)$ with a K-point $b_0 \in C_0 \cap B$. Let $x_0 \in U$ be a pointed K-scheme, and let $h\colon W \to U$ be a morphism such that $h(B) = \{x_0\}$. Then there is an open subset $D^0 \subset D$ such that, for every $d \in D^0$ and for every K-point $b_d \in C_d \cap B$,*

$$\operatorname{im}[\pi_1(C_d, b_d) \to \pi_1(U, x_0)] \supset \operatorname{im}[\pi_1(C_0, b_0) \to \pi_1(U, x_0)].$$

Proof. Choose a normal compactification $\bar{f}\colon \bar{W} \to D$. Let $D^0 \subset D$ be any open subset such that (a) $\bar{f}$ is smooth with irreducible fibers over D^0 and (b) $\bar{W} \setminus W \to D$ is unramified over D^0. Set $W^0 := f^{-1}(D^0)$. Then $\pi_1(C_0, b_0) \to \pi_1(U, x_0)$ factors through $\pi_1(W, b_0) \to \pi_1(U, x_0)$, and it has the same image as $\pi_1(W, b) \to \pi_1(U, x_0)$ for any $b \in B(K)$. By (3.3), $\pi_1(W^0, b) \to \pi_1(W, b)$ is surjective. By (3.1), there is an exact sequence

$$\pi_1(C_d, b_d) \to \pi_1(W_0, b_d) \to \pi_1(D^0, d) \to 1.$$

Let $B^0 \subset B \cap W^0$ be a section of f. Then $\pi_1(B^0, b_d)$ maps onto $\pi_1(D^0, d)$, and the image of $\pi_1(B^0, b_d)$ in $\pi_1(U, x_0)$ is trivial. Hence

$$\operatorname{im}[\pi_1(C_d, b_d) \to \pi_1(U, x_0)] = \operatorname{im}[\pi_1(W^0, b_d) \to \pi_1(U, x_0)]$$

and we are done. □

LEMMA 3.3 (cf. [G, IX.5.6; C, 1.3]). *Let X, Y be normal varieties, $x_0 \in X$ a closed point, and $f: X \to Y$ a dominant morphism. Then* $\operatorname{im}[\pi_1(X, x_0) \to \pi_1(Y, f(x_0))]$ *has finite index in $\pi_1(Y, f(x_0))$. If f is an open immersion then $\pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is surjective.*

4. Proof of the Main Results

The theory of free morphisms of curves (cf. [K2, II.3]) suggests that morphisms $f: \mathbb{P}^1 \to X$ such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$ behave rather predictably; we therefore concentrate on such morphisms. First we establish that there is a unique maximal subgroup of $\pi_1(U)$ obtainable from such a morphism.

LEMMA 4.1. *Let X be a smooth, proper, rationally connected variety over an algebraically closed field of characteristic* 0. *Let $U \subset X$ be an open set and $x_0 \in U$ a point. Then there is a unique finite-index subgroup $H < \pi_1(U, x_0)$ with the following properties.*

(1) *There is a morphism $f: \mathbb{P}^1 \to X$ such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$, $f(0:1) = x_0$, and $H = \operatorname{im}[\pi_1(f^{-1}(U), (0:1)) \to \pi_1(U, x_0)]$.*
(2) *Let g be any morphism $g: \mathbb{P}^1 \to X$ such that*

$$g(0:1) = x_0 \quad \textit{and} \quad H^1(\mathbb{P}^1, g^*T_X(-2)) = 0.$$

Then $H \supset \operatorname{im}[\pi_1(g^{-1}(U), (0:1)) \to \pi_1(U, x_0)]$.

Proof. First we find one morphism as in (2) such that

$$\operatorname{im}[\pi_1(g^{-1}(U), (0:1)) \to \pi_1(U, x_0)]$$

has finite index in $\pi_1(U, x_0)$.

Let $F: \mathbb{P}^1 \times P \to X$ be as in (2.1(4)). Let $P^0 \subset P$ be an open subset such that $F^{-1}(X \setminus U) \to P$ is étale over P^0. Pick any point $p \in P^0$. By (3.1) there is an exact sequence

$$\pi_1(F_p^{-1}(U), (0:1) \times \{p\}) \to \pi_1((\mathbb{P}^1 \times P^0) \cap F^{-1}(U), (0:1) \times \{p\}) \\ \to \pi_1(P^0, p) \to 1.$$

The section $(0:1) \times P^0$ is mapped to a point by F, so

$$\pi_1(F_p^{-1}(U), (0:1) \times \{p\}) \quad \text{and} \quad \pi_1((\mathbb{P}^1 \times P^0) \cap F^{-1}(U), (0:1) \times \{p\})$$

have the same image in $\pi_1(U, x_0)$. The latter image has finite index by (3.3); thus $g := F_p: \mathbb{P}^1 \to X$ is as desired.

In order to finish, it is sufficient to prove that if $g_1, g_2\colon \mathbb{P}^1 \to X$ are as in (2) then there is a third morphism $g\colon \mathbb{P}^1 \to X$ such that

$$\operatorname{im}[\pi_1(g^{-1}(U), (0:1)) \to \pi_1(U, x_0)] \supset \operatorname{im}[\pi_1(g_i^{-1}(U), (0:1)) \to \pi_1(U, x_0)]$$

for $i = 1, 2$. To do this, let $S = \{1, 2\}$ be a 2-point scheme and take $f_0\colon \mathbb{P}^1_S \to X$ to be g_i on $\{i\} \times \mathbb{P}^1$. Construct $h\colon Y \to D$ and $F\colon Y \to X$ as in (2.4). Set $W := F^{-1}(U)$ and apply (3.2) twice for $i = 1, 2$ with $C_0 := f_i^{-1}(U)$. Then, for any $d \in D^0$, $g := F_d$ has the required property. $\square$

4.2. Proof of Theorem 1.1. Let $H < \pi_1(U, x_0)$ be the subgroup obtained in (4.1). We are done if $H = \pi_1(U, x_0)$. Otherwise, there is a corresponding irreducible étale cover $(x_0' \in U') \to (x_0 \in U)$. Let $w\colon X' \to X$ be the normalization of X in the function field of U'; X is simply connected by (2.3) and so, by the purity of branch loci (cf. [G, X.3.1]), there is a divisor $D' \subset X'$ such that w ramifies along D'. Let $D \subset X$ be the image of D'. By construction, $D \subset X \setminus U$. We derive a contradiction as follows.

Let $f\colon \mathbb{P}^1 \to X$ be a morphism such that $f(0:1) = x_0$ and $f(\mathbb{P}^1)$ intersects D transversally at a point $x_1 := f(1:1)$. This implies that the local fundamental group of $\mathbb{P}^1 \setminus \{(1:1)\}$ at $(1:1)$ surjects onto the local fundamental group of $X \setminus D$ at x_1. (For the local fundamental group of a divisor in a variety see [GM], where it is called the fundamental group of the formal neighborhood of a divisor.) Therefore, if f lifts to $f'\colon \mathbb{P}^1 \to X'$ then $X' \to X$ is étale at $f'(1:1)$. If $X' \to X$ is a Galois extension then we already have a contradiction, since $X' \to X$ ramifies everywhere above D. In the non-Galois case, p may be unramified along some of the irreducible components of $p^{-1}(D)$ and we have a contradiction only if the image of f' intersects D'.

If $X' \to X$ is not Galois, we need to proceed in a somewhat roundabout way. First I give the outlines; a precise version is given afterwards.

It is clear from the definition that there are many maps $\mathbb{P}^1 \to X'$, so X' (or rather any desingularization of X') should be rationally connected. This indeed follows from (4.3) applied to any $F\colon \mathbb{P}^1 \times P \to X$ as in (2.1(4)). Thus by (2.1(5)) there is a morphism $f'\colon \mathbb{P}^1 \to X'$ that passes through x_0' and intersects D' at a smooth point x_1'. We obtain a contradiction if $w \circ f'$ is the limit of a sequence of maps $f_t\colon \mathbb{P}^1 \to X$ such that:

(1) f_t lifts to $f_t'\colon \mathbb{P}^1 \to X'$ and f' is the limit of the maps f_t'; and
(2) the image of f_t intersects D transversally.

First we prove a general lifting property for families of morphisms and then we proceed to construct the morphism f'.

Lemma 4.3. *Let V be a normal variety and let $G\colon \mathbb{P}^1 \times V \to X$ be a morphism such that $G((0:1) \times V) = \{x_0\}$. Assume that $H^1(\mathbb{P}^1, G_v^* T_X(-2)) = 0$ for some $v \in V$. Then G can be lifted to $G'\colon \mathbb{P}^1 \times V \to X'$.*

Proof. First we show that such a lifting exists over an open subset of $\mathbb{P}^1 \times V$. Choose $V^0 \subset V$ such that $H^1(\mathbb{P}^1, G_v^* T_X(-2)) = 0$ for every $v \in V^0$ (this is possible by the upper semicontinuity of cohomology groups) and such that we have an exact sequence

$$\pi_1(G_v^{-1}(U), (0:1)\times v) \to \pi_1((G^0)^{-1}(U), (0:1)\times v) \to \pi_1(V^0, v) \to 1$$

for every $v \in V^0$, where $G^0 := G\big|_{\mathbb{P}^1\times V^0}$ (this is possible by (3.1)). Then

$$\operatorname{im}[\pi_1((G^0)^{-1}(U), (0:1)\times v) \to \pi_1(U, x_0)] \subset H$$

and so $G^0\big|_{(G^0)^{-1}(U)}$ can be lifted to $G'\colon (G^0)^{-1}(U) \to U'$. This extends to a rational map $G'\colon \mathbb{P}^1 \times V^0 \dashrightarrow X'$.

Next let $\Gamma \subset (\mathbb{P}^1 \times V) \times X$ be the graph of G with $\Gamma^* \subset (\mathbb{P}^1 \times V) \times X'$ its preimage. The rational lifting G' corresponds to an irreducible component $\Gamma' \subset \Gamma^*$ such that the projection $\Gamma' \to (\mathbb{P}^1 \times V)$ is birational. Since $X' \to X$ is finite, so is $\Gamma^* \to \Gamma$. Thus $\Gamma' \to \Gamma \to \mathbb{P}^1 \times V$ is both finite and birational and hence is an isomorphism. □

Let $\phi\colon X'' \to X'$ be any desingularization, and let $x_1' \in D'$ be a point such that (a) D' and X' are smooth at x_1' and (b) ϕ^{-1} is a local isomorphism near x_1'. By (2.1(5)), there is a dominant morphism $F\colon \mathbb{P}^1 \times P \to X'$ such that:

(1) $F((0:1) \times P) = \{x_0'\}$;
(2) $F((1:1) \times P) = \{x_1'\}$; and
(3) the image of $F_p\colon \mathbb{P}^1 \to X'$ is transversal to D' at x_1' for every $p \in P$.

Let us now consider the dominant morphism $w \circ F\colon \mathbb{P}^1 \times P \to X$. By the first part of (2.2), there is a $p_0 \in P$ such that $H^1(\mathbb{P}^1, (w \circ F_{p_0})^* T_X(-2)) = 0$. Thus, by the second part of (2.2), there is a pointed variety $q_0 \in Q$ and a morphism $G\colon \mathbb{P}^1 \times Q \to X$ such that $G_{q_0} = w \circ F_{p_0}$ and G is smooth away from $(0:1) \times Q$. In particular, $G^{-1}(D) \subset \mathbb{P}^1 \times Q$ is a generically smooth divisor; hence there is a dense open set $Q^0 \subset Q$ such that the projection $G^{-1}(D) \to Q$ is smooth over Q^0. This means that the image of $G_q\colon \mathbb{P}^1 \to X$ intersects D transversally for every $q \in Q^0$. (Note that $G^{-1}(D)$ denotes the inverse image scheme.)

By (4.3), G can be lifted to $G'\colon \mathbb{P}^1 \times Q \to X'$. On $\mathbb{P}^1 \times \{q_0\}$ the lifting agrees with F_{p_0}, hence $G'(\mathbb{P}^1 \times \{q_0\})$ intersects D' at the point x_1'. This implies that $(G')^{-1}(D')$ is a divisor and so, by shrinking Q, we may assume that the image of G_q' intersects D' for every $q \in Q$. Thus $G_q = w \circ G_q'$ is never transversal to D, a contradiction.

Condition (1.1(1)) holds by construction, and a general choice of f satisfies (1.1(2)) by [K2, II.3.14]. □

4.4. PROOF OF COROLLARY 1.2. Pick $f_1\colon V_1 \to U_{\bar K}$ defined over $\bar K$ such that $\pi_1(V_1, 0) \to \pi_1(U_{\bar K}, x_0)$ is surjective and $H^1(\mathbb{P}^1, \bar f_1^* T_X(-2)) = 0$. Then f_1 is defined over a finite Galois extension $L \supset K$; let $f_i\colon V_i \to U_{\bar K}$ be its conjugates. Each of these extends to a morphism $\bar f_i\colon \mathbb{P}^1 \to X_{\bar K}$. Let $S = \operatorname{Spec}_K L$. We can view the morphisms f_i as one morphism $f_0\colon \mathbb{P}^1_S \to X$ defined over K. By (2.4) we obtain $h\colon Y \to D$ and $F\colon Y \to X$, all defined over K. Let $d \in D^0(\bar K)$ be any point. Then, by (3.2) we see that

$$\mathrm{im}[\pi_1(Y_d, 0) \xrightarrow{F_d} \pi_1(U_{\bar{K}}, x_0)] \supset \mathrm{im}[\pi_1(V_1, 0) \xrightarrow{f_1} \pi_1(U_{\bar{K}}, x_0)],$$

and the latter image is $\pi_1(U_{\bar{K}}, x_0)$ by assumption. Now D^0 is a Zariski open set in a curve D with a smooth K-point 0. By (1.3), this implies that $D(K)$ is dense in D, hence $D^0(K) \neq \emptyset$. By choosing $d \in D^0(K)$ we obtain an open subset $0 \in V \subset \mathbb{A}^1$ and a morphism $f: V \to U$ (all defined over K) such that $f(0) = x_0$ and

$$\pi_1(V_{\bar{K}}, 0) \twoheadrightarrow \pi_1(U_{\bar{K}}, x_0) \ \text{ is surjective.}$$

The fundamental group of a K-scheme W is related to the fundamental group of $W_{\bar{K}}$ by the exact sequence (cf. [G, IX.6.1])

$$1 \to \pi_1(W_{\bar{K}}, 0) \to \pi_1(W, 0) \to \mathrm{Gal}(\bar{K}/K) \to 1.$$

This implies that

$$\pi_1(V, 0) \twoheadrightarrow \pi_1(U, x_0) \ \text{ is also surjective.} \qquad \square$$

EXAMPLE 4.5. Here, for every $n \geq 4$, I give an example of a rational threefold X, a normal crossing divisor $F \subset X$, and a smooth rational curve $B \subset X$ such that B intersects F everywhere transversally, B intersects every irreducible component of F, and the image of $\pi_1(B \setminus F) \to \pi_1(X \setminus F)$ has index n in $\pi_1(X \setminus F)$.

Let us start with a similar surface example. Let $g_1: \mathbb{P}^1 \to \mathbb{P}^1$ be a degree-n morphism with critical points $x_1, \dots, x_{2n-2} \in \mathbb{P}^1$ and different critical values $y_1, \dots, y_{2n-2}$. Choose three other points $x_{2n-1}, x_{2n}, x_{2n+1}$ in $\mathbb{P}^1$ such that (a) $g_1(x_{2n-1})$ and $g_1(x_{2n})$ are different critical values of g_1 and (b) $g_1(x_{2n+1})$ is not a critical value of g_1. Let $g_2: \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism, with $g_2(x_i) = x_i$ for $i \leq 2n + 1$. Consider the morphism $h : (g_2, g_1): \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. Set $D = \mathbb{P}^1 \times \{y_1, \dots, y_{2n-2}\}$ and $U = \mathbb{P}^1 \times \mathbb{P}^1 \setminus D$. Then $\mathrm{im}\, h$ intersects every irreducible component of D transversally at $n - 2$ points, and the image of $\pi_1(h^{-1}(U)) \to \pi_1(U)$ has index n in $\pi_1(U)$.

Here $\mathrm{im}\, h$ is also tangent to every irreducible component of D. The tangencies can be resolved by $2n$ blow-ups, but then the birational transform of $\mathrm{im}\, h$ does not intersect every boundary component. To remedy this situation, take another morphism $g_3: \mathbb{P}^1 \to \mathbb{P}^1$ with $g_3(x_i) = x_i$ for $i \leq 2n+1$. Set $Y := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $F = D \times \mathbb{P}^1$, and $H := (g_3, g_2, g_1): \mathbb{P}^1 \to Y$. Again the only problem is that $\mathrm{im}\, H$ is tangent to every irreducible component of F. These can be resolved by blowing up two suitable smooth curves. First we take a smooth curve that passes through every point of tangency and also through the point $H(x_{2n-1})$. After blow up, the birational transform of $\mathrm{im}\, H$ intersects every boundary component transversally, but above each point there is a point common to two boundary components and to the birational transform of $\mathrm{im}\, H$. Next take a smooth curve that passes through all these points with a general tangent direction there and also through $H(x_{2n})$. We can also assume that neither of the two curves passes through $H(x_{2n+1})$. Doing two such blow-ups creates two new boundary components, and the birational transform of $\mathrm{im}\, H$ intersects both of them. The fundamental group computation is unchanged.

Varying g_2 and g_3 we obtain many morphisms, all of which pass through the point $H(x_{2n+1})$.

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