Cent. Eur. J. Phys.
DOI: 10.2478/s11534-013-0272-7

VERSITA

Central European Journal of **Physics**

Fundamental solutions to time-fractional heat conduction equations in two joint half-lines

Research Article

Yuriy Povstenko[1] *

1 *Institute of Mathematics and Computer Science,*
Jan Długosz University in Częstochowa,
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland

Received 16 January 2013; accepted 23 June 2013

Abstract: Heat conduction in two joint half-lines is considered under the condition of perfect contact, i.e. when the temperatures at the contact point and the heat fluxes through the contact point are the same for both regions. The heat conduction in one half-line is described by the equation with the Caputo time-fractional derivative of order α, whereas heat conduction in another half-line is described by the equation with the time derivative of order β. The fundamental solutions to the first and second Cauchy problems as well as to the source problem are obtained using the Laplace transform with respect to time and the cos-Fourier transform with respect to the spatial coordinate. The fundamental solutions are expressed in terms of the Mittag-Leffler function and the Mainardi function.

PACS (2008): 02.30.Gp, 44.10+i, 66.30.-h

Keywords: non-Fourier heat conduction • non-Fickian diffusion • fractional calculus • Mittag-Leffler function • Mainardi function

1. Introduction

The classical theory of heat conduction is based on the Fourier law

$$\mathbf{q} = -k \,\mathrm{grad}\, T, \tag{1}$$

where $\mathbf{q}$ is the heat flux vector, T denotes the temperature, and k is the thermal conductivity. In combination with the law of conservation of energy, the standard Fourier law results in the parabolic heat conduction equation.

*E-mail: j.povstenko@ajd.czest.pl

It is well known that from the mathematical viewpoint, the Fourier law in the theory of heat conduction and the Fick law in the theory of diffusion, are identical. In this paper we discuss heat conduction, but it is obvious that the discussion also concerns diffusion.

The time-nonlocal dependence between the heat flux vector and a temperature gradient with the "long-tail" power kernel [1–4] can be interpreted in terms of fractional calculus:

$$\mathbf{q}(t) = -k D_{RL}^{1-\alpha} \mathrm{grad}\, T(t), \qquad 0 < \alpha \leq 1, \tag{2}$$

$$\mathbf{q}(t) = -k I^{\alpha-1} \mathrm{grad}\, T(t), \qquad 1 < \alpha \leq 2. \tag{3}$$

Here $I^{\alpha} f(t)$ and $D_{RL}^{\alpha} f(t)$ are the Riemann–Liouville fractional integral and derivative of the order α, respectively, [5–8]:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad \alpha > 0, \tag{4}$$

$$D^\alpha_{RL} f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\,\mathrm{d}\tau\right], \quad m-1<\alpha<m, \tag{5}$$

where $\Gamma(\alpha)$ is the Gamma function.

It should be noted that in fractional calculus, where integrals and derivatives of arbitrary (not integer) order are considered, there is no sharp boundary between integration and differentiation. For this reason, some authors [7, 9] do not use a separate notation for the fractional integral $I^\alpha f(t)$. The fractional integral $I^\alpha f(t)$ of the order $\alpha > 0$ is denoted as $D^{-\alpha}_{RL} f(t)$. Using this notation, Eqs. (2) and (3) can be rewritten as one dependence

$$\mathbf{q}(t) = -k D^{1-\alpha}_{RL} \operatorname{grad} T(t), \qquad 0 < \alpha \le 2. \tag{6}$$

In combination with the law of concervation of energy, the constitutive equation (6) leads to the time fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\Delta T, \qquad 0<\alpha\le 2, \tag{7}$$

with the Caputo fractional derivative

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\mathrm{d}^m f(\tau)}{\mathrm{d}\tau^m}\,\mathrm{d}\tau, \quad m-1<\alpha<m. \tag{8}$$

The details of obtaining the time-fractional heat conduction equation (7) from the constitutive equation (6) can be found in [10]. In the case $0 < \alpha < 1$, the fractional heat conduction equation interpolates the elliptic Helmholtz equation ($\alpha \to 0$) and the parabolic heat conduction equation ($\alpha = 1$). When $1 < \alpha < 2$, the fractional heat conduction equation interpolates the standard heat conduction equation ($\alpha = 1$) and the hyperbolic wave equation ($\alpha = 2$).

Starting from the pioneering papers [11–15], considerable interest has been shown in solutions to Eq. (7). Different kinds of boundary conditions for time-fractional heat conduction equation were analysed in [16, 17]. If the surfaces of two solids are in perfect thermal contact, the temperatures on the contact surface and the heat fluxes through the contact surface are the same for both solids, and we obtain the boundary conditions of the fourth kind:

$$T_1\Big|_S = T_2\Big|_S, \tag{9}$$

$$k_1 D^{1-\alpha}_{RL} \frac{\partial T_1}{\partial n}\bigg|_S = k_2 D^{1-\beta}_{RL} \frac{\partial T_2}{\partial n}\bigg|_S, \qquad 0<\alpha\le 2, \quad 0<\beta\le 2, \tag{10}$$

where the subscripts 1 and 2 refer to solids 1 and 2, respectively, and n is the common normal at the contact surface.

To the best of our knowledge, no prior solutions of the fractional heat conduction (diffusion) equation in composite media have been obtained. In the previous paper [17], the problem of fractional heat conduction in two semi-infinite regions, $x > 0$ and $x < 0$, was considered. The heat conduction in the region $x > 0$ was described by the heat conduction equation with the Caputo time-fractional derivative of order α, whereas the heat conduction in the region $x < 0$ was described by the heat conduction equation with the derivative of order β. A particular case of initial condition was investigated where the region $x > 0$ was at initial uniform temperature T_0 and the region $x < 0$ was at initial zero temperature. In the present paper, the fundamental solutions to the first and second Cauchy problems as well as to the source problem are obtained using the Laplace transform with respect to time t and the cos-Fourier transform with respect to the spatial coordinate x. The fundamental solutions are expressed in terms of the Mittag-Leffler function and the Mainardi function.

2. Preliminaries

Recall the Laplace transform rules for fractional integrals and derivatives [6–8]:

$$\mathcal{L}\left\{I^\alpha f(t)\right\} = \frac{1}{s^\alpha} f^*(s), \tag{11}$$

$$\mathcal{L}\left\{\mathrm{D}^\alpha_{RL} f(t)\right\} = s^\alpha f^*(s) - \sum_{k=0}^{m-1} D^k I^{m-\alpha} f(0^+) s^{m-1-k}, \quad m-1<\alpha<m, \tag{12}$$

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha f^*(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad m-1<\alpha<m. \tag{13}$$

Here s is the Laplace transform variable, and the asterisk denotes the transform.

In what follows we shall use the cos-Fourier transforms (denoted by the tilde) for the region $x > 0$:

$$\mathcal{F}_c\left\{f(x)\right\} = \tilde{f}(\xi) = \int_0^\infty f(x)\,\cos(x\xi)\,\mathrm{d}x, \tag{14}$$

$$\mathcal{F}_c^{-1}\left\{\tilde{f}(\xi)\right\} = f(x) = \frac{2}{\pi}\int_0^\infty \tilde{f}(\xi)\,\cos(x\xi)\,\mathrm{d}\xi, \tag{15}$$

$$\mathcal{F}_c\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2\tilde{f}(\xi) - \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0^+} \tag{16}$$

and for the region $x < 0$:

$$\mathcal{F}_c\left\{f(x)\right\} = \tilde{f}(\xi) = \int_{-\infty}^{0} f(x)\,\cos(x\xi)\,\mathrm{d}x, \tag{17}$$

$$\mathcal{F}_c^{-1}\left\{\tilde{f}(\xi)\right\} = f(x) = \frac{2}{\pi}\int_{-\infty}^{0} \tilde{f}(\xi)\,\cos(x\xi)\,\mathrm{d}\xi, \tag{18}$$

$$\mathcal{F}_c\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2\tilde{f}(\xi) + \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0^-}. \tag{19}$$

The Mittag-Leffler function in one parameter α [6–8]:

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha > 0, \quad z \in C, \tag{20}$$

provides a generalization of the exponential function.
The Mittag-Leffler type function in two parameters α and β [6–8] is described by the following series representation:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \;\; \beta > 0, \;\; z \in C. \tag{21}$$

The essential role of the Mittag-Leffler functions in fractional calculus results from the formula for the inverse Laplace transform [7]:

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha + b}\right\} = t^{\beta-1}\,E_{\alpha,\beta}(-bt^\alpha). \tag{22}$$

The Wright function is defined as [7, 8, 14, 15, 18]

$$W(\alpha,\beta;z) = \sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(\alpha k+\beta)}, \quad \alpha > -1, \quad z \in C. \tag{23}$$

The Wright function is a generalization of the exponential function and the Bessel functions.
The Mainardi function [7, 14, 15] $M(\alpha; z)$ is a particular case of the Wright function:

$$M(\alpha;z) = W(-\alpha, 1-\alpha; -z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!\,\Gamma[-\alpha k + (1-\alpha)]}, \quad 0 < \alpha < 1, \quad z \in C. \tag{24}$$

The Mainardi and Wright functions appear in the formulae for the inverse Laplace transform (see [14, 15, 19–23]):

$$\mathcal{L}^{-1}\left\{\exp\left(-\lambda s^\alpha\right)\right\} = \frac{\alpha\lambda}{t^{\alpha+1}}M\left(\alpha;\lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0, \tag{25}$$

$$\mathcal{L}^{-1}\left\{s^{\alpha-1}\exp\left(-\lambda s^\alpha\right)\right\} = t^{-\alpha}M\left(\alpha;\lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0, \tag{26}$$

$$\mathcal{L}^{-1}\left\{s^{-\beta}\exp\left(-\lambda s^\alpha\right)\right\} = t^{\beta-1}W\left(-\alpha,\beta;-\lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0. \tag{27}$$

The Mittag-Leffler function and the Mainardi function are related by the pair of the cos-Fourier transform:

$$\mathcal{F}_c\left\{M\left(\frac{\alpha}{2};x\right)\right\} = E_\alpha\left(-\xi^2\right), \quad 0 < \alpha < 2, \tag{28}$$

$$\mathcal{F}_c^{-1}\left\{E_\alpha\left(-\xi^2\right)\right\} = M\left(\frac{\alpha}{2};x\right), \quad 0 < \alpha < 2. \tag{29}$$

Similarly,

$$\mathcal{F}_c\left\{W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -x\right)\right\} = E_{\alpha,2}\left(-\xi^2\right), \quad 0 < \alpha < 2, \tag{30}$$

$$\mathcal{F}_c^{-1}\left\{E_{\alpha,2}\left(-\xi^2\right)\right\} = W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -x\right), \quad 0 < \alpha < 2, \tag{31}$$

and

$$\mathcal{F}_c\left\{W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -x\right)\right\} = E_{\alpha,\alpha}\left(-\xi^2\right), \quad 0 < \alpha < 2, \tag{32}$$

$$\mathcal{F}_c^{-1}\left\{E_{\alpha,\alpha}\left(-\xi^2\right)\right\} = W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -x\right), \quad 0 < \alpha < 2. \tag{33}$$

3. Statement of the problem

The general mathematical formulation of the problem is stated as follows: to solve the time-fractional heat conduction equations:

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1\frac{\partial^2 T_1}{\partial x^2} + Q_1(x,t), \quad x > 0, \;\; t > 0, \quad 0 < \alpha \le 2, \tag{34}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2\frac{\partial^2 T_2}{\partial x^2} + Q_2(x,t), \quad x < 0, \;\; t > 0, \quad 0 < \beta \le 2, \tag{35}$$

under the initial conditions:

$$t = 0: \quad T_1 = f_1(x), \quad x > 0, \quad 0 < \alpha \le 2, \tag{36}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = F_1(x), \quad x > 0, \quad 1 < \alpha \le 2, \tag{37}$$

$$t = 0: \quad T_2 = f_2(x), \quad x < 0, \quad 0 < \beta \le 2, \tag{38}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = F_2(x), \quad x < 0, \quad 1 < \beta \le 2, \tag{39}$$

and the boundary conditions of perfect thermal contact

$$\left.T_1(x,t)\right|_{x=0^+} = \left.T_2(x,t)\right|_{x=0^-}, \quad t > 0, \tag{40}$$

$$k_1 D_{RL}^{1-\alpha}\frac{\partial T_1(x,t)}{\partial x}\bigg|_{x=0^+} = k_2 D_{RL}^{1-\beta}\frac{\partial T_2(x,t)}{\partial x}\bigg|_{x=0^-}, \quad t>0, \quad 0<\alpha\le 2, 0<\beta\le 2, \tag{41}$$

which state that two bodies in contact must have the same temperature at the contact point and the heat fluxes through the contact point must be the same. It should be emphasized that the equation for the heat flux (6) is formulated in terms of the Riemann-Liouville derivative, but such a constitutive equation results in the heat conduction equation with the Caputo derivative. For this reason, in Eqs. (42) and (43) there appear the fractional Caputo derivatives, but in the condition of perfect thermal contact (41) we have the Riemann-Liouville fractional derivatives (see also [10]).

3.1. The fundamental solution to the first Cauchy problem

In this case the following initial-boundary-value problem is solved:

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1\frac{\partial^2 T_1}{\partial x^2}, \quad x>0, \quad t>0, \quad 0<\alpha\le 2, \tag{42}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2\frac{\partial^2 T_2}{\partial x^2}, \quad x<0, \quad t>0, \quad 0<\beta\le 2, \tag{43}$$

under the initial conditions

$$t=0: \quad T_1 = p_0\delta(x-\rho), \quad x>0, \quad 0<\alpha\le 2, \tag{44}$$

$$t=0: \quad \frac{\partial T_1}{\partial t} = 0, \quad x>0, \quad 1<\alpha\le 2, \tag{45}$$

$$t=0: \quad T_2 = 0, \quad x<0, \quad 0<\beta\le 2, \tag{46}$$

$$t=0: \quad \frac{\partial T_2}{\partial t} = 0, \quad x<0, \quad 1<\beta\le 2. \tag{47}$$

For the sake of convenience and to obtain the nondimensional quantities used in the calculations we have introduced the constant multiplier p_0 in Eq. (44).

The boundary condition of perfect thermal contact (41) is rewritten as

$$k_1 D_{RL}^{1-\alpha}\frac{\partial T_1(x,t)}{\partial x}\bigg|_{x=0^+} = \varphi(t), \quad t>0, \quad 0<\alpha\le 2, \tag{48}$$

$$k_2 D_{RL}^{1-\beta}\frac{\partial T_2(x,t)}{\partial x}\bigg|_{x=0^-} = \varphi(t), \quad t>0, \quad 0<\beta\le 2, \tag{49}$$

where $\varphi(t)$ is the unknown function which should be found from the condition (40) (see below).

The Laplace transform with respect to time t (for simplicity neglecting the initial value of the temperature gradient) and the cos-Fourier transforms (14) and (17) with respect to the spatial coordinates $x>0$ and $x<0$ give

$$\tilde{T}_1^*(\xi,s) = \frac{s^{\alpha-1}}{s^\alpha + a_1\xi^2}\left[p_0\cos(\rho\xi) - \frac{a_1}{k_1}\varphi^*(s)\right], \tag{50}$$

$$\tilde{T}_2^*(\xi,s) = \frac{a_2}{k_2}\frac{s^{\beta-1}}{s^\beta + a_2\xi^2}\varphi^*(s). \tag{51}$$

Inversion of the cos-Fourier transform, taking into account that [24]

$$\int_0^\infty \frac{\cos(x\xi)}{\xi^2+c^2}\,\mathrm{d}\xi = \frac{\pi}{2c}\,\mathrm{e}^{-c|x|}, \quad c>0, \tag{52}$$

results in:

$$T_1^*(x,s) = \frac{p_0}{2\sqrt{a_1}}s^{\alpha/2-1}\left[\exp\left(-\frac{x+\rho}{\sqrt{a_1}}s^{\alpha/2}\right) + \exp\left(-\frac{|x-\rho|}{\sqrt{a_1}}s^{\alpha/2}\right)\right] - \frac{\sqrt{a_1}}{k_1}\varphi^*(s)\,s^{\alpha/2-1}\exp\left(-\frac{x}{\sqrt{a_1}}s^{\alpha/2}\right), \quad x\ge 0, \tag{53}$$

$$T_2^*(x,s) = \frac{\sqrt{a_2}}{k_2}\varphi^*(s)\,s^{\beta/2-1}\exp\left(-\frac{|x|}{\sqrt{a_2}}s^{\beta/2}\right), \quad x\le 0. \tag{54}$$

The requirement that the temperatures at the two sides of contact are the same ($T_1^*(0,s) = T_2^*(0,s)$) allows us to find the function $\varphi^*(s)$:

$$\varphi^*(s) = \frac{p_0 k_1 k_2}{\sqrt{a_1}}\frac{s^{\alpha/2}}{k_2\sqrt{a_1}s^{\alpha/2} + k_1\sqrt{a_2}s^{\beta/2}}\exp\left(-\frac{\rho}{\sqrt{a_1}}s^{\alpha/2}\right). \tag{55}$$

Invertion of the Laplace transform in (55) depends on relation between the orders α and β. For $\alpha<\beta$ we have

$$\varphi(t) = \frac{\alpha\rho p_0 k_2}{2a_1\sqrt{a_2}} \int_0^t \frac{(t-\tau)^{\beta/2-\alpha/2-1}}{\tau^{1+\alpha/2}}\, M\left(\frac{\alpha}{2}; \frac{\rho}{\sqrt{a_1}\tau^{\alpha/2}}\right) E_{\beta/2-\alpha/2,\,\beta/2-\alpha/2}\left[-\frac{(t-\tau)^{\beta/2-\alpha/2}}{\gamma}\right] \mathrm{d}\tau, \tag{56}$$

where

$$\gamma = \frac{k_1\sqrt{a_2}}{k_2\sqrt{a_1}}.$$

For $\alpha > \beta$ we obtain

$$\begin{aligned}\varphi(t) \;&= \frac{\alpha\rho p_0 k_1}{2a_1^{3/2}}\left[\frac{1}{t^{1+\alpha/2}} M\left(\frac{\alpha}{2}; \frac{\rho}{\sqrt{a_1}t^{\alpha/2}}\right) - \gamma \int_0^t \frac{(t-\tau)^{\alpha/2-\beta/2-1}}{\tau^{1+\alpha/2}}\, M\left(\frac{\alpha}{2}; \frac{\rho}{\sqrt{a_1}\tau^{\alpha/2}}\right)\right. \\ &\quad \left.\times\, E_{\alpha/2-\beta/2,\alpha/2-\beta/2}\left[-\gamma(t-\tau)^{\alpha/2-\beta/2}\right] \mathrm{d}\tau\right].\end{aligned} \tag{57}$$

Inversion of the Laplace transform, taking into account (26), produces:

$$T_1(x,t) = \frac{p_0}{2\sqrt{a_1}t^{\alpha/2}}\left[M\left(\frac{\alpha}{2}; \frac{x+\rho}{\sqrt{a_1}t^{\alpha/2}}\right) + M\left(\frac{\alpha}{2}; \frac{|x-\rho|}{\sqrt{a_1}t^{\alpha/2}}\right)\right] - \frac{\sqrt{a_1}}{k_1}\int_0^t \frac{\varphi(t-\tau)}{\tau^{\alpha/2}}\, M\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a_1}\tau^{\alpha/2}}\right) \mathrm{d}\tau, \quad x \geq 0, \tag{58}$$

$$T_2(x,t) = \frac{\sqrt{a_2}}{k_2}\int_0^t \frac{\varphi(t-\tau)}{\tau^{\beta/2}}\, M\left(\frac{\beta}{2}; \frac{|x|}{\sqrt{a_2}\tau^{\beta/2}}\right) \mathrm{d}\tau, \quad x \leq 0. \tag{59}$$

Let us consider several particular cases of the obtained solution. For $\alpha = \beta$ we have

$$T_1(x,t) = \frac{p_0}{2\sqrt{a_1}t^{\alpha/2}}\left[M\left(\frac{\alpha}{2}; \frac{|x-\rho|}{\sqrt{a_1}t^{\alpha/2}}\right) + \frac{\gamma-1}{\gamma+1} M\left(\frac{\alpha}{2}; \frac{x+\rho}{\sqrt{a_1}t^{\alpha/2}}\right)\right], \quad x \geq 0, \tag{60}$$

$$T_2(x,t) = \frac{p_0\gamma}{(\gamma+1)\sqrt{a_1}t^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{|x|}{\sqrt{a_2}t^{\alpha/2}} + \frac{\rho}{\sqrt{a_1}t^{\alpha/2}}\right), \quad x \leq 0; \tag{61}$$

in particular, for $\alpha = \beta = 2$:

$$T_1(x,t) = \frac{p_0}{2}\left[\delta(x-\rho+\sqrt{a_1}t) + \delta(x-\rho-\sqrt{a_1}t) + \frac{\gamma-1}{\gamma+1}\,\delta(x+\rho-\sqrt{a_1}t)\right], \quad x \geq 0, \tag{62}$$

$$T_2(x,t) = \frac{p_0\gamma}{\gamma+1}\,\delta\left(\frac{\sqrt{a_1}}{\sqrt{a_2}}|x| + \rho - \sqrt{a_1}t\right), \quad x \leq 0. \tag{63}$$

For $\alpha = 1$, $\beta = 2$, we get:

$$\begin{aligned}T_1(x,t) &= \frac{p_0}{2\sqrt{\pi a_1 t}}\left\{\exp\left[-\frac{(x+\rho)^2}{4a_1 t}\right] + \exp\left[-\frac{(x-\rho)^2}{4a_1 t}\right]\right\} \\ &\quad - \frac{p_0}{\gamma\sqrt{a_1}} \exp\left(\frac{x+\rho}{\gamma\sqrt{a_1}} + \frac{t}{\gamma^2}\right) \operatorname{erfc}\left(\frac{x+\rho}{2\sqrt{a_1 t}} + \frac{\sqrt{t}}{\gamma}\right), \quad x \geq 0,\end{aligned} \tag{64}$$

$$T_2(x,t) = \begin{cases} \dfrac{p_0}{\sqrt{a_1}}\left\{-\dfrac{1}{\gamma}\exp\left(\dfrac{\rho}{\sqrt{a_1}\gamma} + \dfrac{t+x/\sqrt{a_2}}{\gamma^2}\right) \operatorname{erfc}\left[\dfrac{\rho}{2\sqrt{a_1(t+x/\sqrt{a_2})}} + \dfrac{\sqrt{t+x/\sqrt{a_2}}}{\gamma}\right] \right. \\ \left. + \dfrac{1}{\sqrt{\pi(t+x/\sqrt{a_2})}} \exp\left[-\dfrac{\rho^2}{4a_1(t+x/\sqrt{a_2})}\right]\right\}, & -\sqrt{a_2}t < x \leq 0, \\ 0, & -\infty < x < -\sqrt{a_2}t, \end{cases} \tag{65}$$

where erfc (x) is the complementary error function.
When $\alpha = 2$, $\beta = 1$, we arrive at:

$$T_1(x,t) = \frac{p_0}{2}[\delta(x-\rho-\sqrt{a_1}t)+\delta(x-\rho+\sqrt{a_1}t)-\delta(x+\rho-\sqrt{a_1}t)$$
$$+\begin{cases} \dfrac{p_0\gamma}{\sqrt{a_1}}\left\{\dfrac{1}{\sqrt{\pi[t-(x+\rho)/\sqrt{a_1}]}}\right. \\ \left. -\gamma\exp\left[\gamma^2\left(t-\dfrac{x+\rho}{\sqrt{a_1}}\right)\right]\operatorname{erfc}\left(\gamma\sqrt{t-\dfrac{x+\rho}{\sqrt{a_1}}}\right)\right\}, & 0 \le x < \sqrt{a_1}t-\rho, \\ 0, & \sqrt{a_1}t-\rho < x < \infty, \end{cases} \tag{66}$$

$$T_2(x,t) = \begin{cases} \dfrac{p_0\gamma}{\sqrt{a_1}}\left\{\dfrac{1}{\sqrt{\pi(t-\rho/\sqrt{a_1})}}\exp\left[-\dfrac{x^2}{4a_2(t-\rho/\sqrt{a_1})}\right]\right. \\ \left. -\gamma\exp\left[\dfrac{\gamma|x|}{\sqrt{a_2}}+\gamma^2\left(t-\dfrac{\rho}{\sqrt{a_1}}\right)\right]\operatorname{erfc}\left[\dfrac{|x|}{2\sqrt{a_2(t-\rho/\sqrt{a_1})}}+\gamma\sqrt{t-\dfrac{\rho}{\sqrt{a_1}}}\right]\right\}, & \sqrt{a_1}t>\rho, \\ 0, & \sqrt{a_1}t<\rho. \end{cases} \tag{67}$$

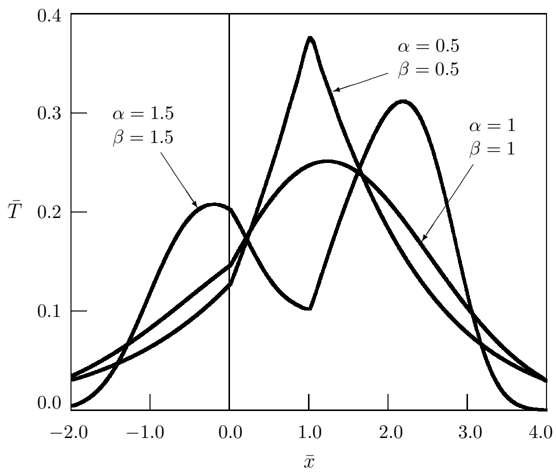

Figure 1. Dependence of the fundamental solution to the first Cauchy problem on distance; $\kappa = 1$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$.

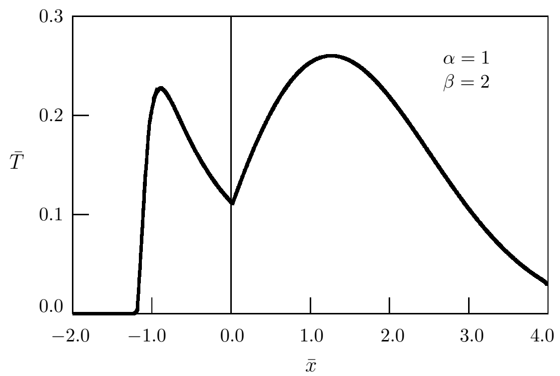

Figure 2. Dependence of the fundamental solution to the first Cauchy problem on distance; $\kappa = 1$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$.

The results of numerical calculations of the fundamental solution to the first Cauchy problem are shown in Figs. 1–3. We have introduced the following nondimensional quantities:

$$\bar{x} = \frac{x}{\rho}, \qquad \kappa = \frac{\sqrt{a_1}t^{\alpha/2}}{\rho}, \qquad \bar{\gamma} = \gamma t^{\alpha/2-\beta/2},$$
$$\epsilon = \frac{\sqrt{a_1}}{\sqrt{a_2}}t^{\alpha/2-\beta/2}, \qquad \bar{T} = \frac{\rho}{p_0}T. \tag{68}$$

3.2. The fundamental solution to the second Cauchy problem

Now we solve the following initial-boundary-value problem:

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1\frac{\partial^2 T_1}{\partial x^2}, \quad x>0, \quad t>0, \quad 1<\alpha\le 2, \tag{69}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2\frac{\partial^2 T_2}{\partial x^2}, \quad x<0, \quad t>0, \quad 0<\beta\le 2, \tag{70}$$

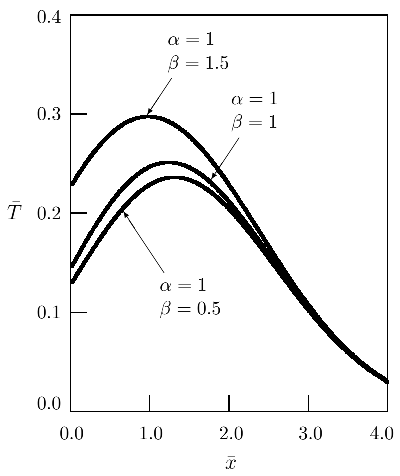

Figure 3. Dependence of the fundamental solution to the first Cauchy problem on distance; $\kappa = 1$, $\bar{\gamma} = 0.5$.

under the initial conditions:

$$t = 0: \quad T_1 = 0, \quad x > 0, \quad 1 < \alpha \le 2, \tag{71}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = w_0 \delta(x - \rho), \quad x > 0, \quad 1 < \alpha \le 2, \tag{72}$$

$$t = 0: \quad T_2 = 0, \quad x < 0, \quad 0 < \beta \le 2, \tag{73}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = 0, \quad x < 0 \quad 1 < \beta \le 2, \tag{74}$$

and the conditions of perfect contact (40) and (41) stating the equality of temperatures and fluxes at the contact point.

It should be emphasized that the second Cauchy problem for the fractional heat conduction equation (69) in the domain $x > 0$ is formulated for $1 < \alpha \le 2$, whereas in the general case heat conduction in the domain $x < 0$ can occur not only for $1 < \beta \le 2$, but also for $0 < \beta \le 1$.

The solution is obtained in a similar manner and reads:

$$T_1(x,t) = \frac{w_0}{2\sqrt{a_1}\, t^{\alpha/2-1}} \left[W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{x+\rho}{\sqrt{a_1}\, t^{\alpha/2}}\right) + W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{|x-\rho|}{\sqrt{a_1}\, t^{\alpha/2}}\right)\right]$$
$$- \frac{\sqrt{a_1}}{k_1} \int_0^t \frac{\varphi(t-\tau)}{\tau^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a_1}\,\tau^{\alpha/2}}\right) \mathrm{d}\tau, \quad x \ge 0, \tag{75}$$

$$T_2(x,t) = \frac{\sqrt{a_2}}{k_2} \int_0^t \frac{\varphi(t-\tau)}{\tau^{\beta/2}} M\left(\frac{\beta}{2}; \frac{|x|}{\sqrt{a_2}\,\tau^{\beta/2}}\right) \mathrm{d}\tau, \quad x \le 0, \tag{76}$$

where

$$\varphi(t) = \frac{\alpha \rho w_0 k_1}{2 a_1^{3/2}} \int_0^t \frac{1}{\tau^{1+\alpha/2}} M\left(\frac{\alpha}{2}; \frac{\rho}{\sqrt{a_1}\,\tau^{\alpha/2}}\right) \left\{1 - E_{\beta/2-\alpha/2}\left[-\frac{(t-\tau)^{\beta/2-\alpha/2}}{\gamma}\right]\right\} \mathrm{d}\tau \tag{77}$$

for $\alpha < \beta$ and

$$\varphi(t) = \frac{\alpha w_0 k_1 \rho}{2 a_1^{3/2}} \int_0^t \frac{1}{\tau^{1+\alpha/2}} M\left(\frac{\alpha}{2}; \frac{\rho}{\sqrt{a_1}\,\tau^{\alpha/2}}\right) E_{\alpha/2-\beta/2}\left[-\gamma (t-\tau)^{\alpha/2-\beta/2}\right] \mathrm{d}\tau \tag{78}$$

for $\alpha > \beta$.

In particular, if $\alpha = \beta$, then:

$$T_1(x,t) = \frac{w_0}{2\sqrt{a_1}\, t^{\alpha/2-1}} \left[W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{|x-\rho|}{\sqrt{a_1}\, t^{\alpha/2}}\right) + \frac{\gamma-1}{\gamma+1} W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{x+\rho}{\sqrt{a_1}\, t^{\alpha/2}}\right)\right], \quad x \ge 0, \tag{79}$$

$$T_2(x,t) = \frac{w_0 \gamma}{(\gamma+1)\sqrt{a_1}\, t^{\alpha/2-1}} W\left[-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\left(\frac{|x|}{\sqrt{a_2}\, t^{\alpha/2}} + \frac{\rho}{\sqrt{a_1}\, t^{\alpha/2}}\right)\right], \quad x \le 0. \tag{80}$$

If $\alpha = \beta = 2$, then:

$$T_1(x,t) = \frac{w_0}{4\sqrt{a_1}}\left\{\operatorname{sgn}\left(x-\rho+\sqrt{a_1}t\right) - \operatorname{sgn}\left(x-\rho-\sqrt{a_1}t\right) + \frac{\gamma-1}{\gamma+1}\left[1-\operatorname{sgn}\left(x+\rho-\sqrt{a_1}t\right)\right]\right\}, \quad x \geq 0, \tag{81}$$

$$T_2(x,t) = \frac{\gamma w_0}{2(\gamma+1)\sqrt{a_1}}\left[1-\operatorname{sgn}\left(\frac{\sqrt{a_1}}{\sqrt{a_2}}|x| + \rho - \sqrt{a_1}t\right)\right], \quad x \leq 0. \tag{82}$$

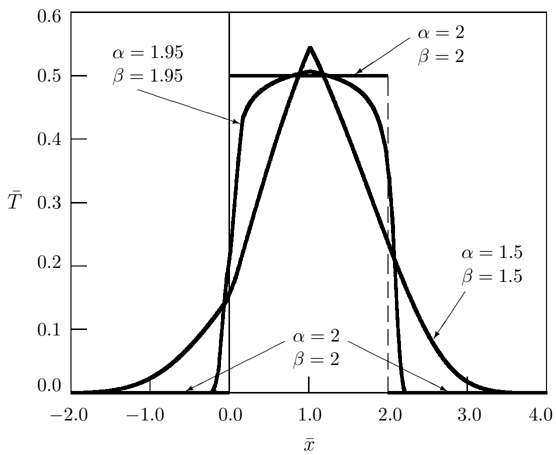

Figure 4. Dependence of the fundamental solution to the second Cauchy problem on distance; $\kappa = 1$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$.

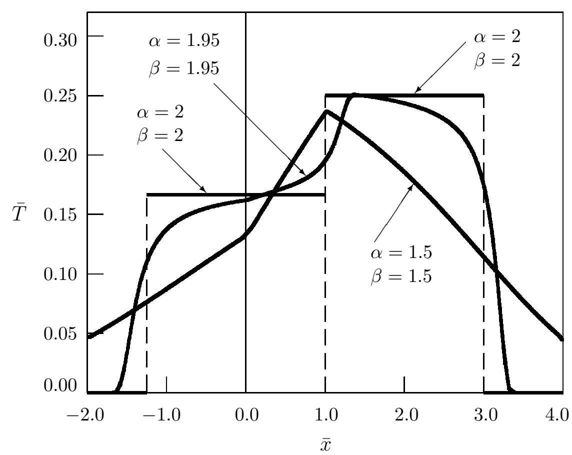

Figure 5. Dependence of the fundamental solution to the second Cauchy problem on distance; $\kappa = 2$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$.

Figures 4 and 5 present the dependence of the fundamental solution to the second Cauchy problem on distance. In this case $\bar{T} = \rho T/(w_0 t)$, and the other nondimensional quantities are the same as in (68).

3.3. The fundamental solution to the source problem

Consider the time-fractional heat conduction equation with the source term:

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2} + q_0 \delta(x-\rho)\,\delta(t), \quad x > 0, \quad t > 0, \quad 0 < \alpha \leq 2, \tag{83}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2}, \quad x < 0, \quad t > 0, \quad 0 < \beta \leq 2, \tag{84}$$

under the zero initial conditions:

$$t = 0: \quad T_1 = 0, \quad x > 0, \quad 0 < \alpha \leq 2, \tag{85}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = 0, \quad x > 0, \quad 1 < \alpha \leq 2, \tag{86}$$

$$t = 0: \quad T_2 = 0, \quad x < 0, \quad 0 < \beta \leq 2, \tag{87}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = 0, \quad x < 0 \quad 1 < \beta \leq 2, \tag{88}$$

and the conditions of perfect contact (40) and (41) stating that at the contact point $x = 0$ the temperatures are equal and the heat fluxes are the same.

The solution has the following form:

$$T_1(x,t) = \frac{q_0 t^{\alpha/2-1}}{2\sqrt{a_1}} \left[W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{x+\rho}{\sqrt{a_1}t^{\alpha/2}}\right) + W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{|x-\rho|}{\sqrt{a_1}t^{\alpha/2}}\right)\right]$$
$$-\frac{\sqrt{a_1}}{k_1}\int_0^t \frac{\varphi(t-\tau)}{\tau^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a_1}\tau^{\alpha/2}}\right) \mathrm{d}\tau, \quad x \geq 0, \tag{89}$$

$$T_2(x,t) = \frac{\sqrt{a_2}}{k_2}\int_0^t \frac{\varphi(t-\tau)}{\tau^{\beta/2}} M\left(\frac{\beta}{2}; \frac{|x|}{\sqrt{a_2}\tau^{\beta/2}}\right) \mathrm{d}\tau, \quad x \leq 0, \tag{90}$$

where

$$\varphi(t) = \frac{q_0 k_2}{\sqrt{a_1 a_2}} \int_0^t \frac{(t-\tau)^{\beta/2-\alpha/2-1}}{\tau^{2-\alpha}} W\left(-\frac{\alpha}{2}, \alpha-1; -\frac{\rho}{\sqrt{a_1}\tau^{\alpha/2}}\right) E_{\beta/2-\alpha/2,\,\beta/2-\alpha/2}\left[-\frac{(t-\tau)^{\beta/2-\alpha/2}}{\gamma}\right] \mathrm{d}\tau \tag{91}$$

for $\alpha < \beta$ and

$$\varphi(t) = \frac{q_0 k_1}{a_1} \int_0^t \frac{(t-\tau)^{\alpha/2-\beta/2-1}}{\tau^{2-\alpha/2-\beta/2}} W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}+\frac{\beta}{2}-1; -\frac{\rho}{\sqrt{a_1}\tau^{\alpha/2}}\right) E_{\alpha/2-\beta/2,\,\alpha/2-\beta/2}\left[-\gamma(t-\tau)^{\alpha/2-\beta/2}\right] \mathrm{d}\tau \tag{92}$$

for $\alpha > \beta$.

Consider several particular cases of the solution. For $\alpha = \beta$:

$$T_1(x,t) = \frac{q_0 t^{\alpha/2-1}}{2\sqrt{a_1}} \left[W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{|x-\rho|}{\sqrt{a_1}t^{\alpha/2}}\right) + \frac{\gamma-1}{\gamma+1} W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{x+\rho}{\sqrt{a_1}t^{\alpha/2}}\right)\right], \quad x \geq 0, \tag{93}$$

$$T_2(x,t) = \frac{\gamma q_0 t^{\alpha/2-1}}{(\gamma+1)\sqrt{a_1}} W\left[-\frac{\alpha}{2}, \frac{\alpha}{2}; -\left(\frac{|x|}{\sqrt{a_2}t^{\alpha/2}} + \frac{\rho}{\sqrt{a_1}t^{\alpha/2}}\right)\right], \quad x \leq 0. \tag{94}$$

It is evident that in the case $\alpha = \beta = 2$ the solutions to the second Cauchy problem and to the source problem coincide and are described by (81) and (82).

For $\alpha = 1$ and $\beta = 2$ the solution to the source problem coincides with the corresponding solution (64), (65) to the first Cauchy problem.

In the case $\alpha = 2$, $\beta = 1$, we get:

$$T_1 = \frac{q_0}{4\sqrt{a_1}}\left[1 - \mathrm{sgn}\,(x - \sqrt{a_1}t + \rho) - \mathrm{sgn}\,(x - \sqrt{a_1}t - \rho) + \mathrm{sgn}\,(x + \sqrt{a_1}t - \rho)\right]$$
$$-\frac{q_0}{2\sqrt{a_1}} \exp\left[\gamma^2\left(t - \frac{x+\rho}{\sqrt{a_1}}\right)\right] \mathrm{erfc}\left(\gamma\sqrt{t - \frac{x+\rho}{\sqrt{a_1}}}\right)\left[1 - \mathrm{sgn}\,(x - \sqrt{a_1}t + \rho)\right], \quad x \geq 0, \tag{95}$$

$$T_2(x,t) = \begin{cases} \dfrac{q_0}{\sqrt{a_1}}\left\{ -\exp\left[\dfrac{\gamma|x|}{\sqrt{a_2}} + \gamma^2\left(t - \dfrac{\rho}{\sqrt{a_1}}\right)\right] \mathrm{erfc}\left[\dfrac{|x|}{2\sqrt{a_2(t-\rho/\sqrt{a_1})}} + \gamma\sqrt{t - \dfrac{\rho}{\sqrt{a_1}}}\right]\right. \\ \left. + \mathrm{erfc}\left[\dfrac{|x|}{2\sqrt{a_2(t-\rho/\sqrt{a_1})}}\right]\right\}, & t > \rho/\sqrt{a_1}, \quad x \leq 0, \\ 0, & t < \rho/\sqrt{a_1}, \quad x \leq 0. \end{cases} \tag{96}$$

Dependence of the fundamental solution to the source problem $\bar{T} = \rho t^{1-\alpha} T/q_0$ on distance $\bar{x} = x/\rho$ is depicted

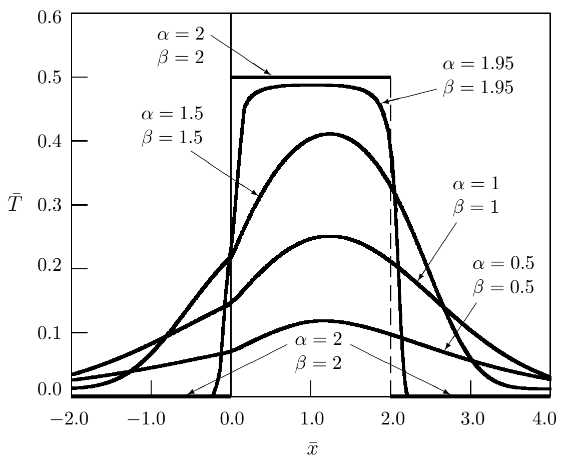

Figure 6. Dependence of the fundamental solution to the source problem on distance; $\kappa = 1$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$.

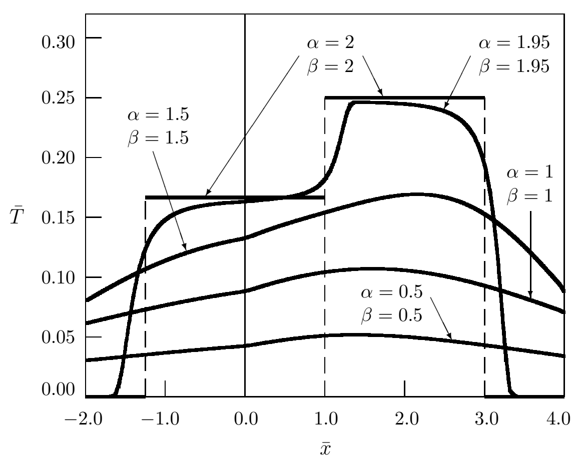

Figure 7. Dependence of the fundamental solution to the source problem on distance; $\kappa = 2$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$.

in Figs. 6 and 7.

4. Conclusion

We have investigated the fundamental solutions of the time-fractional heat conduction equations with different orders of time derivatives in a composite medium consisting of two regions in perfect thermal contact. The Laplace transform with respect to time and the cos-Fourier transform with respect to the spatial coordinate have been used. The fundamental solutions are expressed in terms of the function $\varphi(t)$ describing the heat flux at the boundary. The function $\varphi(t)$ has been found from the condition that the temperatures at the contact point are the same for both solids (the condition of perfect thermal contact). We have considered the fundamental solution in the case when delta function terms appear in the region $x > 0$. Solutions for problems with delta function terms in the region $x < 0$ can be obtained in similar manner (by changing α and β and the indices 1 and 2). The parameter κ describes the non-dimensional time. In the case of the wave equation ($\alpha = 2$) the values $0 < \kappa < 1$, $\kappa = 1$ and $1 < \kappa$ correspond to three characteristic events: the wave front has not yet arrived at the contact point, the wave front has arrived at the contact point, and the wave front has reflected from the contact point for $x > 0$ and has transmitted into the domain $x < 0$. For the same orders of time-derivative ($\alpha = \beta$), it is worth comparing the results of the present paper and the corresponding results for the uniform line (see Fig. 1 from [1] and Fig. 1 from [25]). The results are similar, but for two joint half-lines the corresponding curves have jogs at the contact point $x = 0$. When $\alpha = \beta = 2$, the fundamental solutions to the second Cauchy problem and to the source problem coincide, but with $1 < \alpha < 2$ increasing and approaching 2 approximation of this solution occurs in quite different ways. It should be emphasized that in the case of the second Cauchy problem $1 < \alpha \leq 2$, whereas in the case of the source problem $0 < \alpha \leq 2$ (compare Figs. (4) and (6) and Figs. (5) and (7), respectively).

References

[1] Y. Z. Povstenko, J. Therm. Stresses 28, 83 (2005)
[2] Y. Povstenko, Math. Meth. Phys.-Mech. Fields 51, 239 (2008)
[3] Y. Povstenko, J. Math. Sci. 162, 296 (2009)
[4] Y. Povstenko, Phys. Scr. T 136, 014017 (2009)
[5] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives, Theory and Applications (Gordon and Breach, Amsterdam, 1993)
[6] R. Gorenflo, F. Mainardi, In: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics (Springer-Verlag, New York, 1997) 223
[7] I. Podlubny, Fractional Differential Equations (Academic Press, San Diego, 1999)
[8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations (Elsevier, Amsterdam, 2006)
[9] R. L. Magin, Fractional Calculus in Bioengineering (Begell House Publishers, Inc, Connecticut, 2006)
[10] Y. Povstenko, Fract. Calc. Appl. Anal. 14, 418 (2011)
[11] W. Wyss, J. Math. Phys. 27, 2782 (1986)

[12] W. R. Schneider, W. Wyss, J. Math. Phys. 30, 134 (1989)

[13] Y. Fujita, Osaka J. Math. 27, 309 (1990)

[14] F. Mainardi, Appl. Math. Lett. 9, 23 (1996)

[15] F. Mainardi, Chaos Soliton. Fract. 7, 1461 (1996)

[16] Y. Povstenko, In: I. Petraš, I. Podlubny, K. Kostúr, J. Kačúr, A. Mojžišová (Eds.), 13th International Carpathian Control Conference, May 28–31, 2012, Podbanské, Slovak Republic (Institute of Electrical and Electronics Engineers (IEEE), 2012)

[17] Y. Povstenko, J. Therm. Stresses 36, 351 (2013)

[18] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, Higher Transcendental Functions, vol. 3 (McGraw-Hill, New York, 1955)

[19] J. Mikusiński, Stud. Math. 12, 208 (1951)

[20] J. Mikusiński, Stud. Math. 18, 191 (1959)

[21] B. Stanković, Publ. Inst. Math. 10 (24), 113 (1970)

[22] L. Gajić, B. Stanković, Publ. Inst. Math. 20 (34), 91 (1976)

[23] R. Gorenflo, Y. Luchko, F. Mainardi, Fract. Calc. Appl. Anal. 2, 383 (1999)

[24] I. S. Gradshtein, I. M. Ryzhik, Tables of Integrals, Series and Products (Academic Press, New York, 1980)

[25] Y. Povstenko, Int. J. Engrg Sci. 43, 977 (2005)