FUNDAMENTAL STRUCTURAL-ACOUSTIC IDEALIZATIONS FOR STRUCTURES WITH FUZZY INTERNALS

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ABSTRACT

Fundamental issues relative to structural vibration and to scattering of sound from structures with imprecisely known internals are explored, with the master structure taken as a rectangular plate in a rigid baffle, which faces an unbounded fluid medium on the external side. On the internal side is a fuzzy structure, consisting of a random array of point-attached spring-mass systems. The theory predicts that the fuzzy internal structure can be approximated by a statistical average in which the only relevant property is a function $\bar{m}_F(\Omega)$ which gives a smoothed-out total mass, per unit plate area, of all those attached oscillators which have their natural frequencies less than a given value Ω. The theory also predicts that the exact value of the damping in the fuzzy structure is of little importance, because the structure, even in the limit of zero damping, actually absorbs energy with an apparent frequency-dependent damping constant proportional to $d\bar{m}_F(\omega)/d\omega$ incorporated into the dynamical description of the master structure. A small finite value of damping within the internals will cause little appreciable change to this limiting value.

I. INTRODUCTION

Various large man-made structures of practical interest consist of an outer metallic shell with a somewhat complicated internal structure. The outer shell is reasonably easy to describe and to model in regard to dynamic and structural acoustic behavior. The interior structure, in contrast, presents formidable difficulties, even to one who has access to detailed blueprints. In particular, one has relatively little hope of knowing at the outset all of the parameters that would be appropriate to fully account for the dissipation of vibratory energy within the structure.

Insofar as one is mainly interested only in larger scale vibrations of the outer shell of the structure and in somewhat gross predictions of the radiation of sound by the structure and of how the overall structure scatters external sound fields, there is some hope that a detailed knowledge and modeling of the internal structure is not really needed; some rough quantitative descriptors may suffice. One such descriptor is certainly the total internal mass, but it is evident that quantity alone would be insufficient for any prediction over a wide range of frequencies, especially if the internal structure has a large number of internal degrees of freedom. One cannot, for example, assume that any portion of the internal mass is rigidly bound to the outer shell if such mass is associated with a natural resonance frequency that is substantially lower than the frequency with which the structure is being driven. Such mass is occasionally referred to as the "sprung mass." The present authors are aware of various studies [Hwang, 1979a, b; Achenbach, Bjarnason, and Igusa, 1992; Felsen and Guo, 1991; Bjarnason, Achenbach, and Igusa, 1992; Guo, 1992; Guo, 1993], in which the influence of internal degrees of freedom have been considered. In some cases, the considered number of degrees of freedom is relatively small, and in others the internal is a standard sort of structure such as a plate or a beam. It does appear, however, that there is still a need for relatively simple, albeit approximate, methods for handling cases for the internal sprung masses when there are many internal degrees of freedom and when the internal structure is of a complicated form. In such cases statistical averages would seem appropriate, and here again there is a rich prior literature. The present paper is directed primarily toward the prediction of the apparent damping of structures as contributed by the internals, and yields a result that, to the best of the present authors' knowledge, has been unknown up to the present.

In formulating the approach taken in the present paper, the authors have been inspired somewhat by a sequence of papers by Soize and others [Soize (1986), Chabas, Desanti, and Soize (1986), Soize, Hutin, Desanti, David, and Chabas (1986), Soize (1993)] at ONERA in France who coined the suggestive term "fuzzy structure." The overall structure is divided conceptually into a master structure and a fuzzy substructure. The dynamical properties of the master structure are known; those of the fuzzy substructure are known only in some statistical sense. Soize gives a formulation of how this statistical description can be taken into account in the prediction of some suitably averaged dynamical responses of the master structure, with the terminology and mathematical steps phrased in the context of a finite element idealization of the master structure. The latter tends to obscure the basic physical ideas used in the formulation and has tended to impede their comprehension by the larger structural acoustics community. In any event, there is nothing especially sacrosanct concerning Soize's formulation, so the present paper begins afresh, although freely drawing on Soize's innovative idea of dividing a structure into a master structure and a fuzzy substructure.

The principal conceptual hurdle of carrying out a formulation for some average dynamical behavior of a fuzzy structure is that of just how averages should be taken. In the present paper, such averages are taken over forcing terms, with time held fixed, in a dynamical model for general transient response. The appropriate dynamical model governing the behavior of the system at a fixed frequency is then found by examination of the Fourier transform of the resulting smeared-out transient model. It is not clear that such a round-about procedure is absolutely necessary, although it avoids paradoxes that one might encounter when one considers the possibility that the driving frequency might coincide with the resonance frequency of an internal degree of freedom for which there is very little damping. [Some recent work we have seen by Igusa and Tang (1992) and by Xu and Igusa (1992) , which refers in turn to earlier work by Skudrzyk (1968, 1980) and by Dowell and Kubota (1985), suggests that a comparable formulation can be carried out without recourse to the transient case, providing one assumes the damping of the internal oscillations is sufficiently large. A formulation following such a train of thought, however, would tend to introduce some confusion concerning one of the present paper's main conclusions - that the response of the master structure, in the limit of small damping of the fuzzy internal structure, is independent of the magnitude of the damping.]

Another distinction from Soize's earlier work is that the present paper is concerned with a specific example, rather than with a somewhat general class of cases. The specific example considered here, of a rectangular plate with an attached system of spring-mass oscillators, is regarded as a prototype of a structural acoustics system. Because it is a relatively explicit example, it is hoped that the reader will be able to perceive the basic ideas of the formulation relatively quickly. Once such are perceived, their application to more general systems should be apparent.

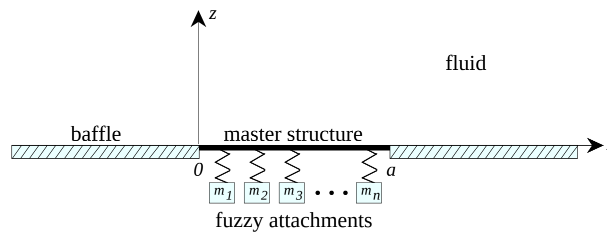


FIGURE 1: IDEALIZED FUZZY STRUCTURE STUDIED IN PRESENT PAPER. THE MASTER STRUCTURE IS A RECTANGULAR PLATE IN AN INFINITE BAFFLE WITH A COMPRESSIBLE FLUID ON ONE SIDE. THE FUZZY SUBSTRUCTURE CONSISTS OF A LARGE NUMBER OF ATTACHED OSCILLATORS WITH SMALL DAMPING.

II. DESCRIPTION OF BASIC MODEL

To introduce and to explore basic mathematical and physical concepts, a simplified model is considered here. The structure that undergoes vibrations consists in major part of a rectangular elastic plate (Fig. 1), which has a width a and length b and which is mounted in a rigid baffle. (The plate has some attachments on its backside, which are described further below.) The nominal location of the front face of the plate and of the baffle surface is the plane $z = 0$, and the plate extends from $x = 0$ to $x = a$, from $y = 0$ to $y = b$.

To demonstrate that the principal results of the paper hold regardless of whether or not there is fluid loading, the region in front of the plate, which is the region $z > 0$, is taken to be a semi-infinite halfspace filled with compressible fluid. The fluid has ambient density ρ and sound speed c. The vibrations of the structure and perhaps some external disturbance give rise to a fluctuating pressure field in this halfspace, which is described by a pressure field $p(x, y, z, t)$. If the plate were rigid, then the surface would be a perfectly reflecting rigid plane, and the (external) pressure field which results in this limiting case is denoted by $p_{\text{ext}}(x, y, z, t)$. Because this external field causes the plate to vibrate there is an additional contribution, here termed the radiated wave, to the overall pressure disturbance, so that

$$p\left(x,y,z,t\right) = p_{\text{ext}}\left(x,y,z,t\right) + p_{\text{rad}}\left(x,y,z,t\right) \tag{1}$$

This radiated wave can be regarded as caused by the plate's vibrations, in the sense that if one knew the z-component of the plate displacement $w(x, y, t)$ explicitly, then one could calculate the radiated wave by an appropriate version (Pierce, 1989) of the Rayleigh integral

$$p_{\text{rad}}\left(x,y,z,t\right) = \frac{\rho}{2\pi}\frac{\partial^2}{\partial t^2}\int_0^a\int_0^b \frac{w\left(\xi,\eta,t-c^{-1}R\right)}{R}d\xi d\eta \tag{2}$$

where

$$R = \left[(x - \xi)^2 + (y - \eta)^2 + z^2 \right]^{1/2} \tag{3}$$

is the distance from the integration point on the plate to the listener point.

The plate is idealized as an Euler-Bernoulli-Kirchhoff plate with a mass m_{pl} per unit area and a plate bending modulus B_{pl}. With N attachments on the back side of the plate, the plate dynamics are consequently governed by

$$m_{\mathrm{pl}} \frac{\partial^2 w}{\partial t^2} + B_{\mathrm{pl}} \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\}^2 w = -p\,(x, y, 0, t) - \sum_{n=1}^{N} F_n\,(t)\,\delta\,(x - x_n)\,\delta\,(y - y_n) \tag{4}$$

this holding for x between 0 and a and for y between 0 and b. Here, as in Eq. (2), $w(x, y, t)$ is the plate displacement in the $+z$ direction, and $p(x, y, 0, t)$ is the pressure on the front side of the plate. The quantity F_n is the force exerted on the plate in the negative-z-direction by an attachment at $x = x_n$ and $y = y_n$. Equivalently, F_n is the force in the positive-z-direction exerted on the n-th attachment by the plate, in accord with Newton's third law. These attachments are concentrated at points, and such is indicated by the product of Dirac delta functions $\delta(x - x_n)$ and $\delta(x - y_n)$, which appear as multiplicative factors in each of the attachment force terms.

The attachments themselves are simple spring-mass systems with very light damping. The n-th attachment is characterized by a mass M_n. It is also characterized by a spring constant K_n, which corresponds to incremental force per unit incremental elongation. Alternately, one can replace K_n by $\Omega_n^2 M_n$ where Ω_n is the corresponding natural frequency in radians per second. The damping within any given attachment is characterized by a fraction of critical damping ζ, so that the dash pot constant is $2\zeta_n M_n \Omega_n$. If $z_n(t)$ is the displacement of the mass M_n in the positive z-direction, then the force balance and constitutive relations yield the equation of motion

$$M_n \frac{d^2 z_n}{dt^2} + 2\zeta_n M_n \Omega_n \left\{ \frac{dz_n}{dt} - \frac{dw\,(x_n, y_n, t)}{dt} \right\} + M_n \Omega_n^2 \left[z_n - w\,(x_n, y_n, t) \right] = 0 \tag{5}$$

and the force $F_n(t)$ is identified as

$$F_n\,(t) = M_n \frac{d^2 z_n}{dt^2} \tag{6}$$

If the plate displacement $w_n(t) = w(x_n, y_n, t)$ is regarded as given, then one can readily derive a solution to Eq. (5) for z_n, this being a linear superposition of responses to the consecutive values of w during previous times,

$$z_n\,(t) = \int_{-\infty}^{t} w_n\,(\tau)\,G\left(t - \tau, \Omega_n, \zeta_n\right) d\tau \tag{7}$$

where

$$G\left(t, \Omega, \zeta\right) = \left\{ \Omega + 2\zeta \frac{d}{dt} \right\} e^{-\zeta \Omega t} \frac{\sin\left(\Omega \left[1 - \zeta^2 \right]^{\frac{1}{2}} t \right)}{\left[1 - \zeta^2 \right]^{\frac{1}{2}}} \tag{8}$$

The force F_n is consequently given by

$$F_n\,(t) = M_n \frac{d^2}{dt^2} \int_{-\infty}^{t} w_n\,(\tau)\,G\left(t - \tau, \Omega_n, \zeta_n\right) d\tau \tag{9}$$

The overall problem of determining the plate vibrations is posed by Eqs. (1), (2), (3), (4), (8), and (9). These can be combined to give the somewhat cumbersome equation

$$\begin{aligned} m_{\mathrm{pl}} \frac{\partial^2 w}{\partial t^2} + B_{\mathrm{pl}} \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\}^2 w \quad = \\ - p_{\mathrm{ext}}\,(x, y, 0, t) \\ - \frac{\rho}{2\pi} \frac{\partial^2}{\partial t^2} \int_0^a \int_0^b \frac{w\left(\xi, \eta, t - c^{-1} R\right)}{R} d\xi d\eta \\ - \sum_{n=1}^{N} \delta\,(x - x_n)\,\delta\,(y - y_n)\; M_n \\ \times \frac{d^2}{dt^2} \int_{-\infty}^{t} w_n\,(\tau)\,G\left(t - \tau, \Omega_n, \zeta_n\right) d\tau \end{aligned} \tag{10}$$

Here, in the radiative term, the source-listener separation distance R is understood to be evaluated at $z = 0$.

The appropriate initial conditions are those imposed by causality, so that $w(x, y, t)$ and $\partial w / \partial t$ vanish at all times before the external pressure field first begins to excite the plate. Once $w(x, y, t)$ is determined, one can determine the radiated pressure field from Eq. (2). The interest in the present paper is almost exclusively with the last term in Eq. (10), this term being what accounts for the effects of the internal structure on the plate vibrations and on the radiated sound field.

III. MASTER STRUCTURE WITH FUZZY ATTACHMENTS

The overall structure introduced above consists of a plate with a set of attached oscillators. This is here considered as a fuzzy structure in the sense used by Soize. The plate per se is taken to be what Soize terms a master structure, and its properties (namely, the width a, the length b, the mass density m_{pl}, and the bending modulus B_{pl}) are taken to be known. The properties of the spring-mass attachments are imperfectly known, and this portion of the structure is referred to as the fuzzy substructure. Individual members of this set of attachments are referred to as fuzzy internals or simply as fuzzies. Thus one does not necessarily know the total number N of attachments, the locations (x_n, y_n)

at which they are attached, the individual masses M_n, the damping parameters, ζ_n, or the corresponding natural frequencies Ω_n.

Certain statistical properties of the fuzzy substructure, however, are taken to be known at the outset. One can conceive of a statistical ensemble of such substructures, from which any particular realization is drawn according to particular rules. The most pertinent statistical average which may be presumed known is the mass $(ab)\bar{m}_F(\Omega)$ of all those portions of the substructure which have natural frequencies less than any specified angular frequency Ω. (The plate area (ab) is included as a multiplicative factor in this definition, so that $\bar{m}_F(\Omega)$ will represent the fuzzy mass per unit area of master structure surface that on the average corresponds to natural frequencies less than Ω. Note that we are requiring a priori knowledge of a function rather than of just one or two parameters.) The total mass of the substructure is $(ab)\bar{m}_F(\infty)$. The mass of all the attachments that have natural frequencies lying within a given frequency band between Ω and $\Omega + d\Omega$ is $(ab)(d\bar{m}_F/d\Omega)d\Omega$.

Apart from the information concerning masses, the statistical assumptions made here concerning the fuzzy substructure conform to what may be termed the principle of maximum ignorance. The number N of attachments, although unknown, is presumed very large. For any given attachment, the probability of its being attached within any area element $\Delta x \Delta y$ is simply $\Delta x \Delta y/(ab)$.

One can easily convince oneself that the exact value of the total number of attachments N, given that it is large, is not important by considering the case of two identical attachments, each of mass M, and each having natural frequency Ω. If these two attachments are at identically the same point or in very close proximity, then their net effect is the same as that of a single attachment, also with natural frequency Ω, but with twice the mass. Thus the concept of modal density, which plays a dominant role in the theory commonly referred to as statistical energy analysis, plays no role in the present formulation.

IV. SMEARED FUZZIES

The general principles described above concerning the fuzzy attachments allow one to make a wide-sweeping and very important approximation for the fuzzy attachment term on the right side of Eq. (10). This term is rewritten here for convenience of referral as

$$\begin{aligned} \text{Fuzzy} = &- \sum_{n=1}^{N} \delta\left(x - x_n\right) \delta\left(y - y_n\right) M_n \\ &\times \frac{d^2}{dt^2} \int_{-\infty}^{t} w_n\left(\tau\right) G\left(t - \tau, \Omega_n, \zeta_n\right) d\tau \end{aligned} \quad (11)$$

To decompose the sum in the above expression, one supposes first that the range of natural frequencies is broken up into small frequency bands, individual bands denoted by the subscript B. If Ω_B is the upper limit of frequency band B, then the B-th band consists of frequencies between Ω_{B-1} and Ω_B, and it has a bandwidth

$$(\Delta\Omega)_B = \Omega_B - \Omega_{B-1} \quad (12)$$

One also supposes that the area of the plate is broken up into small rectangles, individual elements denoted by the subscripts i and j, there being $N_I N_J$ elements in all. The partitioning in width and length is similar to that for frequency, with the i-th width interval having a width

$$(\Delta x)_i = x_i - x_{i-1} \quad (13)$$

and with analogous nomenclature for the j-th length interval. This allows one to write

$$\text{Fuzzy} = \sum_{B=1}^{\infty} \sum_{i=1}^{N_I} \sum_{j=1}^{N_J} (\text{Fuzzy})_{B,i,j} \quad (14)$$

where

$$\begin{aligned} (\text{Fuzzy})_{B,i,j} = &- \sum{}' \delta\left(x - x_n\right) \delta\left(y - y_n\right) M_n \\ &\times \frac{d^2}{dt^2} \int_{-\infty}^{t} w_n\left(\tau\right) G\left(t - \tau, \Omega_n, \zeta_n\right) d\tau \\ \approx &- \frac{d^2}{dt^2} \int_{-\infty}^{t} w\left(x_i, y_i, \tau\right) \bar{G}_\zeta\left(t - \tau, \Omega_B\right) d\tau \\ &\times \sum{}' \delta\left(x - x_n\right) \delta\left(y - y_n\right) M_n \end{aligned} \quad (15)$$

where the prime on the summation implies the restriction to values of n for which x_n is in the i-th width interval, y_n is in the j-th length interval, and Ω is in the B-th frequency interval. We have also abbreviated

$$\bar{G}_\zeta\left(t - \tau, \Omega\right) = \int G\left(t - \tau, \Omega, \zeta'\right) p_\zeta\left(\zeta'\right) d\zeta' \quad (16)$$

where $p_\zeta(\zeta')$ is the probability density function for the fraction ζ of critical damping. It is assumed that the latter is statistically independent of the location of the point of attachment, of the mass, and of the natural frequency.

The latter version of Eq. (15) assumes the area element width and length, and the frequency bandwidth, are sufficiently narrow that, for all the terms in the sub-sum, all of the Ω_n can be well-approximated by Ω_B, and $w_n(\tau)$ can be approximated by $w(x_i, y_j, \tau)$. Here we are tacitly assuming that the bending stiffness of the plate is sufficiently high that the displacement $w(x, y, t)$ varies negligibly over the dimensions of any small area element. The nature of the plate model, with a fourth derivative rather than, say, a second derivative, allows such an assumption to be made.

Recalling the manner in which the delta function is used in mathematical physics, one concludes, for reasons similar to those alluded to above, that it is consistent to replace the weighted sum over delta function products in (15) by a smeared-out version which has the same integral

$$\sum{}' \delta(x-x_n)\,\delta(y-y_n)\,M_n \rightarrow \frac{1}{(\Delta x)_i(\Delta y)_j}\sum{}' M_n U_i(x)\,U_y(y) \times \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\delta(x-x_n)\,\delta(y-y_n)\,dxdy = \frac{M_{B,i,j}}{(\Delta x)_i(\Delta y)_j}U_i(x)\,U_j(y) \quad (17)$$

where $M_{B,i,j}$ is the total mass of all the attachments which are (i) attached in width interval i, (ii) attached in length interval j, and (iii) have natural frequencies lying in frequency band B. The quantity $U_i(x)$ is unity if x lies in the interval $(\Delta x)_i$ and is zero otherwise.

According to the statistical assumptions outlined above, on the average one expects

$$M_{B,i,j} = (\Delta x)_i(\Delta y)_j\frac{d\bar{m}_F}{d\Omega}(\Delta\Omega)_B \quad (18)$$

for the mass in frequency band B and in the area element $\Delta x_i \Delta y_j$. With the substitution of Eqs. (15) and (17) into (14), and with the subsequent passage to a limit, one arrives at the result

$$\text{Fuzzy} = -\int_0^\infty \frac{d\bar{m}_F}{d\Omega}\left\{\frac{\partial^2}{\partial t^2}\int_{-\infty}^t w(x,y,\tau)\,\bar{G}_\zeta(t-\tau,\Omega)\,d\tau\right\}d\Omega \quad (19)$$

Alternately, after an interchange of the order of integration, this becomes

$$\text{Fuzzy} = -\bar{m}_F(\infty)\frac{\partial^2}{\partial t^2}\int_{-\infty}^t w(x,y,\tau)\,S_F(t-\tau)\,d\tau \quad (20)$$

where

$$S_F(t) = \frac{1}{\bar{m}_F(\infty)}\int_0^\infty \frac{d\bar{m}_F}{d\Omega}\bar{G}_\zeta(t,\Omega)\,d\Omega \quad \text{if} \quad t>0$$
$$=0 \quad \text{if} \quad t<0 \quad (21)$$

is what may be termed the fuzzy temporal memory function.

As a result of the substitution (20), the overall equation (10) for the plate vibrations, previously described as somewhat cumbersome, now becomes slightly less cumbersome, with the result

$$m_{\text{pl}}\frac{\partial^2 w}{\partial t^2} + B_{\text{pl}}\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}^2 w = -p_{\text{ext}}(x,y,0,t) - \frac{\rho}{2\pi}\frac{\partial^2}{\partial t^2}\int_0^a\int_0^b \frac{w\left(\xi,\eta,t-c^{-1}R\right)}{R}d\xi d\eta - \bar{m}_F(\infty)\frac{\partial^2}{\partial t^2}\int_{-\infty}^t w(x,y,\tau)\,S_F(t-\tau)\,d\tau \quad (22)$$

The principal achievement at this point is that an equation involving imprecisely known parameters has been replaced by one for which all of the relevant parameters are presumed known and whose solution must be deterministic.

V. FOURIER TRANSFORM DESCRIPTION

Because interest is often in vibrations and sound radiation of nearly constant frequency, we here reexpress our governing equation (22) in the frequency domain. For convenience, we regard our excitation and the resulting structural disturbance as being transient and as being such that all requisite Fourier transforms exist. (This assures that the results, even if used for long term constant frequency excitation or narrow-band noise excitation, will conform to the causality requirement.)

One can consequently set

$$w(x,y,t) = \int_{-\infty}^\infty \hat{w}(x,y,\omega)\,e^{-i\omega t}d\omega \quad (23)$$

so that, in accord with the Fourier integral theorem,

$$\hat{w}(x,y,\omega) = \frac{1}{2\pi}\int_{-\infty}^\infty w(x,y,t)\,e^{i\omega t}dt \quad (24)$$

Application of the operator implicit on the right side of the latter to Eq. (22) consequently yields

$$-\omega^2 m_{\text{pl}}\hat{w} + B_{\text{pl}}\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}^2\hat{w} = -\hat{p}_{\text{ext}}(x,y,0,\omega) + \omega^2\frac{\rho}{2\pi}\int_0^a\int_0^b\frac{\hat{w}(\xi,\eta,\omega)}{R}e^{ikR}d\xi d\eta - \frac{\bar{m}_F(\infty)}{2\pi}\int_{-\infty}^\infty e^{i\omega t}\frac{\partial^2}{\partial t^2}\left(\int_{-\infty}^0 \times\int_{-\infty}^\infty \hat{w}\left(x,y,\omega'\right)e^{-i\omega'\left[\tau'+t\right]}d\omega' \times S_F\left(-\tau'\right)d\tau'\right)dt \quad (25)$$

where, in the latter term, τ has been replaced by $t+\tau'$. The quantity k is the acoustic wavenumber ω/c. Carrying out the time integration in the last integral, and recognizing the inverse transform of $\delta(\omega-\omega')$, then doing the ω' integration, yields

$$-\omega^2 m_{\text{pl}}\hat{w} + B_{\text{pl}}\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}^2\hat{w} = -\hat{p}_{\text{ext}}(x,y,0,\omega) + \omega^2\frac{\rho}{2\pi}\int_0^a\int_0^b\frac{\hat{w}(\xi,\eta,\omega)}{R}e^{ikR}d\xi d\eta + \omega^2\bar{m}_F(\infty)\hat{w}(x,y,\omega)\int_{-\infty}^0 e^{-i\omega\tau'}S_F\left(-\tau'\right)d\tau' \quad (26)$$

VI. FUZZY DAMPING

The last term on the right side of Eq. (26) requires somewhat of an extended discussion as the requisite integration is nontrivial, and the simple results that emerge have important interpretations.

Our interest here is actually in situations where the fraction of critical damping ζ is very small for all oscillators. Since this is a quantity that is hardly ever known, even in a statistical sense, it is here assumed for simplicity that all oscillators have the same value for ζ, so that

$$\bar{G}_\zeta(t,\Omega) = \frac{1}{\left[1-\zeta^2\right]^{\frac{1}{2}}}\left\{\Omega + 2\zeta\frac{d}{dt}\right\}e^{-\zeta\Omega t}\sin\left(\Omega\left[1-\zeta^2\right]^{\frac{1}{2}}t\right) \tag{27}$$

$$S_F(t) = \frac{1}{\bar{m}_F(\infty)}\int_0^\infty \frac{d\bar{m}_F}{d\Omega}\left\{\frac{1}{\left[1-\zeta^2\right]^{\frac{1}{2}}}\left\{\Omega + 2\zeta\frac{d}{dt}\right\} \times e^{-\zeta\Omega t}\sin\left(\Omega\left[1-\zeta^2\right]^{\frac{1}{2}}t\right)\right\}d\Omega \tag{28}$$

The latter holds only for $t > 0$, it being understood that $S_F(t) = 0$ when $t < 0$.

With the idealizations just described, one proceeds to seek a simplified expression for

$$\begin{aligned}
2\pi\hat{S}_F(\omega) &= \int_{-\infty}^0 e^{-i\omega\tau'}S_F\left(-\tau'\right)d\tau' \\
&= \int_{-\infty}^\infty e^{i\omega t}S_F(t)\,dt \\
&= \frac{1}{\bar{m}_F(\infty)}\int_0^\infty \frac{d\bar{m}_F}{d\Omega}\frac{\Omega}{2i\left[1-\zeta^2\right]^{1/2}} \\
&\quad \times\left\{\int_0^\infty\left(B_1e^{iA_1t} - B_2e^{iA_2t}\right)dt\right\}d\Omega \\
&= \frac{1}{\bar{m}_F(\infty)}\int_0^\infty \frac{d\bar{m}_F}{d\Omega}\frac{\Omega}{2\left[1-\zeta^2\right]^{1/2}} \\
&\quad \times\left(\frac{B_1}{A_1} - \frac{B_2}{A_2}\right)d\Omega
\end{aligned} \tag{29}$$

where

$$A_{1,2} = i\zeta\Omega + \omega \pm \Omega\left[1-\zeta^2\right]^{1/2} \tag{30}$$

$$B_{1,2} = 1 - 2\zeta^2 \pm 2i\zeta\left[1-\zeta^2\right]^{1/2} \tag{31}$$

Further algebra then yields the result

$$\begin{aligned}
2\pi\hat{S}_F(\omega) &= \frac{1}{\bar{m}_F(\infty)}\int_0^\infty \frac{d\bar{m}_F}{d\Omega}\left\{\frac{\Omega\left[\Omega - 2i\zeta\omega\right]}{\Omega^2 - \omega^2 - 2i\zeta\omega\Omega}\right\}d\Omega \\
&= \frac{1}{\bar{m}_F(\infty)}\int_0^\infty \frac{d\bar{m}_F}{d\Omega} \\
&\quad \times\left\{\frac{\Omega^2\left[\Omega^2 - \omega^2 + 4\omega^2\zeta^2\right] + 2i\zeta\omega^3\Omega}{\left[\Omega^2-\omega^2\right]^2 + \left[2\zeta\omega\Omega\right]^2}\right\}d\Omega
\end{aligned} \tag{32}$$

With the above result in hand, one can now rewrite the Eq. (26) governing the plate dynamics in the following suggestive form

$$\begin{aligned}
-\omega^2\left[m_{\rm pl} + m_{F,\rm appar}\right]\hat{w} - i\omega R_{F,\rm appar}\hat{w}& \\
+ B_{\rm pl}\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}^2\hat{w} &= -\hat{p}_{\rm ext}(x,y,0,\omega) \\
&\quad + \omega^2\frac{\rho}{2\pi}\int_0^a\int_0^b \frac{\hat{w}(\xi,\eta,\omega)}{R}e^{ikR}d\xi d\eta
\end{aligned} \tag{33}$$

Here

$$m_{F,\rm appar} = \int_0^\infty \frac{d\bar{m}_F}{d\Omega}\left\{\frac{\Omega^2\left[\Omega^2-\omega^2+4\omega^2\zeta^2\right]}{\left[\Omega^2-\omega^2\right]^2 + \left[2\zeta\omega\Omega\right]^2}\right\}d\Omega \tag{34}$$

is the apparent extra mass per unit area contributed to the master structure by the fuzzy substructure, while

$$R_{F,\rm appar} = \int_0^\infty \frac{d\bar{m}_F}{d\Omega}\left\{\frac{2\zeta\omega^4\Omega}{\left[\Omega^2-\omega^2\right]^2 + \left[2\zeta\omega\Omega\right]^2}\right\}d\Omega \tag{35}$$

is the apparent damping (units of force per unit area divided by velocity) imposed on the master structure by the added substructure. Thus, with the approximations made so far, the influence of the fuzzy substructure can be accounted for solely by replacing the mass of the master structure by a frequency dependent mass and by adding a frequency dependent damping to the dynamics of the master structure.

VII. SMALL-DAMPING LIMIT

For sufficiently small damping, the two parameters $m_{F,\rm appar}$ and $R_{F,\rm appar}$ can be replaced by asymptotic limits that result when one lets $\zeta \to 0$. The resulting expressions, which are derived further below, are

$$m_{F,\rm appar} = {\rm Pr}\int_0^\infty \frac{d\bar{m}_F}{d\Omega}\frac{\Omega^2}{\Omega^2-\omega^2}d\Omega \tag{36}$$

$$R_{F,\rm appar} = \frac{\pi\omega^2}{2}\frac{d\bar{m}_F}{d\omega} \tag{37}$$

where Pr indicates that the principal value of the integral is to be taken.

To derive Eq. (36) from (34), one considers a nonzero value of ω and lets ϵ be a fixed positive parameter that is much less than unity and examines the integral

$$I(\epsilon,\zeta) = \int_{(1-\epsilon)\omega}^{(1+\epsilon)\omega} \frac{d\bar{m}_F}{d\Omega}\left\{\frac{\Omega^2\left[\Omega^2-\omega^2+4\omega^2\zeta^2\right]}{\left[\Omega^2-\omega^2\right]^2 + \left[2\zeta\omega\Omega\right]^2}\right\}d\Omega \tag{38}$$

A change of variable to u, where $\Omega - \omega = \omega u$, transforms this to a form which can be further manipulated to

$$I\left(\epsilon, \zeta\right) = \omega \int_{-\epsilon}^{\epsilon} \left(\frac{d\bar{m}_F}{d\Omega}\right)_{\Omega=(1+u)\omega} \times \left\{ \frac{N_e\left(u^2, \zeta^2\right) + u N_o\left(u^2, \zeta^2\right)}{D\left(u^2, \zeta^2\right)} \right\} du \quad (39)$$

where the quantities N_e, N_o, and D (N for numerator, and D for denominator) are simple polynomials in u^2 and ζ^2. To leading order in u^2 and ζ^2, these quantities (with the omission of a common factor of $4(\zeta^2 + u^2)$) are

$$N_e\left(u^2, \zeta^2\right) \sim 4\zeta^2 + 5u^2 \quad (40)$$

$$N_o\left(u^2, \zeta^2\right) \sim 2 \quad (41)$$

$$D\left(u^2, \zeta^2\right) \sim 4\left(\zeta^2 + u^2\right) \quad (42)$$

Here the desirable aspect of having the integration limits in (38) symmetrically placed about ω becomes evident. The quantity uN_o/D has no upper bound in the limit as ζ and u successively go to zero. However, this quantity is odd in u and consequently integrates out to zero for arbitrary nonzero ζ. Also, both N_e/D and $u^2 N_o/D$ are bounded and approach limiting values of 5/4 and 1/2 as $\zeta \to 0$, then $u \to 0$. Consequently, one concludes that

$$\lim_{\zeta \to 0} I\left(\epsilon, \zeta\right) = 2\epsilon \left\{ \frac{5}{4}\omega \frac{d\bar{m}_F}{d\omega} + \frac{1}{2}\omega^2 \frac{d^2\bar{m}_F}{d\omega^2} \right\} + O\left(\epsilon^3\right) \quad (43)$$

What is important from the standpoint of the derivation of Eq. (36) is that the ordered double limit, first $\zeta \to 0$, then $\epsilon \to 0$, exists and is identically zero.

One can always split the integral in (34) into three integrals, from 0 to $(1-\epsilon)\omega$, from $(1-\epsilon)\omega$ to $(1+\epsilon)\omega$, and from $(1+\epsilon)\omega$ to ∞. For the first and last segments, the limit as $\zeta \to 0$ is meaningful and exists, as long as $\epsilon \neq 0$. For the middle segment, the same limit is $O(\epsilon)$. Consequently, the limit as $\zeta \to 0$ for the entire integral in (34) has to be the same as the limit of the sum of the $\zeta \to 0$ limits of just the first and third segments, with the limit $\epsilon \to 0$ applied to the sum of the two (rather than to each separately). The latter, however, is just what is implied by the operation of taking a principal value in Eq. (36), so the assertion is verified.

In regard to the apparent damping constant $R_{F,\mathrm{appar}}$, because it may appear surprising that it is actually nonzero in the limit $\zeta \to 0$, it is of interest to carry out the derivation of the approximation for small ζ to O(ζ), rather than O(1). Doing so will allow one to assess when it is adequate to use the small damping limit. Deriving the explicit limiting expressions is an exercise in singular perturbation theory and involves local and global analysis (Bender and Orszag, 1978). One recognizes the integrand factor

$$J\left(\zeta, \Omega, \omega\right) = \frac{2\omega^3\Omega}{\left[\Omega^2 - \omega^2\right]^2 + \left[2\zeta\omega\Omega\right]^2} \quad (44)$$

and notes that, when $\zeta \to 0$, it is singular at $\Omega = \omega$ and nonintegrable. To extract an integrand which is integrable in this limit one subtracts off from $d\bar{m}_F/d\Omega$ whatever is necessary to obtain a second integrand factor that goes to zero as $(\Omega - \omega)^2$ at the other factor's singularity. With reference to a Taylor's series of $d\bar{m}_F/d\Omega$, one recognizes the ordering

$$\frac{d\bar{m}_F}{d\Omega} - \frac{d\bar{m}_F}{d\omega} - (\Omega - \omega)\frac{d^2\bar{m}_F}{d\omega^2} = O\left([\Omega - \omega]^2\right) \quad (45)$$

Consequently, the aforementioned objective is accomplished with the decomposition

$$\begin{aligned} R_{F,\mathrm{appar}} = \\ &\omega\zeta \int_0^\infty \left\{ \frac{d\bar{m}_F}{d\Omega} - \frac{d\bar{m}_F}{d\omega} - (\Omega - \omega)\frac{d^2\bar{m}_F}{d\omega^2} \right\} J\left(\zeta, \Omega, \omega\right) d\Omega \\ &+ \omega\zeta \frac{d\bar{m}_F}{d\omega} \int_0^\infty J\left(\zeta, \Omega, \omega\right) d\Omega \\ &+ \omega\zeta \frac{d^2\bar{m}_F}{d\omega^2} \int_0^\infty (\Omega - \omega) J\left(\zeta, \Omega, \omega\right) d\Omega \end{aligned} \quad (46)$$

In the first integral, the overall integrand is finite at $\Omega - \omega$ even when $\zeta = 0$, so it is all right to here set ζ to 0 in the argument of the integrand factor $J(\zeta, \Omega, \omega)$. This is acceptable for the desired approximation, because what results is correct to $O(\zeta)$. Such cannot be done, however, for the other two integral terms, although each of these integrals is independent of the detailed form of the function $d\bar{m}_F/d\omega$.

To evaluate the latter two integrals to $O(\zeta)$, one breaks each integral into three segments: from 0 to $(1-\epsilon)\omega$, from $(1-\epsilon)\omega$ to $(1+\epsilon)\omega$, and from $(1+\epsilon)\omega$ to ∞. The quantity ϵ is regarded as much less than unity but much larger than ζ. We take it to be of the order of $\zeta^{1/2}$. In the integrations over the first and third segments, one expands the integrand, with Ω held fixed, in a series in ζ, keeping only the first two terms, and changes the variable of integration to $u = \Omega/\omega$. (The integration limits then become 0 and $1-\epsilon$ for the first segment, and they become $1+\epsilon$ and ∞ for the third segment.) For the integration over the middle segment, one rescales by changing the variable of integration to $v = (\Omega - \omega)/\zeta\omega$, then expands the integrand in a series in ζ, but now holding v fixed. (The integration limits for this middle segment then become $-\epsilon/\zeta$ and ϵ/ζ.) In the two cases of integration variable change, one has, respectively,

$$(1/\omega)\,\zeta J d\Omega \sim \frac{2u\zeta du}{\left(u^2 - 1\right)^2} - \frac{8u^3\zeta^3 du}{\left(u^2 - 1\right)^4} \quad (47)$$

$$\left(1/\omega^2\right)\zeta\left(\Omega-\omega\right)Jd\Omega \sim \frac{2u\left(u-1\right)\zeta du}{\left(u^2-1\right)^2}-\frac{8u^3\left(u-1\right)\zeta^3 du}{\left(u^2-1\right)^4} \tag{48}$$

$$\left(1/\omega\right)\zeta Jd\Omega \sim \frac{1}{2}\frac{1}{\left(v^2+1\right)}\left\{1-\frac{\zeta v}{v^2+1} + \frac{\zeta^2 v^2\left(4-v^2-v^4\right)}{4\left(v^2+1\right)^2}\right\}dv \tag{49}$$

$$\left(1/\omega^2\right)\zeta\left(\Omega-\omega\right)Jd\Omega \sim \frac{1}{2}\frac{\zeta v}{\left(v^2+1\right)}\left\{1-\frac{\zeta v}{v^2+1}\right\}dv \tag{50}$$

Here one notes that the second terms of (47) and (48) are of no consequence in the considered order of approximation because $\zeta^3/\epsilon^3 = o(\zeta)$ and because $\zeta^3/\epsilon^2 = o(\zeta)$. The second term of (49) and the first term of (50) are of no consequence because these are both odd in v. The third term of (49) and the second term of (50) are of no consequence because $\zeta\epsilon = o(\zeta)$ and $\zeta^2/\epsilon = o(\zeta)$.

These afore-mentioned deletions lead one to the integral expressions

$$\left(1/\omega\right)\zeta\int_0^\infty Jd\Omega \sim \zeta\int_0^{1-\epsilon}\frac{2udu}{\left(u^2-1\right)^2}+\int_{-\epsilon/\zeta}^{\epsilon/\zeta}\frac{1}{2}\frac{vdv}{\left(v^2+1\right)} + \zeta\int_{1+\epsilon}^\infty\frac{2udu}{\left(u^2-1\right)^2}+o\left(\zeta\right) \tag{51}$$

$$\left(1/\omega^2\right)\zeta\int_0^\infty(\Omega-\omega)Jd\Omega \sim \zeta\int_0^{1-\epsilon}\frac{2u\left(u-1\right)du}{\left(u^2-1\right)^2} + \zeta\int_{1+\epsilon}^\infty\frac{2u\left(u-1\right)du}{\left(u^2-1\right)^2}+o\left(\zeta\right) \tag{52}$$

One notes, moreover, that

$$\begin{aligned}\int_0^{1-\epsilon}\frac{2u\left(u-1\right)du}{\left(u^2-1\right)^2} &= \int_0^{1-\epsilon}\frac{du}{\left(u+1\right)^2}+\frac{1}{2}\int_0^{1-\epsilon}\frac{du}{u-1} - \frac{1}{2}\int_0^{1-\epsilon}\frac{du}{u+1} \\ &= 1-\frac{1}{2-\epsilon}+\frac{1}{2}\ln\epsilon-\frac{1}{2}\ln\left(2-\epsilon\right)\end{aligned} \tag{53}$$

$$\begin{aligned}\int_{1+\epsilon}^\infty\frac{2u\left(u-1\right)du}{\left(u^2-1\right)^2} &= \int_{1+\epsilon}^\infty\frac{du}{\left(u+1\right)^2} + \lim_{K\to\infty}\frac{1}{2}\left\{\int_{1+\epsilon}^K\frac{du}{u-1}-\int_{1+\epsilon}^K\frac{du}{u+1}\right\} \\ &= \frac{1}{2+\epsilon}-\frac{1}{2}\ln\epsilon+\frac{1}{2}\ln\left(2+\epsilon\right) + \frac{1}{2}\lim_{K\to\infty}\ln\left[\frac{K-1}{K+1}\right]\end{aligned} \tag{54}$$

where the last term in the second version of Eq. (54) is identically zero. Thus the integrals in Eqs. (51) and (52) become

$$\frac{\zeta}{\omega}\int_0^\infty Jd\Omega = \frac{\zeta}{1-\left(1-\epsilon\right)^2}-\zeta+\tan^{-1}\left(\epsilon/\zeta\right) + \frac{\zeta}{\left(1+\epsilon\right)^2-1}+o\left(\zeta\right) \tag{55}$$

$$\frac{\zeta}{\omega^2}\int_0^\infty\left(\Omega-\omega\right)Jd\Omega = \zeta\left\{1-\frac{1}{2-\epsilon}\right\}+\frac{\zeta}{2}\ln\left[\frac{\epsilon}{2-\epsilon}\right] + \frac{\zeta}{2+\epsilon}+\frac{\zeta}{2}\ln\left[\frac{2+\epsilon}{\epsilon}\right]+o\left(\zeta\right) \tag{56}$$

The integrals evaluated above can be further simplified with the replacements

$$\tan^{-1}\left(\epsilon/\zeta\right)\to\frac{\pi}{2}-\frac{\zeta}{\epsilon};\qquad \ln\left(2\pm\epsilon\right)\to\ln 2; \qquad \frac{1}{\left(1\pm\epsilon\right)^2-1}\to\pm\frac{1}{2\epsilon}-\frac{1}{4} \tag{57}$$

which are consistent with the intent that the overall expressions be correct only to $O(\zeta)$. Thus one arrives at the results

$$\frac{\zeta}{\omega}\int_0^\infty Jd\Omega = \frac{\pi}{2}-\zeta+o\left(\zeta\right) \tag{58}$$

$$\frac{\zeta}{\omega^2}\int_0^\infty\left(\Omega-\omega\right)Jd\Omega = \zeta+o\left(\zeta\right) \tag{59}$$

These in turn, when substituted into Eq. (46), yield the asymptotic approximation

$$\begin{aligned}R_{F,\mathrm{appar}} \sim{}& \frac{\pi\omega^2}{2}\frac{d\bar{m}_F}{d\omega}-\zeta\omega^2\frac{d\bar{m}_F}{d\omega}+\zeta\omega^3\frac{d^2\bar{m}_F}{d\omega^2} \\ &+2\omega^4\zeta\int_0^\infty\left\{\frac{d\bar{m}_F}{d\Omega}-\frac{d\bar{m}_F}{d\omega}-\left(\Omega-\omega\right)\frac{d^2\bar{m}_F}{d\omega^2}\right\} \\ &\times\frac{\Omega}{\left(\Omega^2-\omega^2\right)^2}\,d\Omega\end{aligned} \tag{60}$$

The surprising aspect of this result is that it is not identically zero when $\zeta = 0$.

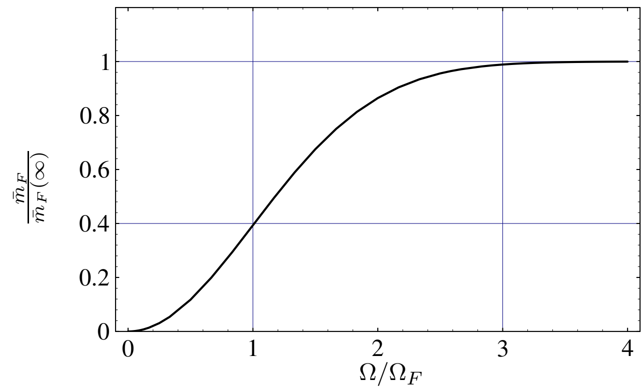


FIGURE 2: PROTOTYPE DISTRIBUTION FUNCTION FOR ATTACHED MASSES AMONG NATURAL FREQUENCIES. THE QUANTITY $\bar{m}_F(\Omega)$ IS THE MASS PER UNIT AREA OF ALL OSCILLATORS WHICH HAVE NATURAL FREQUENCIES LESS THAN Ω. THE QUANTITY Ω_F IS A CHARACTERISTIC FREQUENCY OF THE FUZZY SUBSTRUCTURE AND CORRESPONDS TO THAT NATURAL FREQUENCY FOR WHICH THE DENSITY FUNCTION $d\bar{m}_F(\Omega)/d\Omega$ HAS A MAXIMUM.

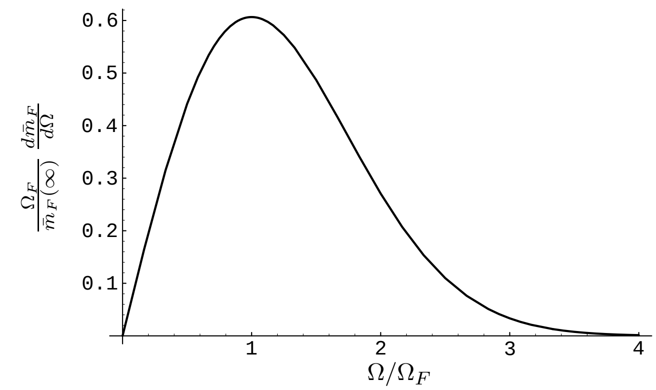


FIGURE 3: PROTOTYPE DENSITY FUNCTION FOR ATTACHED MASSES AMONG NATURAL FREQUENCIES. THE QUANTITY $[d\bar{m}_F(\Omega)/d\Omega]\triangle\Omega$ FOR SMALL $\triangle\Omega$ CORRESPONDS TO THE ATTACHED MASS PER UNIT AREA WHICH IS ASSOCIATED WITH OSCILLATORS WHOSE NATURAL FREQUENCIES ARE BETWEEN Ω AND $\Omega+\triangle\Omega$. THE ADOPTED PROTOTYPE DENSITY FUNCTION HAS A SINGLE MAXIMUM, THIS OCCURRING WHEN $\Omega = \Omega_F$, WHERE Ω_F IS A CHARACTERISTIC FREQUENCY OF THE FUZZY SUBSTRUCTURE.

VIII. PROTOTYPE MASS-NATURAL-FREQUENCY DISTRIBUTION

Although the function $\bar{m}_F(\Omega)$ is not necessarily known or of any simple form, it would seem advantageous from the standpoint of a quantitative assessment that one have some prototype function depending on a relatively small number of parameters as a reference. With this in mind, the following (Fig. 2) is suggested

$$\bar{m}_F(\Omega) = \bar{m}_F(\infty)\left[1 - e^{-\Omega^2/2\Omega_F^2}\right] \tag{61}$$

so that

$$\frac{d\bar{m}_F(\Omega)}{d\Omega} = \bar{m}_F(\infty)\frac{\Omega}{\Omega_F^2}e^{-\Omega^2/2\Omega_F^2};$$
$$\frac{d^2\bar{m}_F(\Omega)}{d\Omega^2} = \bar{m}_F(\infty)\frac{\Omega_F^2-\Omega^2}{\Omega_F^4}e^{-\Omega^2/2\Omega_F^2} \tag{62}$$

The quantity Ω_F is here termed the most probable resonance frequency of the fuzzy internals and is equal to that resonance frequency at which the mass per unit resonance frequency bandwidth is a maximum. The suggested density distribution of mass among resonance frequencies, given here by Eq. (62), is plotted in Fig. 3 This is in a dimensionless form, with $[\Omega_F/m_F(\infty)]d\bar{m}_F(\Omega)/d\Omega$ regarded as a function of the frequency ratio Ω/Ω_F.

With the above introduced quantity substituted into Eq. (36), the apparent frequency-dependent mass per unit area added by the fuzzy substructure, in the $\zeta = 0$ limit, is

$$m_{F,\text{appar}} = \bar{m}_F(\infty)\,\text{Pr}\int_0^\infty \frac{\Omega^2}{\Omega^2-\omega^2}\frac{\Omega}{\Omega_F^2}e^{-\Omega^2/2\Omega_F^2}d\Omega$$
$$= \bar{m}_F(\infty)\,\text{Pr}\int_0^\infty \frac{u}{u-\eta}e^{-u}du \tag{63}$$

where in the latter expression the integration variable has been changed to $u = \Omega^2/2\Omega_F^2$, and η abbreviates

$$\eta = \frac{\omega^2}{2\Omega_F^2} \tag{64}$$

The integral in the latter version of (63) is readily expressed in terms of the exponential integral, so that

$$m_{F,\text{appar}} = \bar{m}_F(\infty)\left[1 - \eta e^{-\eta}\text{Ei}(\eta)\right] \tag{65}$$

where

$$\text{Ei}(\eta) = -\text{Pr}\int_{-\eta}^\infty \frac{e^{-\xi}}{\xi}d\xi \tag{66}$$

The expression in Eq. (65) is plotted in Fig. 4 in a dimensionless form, with the ratio $m_{F,\text{appar}}/\bar{m}_F(\infty)$ regarded as a function of the frequency ratio ω/Ω_F. Although the choice of the mass density function in Eq. (36) is somewhat ad hoc, the qualitative shape of the curve in Fig. 4 is expected to be representative of realistic cases. In the limit of zero frequency, the apparent mass is the same as the total mass of the internals, as indicated by the value of unity for $m_{F,\text{appar}}/\bar{m}_F(\infty)$ when ω/Ω_F is zero. This must be so because, at such a frequency, all the springs appear so stiff that the fuzzy internals must move as if they were rigidly welded to the master structure. However, as the the frequency first increases beyond zero, the apparent mass to be added to the master structure is predicted to be greater than the mass of the internals, as is

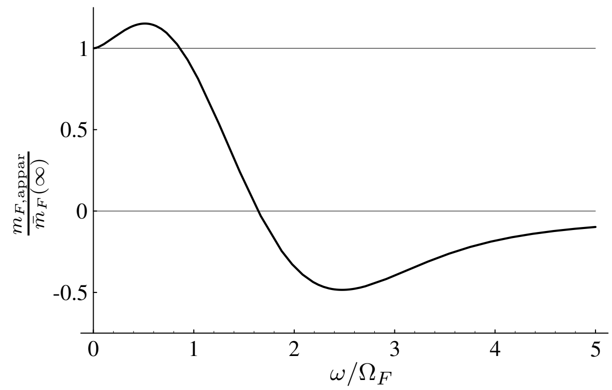


FIGURE 4: APPARENT (FREQUENCY-DEPENDENT) MASS PER UNIT AREA ADDED TO THE MASTER STRUCTURE (RECTANGULAR PLATE) BY THE FUZZY STRUCTURE WHEN THE MASTER STRUCTURE IS OSCILLATING AT ANGULAR FREQUENCY ω. THE RATIO OF APPARENT SUBSTRUCTURE MASS TO TOTAL SUBSTRUCTURE MASS RETURNS TO VALUE OF UNITY WHEN $\omega = 0.863\Omega_F$ AND GOES FROM POSITIVE TO NEGATIVE WHEN $\omega = 1.641\Omega_F$.

indicated by the ratio $m_{F,\text{appar}}/\bar{m}_F(\infty)$ being larger than unity. In this lower frequency range, the majority of the internal oscillators are being driven at below their resonance frequencies, so they are moving in phase with the master structure. Their displacements are larger than that of their attachment points, so their kinetic energies are larger than would be expected were they rigidly bound to the master structure. As the frequency becomes larger this mass enhancement reaches a maximum, and then drops below the total internal mass when $\omega = 0.863\Omega_F$. The function goes negative when $\omega = 1.641\Omega_F$ and remains negative at all higher frequencies. The reason for this latter behavior is that the major portion of the internal mass is being driven at a frequency higher than their associated resonance frequencies, so that the internal mass motion tends to be 180^o out of phase with the plate motion. This phase opposition means that the forces exerted by the internals on the plate tend to be in the same direction as the plate acceleration, so the net effect is as if mass were being subtracted from the plate.

With reference to the known asymptotic behavior of the exponential integral one infers that in the low frequency limit

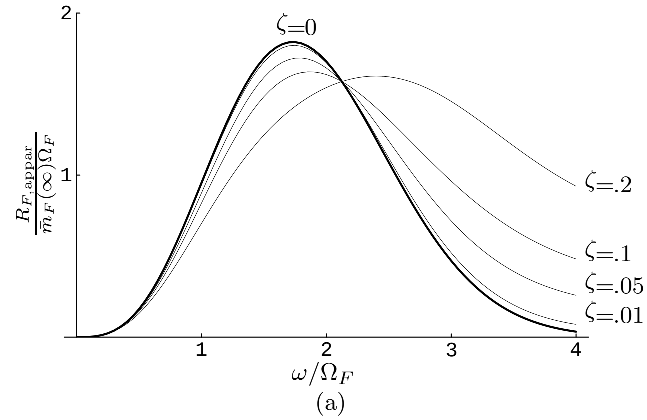
$$m_{F,\text{appar}} \sim \bar{m}_F(\infty)\left[1 + \eta \ln\{e^{-\gamma}/\eta\}\right] \tag{67}$$

where $\gamma \approx .57721$ is the Euler-Mascheroni constant. In the high frequency limit

$$m_{F,\text{appar}} \sim -\frac{\bar{m}_F(\infty)}{\eta} \tag{68}$$

The asymptotic validity of both of these limiting expressions is supported by the numerical values plotted in Fig. 4.

The apparent damping R_F is plotted versus frequency for several values of the critical damping ratio ζ in Fig. 5. The



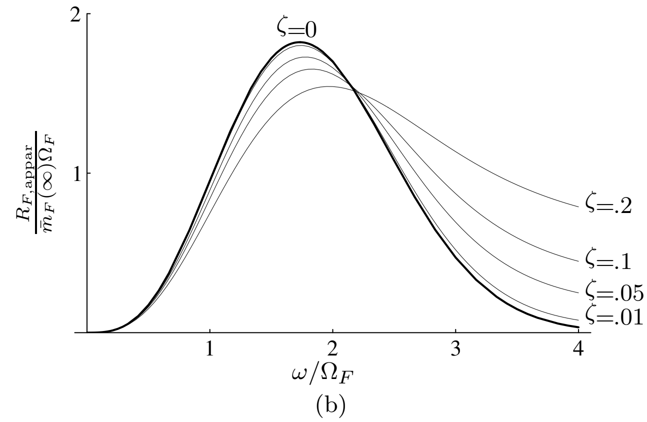


FIGURE 5: APPARENT (FREQUENCY-DEPENDENT) DAMPING CONSTANT ADDED TO THE MASTER STRUCTURE (RECTANGULAR PLATE) BY THE FUZZY SUBSTRUCTURE WHEN THE MASTER STRUCTURE IS OSCILLATING AT ANGULAR FREQUENCY ω. THE GENERAL THEORY PREDICTS THE "PLATE DASHPOT CONSTANT" R_F (DAMPING FORCE PER UNIT AREA PER UNIT OSCILLATION VELOCITY) TO VARY WITH FRACTION OF CRITICAL DAMPING ζ OF ATTACHED OSCILLATORS IN THE LIMIT OF SMALL ζ AS THE SUM OF A NON-ZERO QUANTITY INDEPENDENT OF ζ PLUS ANOTHER QUANTITY DIRECTLY PROPORTIONAL TO ζ. (a) CURVES FOR VARIOUS VALUES OF ζ BASED ON THE TWO TERM APPROXIMATION. (b) CURVES BASED ON DIRECT NUMERICAL EVALUATION OF THE INTEGRAL EXPRESSION THAT HOLDS FOR ARBITRARY ζ.

upper version of this figure is based on the approximate expression of Eq. (60); the lower version was evaluated by direct numerical integration of Eq. (35). The limiting case of $\zeta = 0$ corresponds in both parts of the figure to Eq. (37). One should note that the derived small-damping limit is a very good approximation up to $\zeta = 0.2$. The coefficient of the ζ-correction term (all terms in Eq. (60) that are directly proportional to ζ) is plotted in Fig. 6. This latter

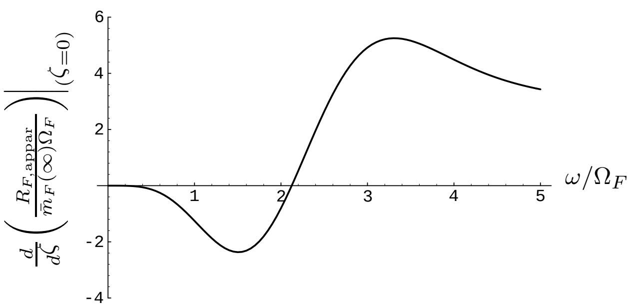


FIGURE 6: COEFFICIENT OF CORRECTION TERM FOR NON-ZERO FRACTION ζ OF CRITICAL DAMPING THAT IS TO BE ADDED TO THE VALUE FOR $\zeta = 0$ OF THE APPARENT (FREQUENCY-DEPENDENT) DAMPING CONSTANT ADDED TO THE MASTER STRUCTURE (RECTANGULAR PLATE) BY THE FUZZY SUBSTRUCTURE WHEN THE MASTER STRUCTURE IS OSCILLATING AT ANGULAR FREQUENCY ω. THE GENERAL THEORY PREDICTS THE "PLATE DASHPOT CONSTANT" R_F (DAMPING FORCE PER UNIT AREA PER UNIT OSCILLATION VELOCITY) TO VARY WITH FRACTION OF CRITICAL DAMPING ζ OF ATTACHED OSCILLATORS IN THE LIMIT OF SMALL ζ AS THE SUM OF A NON-ZERO QUANTITY INDEPENDENT OF ζ PLUS ANOTHER QUANTITY DIRECTLY PROPORTIONAL TO ζ.

figure is also in a dimensionless form, and one can note that the so-indicated dimensionless group involving the coefficient is at most of the order of 6.

One should take particular note from both versions of Fig. 5 that the $\zeta = 0$ approximate version of R_F, that expression being $[\pi\omega^2/2]d\bar{m}_F/d\omega$, is ordinarily an excellent approximation for any realistic value of ζ (up to, say, $\zeta = 0.1$).

IX. CONCLUDING REMARKS

The results in the preceding section support the assertion that the exact values of the damping in the fuzzy internal substructure are of little importance. The theoretical development presented here is consistent with the viewpoint that the actual absorption of energy by internal oscillators tends to be independent of the damping, given that the damping is small. There are undoubtedly a variety of mathematically equivalent ways of giving a "physical interpretation" to this result, and seeking such is important because any such cogently phrased interpretation will help in the formulation of future research and in understanding the limitations of the result. A rough physical explanation offered here for this behavior is that the most important contributions come from those masses which are being driven near their resonance frequencies. The closer the driving frequency to the resonance frequency and the smaller the oscillator's actual damping, the higher the limiting amplitude. However, the actual force exerted on the plate by the highly oscillating lightly damped oscillators driven at close to resonance remains finite, at least in an averaged sense, and approaches a limit when the number of oscillators becomes sufficiently large and when they are randomly dispersed on the surface. This "averaged force" is sufficient to account for the local effect of the fuzzy substructure on the master structure.

In a certain sense, the results developed here have a relationship to an exact description analogous to that of fluid dynamic pressure to the dynamics of a large collection of small spherical balls moving within a large container with nearly rigid walls. The forces exerted by individual collisions with the walls are large and in rapid succession, but the laws of large numbers lead to the viewpoint that there is a continuous force smeared out over the walls, so that one can speak of a force per unit area. The attractive features of this thermodynamic asymptotic limit are (i) simplicity, and (ii) insensitivity to details of the overall system description. The result here has comparable features. A casual perusal and comparison of texts on thermodynamics with texts on statistical mechanics suggests that really understanding just how well the limiting results derived here approximate specific systems will not be trivial. This does not mean the seeking of such an understanding should be abandoned at the outset, but one should be prepared to make tentative use (with possible support from laboratory and field experiment and from computational simulation experiments) of the limiting asymptotic results in the absence of any formulation of comparable simplicity or in the absence of explicit system-parameter information that might be required by a more exact formulation, but which is difficult to obtain at the time of the immediate analysis.

The model of a rectangular plate with point-attached masses was used in the present paper primarily to assist the pedagogical development. The authors believe the basic ideas presented here can be applied to a wide variety of structures, provided one is willing at the outset to look for asymptotic limits that apply when the number of degrees of freedom of the internal structure is large and when the details of the structure are taken as randomly distributed. The master structure, for example, could be taken as a cylindrical shell. The fuzzy substructure could be interconnected, masses could vibrate parallel to the surface of the master structure, so that traveling longitudinal and shear waves would excite the internal structure. The authors foresee a wealth of interesting and practically important versions of the embryonic model introduced here and invite other members of the structural acoustics community to participate in their development.

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