

Fundamental quantum limits to waveform detection

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Ever since the inception of gravitational-wave detectors, limits imposed by quantum mechanics to the detection of time-varying signals have been a subject of intense research and debate. Drawing insights from quantum information theory, quantum detection theory, and quantum measurement theory, here we prove lower error bounds for waveform detection via a quantum system, settling the long-standing problem. In the case of optomechanical force detection, we derive analytic expressions for the bounds in some cases of interest and discuss how the limits can be approached using quantum control techniques.

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I. INTRODUCTION

The study of quantum measurement has come a long way since the proposal of wavefunction collapse by Heisenberg and von Neumann, the philosophical debates by Bohr and Einstein, and the cat experiment hypothesized by Schrödinger. With more and more experimental demonstrations of bizarre quantum effects being realized in laboratories, many researchers have shifted their focus to the practical implications of quantum mechanics for precision measurements, such as gravitational-wave detection, optical interferometry, atomic clocks, and magnetometry [1–4]. Braginsky, Thorne, Caves, and others pioneered the application of quantum measurement theory to gravitational-wave detectors [5–7], while Holevo, Yuen, Helstrom, and others have developed a beautiful theory of quantum detection and estimation [8,9] based on the more abstract notions of quantum states, effects, and operations [10]. Although the approach of Holevo was able to produce rigorous proofs of quantum limits to various information processing tasks, so far it has been applied mainly to simple quantum systems with trivial dynamics measured destructively to extract static parameters. Applying such an approach to gravitational-wave detection, or optomechanical force detection in general [11], proved to be far trickier; the signal of interest there is time-varying (commonly called a waveform in engineering literature [12]), the detector is a dynamical system, and the measurements are nondestructive and continuous [5–7]. Quantum limits to such detectors had been a subject of debate [13–15], with no definitive proof that any limit exists. In more recent years, the rapid progress in experimental quantum technology suggests that quantum effects are becoming relevant to metrological applications and has given the study of quantum limits a renewed impetus [1–3,11].

Generalizing the quantum Cramér-Rao bound first proposed by Helstrom [8], Tsang, Wiseman, and Caves recently derived a quantum limit to waveform estimation [16], which represents the first step toward a rigorous treatment of quantum limits to a waveform sensor. That work assumes that one is interested in estimating an existing waveform accurately, so that the mean-square error is an appropriate error measure.

The first goal of gravitational-wave detectors is not estimation, however, but detection of the existence of gravitational waves, in which case the miss and false-alarm probabilities are the more relevant error measures [12] and the existence of quantum limits remains an open problem. Here we settle this long-standing question by proving lower error bounds for the quantum waveform detection problem. To illustrate our results, we apply them to optomechanical force detection, demonstrating a fundamental tradeoff between force detection performance and precision in detector position, and discuss how the limits can be approached in some cases of interest using a quantum-noise cancellation (QNC) technique [17–22] and an appropriate optical receiver, such as the ones proposed by Kennedy and Dolinar [8,23]. Merging the continuous quantum measurement theory pioneered by Braginsky *et al.* and the quantum detection theory pioneered by Holevo *et al.*, these results are envisaged to play an influential role in quantum metrological techniques of the future.

II. QUANTUM DETECTION OF A CLASSICAL WAVEFORM

Let $P[y|\mathcal{H}_0]$ be the probability functional of an observation process $y(t)$ under the null hypothesis \mathcal{H}_0 , and

$$P[y|\mathcal{H}_1] = \int Dx P[x]P[y|x,\mathcal{H}_1] \quad (2.1)$$

be the probability functional under the alternative hypothesis \mathcal{H}_1 . $x(t)$ is a classical waveform, $P[x]$ is its prior probability functional, and $P[y|x,\mathcal{H}_1]$ is the likelihood functional under \mathcal{H}_1 . To perform hypothesis testing given a record of $y(t)$, one separates the observation space into two decision regions Υ_0 and Υ_1 , such that \mathcal{H}_0 is chosen if y falls in Υ_0 and \mathcal{H}_1 is chosen if y falls in Υ_1 . The miss probability is defined as

$$P_{01} \equiv \int_{\Upsilon_0} Dy P[y|\mathcal{H}_1] \quad (2.2)$$

and the false-alarm probability is

$$P_{10} \equiv \int_{\Upsilon_1} Dy P[y|\mathcal{H}_0]. \quad (2.3)$$

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Two popular decision strategies are the Bayes criterion, which minimizes the average error probability

$$P_e \equiv P_{10}P_0 + P_{01}P_1 \quad (2.4)$$

given the prior hypothesis probabilities P_0 and $P_1 = 1 - P_0$, and the Neyman-Pearson criterion, which minimizes P_{01} for an allowable P_{10} , or vice versa [12].

To introduce quantum mechanics to the problem, assume that $x(t)$ perturbs the dynamics of a quantum system under \mathcal{H}_1 and $y(t)$ results from measurements of the system. Without any loss of generality, we model $P[y|\mathcal{H}_0]$ and $P[y|x, \mathcal{H}_1]$ by considering a large enough Hilbert space, such that the initial quantum state $|\psi\rangle$ at time t_i is pure, the evolution in the Schrödinger picture is unitary, and measurements are modeled by a positive-operator-valued measure (POVM) $E[y]$ at the final time t_f via the principle of deferred measurement [10,16,24]:

$$P[y|\mathcal{H}_0] = \text{tr}\{E[y]U_0(t_f, t_i)|\psi\rangle\langle\psi|U_0^\dagger(t_f, t_i)\}, \quad (2.5)$$

$$P[y|x, \mathcal{H}_1] = \text{tr}\{E[y]U_1(t_f, t_i)|\psi\rangle\langle\psi|U_1^\dagger(t_f, t_i)\}, \quad (2.6)$$

where only the unitaries U_0 and U_1 are assumed to differ and U_1 depends on x . Assume further that

$$U_0(t_f, t_i) = \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_{t_i}^{t_f} dt H_0(t) \right], \quad (2.7)$$

$$U_1(t_f, t_i) = \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_{t_i}^{t_f} dt H_1(x(t), t) \right], \quad (2.8)$$

$$H_1(x(t), t) = H_0(t) + \Delta H(x(t), t), \quad (2.9)$$

where \mathcal{T} denotes time-ordering and $\Delta H(x(t), t)$ is the Hamiltonian term responsible for the coupling of the waveform $x(t)$

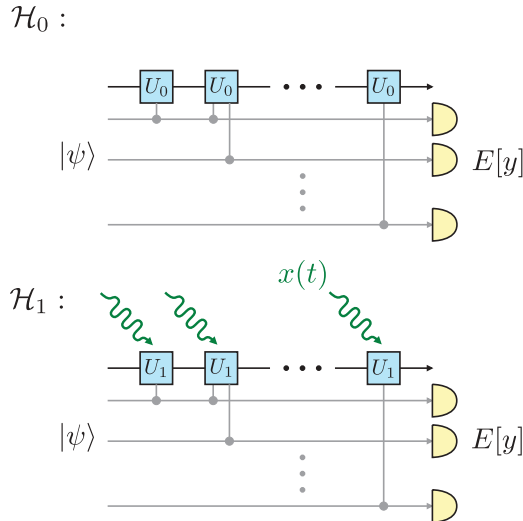


FIG. 1. (Color online) Quantum-circuit diagrams for the waveform detection problem. The quantum system is modeled as a pure state $|\psi\rangle$ with unitary evolution (U_0 or U_1) under each hypothesis (\mathcal{H}_0 or \mathcal{H}_1) in a large enough Hilbert space for a given classical waveform $x(t)$, which perturbs the evolution under \mathcal{H}_1 . If $x(t)$ is stochastic, the final quantum state under \mathcal{H}_1 is mixed. Measurements are modeled as a positive-operator-valued measure (POVM) $E[y]$ at the final time through the principle of deferred measurement.

to the quantum detector. Figure 1 shows the quantum-circuit diagrams [25] that depict the problem.

This setup can now be cast as a problem of quantum state discrimination between a pure state

$$\rho_0 \equiv U_0|\psi\rangle\langle\psi|U_0^\dagger \quad (2.10)$$

and a mixed state

$$\rho_1 \equiv \int Dx P[x]U_1|\psi\rangle\langle\psi|U_1^\dagger. \quad (2.11)$$

Let $|\Psi_j\rangle$ be a purification of ρ_j in a larger Hilbert space $\mathbb{H}_A \otimes \mathbb{H}_B$, such that $\rho_j = \text{tr}_B |\Psi_j\rangle\langle\Psi_j|$ and $\text{tr}\{E[y]\rho_j\} = \text{tr}[(E[y] \otimes 1_B)|\Psi_j\rangle\langle\Psi_j|]$, where 1_B denotes the identity operator with respect to \mathbb{H}_B . The average error probability is thus lower-bounded by [8]

$$P_e \geq \frac{1}{2}(1 - \sqrt{1 - 4P_0P_1|\langle\Psi_0|\Psi_1\rangle|^2}), \quad (2.12)$$

which is valid for any purification. Hence

$$P_e \geq \frac{1}{2}(1 - \sqrt{1 - 4P_0P_1 \max_{|\Psi_0\rangle, |\Psi_1\rangle} |\langle\Psi_0|\Psi_1\rangle|^2}) \quad (2.13)$$

$$= \frac{1}{2}(1 - \sqrt{1 - 4P_0P_1F}), \quad (2.14)$$

where F is the quantum fidelity by Uhlmann's theorem [24]:

$$F(\rho_0, \rho_1) \equiv (\text{tr} \sqrt{\sqrt{\rho_1}\rho_0\sqrt{\rho_1}})^2. \quad (2.15)$$

As ρ_0 is pure, the fidelity is given by

$$F = \langle\psi|U_0^\dagger\rho_1U_0|\psi\rangle = \mathbb{E}(F_x), \quad (2.16)$$

$$F_x \equiv |\langle U_0^\dagger(t_f, t_i)U_1(t_f, t_i)\rangle|^2, \quad (2.17)$$

where we have defined classical and quantum averages by

$$\mathbb{E}(\cdot) \equiv \int Dx P[x](\cdot), \quad (2.18)$$

$$\langle\cdot\rangle \equiv \langle\psi|\cdot|\psi\rangle. \quad (2.19)$$

By similar arguments, a quantum bound on the miss probability P_{01} for a given allowable false-alarm probability P_{10} can be derived from the bound for the pure-state case [8]:

$$P_{01} \geq \begin{cases} 1 - [\sqrt{P_{10}F} + \sqrt{(1 - P_{10})(1 - F)}]^2, & P_{10} \leq F; \\ 0, & P_{10} \geq F. \end{cases} \quad (2.20)$$

Note that the latter bound is equally valid if we interchange P_{01} and P_{10} ; for example, fixing $P_{01} = 0$ means $P_{10} \geq F$. Equations (2.14) and (2.20) are valid for any POVM and achievable if $x(t)$ is known *a priori*, such that both ρ_0 and ρ_1 are pure [8].

In terms of related prior work at this point, Ou [26] and Paris [27] studied quantum limits to interferometry in the context of detection, while Childs *et al.* [28], Acín *et al.* [29,30], and D'Ariano *et al.* [31] also studied unitary or channel discrimination, but none of them considered time-dependent Hamiltonians, which are the subject of interest here.

A key step toward simplifying Eq. (2.17) is to recognize that

$$U_0^\dagger(t_f, t_i)U_1(t_f, t_i) = \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_{t_i}^{t_f} dt \Delta H_0(x(t), t) \right], \quad (2.21)$$

where

$$\Delta H_0(x(t), t) \equiv U_0^\dagger(t, t_i) \Delta H(x(t), t) U_0(t, t_i) \quad (2.22)$$

is ΔH in the *interaction picture* [5]. In general, Eq. (2.17) can then be expanded in a Dyson series and evaluated using perturbation theory [32]. To derive analytic expressions, however, we shall be more specific about the Hamiltonians and the initial quantum state.

III. FORCE DETECTION WITH A LINEAR GAUSSIAN SYSTEM

Assume that x is a force on a quantum object with position operator q , so that

$$\Delta H = -qx, \quad (3.1)$$

and the conditional fidelity F_x becomes

$$F_x = \left| \left\langle \mathcal{T} \exp \left[\frac{i}{\hbar} \int_{t_i}^{t_f} dt q_0(t) x(t) \right] \right\rangle \right|^2, \quad (3.2)$$

with $q_0(t)$ obeying equations of motion under the null hypothesis \mathcal{H}_0 in the interaction picture. The $\langle \cdot \rangle$ expression in Eq. (3.2) is a noncommutative version of the characteristic functional [33]. To simplify it, assume further that H_0 consists of terms at most quadratic with respect to canonical position or momentum operators, such that the equations of motion are linear and $q_0(t)$ depends linearly on the initial-time canonical operators. Let $Z(t)$ be a column vector of canonical position and momentum operators, including $q_0(t)$, that obey the equation of motion

$$\frac{dZ(t)}{dt} = G(t)Z(t) + J(t) \quad (3.3)$$

under hypothesis \mathcal{H}_0 , where $G(t)$ is a drift matrix and $J(t)$ is a source vector, both consisting of real numbers. $q_0(t)$ can then be written as

$$q_0(t) = V_q(t, t_i) Z(t_i) + \int_{t_i}^t d\tau V_q(t, \tau) J(\tau), \quad (3.4)$$

where $V_q(t, t_i)$ is a row vector and a function of $G(t)$. This gives

$$\frac{1}{\hbar} \int_{t_i}^{t_f} dt q_0(t) x(t) = \kappa^\top Z(t_i) + \phi, \quad (3.5)$$

$$\kappa^\top \equiv \frac{1}{\hbar} \int_{t_i}^{t_f} dt x(t) V_q(t, t_i), \quad (3.6)$$

$$\phi \equiv \frac{1}{\hbar} \int_{t_i}^{t_f} dt x(t) \int_{t_i}^t d\tau V_q(t, \tau) J(\tau). \quad (3.7)$$

With F_x now given by

$$F_x = |\langle \mathcal{T} \exp[i\kappa^\top Z(t_i) + i\phi] \rangle|^2, \quad (3.8)$$

the time-ordering operator becomes redundant:

$$F_x = |\langle \psi | \exp[i\kappa^\top Z(t_i)] | \psi \rangle|^2. \quad (3.9)$$

This expression can be simplified using the Wigner representation $W(z, t_i)$ of $|\psi\rangle$, which has the following property [34]:

$$\langle \psi | \exp[i\kappa^\top Z(t_i)] | \psi \rangle = \int dz W(z, t_i) \exp(i\kappa^\top z), \quad (3.10)$$

where z is a column vector of phase-space variables. Assuming further that $W(z, t_i)$ is Gaussian with mean vector \bar{z} and covariance matrix Σ , we obtain an analytic expression for F_x :

$$F_x = \left| \int dz W(z, t_i) \exp(i\kappa^\top z) \right|^2 \quad (3.11)$$

$$= \exp(-\kappa^\top \Sigma \kappa) \quad (3.12)$$

$$= \exp \left[-\frac{1}{\hbar^2} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' x(t) \Sigma_q(t, t') x(t') \right], \quad (3.13)$$

$$\Sigma_q(t, t') \equiv V_q(t, t_i) \Sigma V_q^\top(t', t_i). \quad (3.14)$$

The covariance matrix is given by the Weyl-ordered second moment:

$$\Sigma_{jk} = \frac{1}{2} \langle Z_j(t_i) Z_k(t_i) + Z_k(t_i) Z_j(t_i) \rangle - \langle Z_j(t_i) \rangle \langle Z_k(t_i) \rangle. \quad (3.15)$$

Hence

$$\Sigma_q(t, t') = \frac{1}{2} \langle q_0(t) q_0(t') + q_0(t') q_0(t) \rangle - \langle q_0(t) \rangle \langle q_0(t') \rangle. \quad (3.16)$$

It is interesting to note that the expression given by $-\ln F_x$ in Eq. (3.13) coincides with the one proposed in Refs. [5,35] as an upper quantum limit on the force-sensing signal-to-noise ratio, and $4\Sigma_q(t, t')/\hbar^2$ is equal to the quantum Fisher information in the quantum Cramér-Rao bound for waveform estimation [16]. The relation of this expression to the fidelity and the detection error bounds is a novel result here, however.

If the statistics of $q_0(t)$ can be approximated as stationary, viz.,

$$\Sigma_q(t, t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_q(\omega) \exp[-i\omega(t - t')], \quad (3.17)$$

F_x becomes

$$F_x = \exp \left[-\frac{1}{\hbar^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_q(\omega) |x(\omega)|^2 \right], \quad (3.18)$$

$$x(\omega) \equiv \int_{t_i}^{t_f} dt x(t) \exp(i\omega t). \quad (3.19)$$

For example, if

$$x(t) = X \cos(\Omega t + \theta) \quad (3.20)$$

is a sinusoid,

$$F_x \approx \exp \left[-\frac{T}{\hbar^2} S_q(\Omega) X^2 \right], \quad T \equiv t_f - t_i. \quad (3.21)$$

These expressions for the fidelity suggest that, for a given $x(t)$, there is a fundamental tradeoff between force detection performance and precision in detector position.

IV. OPTOMECHANICS

Suppose now that the mechanical object is a moving mirror of an optical cavity probed by a continuous-wave optical beam, the phase of which is modulated by the object position and the intensity of which exerts a measurement back action via radiation pressure on the object, as depicted in Fig. 2. This setup provides a basic and often sufficient model for more

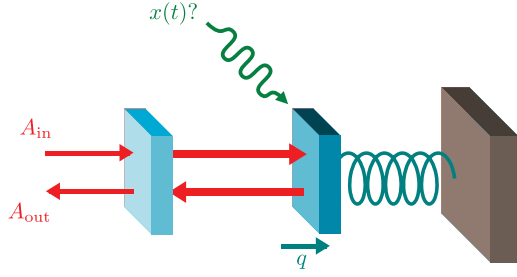


FIG. 2. (Color online) A cavity optomechanical force detector. An optical cavity with a moving mirror is pumped on-resonance with an input field A_{in} , and the output field A_{out} is measured to infer whether a force $x(t)$ has perturbed the motion of the mirror.

complex optomechanical force detectors. Let the output field operator under hypothesis \mathcal{H}_j be

$$A_{\text{out}j}(t) \approx K_1(t) * A_{\text{in}}(t) + i\mathcal{A}K_2(t) * q_j(t), \quad (4.1)$$

where

$$a(t) * b(t) \equiv \int_{-\infty}^{\infty} d\tau a(t - \tau)b(\tau) \quad (4.2)$$

denotes convolution,

$$K_n(t) \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} K_n(\omega) \exp(-i\omega t) \quad (4.3)$$

is an impulse-response function with

$$K_1(\omega) \equiv \frac{i\omega + \gamma}{-i\omega + \gamma}, \quad (4.4)$$

$$K_2(\omega) \equiv \frac{2\omega_0}{L} \frac{1}{-i\omega + \gamma} \quad (4.5)$$

in the frequency domain, \mathcal{A} is the input mean field, ω_0 is the optical carrier frequency, L is the cavity length, and γ is the optical cavity decay rate [17]. $q_j(t)$ is the position operator under each hypothesis, which can be written as [17]

$$q_0(t) \approx K_3(t) * \hbar K_2(t) * \xi(t), \quad (4.6)$$

$$q_1(t) \approx K_3(t) * [\hbar K_2(t) * \xi(t) + x(t)], \quad (4.7)$$

where $K_3(t)$ is another impulse response function that transfers a force to the position,

$$\xi \approx \mathcal{A}^* \Delta A_{\text{in}}(t) + \mathcal{A} \Delta A_{\text{in}}^\dagger(t) \quad (4.8)$$

is the back action noise, and the transient solutions are assumed to have decayed to zero. Defining

$$K_4(t) \equiv K_3(t) * K_2(t), \quad (4.9)$$

such that the position power spectral density is

$$S_q(\omega) = \hbar^2 |K_4(\omega)|^2 S_\xi(\omega), \quad (4.10)$$

we obtain

$$F_x = \exp \left[- \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_\xi(\omega) |K_4(\omega)x(\omega)|^2 \right]. \quad (4.11)$$

The back action noise ξ that appears in the output field, in addition to the shot noise in A_{in} , can limit the detection performance at the so-called standard quantum limit [5–7,15]. This does not seem to agree with the fundamental quantum

limits in terms of Eq. (4.11), which suggest that increased fluctuations in $q_0(t)$ due to $\xi(t)$ can improve the detection. Fortunately, it is now known that the back action noise can be removed from the output field [13,14,17–22,36]. One method, called quantum-noise cancellation, involves passing the optical beam through another quantum system that has the effective dynamics of an optomechanical system with negative mass [17,18,20–22]. With the back action noise removed, the output fields become

$$A_{\text{out}0}(t) \approx K_1(t) * A_{\text{in}}(t), \quad (4.12)$$

$$A_{\text{out}1}(t) \approx K_1(t) * A_{\text{in}}(t) + i\mathcal{A}K_2(t) * K_3(t) * x(t). \quad (4.13)$$

If the phase quadrature of $A_{\text{out}j}(t)$ is measured by homodyne detection, the outputs can be written as

$$y_0(t) \approx \eta(t), \quad (4.14)$$

$$y_1(t) \approx \eta(t) + K_2(t) * K_3(t) * x(t), \quad (4.15)$$

$$\eta(t) \equiv \frac{1}{2i|\mathcal{A}|^2} [\mathcal{A}^* K_1(t) * \Delta A_{\text{in}}(t) - \mathcal{A} K_1^*(t) * \Delta A_{\text{in}}^\dagger(t)]. \quad (4.16)$$

The power spectral densities of $\xi(t)$ and $\eta(t)$ satisfy an uncertainty relation [5]:

$$S_\xi(\omega) S_\eta(\omega) \geq \frac{1}{4}. \quad (4.17)$$

The detection problem described by Eqs. (4.14) and (4.15) becomes a classical one with additive Gaussian noise, a scenario that has been studied extensively in gravitational-wave detection [37,38].

V. ERROR BOUNDS FOR DETERMINISTIC WAVEFORM DETECTION

Suppose that $x(t)$ is known *a priori*. It is then well known that the error probabilities for the detection problem described by Eqs. (4.14) and (4.15) using a likelihood-ratio test are [12]

$$P_{10,\text{hom}} = \frac{1}{2} \operatorname{erfc} \left(\sigma + \frac{\lambda}{4\sigma} \right), \quad (5.1)$$

$$P_{01,\text{hom}} = \frac{1}{2} \operatorname{erfc} \left(\sigma - \frac{\lambda}{4\sigma} \right), \quad (5.2)$$

where

$$\operatorname{erfc} u \equiv \frac{2}{\sqrt{\pi}} \int_u^\infty dv \exp(-v^2), \quad (5.3)$$

λ is the threshold in the likelihood-ratio test, which can be adjusted according to the desired criterion, and σ is a signal-to-noise ratio given by

$$\sigma^2 \approx \frac{1}{8} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{|K_4(\omega)x(\omega)|^2}{S_\eta(\omega)} \quad (5.4)$$

for a long observation time relative to the duration of $x(t)$ plus the decay time of $K_4(t)$. To compare homodyne detection with the quantum limits, suppose that the duration of $x(t)$ is long and σ^2 increases at least linearly with T , so that we can define an error exponent as the asymptotic decay rate of an error probability in the long-time limit. For simplicity, we consider

here only the exponent of the higher error probability:

$$\Gamma \equiv - \lim_{T \rightarrow \infty} \frac{1}{T} \ln \max \{P_{10}, P_{01}\}. \quad (5.5)$$

Although this asymptotic limit may not be relevant to gravitational-wave detectors in the near future, the error probabilities for which are anticipated to remain high, we focus on this limit to obtain simple analytic results, which allow us to gain useful insight into the fundamental physics. More precise calculations of error probabilities are more tedious but should be straightforward following the theory outlined here.

For homodyne detection, the error exponent is

$$\Gamma_{\text{hom}} = \frac{\sigma^2}{T} = \frac{1}{8T} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{|K_4(\omega)x(\omega)|^2}{S_{\eta}(\omega)}. \quad (5.6)$$

The quantum limit, on the other hand, is

$$\begin{aligned} & - \lim_{T \rightarrow \infty} \frac{1}{T} \ln \max \{P_{10}, P_{01}\} \\ & \leq - \lim_{T \rightarrow \infty} \frac{1}{T} \ln \max_{P_e} P_e \leq - \lim_{T \rightarrow \infty} \frac{1}{T} \ln F \equiv \Gamma_F, \end{aligned} \quad (5.7)$$

which gives

$$\Gamma_F = \frac{1}{T} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{\xi}(\omega) |K_4(\omega)x(\omega)|^2. \quad (5.8)$$

Using the uncertainty relation between S_{ξ} and S_{η} in Eq. (4.17), it can be seen that

$$\Gamma_{\text{hom}} \leq \frac{\Gamma_F}{2}, \quad (5.9)$$

that is, the homodyne error exponent is at most half the optimal value. This fact is well known in the context of coherent-state discrimination [8,23,39,40]. The suboptimality of homodyne detection here should be contrasted with the conclusion of Ref. [16], which states that homodyne detection together with QNC are sufficient to achieve the quantum limit for the task of waveform estimation.

To see how one can get closer to the quantum limits, let us go back to Eqs. (4.12) and (4.13). Observe that if the input field is in a coherent state, the output field is also in a coherent state (in the Schrödinger picture) under each hypothesis. This means that existing results for coherent-state discrimination can be used to construct an optimal receiver. The Kennedy receiver, for example, displaces the output field so that it becomes vacuum under \mathcal{H}_0 and then detects the presence of any output photon [8]. Any detected photon means that \mathcal{H}_1 must be true. Deciding on \mathcal{H}_0 if no photon is detected and \mathcal{H}_1 otherwise, the false-alarm probability P_{10} is zero, while the miss probability is the probability of detecting no photon given \mathcal{H}_1 , or

$$P_{01, \text{Ken}} = \exp \left[- \int_{t_i}^{t_f} dt |AK_2(t) * K_3(t) * x(t)|^2 \right]. \quad (5.10)$$

For a long observation time with $S_{\xi} = |\mathcal{A}|^2$ for a coherent state,

$$P_{01, \text{Ken}} \approx \exp \left[- \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{\xi} |K_4(\omega)x(\omega)|^2 \right] = F, \quad (5.11)$$

which makes the Kennedy receiver optimal under the Neyman-Pearson criterion in the case of $P_{10} = 0$ according to Eq. (2.20)

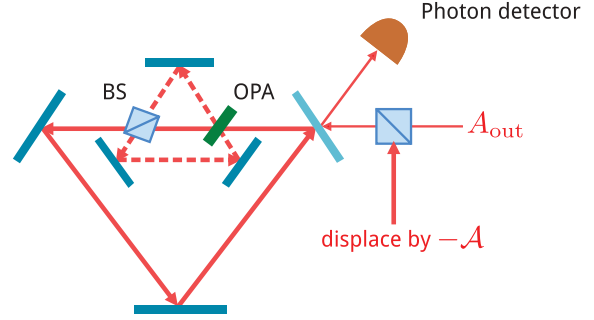


FIG. 3. (Color online) An integrated QNC-Kennedy receiver. The output field A_{out} from the optomechanical force detector in Fig. 2 is displaced by $-\mathcal{A}$ and then passed through an optical setup that removes the measurement back action noise. The dashed arrows represent a red-detuned optical cavity mode that mimics a negative-mass oscillator and interacts with the optical probe field via a beam splitter (BS) and a two-mode optical parametric amplifier (OPA). Details of how this setup works can be found in Refs. [17,18]. If the field A_{in} is in a coherent state, the final output field should be in a vacuum state under the null hypothesis \mathcal{H}_0 . Any photon detected at the output indicates that \mathcal{H}_1 must be true.

and also achieve the optimal error exponent:

$$\Gamma_{\text{Ken}} = - \lim_{T \rightarrow \infty} \frac{1}{T} \ln P_{01, \text{Ken}} = \Gamma_F. \quad (5.12)$$

The Kennedy receiver can be integrated with the QNC setup; an example is shown in Fig. 3. The Dolinar receiver, which updates the displacement field continuously according to the measurement record, can further improve the average error probability slightly to saturate the lower limit given by Eq. (2.14) [8,23]. Other more recently proposed receivers may also be used here to beat the homodyne limit [39,40].

VI. ERROR BOUNDS FOR STOCHASTIC WAVEFORM DETECTION

Consider now a stochastic $x(t)$, which should be relevant to the detection of stochastic backgrounds of gravitational waves [41]. Since F_x is Gaussian,

$$F = \int Dx P[x] \exp \left[- \frac{1}{\hbar^2} \int dt dt' x(t) \Sigma_q(t, t') x(t') \right] \quad (6.1)$$

can be computed analytically if the prior $P[x]$ is also Gaussian. Here we shall use a discrete-time approach and take the continuous limit at the end of our calculations. If $x(t)$ is a zero-mean Gaussian process with covariance

$$\Sigma_x(t, t') \equiv \mathbb{E}[x(t)x(t')], \quad (6.2)$$

it can be discretized as

$$x \equiv (x_0, \dots, x_{N-1})^T, \quad (6.3)$$

$$\begin{aligned} Dx P[x] & \approx dx_0 \cdots dx_{N-1} \frac{1}{\sqrt{(2\pi)^N \det \Sigma_x}} \\ & \times \exp \left(- \frac{1}{2} x^T \Sigma_x^{-1} x \right), \end{aligned} \quad (6.4)$$

$$\Sigma_x \equiv \mathbb{E}[xx^T]. \quad (6.5)$$

The fidelity then becomes a finite-dimensional Gaussian integral:

$$F \approx \int dx_0 \cdots dx_{N-1} \frac{1}{\sqrt{(2\pi)^N \det \Sigma_x}} \times \exp\left(-\frac{1}{2}x^\top \Sigma_x^{-1}x - \frac{\delta t^2}{\hbar^2}x^\top \Sigma_q x\right) \quad (6.6)$$

$$= \sqrt{\frac{\det(\Sigma_x^{-1} + 2\delta t^2 \Sigma_q / \hbar^2)^{-1}}{\det \Sigma_x}} \quad (6.7)$$

$$= \left[\det\left(I + \frac{2\delta t^2}{\hbar^2} \Sigma_q \Sigma_x\right) \right]^{-1/2} \quad (6.8)$$

$$= \exp\left[-\frac{1}{2} \text{tr} \ln\left(I + \frac{2\delta t^2}{\hbar^2} \Sigma_q \Sigma_x\right)\right] \quad (6.9)$$

$$= \exp\left(-\frac{1}{2} \sum_{\omega} \ln \lambda_{\omega}\right), \quad (6.10)$$

where λ_{ω} are the eigenvalues of the matrix

$$C \equiv I + \frac{2\delta t^2}{\hbar^2} \Sigma_q \Sigma_x. \quad (6.11)$$

If $\Sigma_q(t, t')$ and $\Sigma_x(t, t')$ are both stationary, viz.,

$$\Sigma_q(t, t') = \sigma_q(t - t'), \quad (6.12)$$

$$\Sigma_x(t, t') = \sigma_x(t - t'), \quad (6.13)$$

they can be modeled as circulant matrices in discrete time, so that C is also circulant, with eigenvalues given by the discrete Fourier transform of a row or column vector of the matrix. Taking the continuous-time limit using

$$\sum_{\omega} \rightarrow T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}, \quad (6.14)$$

we get

$$F = \exp(-\Gamma_F T), \quad (6.15)$$

$$\Gamma_F = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \ln \left[1 + \frac{2}{\hbar^2} S_q(\omega) S_x(\omega) \right], \quad (6.16)$$

$$S_q(\omega) \equiv \int_{-\infty}^{\infty} dt \sigma_q(t) \exp(i\omega t), \quad (6.17)$$

$$S_x(\omega) \equiv \int_{-\infty}^{\infty} dt \sigma_x(t) \exp(i\omega t). \quad (6.18)$$

This fidelity expression can then be used in the detection error bounds.

For homodyne detection, the error exponent is more complicated for stochastic waveform detection and given by the Chernoff distance [42,43]:

$$\Gamma_{\text{hom}} = \sup_{0 \leq s \leq 1} \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \ln \frac{1 + (1-s)|K_4(\omega)|^2 S_x(\omega)/S_{\eta}(\omega)}{[1 + |K_4(\omega)|^2 S_x(\omega)/S_{\eta}(\omega)]^{1-s}}. \quad (6.19)$$

The performance of homodyne detection relative to the quantum limits then depends on the specific form of $|K_4(\omega)|^2 S_x(\omega)$. The Kennedy receiver, on the other hand, is still applicable

here, as the output is still a coherent state under \mathcal{H}_0 . The false-alarm probability is still zero, and the miss probability is now

$$P_{01, \text{Ken}} \approx \mathbb{E} \exp \left[- \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{\xi}(\omega) |K_4(\omega)x(\omega)|^2 \right] \approx F, \quad (6.20)$$

which means that the Kennedy receiver remains optimal, both in terms of the Neyman-Pearson criterion in the case of $P_{10} = 0$ and the error exponent. Whether other receivers can do even better and saturate the other quantum bounds is a more difficult question, as the output field under \mathcal{H}_1 is now in a mixed state and the fidelity lower bounds may not be achievable.

The use of Kennedy or Dolinar receivers assumes coherent states at the output, which is the case only if the back action noise cancellation is complete and quantum shot noise in the input beam is the only source of noise at the output. Although such assumptions are highly idealistic, especially for current gravitational-wave detectors, the ideal scenario shows that the quantum bounds proposed here are in principle achievable using known optics technology. Optimal discrimination of squeezed or other Gaussian states remains a topic of current research [44–46] and may be useful for future gravitational-wave detectors that use squeezed light [47]. Generalization of the results here to multi-waveform discrimination should also be useful for gravitational-wave astronomy [37,38] and may be done by following Refs. [8,48,49].

VII. OUTLOOK

Now that quantum limits to waveform detection have been discovered, the natural next question to ask is how they can be approached in practice. In the case of optomechanical force detection, the requirements are quantum shot noise as the only source of noise at the output and an appropriate receiver, such as the Kennedy receiver. A proof-of-concept experimental demonstration of waveform detection approaching the shot-noise limits based on Eq. (4.11) should be well within reach of current quantum optics technology. To demonstrate the tradeoff between force detection performance and detector localization suggested by Eq. (3.18) with optomechanics, however, can be much more challenging, as it would require quantum back action noise to dominate the detector position fluctuation but become negligible in the output via QNC. A more promising candidate for this demonstration is atomic spin ensembles, with which back action-noise-canceled magnetometry has already been realized [22]. The likelihood-ratio formulas derived in Ref. [50] should be used in practice instead of the ideal-case decision rules discussed here to account for any excess noise.

In terms of potential further theoretical work, it should be useful to generalize beyond the assumptions of scalar waveform, linear Gaussian systems, optical coherent states, stationary processes, and long observation time used here. The fidelity expressions derived here may also be useful for the study of waveform estimation [16], either as an alternative way of deriving the quantum Fisher information via a Taylor series expansion [51] or used directly in the quantum Ziv-Zakai bound [52].

From a more conceptual point of view, this study, together with the earlier work on waveform estimation [16], shows that the concepts of states, effects, and operations fade into background when dealing with dynamical quantum information systems, and *multitime* quantum statistics, through the use of Heisenberg or interaction picture, take the center stage. It may be interesting to explore whether this perspective has any relevance to other dynamical quantum information systems,

such as quantum computers [24], and the study of quantum correlations [53].

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