

Fundamental solutions of real homogeneous cubic operators of principal type in three dimensions

by

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1. Introduction

1.1. The operator $\partial_1^3 + \partial_2^3 + \partial_3^3$ was considered—to my knowledge—for the first time in 1913 in N. Zeilon’s article [20], wherein he generalizes I. Fredholm’s method of construction of fundamental solutions (see [5]) from homogeneous *elliptic* equations to arbitrary homogeneous equations in three variables with a *real-valued* symbol (cf. [20, II, pp. 14–22], [6, Chapter 11, pp. 146–148]). An explicit formula for a fundamental solution was given in [19]. The objective of this paper is to generalize the calculations in [19] to the operators $\partial_1^3 + \partial_2^3 + \partial_3^3 + 3a\partial_1\partial_2\partial_3$, $a \in \mathbf{R} \setminus \{-1\}$. As discussed below, this class of operators comprises all real homogeneous cubic operators of principal type in three dimensions.

According to Newton’s classification of real elliptic curves, the non-singular real homogeneous polynomials $P(\xi)$ of third order in three variables are divided into two types according to whether the real projective curve $\{[\xi] \in \mathbf{P}(\mathbf{R}^3) : P(\xi) = 0\}$ consists of one or of two connected components, respectively. (For $\xi \in \mathbf{R}^n \setminus \{0\}$, $[\xi] \in \mathbf{P}(\mathbf{R}^n)$ denotes the corresponding projective point, i.e., the line $\{t\xi : t \in \mathbf{R}\}$.) In Hesse’s normal form, all non-singular real cubic curves are—up to linear transformations—given by

$$P_a(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3, \quad a \in \mathbf{R} \setminus \{-1\}$$

(cf. [3, 7.3, Satz 4, p. 379; English transl., p. 293], [4, §7, (10), p. 39], [17, §1.4, p. 19]). Let $X_a := \{[\xi] \in \mathbf{P}(\mathbf{R}^3) : P_a(\xi) = 0\}$ denote the real projective variety defined by P_a . For $a > -1$, X_a is connected, whereas, for $a < -1$, X_a consists of two components (cf. Figure 1). The corresponding operators $P_a(\partial)$ also differ from the physical viewpoint: For $a < -1$, every projective line through $[1, 1, 1]$ intersects X_a in three different projective points, and thus P_a is strongly hyperbolic in the direction $(1, 1, 1)$ ([1, 3.8, p. 129]); for $a > -1$, P_a is not hyperbolic in any direction, nor is it an evolution operator (cf. [15, Example 1, p. 463] for the case of $a=0$).

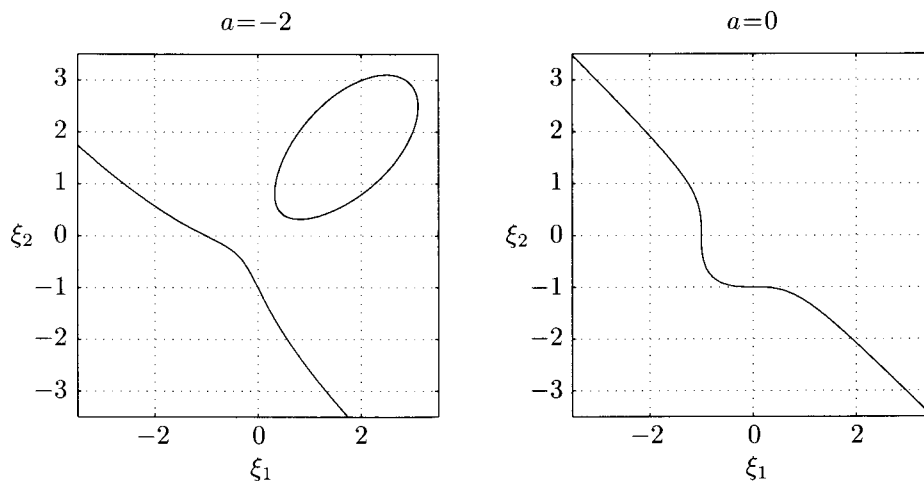


Fig. 1. $\{(\xi_1, \xi_2) : [\xi_1, \xi_2, 1] \in X_a\}$ for $a = -2$ and for $a = 0$.

1.2. In §2 of this paper, we shall define the fundamental solution E_a of $P_a(\partial)$ as Fourier transform of the homogeneous distribution which is of order -3 and has $\text{vp}(1/P_a(\omega)) \in \mathcal{D}'(\mathbf{S}^2)$ as its restriction to the sphere. From theorems on the wave front set of the Fourier transform of a homogeneous distribution ([11, Theorems 8.1.8, 8.4.18]), it immediately results that the (analytic) singular support of E_a is the dual (see [1, p. 154]) of X_a , i.e.,

$$\text{sing supp } E_a = \text{sing supp}_A E_a = \{t\nabla P_a(\xi) : \xi \in \mathbf{R}^3, P_a(\xi) = 0, t \in \mathbf{R}\}.$$

By the classical Plücker formulas (cf. [9, p. 280]), $[\text{sing supp } E_a \setminus \{0\}]$ is an algebraic curve of degree 6. Its complexification has nine cusps, three of which are real in correspondence with the three flexes of X_a (cf. Figure 2). Explicitly, we have $\text{sing supp } E_a = \{x \in \mathbf{R}^3 : A_a(x) = 0\}$, where

$$A_a(x) := 3a(a^3 + 4)x_1^2x_2^2x_3^2 + 4(a^3 + 1)(x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3) + 6a^2x_1x_2x_3(x_1^3 + x_2^3 + x_3^3) - (x_1^3 + x_2^3 + x_3^3)^2. \tag{1}$$

If $a < -1$, then P_a is hyperbolic with respect to $(1, 1, 1)$, and

$$W_a := \{x \in \mathbf{R}^3 : A_a(x) = 0, x_1 + x_2 + x_3 \geq 0\} \quad (a < -1) \tag{2}$$

consists of *two* conical surfaces which are the respective duals of the two components of X_a . Let F_a denote the unique fundamental solution of $P_a(\partial)$ with support in $\{x \in \mathbf{R}^3 : x_1 + x_2 + x_3 \geq 0\}$. Then $E_a = \frac{1}{2}(F_a - \tilde{F}_a)$, where the superscript $\tilde{}$ indicates reflection with

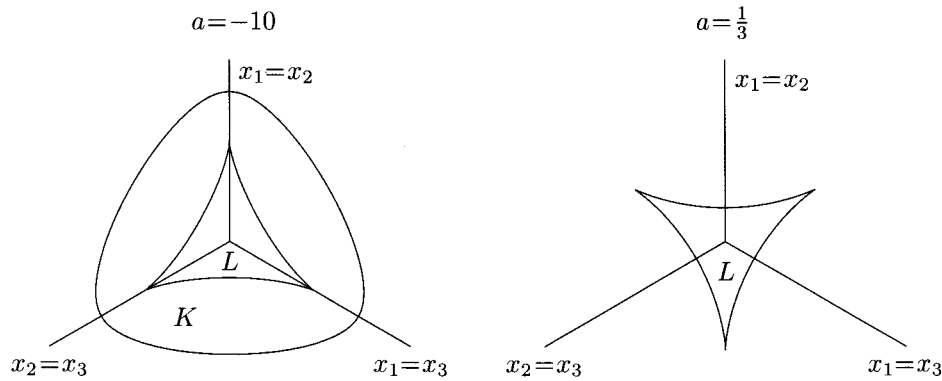


Fig. 2. $\{x \in W_a : x_1 + x_2 + x_3 = 1\}$ for $a = -10$ and for $a = \frac{1}{3}$.

respect to the origin. Further, we denote by K_a the propagation cone of P_a with respect to $(1, 1, 1)$, i.e.,

$$\begin{aligned}
 K_a &:= \text{dual cone of the component of } (1, 1, 1) \text{ in } \{x \in \mathbf{R}^3 : P_a(x) \neq 0\} \\
 &= \text{convex hull of } W_a.
 \end{aligned}
 \tag{3}$$

From the Herglotz–Petrovsky–Leray formula (cf. [1, 7.16, p. 173]), we infer that F_a has a Petrovsky lacuna (in the sense of [1, p. 185]) inside the cone

$$L_a := \{x \in K_a : A_a(x) > 0\} \quad (a < -1). \tag{41}$$

Hence W_a consists of ∂K_a and of ∂L_a , which bound a convex and a non-convex cone, respectively (cf. Figure 2).

If $a > -1$, then still E_a has lacunas inside L_a and $-L_a$, where now we define

$$L_a := \text{component of } (1, 1, 1) \text{ in } \{x \in \mathbf{R}^3 : A_a(x) > 0\} \quad (a > -1) \tag{42}$$

and

$$W_a := \partial L_a \quad (a > -1). \tag{22}$$

In both cases, the fundamental solutions E_a are constant inside L_a and $-L_a$, and we represent these constant values as *complete* elliptic integrals of the first kind. Finally, we show in §2 that E_a is continuous outside the origin.

1.3. In §3, we shall derive an explicit representation for $E_a(x)$ by elliptic integrals of the first kind. Following N. Zeilon, we introduce first one of the complex zeros of the rational integrand in the Herglotz–Petrovsky–Leray formula as a new variable, and, using a substitution (also indicated by N. Zeilon already), we transform the resulting integral

into Weierstrass' canonical form. Then we use the addition theorem for the \wp -function and the qualitative information from §2 in order to find a real-valued representation of E_a symmetric in the variables x_1, x_2, x_3 . The final result is contained in the following theorem. (Y denotes Heaviside's function and \mathcal{F} the Fourier transform, cf. 1.4.)

THEOREM. *Let $a \in \mathbf{R} \setminus \{-1\}$. The limit*

$$T_a := \lim_{\varepsilon \searrow 0} \frac{Y(|\xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3| - \varepsilon)}{\xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3}$$

defines a distribution in $\mathcal{S}'(\mathbf{R}^3)$. If $E_a := (i/2\pi)^3 \mathcal{F}T_a$, and A_a, W_a, L_a and, for $a < -1$, K_a are as in (1), (2₁), (2₂), (4₁), (4₂), (3), respectively, then:

- (a) E_a is a fundamental solution of $\partial_1^3 + \partial_2^3 + \partial_3^3 + 3a\partial_1\partial_2\partial_3$;
- (b) E_a is homogeneous of degree 0;
- (c) E_a is odd and invariant under permutations of the co-ordinates;
- (d) $\text{sing supp } E_a = \text{sing supp}_A E_a = W_a \cup -W_a$;
- (e) E_a is continuous in $\mathbf{R}^3 \setminus \{0\}$;
- (f) If $a < -1$, then $E_a = \frac{1}{2}(F_a - \tilde{F}_a)$, $P_a(\partial)F_a = \delta$, $\text{supp } F_a = K_a$;
- (g) E_a is constant in L_a and in $-L_a$, and the values $E_a|_{L_a}$ are given by the following complete elliptic integrals of the first kind:

$$E_a|_{L_a} = \begin{cases} -\frac{1}{4\sqrt{3}\pi} \int_{\varrho}^{\infty} \frac{du}{\sqrt{p_a(u)}}, & a > -1, \\ -\frac{1}{4\sqrt{3}\pi} \int_{-\infty}^e \frac{2du}{\sqrt{p_a(u)}}, & a < -1, \end{cases}$$

where $p_a(u) := 4(a^3 + 1)u^3 + 9a^2u^2 + 6au + 1$ and ϱ is the smallest real root of $p_a(u)$;

(h) Let $x \in U_a$, where $U_a := \mathbf{R}^3 \setminus (\bar{L}_a \cup -\bar{L}_a)$ if $a > -1$, and $U_a := K_a \setminus (L_a \cup W_a)$ if $a < -1$, and denote by $z(x)$ the only simple real root or, if x belongs to one of the co-ordinate axes, the triple root 0, respectively, of the cubic equation

$$\begin{aligned} Q_a(x, z) := & A_a(x)z^3 + 9(ax_1^2 + x_2x_3)(ax_2^2 + x_1x_3)(ax_3^2 + x_1x_2)z^2 \\ & + [9a^2x_1^2x_2^2x_3^2 + 6a(x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3) + 3x_1x_2x_3(x_1^3 + x_2^3 + x_3^3)]z \\ & + 3ax_1^2x_2^2x_3^2 + x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3 = 0. \end{aligned} \tag{5}$$

Then z is a real-analytic function in U_a , and

$$E_a(x) = \frac{1}{2}Y(-1-a)E_a|_{L_a} + \frac{\text{sign}(\tilde{P}_a(x))}{4\sqrt{3}\pi} \int_{\varrho}^{z(x)} \frac{du}{\sqrt{p_a(u)}}$$

where $\tilde{P}_a(x) := 3[(a^3 - 2)\varrho + a^2]x_1x_2x_3 - (3a\varrho + 1)(x_1^3 + x_2^3 + x_3^3)$.

Remark. Before proceeding, let us comment on some of the properties of the polynomial Q_a , which, outside the lacunas, yields the level sets of E_a .

First, if q_i denote the coefficients of Q_a with respect to z , i.e.,

$$Q_a(x, z) = \sum_{i=0}^3 q_i(a, x)z^i,$$

then

$$q_3 = A_a, \quad q_2 = \frac{3}{4} \cdot \frac{\partial q_3}{\partial a}, \quad q_1 = \frac{1}{3} \cdot \frac{\partial q_2}{\partial a}, \quad q_0 = \frac{1}{6} \cdot \frac{\partial q_1}{\partial a}.$$

Second, let us investigate the relation between Q_a and p_a, \tilde{P}_a . We note that all q_i belong to the four-dimensional subspace V spanned by

$$\begin{aligned} B_1(x) &= x_1^2 x_2^2 x_3^2, & B_2(x) &= x_1^3 x_2^3 + x_1^3 x_3^3 + x_2^3 x_3^3, \\ B_3(x) &= x_1 x_2 x_3 (x_1^3 + x_2^3 + x_3^3), & B_4(x) &= (x_1^3 + x_2^3 + x_3^3)^2 \end{aligned}$$

in the complex vector space of all symmetric polynomials in x_1, x_2, x_3 of degree six. The closure C_a of $\{[Q_a(x, z)] : z \in \mathbf{C}\}$ in $\mathbf{P}(V)$ is a cubic curve: $Q_a(x, z) = \sum_{i=1}^4 \beta_i(z) B_i(x)$ with

$$\begin{aligned} \beta_1(z) &= 3a(a^3 + 4)z^3 + 9(a^3 + 1)z^2 + 9a^2z + 3a, \\ \beta_2(z) &= 4(a^3 + 1)z^3 + 9a^2z^2 + 6az + 1, \\ \beta_3(z) &= 6a^2z^3 + 9az + 3z, \\ \beta_4(z) &= -z^3. \end{aligned}$$

The square polynomials make up a quadric curve S in $\mathbf{P}(V)$, namely

$$S = \text{closure of } \{[P_z(x)^2] : z \in \mathbf{C}\} = \left\{ \left[\sum_{i=1}^4 \alpha_i B_i(x) \right] : \alpha_2 = 0, \alpha_3^2 - 4\alpha_1\alpha_4 = 0 \right\}.$$

The curves C_a and S meet at $[Q_a(x, z)]$ for those z for which $p_a(z) = 0$, since $\beta_2 = p_a$ and

$$\beta_3^2 - 4\beta_1\beta_4 = (12az^3 + 9z^2)\beta_2.$$

The polynomial $\tilde{P}_a(x)$ fulfills $\tilde{P}_a(x)^2 = 4(a^3 + 1)Q_a(x, \varrho)$, and hence $[\tilde{P}_a(x)^2]$ is just one of the three intersection points of C_a and S .

For a discussion of the zeros of $Q_a(x, z)$ with respect to z , we refer to 3.4.

1.4. Let us establish some notations. We consider \mathbf{R}^n as a Euclidean space with the inner product $x \cdot y := x_1 y_1 + \dots + x_n y_n$ and write $|x| := \sqrt{x \cdot x}$. \mathbf{S}^{n-1} denotes the unit

sphere $\{\omega \in \mathbf{R}^n : |\omega|=1\}$ in \mathbf{R}^n and $d\sigma$ the Euclidean measure on \mathbf{S}^{n-1} . We write $\mathbf{P}(V)$ for the projective space corresponding to the vector space V (over \mathbf{R} or \mathbf{C} , respectively), and $[\zeta] \in \mathbf{P}(V)$ for the projective point corresponding to $\zeta \in V \setminus \{0\}$. By



we denote the Cauchy principal value.

When we make use of the theory of distributions, we adopt the notations from [11], [13], [18]. In particular, the Heaviside function is abbreviated by Y , i.e., $Y(t)=1$ for $t>0$ and 0 else, and $\langle \varphi, T \rangle$ stands for the value of the distribution T on the test function φ . We use the Fourier transform \mathcal{F} in the form

$$(\mathcal{F}\varphi)(x) := \int \exp(-ix \cdot \xi) \varphi(\xi) d\xi, \quad \varphi \in \mathcal{S}(\mathbf{R}^n).$$

2. Singular support and lacunas of E_a

2.1. Let us repeat first some elements from [19, §2]. If P is a *real-valued, homogeneous polynomial of principal type* in n variables and of degree m , then $\Phi := \text{vp}(1/P(\omega)) \in \mathcal{D}'(\mathbf{S}^{n-1})$ defined by

$$\left\langle \varphi, \text{vp} \frac{1}{P(\omega)} \right\rangle := \lim_{\varepsilon \searrow 0} \int_{|P(\omega)| > \varepsilon} \frac{\varphi(\omega)}{P(\omega)} d\sigma(\omega), \quad \varphi \in \mathcal{D}(\mathbf{S}^{n-1}),$$

solves the division problem $P(\omega) \cdot \Phi = 1$ on the sphere \mathbf{S}^{n-1} , and

$$T := \text{Pf}_{\lambda=-m} \left[\Phi \left(\frac{\xi}{|\xi|} \right) |\xi|^\lambda \right] \in \mathcal{S}'(\mathbf{R}^n)$$

solves the division problem $P(\xi) \cdot T = 1$ in \mathbf{R}^n . Hence $E := (i^m / (2\pi)^n) \mathcal{F}T$ is a fundamental solution of $P(\partial)$. Theorems 8.1.8, 8.4.18 in [11] yield the representation

$$W := \text{sing supp } E = \text{sing supp}_A E = \{t \nabla P(\xi) : \xi \in \mathbf{R}^n, P(\xi) = 0, t \in \mathbf{R}\}$$

for the singular support of E .

2.2. Let us prove next, similarly as in [1], that, for P as above and *odd* n , the Petrovsky condition on lacunas is valid (cf. [1, 10.3, p. 185]). First, radial integration in the Fourier integral for E yields Borovikov's formulas (cf. [2, (5r), (5b), p. 204; English transl., (5c), (5d), p. 16], [7, Chapter I, 6.2, (5), (6), p. 129]):

$$\langle \varphi, E \rangle = \begin{cases} \frac{(-1)^{(n-1)/2}}{4(2\pi)^{n-1}(m-n)!} \left\langle \int \varphi(x) (\omega \cdot x)^{m-n} \text{sign}(\omega \cdot x) dx, \Phi(\omega) \right\rangle, & m \geq n, \\ \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}} \langle \langle \varphi(x), \delta^{(n-m-1)}(\omega \cdot x) \rangle, \Phi(\omega) \rangle, & m < n, \end{cases} \quad (6)$$

where $\varphi \in \mathcal{S}(\mathbf{R}^n)$ and $\Phi := \text{vp}(1/P(\omega))$ as in 2.1.

Let $x \in \mathbf{R}^n \setminus W$ and set

$$v_x(\xi) := \nabla P(\xi) - \frac{x}{|x|^2} (x \cdot \nabla P(\xi)), \quad \xi \in \mathbf{R}^n.$$

If ε is a small positive number, then $P(\omega \pm i\varepsilon v_x(\omega)) \neq 0$ for all $\omega \in \mathbf{S}^{n-1}$ (since $x \notin W$), and

$$\Phi = \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{P(\omega) + i\varepsilon} + \frac{1}{P(\omega) - i\varepsilon} \right) = \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{P(\omega + i\varepsilon v_x(\omega))} + \frac{1}{P(\omega - i\varepsilon v_x(\omega))} \right).$$

If P is hyperbolic in the direction θ and if F denotes the unique fundamental solution of $P(\partial)$ with support in $\{x \in \mathbf{R}^n : \theta \cdot x \geq 0\}$, then

$$F = \frac{i^m}{(2\pi)^n} \mathcal{F} \left(\text{Pf}_{\lambda=-m} \left[\Psi \left(\frac{\xi}{|\xi|} \right) |\xi|^\lambda \right] \right)$$

with $\Psi = \lim_{\varepsilon \searrow 0} 1/P(\omega + i\varepsilon\theta) \in \mathcal{D}'(\mathbf{S}^{n-1})$ (cf. [12, Theorem 12.5.1, p. 120], [16, Proposition 1, p. 530]), and hence $\Phi = \frac{1}{2}(\Psi + (-1)^m \check{\Psi})$ and $E = \frac{1}{2}(F + (-1)^m \check{F})$.

On the other hand, for arbitrary P as in 2.1 and for a multi-index $\nu \in \mathbf{N}_0^n$ which satisfies $n - m + |\nu| > 0$, we obtain from (6), by differentiation,

$$\partial^\nu E(x) = \frac{(-1)^{(n-1)/2}}{4(2\pi)^{n-1}} \lim_{\varepsilon \rightarrow 0} \sum_{\pm} \left\langle \frac{\omega^\nu}{P(\omega \pm i\varepsilon v_x(\omega))}, \delta^{(n-m+|\nu|-1)}(\omega \cdot x) \right\rangle.$$

Note that the two limits $\lim_{\varepsilon \searrow 0} P(\omega \pm i\varepsilon v_x(\omega))^{-k}$ exist in $\mathcal{D}'(\mathbf{S}_x^{n-2})$ if

$$\mathbf{S}_x^{n-2} := \{\omega \in \mathbf{S}^{n-1} : \omega \cdot x = 0\}$$

and $k \in \mathbf{N}$ (cf. [1, p. 121]). Therefore,

$$\begin{aligned} \partial^\nu E(x) &= \frac{(-1)^{(n-1)/2}}{4(2\pi)^{n-1}} \lim_{\varepsilon \rightarrow 0} \sum_{\pm} \int_{\mathbf{S}_x^{n-2}} \left(-\frac{x}{|x|^2} \cdot \nabla_\omega \right)^{n-m+|\nu|-1} \left(\frac{\omega^\nu}{P(\omega \pm i\varepsilon v_x(\omega))} \right) \frac{d\sigma_x(\omega)}{|x|} \\ &= \frac{(-1)^{(n-1)/2}}{4(2\pi)^{n-1}} \lim_{\varepsilon \rightarrow 0} \sum_{\pm} \int_{\mathbf{S}_x^{n-2}} \left(-\frac{x}{|x|^2} \cdot \nabla_\zeta \right)^{n-m+|\nu|-1} \left(\frac{\zeta^\nu}{P(\zeta)} \right) \Big|_{\zeta=\zeta_\pm(\omega)} \frac{d\sigma_x(\omega)}{|x|} \end{aligned}$$

where $\zeta_\pm(\omega) = \omega \pm i\varepsilon v_x(\omega)$ and $d\sigma_x$ is the surface measure on \mathbf{S}_x^{n-2} . Let $\eta_x(\zeta)$ be the Leray form on $\{\zeta \in \mathbf{C}^n : \zeta \cdot x = 0\}$, i.e.,

$$d(\zeta \cdot x) \wedge \tilde{\eta}_x(\zeta) = \sum_{j=1}^n (-1)^{j-1} \zeta_j d\zeta_1 \wedge \dots \wedge d\zeta_{j-1} \wedge d\zeta_{j+1} \wedge \dots \wedge d\zeta_n + O(\zeta \cdot x),$$

$\eta_x(\zeta)$ being the restriction of $\tilde{\eta}_x(\zeta)$ from \mathbf{C}^n to $\{\zeta \in \mathbf{C}^n : \zeta \cdot x = 0\}$ and $O(\zeta \cdot x) \rightarrow 0$ for $\zeta \cdot x \rightarrow 0$, and put

$$\psi_{x,\nu}(\zeta) := \left(-\frac{x}{|x|^2} \cdot \nabla_\zeta \right)^{n-m+|\nu|-1} \left(\frac{\zeta^\nu}{P(\zeta)} \right) \eta_x(\zeta).$$

Then $\psi_{x,\nu}$ induces a holomorphic (and hence closed) $(n-2)$ -form on $U_x := \{[\zeta] \in \mathbf{P}(\mathbf{C}^n) : \zeta \cdot x = 0, P(\zeta) \neq 0\}$, which we denote by $[\psi_{x,\nu}]$, and

$$\partial^\nu E(x) = \frac{(-1)^{(n-1)/2}}{4(2\pi)^{n-1}} \int_{c_x} [\psi_{x,\nu}], \tag{7}$$

where c_x is the homology class of the $(n-2)$ -chain $s_{x,\varepsilon} + \overline{s_{x,\varepsilon}}$, the cycle $s_{x,\varepsilon}$ being given by

$$s_{x,\varepsilon} : \mathbf{S}_x^{n-2} \rightarrow U_x, \quad \omega \mapsto [\omega + i\varepsilon v_x(\omega)],$$

and ε is small. (We choose η_x as orientation on \mathbf{S}_x^{n-2} .) Essentially, the representation in formula (7) is equivalent to [1, (7.17'), p. 173] (cf. also the proof on p. 176) or to [12, (12.6.10)''', p. 131] for hyperbolic operators. Due to (7), E coincides with a polynomial of the degree $m-n$ in those components of $\mathbf{R}^n \setminus W$ which contain a point x with vanishing c_x in the homology group $H_{n-2}(U_x)$. This is precisely the Petrovsky condition for lacunas.

2.3. We apply the foregoing discussion to $P_a(\xi) := \xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3$, $a \in \mathbf{R} \setminus \{-1\}$. In this case, Φ is odd and thus $\lambda \mapsto \Phi(\xi/|\xi|)|\xi|^\lambda$ is analytic in $\lambda = -3$. Hence $T_a := \Phi(\xi/|\xi|)|\xi|^{-3}$ and $E_a := (i/2\pi)^3 \mathcal{F}T_a$ are also odd and homogeneous of the degrees -3 and 0 , respectively. As in [19, 2.2], we obtain $T_a = \lim_{\varepsilon \searrow 0} Y(|P_a(\xi)| - \varepsilon) / P_a(\xi)$.

For $x = (1, 1, 1)$, all the three zeros of P_a in $\{[\zeta] \in \mathbf{P}(\mathbf{C}^3) : \zeta \cdot x = 0\}$ are real. In fact, they are given by $[-1, 0, 1]$, $[0, 1, -1]$, $[1, -1, 0]$. Moreover, $s_{x,\varepsilon}$ and $\overline{s_{x,\varepsilon}}$ coincide since $v_x(-\omega) = v_x(\omega)$. Hence c_x in 2.2 vanishes (cf. [1, (6.26), p. 167] and Figure 3), and E_a is constant in the two components of $\mathbf{R}^3 \setminus W_a$ containing $(1, 1, 1)$ and $-(1, 1, 1)$, respectively, i.e., in L_a and in $-L_a$. Of course, in the hyperbolic case $a < -1$, moreover E_a vanishes in $\mathbf{R}^3 \setminus (K_a \cup -K_a)$, the so-called trivial lacuna (cf. [1, p. 115]).

In order to obtain an equation for the wave front surface W_a , we take into account that W_a is the set of x where the two equations $\xi \cdot x = 0$, $P_a(\xi) = 0$ have multiple solutions $[\xi] \in P(\mathbf{R}^3)$. Hence W_a is the zero set of the discriminant of the polynomial $P_a(u, -(ux_1 + x_3)/x_2, 1)$ with respect to u . This discriminant is $27A_a(x)/x_2^6$ with A_a as in (1).

2.4. Let us calculate next the constant values $E_a|_{L_a}$. Upon application of some obvious estimates and of Lebesgue's dominated convergence theorem (cf. [19, 2.2]), formula (6) implies that E_a is a locally integrable function given by

$$E_a(x) = -\frac{1}{8\pi^2} \lim_{\varepsilon \searrow 0} \int_{|P_a(u,v,1)| > \varepsilon} \frac{\text{sign}(ux_1 + vx_2 + x_3)}{P_a(u,v,1)} du dv. \tag{8}$$

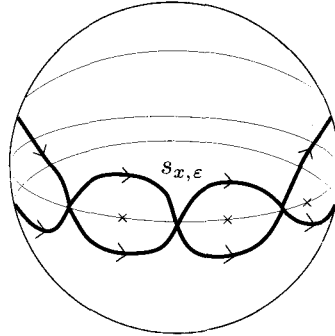


Fig. 3. The path $s_{x, \epsilon}$ in $\{[\zeta] \in \mathbf{P}(\mathbf{C}^3) : \zeta \cdot x = 0\}$ for $x = (1, 1, 1)$. (Each \times denotes one of the three real zeros of P_a in the complex projective line considered.)

Employing the substitution $w = u + v$, we infer

$$E_a(1, 1, 1) = -\frac{1}{8\pi^2} \int_{-\infty}^{\infty} \text{sign}(w+1) dw \oint_{-\infty}^{\infty} \frac{du}{3u^2(w-a) - 3uw(w-a) + w^3 + 1}.$$

The quadratic polynomial of u in the last integral has no real zeros if and only if $(w-a)(w^3 + 3aw^2 + 4)$ is positive, and the inner integral yields

$$\frac{2\pi}{\sqrt{3}} \cdot \frac{\text{sign}(w-a)}{\sqrt{(w-a)(w^3 + 3aw^2 + 4)}}$$

in this case and 0 else, i.e.,

$$E_a(1, 1, 1) = -\frac{1}{4\sqrt{3}\pi} \int_{-\infty}^{\infty} \frac{\text{sign}((w+1)(w-a)) dw}{[(w-a)(w^3 + 3aw^2 + 4)]_+^{1/2}},$$

where $x_+ := Y(x)x$ for $x \in \mathbf{R}$. With $p_a(u) := 4(a^3 + 1)u^3 + 9a^2u^2 + 6au + 1$, the substitution $u = 1/(w-a)$ furnishes

$$E_a(1, 1, 1) = -\frac{1}{4\sqrt{3}\pi} \int_{-\infty}^{\infty} \text{sign}(1 + u(a+1)) p_a(u)_+^{-1/2} du.$$

The discriminant of p_a is $-2^4 \cdot 3^3 \cdot (a^3 + 1)$, and hence p_a has one or three real roots according to the sign of $a+1$.

If $a > -1$, then the only real root ϱ of p_a satisfies $-1/(a+1) < \varrho < 0$, since $p_a(0) = 1$ and $p_a(-1/(a+1)) = -3/(a+1)^2$. Thus $1 + u(a+1) > 0$ if $p_a(u) > 0$ and

$$E_a|_{L_a} = -\frac{1}{4\sqrt{3}\pi} \int_{\varrho}^{\infty} \frac{du}{\sqrt{p_a(u)}} \quad (a > -1).$$

If $a < -1$, then p_a has three real roots. say $\varrho < \sigma < \tau$. From $p_a(0) > 0$, $p'_a(0) < 0$, $p''_a(0) > 0$, and $p_a(-1/(a+1)) < 0$, $p'_a(-1/(a+1)) = -6(a-2)/(a+1) < 0$, $p''_a(-1/(a+1)) = -6(a-2)^2 < 0$, we conclude that $0 < \varrho < \sigma < \tau < -1/(a+1)$, and thus again $1+u(a+1) > 0$ if $p_a(u) > 0$. By [10, 222.2b], this implies

$$E_a|_{L_a} = -\frac{1}{4\sqrt{3}\pi} \left[\int_{-\infty}^{\varrho} + \int_{\sigma}^{\tau} \right] \frac{du}{\sqrt{p_a(u)}} = -\frac{1}{4\sqrt{3}\pi} \int_{-\infty}^{\varrho} \frac{2 du}{\sqrt{p_a(u)}} \quad (a < -1).$$

2.5. Let us finally show in this section that E_a is continuous outside the origin. From formula (8) we infer (substituting $w = u + v$ for v as in 2.4)

$$E_a(x) = -\frac{1}{8\pi^2} \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} \frac{\text{sign}(u(x_1 - x_2) + wx_2 + x_3) du}{3u^2(w - a) - 3uw(w - a) + w^3 + 1}.$$

For real values $\alpha, \beta, \gamma, \delta$, [10, 131.3] yields

$$\int_{\delta}^{\infty} \frac{du}{\alpha u^2 + 2\beta u + \gamma} = \begin{cases} \frac{1}{\sqrt{|\beta^2 - \alpha\gamma|}} \cdot \frac{1}{2} \ln \left| \frac{\beta + \alpha\delta + \sqrt{\beta^2 - \alpha\gamma}}{\beta + \alpha\delta - \sqrt{\beta^2 - \alpha\gamma}} \right|, & \alpha\gamma < \beta^2, \\ \frac{\text{sign } \alpha}{\sqrt{|\beta^2 - \alpha\gamma|}} \cdot \text{arccot} \left(\frac{\beta \text{sign}(\alpha) + \alpha\delta}{\sqrt{\alpha\gamma - \beta^2}} \right), & \alpha\gamma > \beta^2. \end{cases}$$

In our case, $\beta^2 - \alpha\gamma = -\frac{3}{4}(w - a)(w^3 + 3aw^2 + 4)$ and $\delta = (wx_2 + x_3)/(x_2 - x_1)$. If $x_1 \neq x_2$, then Lebesgue's dominated convergence theorem can be applied in order to show that E_a is continuous in x . Since $(1, 1, 1) \notin W_a$, we conclude, by the symmetry of E_a with respect to the co-ordinates x_1, x_2, x_3 , that E_a is continuous in $\mathbf{R}^3 \setminus \{0\}$.

Let us note, by the way, that, for $a = -1$, P_a decomposes:

$$P_{-1}(\xi) = (\xi_1 + \xi_2 + \xi_3)(\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_1\xi_2 - \xi_1\xi_3 - \xi_2\xi_3).$$

From this factorization, one can see that $\lim_{\varepsilon \searrow 0} Y(|P_{-1}(\omega)| - \varepsilon)/P_{-1}(\omega)$ diverges in $\mathcal{D}'(\mathbf{S}^2)$, and hence E_{-1} is not defined. But it is easy to check that

$$\frac{\text{sign}(x_1 + x_2 + x_3)}{12\sqrt{3}\pi} \ln(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$$

is a fundamental solution of $P_{-1}(\partial)$.

3. Representation of E_a by elliptic integrals

3.1. Let us consider now formula (7) in the case of $P = P_a$ and $x \in K_a \setminus L_a$. Then P_a has two complex conjugate zeros in $\{\zeta \in \mathbf{P}(\mathbf{C}^3) : \zeta \cdot x = 0\}$, say p, \bar{p} . The residue theorem

implies

$$\begin{aligned} \partial_j E_a(x) &= -\frac{1}{16\pi^2} \int_{c_x} [\psi_{x,j}] \\ &= \pm \frac{i}{4\pi} (\operatorname{Res}_p [\psi_{x,j}] - \operatorname{Res}_{\bar{p}} [\psi_{x,j}]) = \pm \frac{1}{2\pi} \operatorname{Im}(\operatorname{Res}_p [\psi_{x,j}]), \end{aligned}$$

where $\psi_{x,j} := \zeta_j \eta_x(\zeta) / P_a(\zeta)$. Using ζ_1 as variable on $\{\zeta \in \mathbf{C}^3 : \zeta \cdot x = 0, \zeta_3 = 1\}$ yields successively

$$\begin{aligned} \tilde{\eta}_x(\zeta) &= \frac{1}{|x|^2} \det(\zeta, x, d\zeta), \\ \eta_x(\zeta)|_{\zeta_3=1} &= \frac{1}{|x|^2} \det\left(\zeta, x, \begin{pmatrix} d\zeta_1 \\ -x_1 d\zeta_1/x_2 \\ 0 \end{pmatrix}\right) = -\frac{d\zeta_1}{x_2}, \\ \operatorname{Res}_p [\psi_{x,3}] &= \operatorname{Res}_{y_1}(\psi_{x,3}|_{\zeta_3=1}) = -\frac{1}{x_2 R'(y_1)}, \end{aligned}$$

where $R(u) := P_a(u, -(ux_1 + \lambda)/x_2, 1)$ and $p = [y_1, y_2, 1]$. Next we substitute λ by y_1 (cf. [20, p. 16]) in the integral $E_a(x) = \int^{x_3} (\partial_3 E_a)(x_1, x_2, \lambda) d\lambda$. From

$$\begin{aligned} P_a\left(y_1, \frac{-(y_1 x_1 + \lambda)}{x_2}, 1\right) = 0 &\Rightarrow \left(\partial_1 P_a - \frac{x_1}{x_2} \partial_2 P_a\right) dy_1 = \frac{\partial_2 P_a}{x_2} d\lambda \\ &\Rightarrow \frac{d\lambda}{R'(y_1)} = \frac{x_2 dy_1}{(\partial_2 P_a)(y_1, y_2, 1)} = \frac{x_2 dy_1}{3(y_2^2 + ay_1)} \end{aligned}$$

we infer

$$E_a(x) = \text{constant} \pm \frac{1}{6\pi} \operatorname{Im} \int_{\gamma(x)} \frac{dy_1}{y_2^2 + ay_1},$$

where $\gamma(x)$ is a path in the Riemannian surface $\{(y_1, y_2) \in \mathbf{C}^2 : P_a(y_1, y_2, 1) = 0\}$ ending at the point $(y_1(x), y_2(x))$, which fulfills $y_1(x)x_1 + y_2(x)x_2 + x_3 = 0$, and $\operatorname{Im} y_1(x) > 0$, say.

Let us observe that

$$\Omega := \frac{dy_1}{y_2^2 + ay_1} = \frac{3 dy_1}{\partial P_a(y_1, y_2, 1) / \partial y_2} = -3 \operatorname{P.R.} \left(\frac{dy_1 \wedge dy_2}{P_a(y_1, y_2, 1)} \right)$$

spans the space $\Omega^1(X_a^c)$ of holomorphic 1-forms on the elliptic curve $X_a^c := \{[\zeta] \in \mathbf{P}(\mathbf{C}^3) : P_a(\zeta) = 0\}$ (with the co-ordinates $y_1 = \zeta_1/\zeta_3$, $y_2 = \zeta_2/\zeta_3$), and that E_a can be expressed more symmetrically as $E_a(x) = \text{constant} \pm (1/6\pi) \int_{\{\gamma(x)\}} \Omega$. (Here P.R. denotes the Poincaré residue map as in [9, pp. 147, 221].)

The elliptic integral above is transformed into standard form with the help of the substitution $w=(1+y_2)/y_1$ (cf. [20, p. 60]). In fact,

$$\begin{aligned} y_1^3 + y_2^3 + 1 + 3ay_1y_2 = 0 &\Rightarrow (y_1^2 + ay_2) dy_1 + (y_2^2 + ay_1) dy_2 = 0 \\ &\Rightarrow dw = \frac{dy_2}{y_1} - \frac{(1+y_2) dy_1}{y_1^2} \\ &= -\frac{dy_1}{(y_2^2 + ay_1)y_1^2} [y_1^3 + y_2^3 + 2ay_1y_2 + y_2^2 + ay_1] \\ &= \frac{(1+y_2 - ay_1)(1-y_2)}{(y_2^2 + ay_1)y_1^2} dy_1 \\ &\Rightarrow \frac{dy_1}{y_2^2 + ay_1} = \frac{y_1^2 dw}{(1+y_2 - ay_1)(1-y_2)} \end{aligned}$$

and, on the other hand,

$$\begin{aligned} (w-a)(w^3 + 3aw^2 + 4) &= \frac{1+y_2 - ay_1}{y_1^4} [(1+y_2)^3 + 3ay_1(1+y_2)^2 + 4y_1^3] \\ &= \frac{1+y_2 - ay_1}{y_1^4} [(1+y_2)^3 + 3ay_1(1+y_2)^2 - 4(1+y_2^3) - 12ay_1y_2] \\ &= -3 \frac{(1+y_2 - ay_1)^2(1-y_2)^2}{y_1^4}. \end{aligned}$$

Hence we obtain

$$E_a(x) = \text{constant} \pm \frac{1}{2\sqrt{3}\pi} \operatorname{Re} \int_{\gamma(x)} \frac{dw}{\sqrt{(w-a)(w^3 + 3aw^2 + 4)}}.$$

As in 2.4, we eventually transform this elliptic integral into Weierstrass' form by setting $u=1/(w-a)$. This furnishes

$$\begin{aligned} E_a(x) &= \text{constant} \pm \frac{1}{2\sqrt{3}\pi} \operatorname{Re} \int_{u(x)}^{-1/(a+1)} \frac{du}{\sqrt{p_a(u)}} \\ &= \text{constant} \pm \frac{1}{2\sqrt{3}\pi} \operatorname{Re} \int_{u(x)}^{\infty} \frac{du}{\sqrt{p_a(u)}}, \end{aligned} \tag{10}$$

where p_a is as in (g) of the Theorem, and $u(x)$ is determined by the equations $u(x)=1/(w(x)-a)$, $w(x)=(1+y_2(x))/y_1(x)$, $y_1(x)x_1+y_2(x)x_2+x_3=0$, $P_a(y_1(x), y_2(x), 1)=0$, and by the condition $\operatorname{Im} y_1(x) > 0$.

3.2. In order to obtain an integral representation of E_a over a path on the real axis, let us employ the addition theorem of Weierstrass' \wp -function, i.e.,

$$\wp(s+t) = -\wp(s) - \wp(t) + \frac{1}{4} \left(\frac{\wp'(s) - \wp'(t)}{\wp(s) - \wp(t)} \right)^2, \tag{11}$$

cf. [8, 8.166.2]. Since $\wp'(s) = \sqrt{4\wp(s)^3 - g_2\wp(s) - g_3}$, we have

$$\left[\int_{\sigma}^{\infty} + \int_{\tau}^{\infty} \right] \frac{du}{\sqrt{4u^3 - g_2u - g_3}} = \int_z^{\infty} \frac{du}{\sqrt{4u^3 - g_2u - g_3}} \tag{12}$$

if $\wp(s) = \sigma$, $\wp(t) = \tau$ and $\wp(s+t) = z$, which, by (11), amounts to

$$z = \frac{(4\sigma\tau - g_2)(\sigma + \tau) - 2g_3 - 2\sqrt{4\sigma^3 - g_2\sigma - g_3}\sqrt{4\tau^3 - g_2\tau - g_3}}{4(\sigma - \tau)^2}.$$

Here we suppose that $\sigma \neq \tau$ are sufficiently large real numbers. By a shift of the integration variable, the following slightly more general addition theorem ensues from (12):

$$\left[\int_{\sigma}^{\infty} + \int_{\tau}^{\infty} \right] \frac{du}{\sqrt{q(u)}} = \int_z^{\infty} \frac{du}{\sqrt{q(u)}} \tag{13}$$

where $q(u) = \alpha u^3 + \beta u^2 + \gamma u + \delta$, $\alpha > 0$, $\beta, \gamma, \delta \in \mathbf{R}$, $\sigma, \tau \in \mathbf{C}$, $\text{Re } \sigma, \text{Re } \tau$ are sufficiently large, $\sigma \neq \tau$, and

$$z = \frac{\alpha\sigma\tau(\sigma + \tau) + 2\beta\sigma\tau + \gamma(\sigma + \tau) + 2\delta - 2\sqrt{q(\sigma)q(\tau)}}{\alpha(\sigma - \tau)^2}.$$

We apply formula (13) to (10) with $q = p_a$, $\sigma = u(x)$ and $\tau = \overline{u(x)}$. This yields

$$E_a(x) = \text{constant} \pm \frac{1}{4\sqrt{3}\pi} \int_{z(x)}^{\infty} \frac{du}{\sqrt{p_a(u)}} \tag{14}$$

with

$$\begin{aligned} z(x) &= \frac{4(a^3 + 1)u\bar{u}(u + \bar{u}) + 18a^2u\bar{u} + 6a(u + \bar{u}) + 2 - 2\sqrt{p_a(u)p_a(\bar{u})}}{4(a^3 + 1)(u - \bar{u})^2} \\ &= \frac{2(a^3 + 1)(w + \bar{w} - 2a) + (w - a)(\bar{w} - a)(w + 2a)(\bar{w} + 2a) - S}{2(a^3 + 1)(w - \bar{w})^2} \end{aligned} \tag{15}$$

wherein $u = u(x)$ and $w = w(x)$ are specified at the end of 3.1, and

$$S := \sqrt{(w - a)(w^3 + 3aw^2 + 4)(\bar{w} - a)(\bar{w}^3 + 3a\bar{w}^2 + 4)}. \tag{16}$$

3.3. Let us next derive the cubic equation (5) for $z(x)$. Since

$$x_1y_1(x) + x_2y_2(x) + x_3 = 0 \quad \text{and} \quad P_a(y_1(x), y_2(x), 1) = 0,$$

$y_1(x)$ is a root of the following cubic polynomial in u :

$$x_2^3 P_a(u, -(ux_1 + x_3)/x_2, 1) = x_2^3(u^3 + 1) - (ux_1 + x_3)^3 - 3ax_2^2u(ux_1 + x_3).$$

This implies that

$$w(x) = \frac{1+y_2(x)}{y_1(x)} = -\frac{x_1}{x_2} + \frac{x_2-x_3}{x_2 y_1(x)}$$

is a solution of the cubic equation

$$B(w) := (x_2^2 + x_2 x_3 + x_3^2)w^3 + 3(x_1 x_2 + x_1 x_3 - a x_2 x_3)w^2 + 3x_1(x_1 - a x_2 - a x_3)w + (x_2 - x_3)^2 - 3a x_1^2 = 0.$$

Furthermore, $B(w)=0$ implies

$$(w-a)(w^3 + 3aw^2 + 4) = -\frac{3(w-a)^2(2x_1 + wx_2 + wx_3)^2}{(x_2 - x_3)^2},$$

and thus, for $w=w(x)$, the square root S defined in (16) fulfills

$$S = 3 \frac{(w-a)(\bar{w}-a)(2x_1 + wx_2 + wx_3)(2x_1 + \bar{w}x_2 + \bar{w}x_3)}{(x_2 - x_3)^2}. \quad (17)$$

We now consider B as a polynomial over $K := \mathbf{Q}(a, x_1, x_2, x_3)$, assuming a, x_1, x_2, x_3 transcendental over \mathbf{Q} . If L is a splitting field of B over K , then B has three roots w_1, w_2, w_3 in L , and $z(x)$ is, according to (15) and (17), a rational function of w_1, w_2 say. Although the dimension of L over K is six, $z(x)$ satisfies a *cubic* equation over $\mathbf{Q}(a, x_1, x_2, x_3)$, since $z(x)$ is mapped to itself by that element of the Galois group of L over K which exchanges w_1 and w_2 (cf. [14]).

In order to determine the cubic equation for $z(x)$ over K , we first express $z(x)$ in (15) by w_3 . Since w_1, w_2, w_3 are the roots of B , we have

$$w_1 + w_2 = -w_3 - 3 \frac{x_1 x_2 + x_1 x_3 - a x_2 x_3}{x_2^2 + x_2 x_3 + x_3^2}$$

and

$$w_1 w_2 = -w_3(w_1 + w_2) + 3 \frac{x_1(x_1 - a x_2 - a x_3)}{x_2^2 + x_2 x_3 + x_3^2}.$$

Inserting these equations into (15) and (17), and making use of $B(w_3)=0$, a symbolic calculation program yields

$$z(x) = \frac{x_2 x_3 (x_2^2 + x_2 x_3 + x_3^2) w_3 - 3a x_2^2 x_3^2 - x_1 (x_2 + x_3) (x_2 - x_3)^2}{4(x_1 x_2 + a x_3^2)(x_1 x_3 + a x_2^2) - [(x_2^2 + x_2 x_3 + x_3^2)w_3 + x_1 x_2 + x_1 x_3 - a x_2 x_3]^2}. \quad (18)$$

If N and D denote the numerator and the denominator, respectively, of the quotient in (18), then $z(x)$ is a root of the resultant of $B(w_3)$ and $Dz - N$ with respect to w_3 , which resultant is $-(x_2^3 - x_3^3)^2 (x_2^2 + x_2 x_3 + x_3^2)^2 Q_a(x, z)$.

3.4. Let us finally verify the representation of E_a announced in (h) of the Theorem. Define

$$U_a := \begin{cases} \mathbf{R}^3 \setminus (\bar{L}_a \cup -\bar{L}_a), & a > -1, \\ K_a \setminus (L_a \cup W_a), & a < -1, \end{cases}$$

and consider $x \in U_a$. The discriminant of $Q_a(x, z)$ with respect to z is

$$27(x_1^3 - x_2^3)^2(x_1^3 - x_3^3)^2(x_2^3 - x_3^3)^2 A_a(x),$$

and this is negative in U_a except for the planes $x_1 = x_2$, $x_1 = x_3$ and $x_2 = x_3$. Hence $Q_a(x, z)$ has exactly one real root $z(x)$ if $x \in U_a$ and x does not belong to one of these planes. When $x_2 = x_3$, say, then there is a double root $z = -x_2 / (x_1 + ax_2)$ (note that $A_a(-a, 1, 1) = 0$, so $x_1 + ax_2 \neq 0$), and since the discriminant of $\partial Q_a(x, z) / \partial z$ is

$$36(x_1^3 - x_2^3)^2(x_1 + ax_2)^4 x_2^2,$$

and $(1, 1, 1)$ belongs to L_a , we have precisely one simple zero except on the co-ordinate axes. This simple zero is real-analytic in the whole set U_a , for if say $x_1 = 1$ and $x_2 = x_3$ is small, we have just given the double zero explicitly, and it follows that the simple zero is also analytic there. If ε and δ are small enough, it follows that the simple zero z can be continued uniquely analytically to $\{(x_2, x_3) \in \mathbf{C}^2 : 0 < |x_3| < \varepsilon, |x_2 - x_3| < \delta\}$, and since it is bounded, it extends analytically also to $x_3 = 0$. Therefore, if $z(x)$ is defined as in the Theorem, then it is a real-analytic function of $x \in U_a$.

If ϱ denotes the smallest real root of p_a (cf. 2.4), then a calculation shows that

$$Q_a(x, \varrho) = \frac{\tilde{P}_a(x)^2}{4(a^3 + 1)}, \quad \tilde{P}_a(x) := 3[(a^3 - 2)\varrho + a^2]x_1 x_2 x_3 - (3a\varrho + 1)(x_1^3 + x_2^3 + x_3^3).$$

Hence $z(x) = \varrho$ if $\tilde{P}_a(x) = 0$. Let us investigate \tilde{P}_a . From $(3a\varrho + 1)^2 = -4(a^3 + 1)\varrho^3$, we conclude that $\text{sign } \varrho = -\text{sign}(a + 1)$ and that $3a\varrho + 1 \neq 0$ for $a \in \mathbf{R} \setminus \{-1\}$. Hence

$$\tilde{P}_a = -(3a\varrho + 1)P_{\tilde{a}}, \quad \tilde{a} := -\frac{(a^3 - 2)\varrho + a^2}{3a\varrho + 1},$$

and we have to decide on the sign of $\tilde{a} + 1$. Using two values of a , say $a = 0$ and $a \nearrow -1$, and the continuity of $\varrho(a)$, we obtain that $3a\varrho + 1$ is always positive. $b = \tilde{a}$ is a root of the resultant

$$4(a^3 + 1)(b^3 + 3a^2 b^2 - 3ab + a^3 + 2)$$

of the two polynomials $p_a(u)$ and $(3au + 1)b + (a^3 - 2)u + a^2$ with respect to u , and therefore $\tilde{a} \neq -1$ when $a \neq -1$. A test on two values of a as above reveals that $\tilde{a} + 1$ is negative, and hence $X_{\tilde{a}}$ always consists of two components.

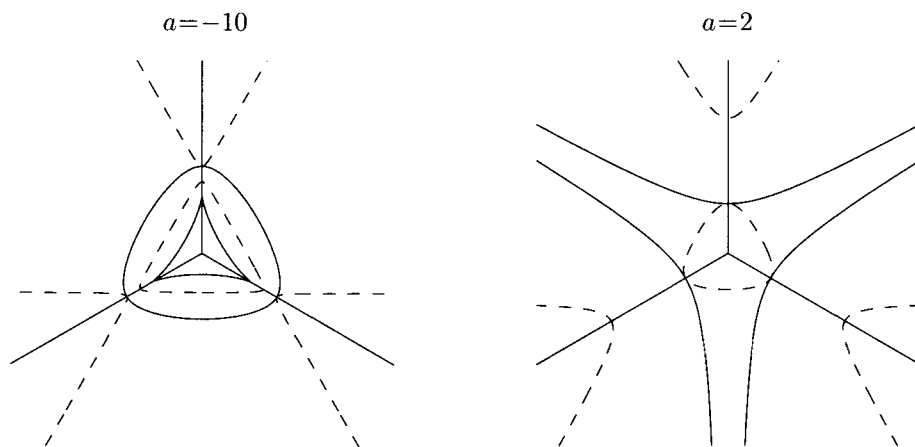


Fig. 4. W_a (solid) and $X_{\bar{a}}$ (dashed) on the plane $x_1+x_2+x_3=1$ for $a=-10$ and for $a=2$.

The curves $[W_a \setminus \{0\}]$ and X_b intersect on the lines $[t, t, 1]$, $t \in \mathbf{R}$, if and only if the resultant of $A_a(t, t, 1)$ and $P_b(t, t, 1)$ vanishes. This resultant is given by

$$108(b^3 + 3a^2b^2 - 3ab + a^3 + 2)(a^3 + 3ab - 2)^3,$$

and thus $[W_a \setminus \{0\}]$ and $X_{\bar{a}}$ touch at points on the three projective lines $x_1=x_2$, $x_1=x_3$ and $x_2=x_3$. Using two values of a as above then shows: If $a > -1$, then the convex component of $X_{\bar{a}}$ lies inside $[L_a \setminus \{0\}]$, and the non-convex component belongs to $[U_a]$; if $a < -1$, then the convex component of $X_{\bar{a}}$ belongs to $[U_a]$, and the non-convex component lies in $\mathbf{P}(\mathbf{C}^3) \setminus [K_a]$ (cf. Figure 4).

Next let us discuss the behaviour of $z(x)$ for x tending to ∂U_a from inside U_a . Evidently, $z(x) \rightarrow \pm\infty$, and we can decide on the sign of the limit by noticing that it coincides with the sign of $Q_a(x, 0)$ since $A_a(x) < 0$ in U_a . For $x = (-a, 1, 1) \in \partial L_a$, $Q_a(x, 0) = a^3 + 1$, and hence

$$z(x) \rightarrow \begin{cases} \infty, & a > -1, \\ -\infty, & a < -1, \end{cases}$$

if $x \rightarrow \partial L_a$ from inside U_a . On the other hand, if $a < -1$ and $x = (0, 0, 1)$, then $z(x) = 0 < \varrho$ and $\tilde{P}_a(x) = -(3a\varrho + 1) < 0$ (whereas $\tilde{P}_a(-a, 1, 1) > 0$), and this implies that $z(x) \rightarrow -\infty$ if $x \rightarrow \partial K_a$ from inside U_a , $a < -1$. Hence, for all $x \in U_a$, $z(x) \geq \varrho$ if $a > -1$, and $z(x) \leq \varrho$ if $a < -1$. From this we conclude that $\int_{\varrho}^{z(x)} du / \sqrt{p_a(u)}$ is real-analytic in $[U_a]$ except possibly on $X_{\bar{a}} \cap [U_a]$. A Taylor series argument as in [19, Remark] shows that

$$\text{sign}(\tilde{P}_a(x)) \int_{\varrho}^{z(x)} \frac{du}{\sqrt{p_a(u)}}$$

is real-analytic on $X_{\bar{a}} \cap [U_a]$ also.

Combining now the continuity of E_a in $\mathbf{R}^3 \setminus \{0\}$, the values of $E_a|_{L_a}$ calculated in 2.4, the limit behaviour of $z(x)$ on the border of U_a analyzed above, $\lim_{x \rightarrow \partial K_a} E_a(x) = 0$ for $a < -1$, and the representation of E_a in (14) with the principle of analytic continuation, furnishes a proof of the assertion (h) of the Theorem.

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