1

Fundamentals of Magnetic Devices

1.1 Introduction

Many electronic circuits require the use of inductors and transformers [1]–[47]. These are usually the largest, heaviest, and most expensive components in a circuit. They are defined by their electromagnetic behavior. The main feature of an inductor is its ability to store magnetic energy in the form of a magnetic field. The important feature of a transformer is its ability to couple magnetic fluxes of different windings and transfer ac energy from the input to the output through the magnetic field. The amount of energy transferred is determined by the operating frequency, flux density, and temperature. Transformers are used to change the ac voltage and current levels as well as to provide dc isolation while transmitting ac signals. They can combine energy from many ac sources by the addition of the magnetic flux and deliver the energy from all the inputs to one or multiple outputs simultaneously. The magnetic components are very important in power electronics and other areas of electrical engineering. Power losses in inductors and transformers are due to skin and proximity effects in windings, as well as to eddy currents and hysteresis in magnetic cores. Failure mechanisms in magnetic components are mostly due to excessive temperature rise. Therefore, these devices should satisfy both magnetic requirements and thermal limitations.

In this chapter, fundamental laws, quantities, and units of the magnetic theory are reviewed. Magnetic relationships are given and an equation for the inductance is derived. Hysteresis and eddy-current losses are studied. There are two kinds of eddy-current effects: skin effect and proximity effect. Both of these effects cause nonuniform distribution of the current density in conductors and increase the conductor ac resistance at high frequencies. The winding and core losses are determined. The winding resistance of magnetic components is studied using Dowell's equation [1]. Three shapes of winding conductors are considered: rectangular, square, and round. Properties of magnetic materials are also discussed.

1.2 Magnetic Relationships

The magnetic field is characterized by magnetomotive force $\mathcal{F}$, magnetic field intensity H, magnetic flux density B, magnetic flux ϕ, and magnetic flux linkage λ.

1.2.1 Magnetomotive Force

An inductor with N turns carrying an ac current i produces the *magnetomotive force* (MMF), which is also called the *magnetomotance*. The MMF is given by

$$\mathcal{F} = Ni \ \ (\mathrm{A} \cdot \mathrm{turns}). \tag{1.1}$$

Its descriptive unit is the ampere-turn (A·t). However, the approved SI unit of the MMF is the ampere (A). The MMF is a source in magnetic circuits. The magnetic flux ϕ is forced to flow in a magnetic circuit by the MMF $\mathcal{F} = Ni$ driving the magnetic circuit. Every time another complete turn with the current i is added, the result of the integration increases by current i. The magnetomotive force is analogous to the electromotive force, which causes a current flow i.

1.2.2 Magnetic Field Intensity

The *magnetic field intensity* (or *magnetic field strength*) is given by

$$H = \frac{\mathcal{F}}{l} = \frac{Ni}{l} \left(\frac{\mathrm{A}}{\mathrm{m}}\right), \tag{1.2}$$

where l is the inductor length and N is the number of turns.

1.2.3 Magnetic Flux

The amount of *magnetic flux* passing through a surface S is given by

$$\phi = \int\int_S \mathbf{B} \cdot d\mathbf{S} \ \ (\mathrm{Wb}). \tag{1.3}$$

The unit of the magnetic flux is the weber. If the magnetic flux is uniform and perpendicular to the surface A, the amount of the magnetic flux passing through the surface A is

$$\phi = AB \quad (\mathrm{Wb}). \tag{1.4}$$

The direction of a magnetic flux ϕ is determined by the right-hand rule. This rule states that if the fingers of the right hand encircle a coil in the direction of the current i, the thumb indicates the direction of the magnetic flux ϕ.

1.2.4 Magnetic Flux Density

The *magnetic flux density*, or *induction*, is given by

$$B = \frac{\phi}{A} \ (\mathrm{T}). \tag{1.5}$$

The relationship between the magnetic flux density and the magnetic field intensity is given by

$$B = \mu H = \mu_r \mu_0 H = \frac{\mu N i}{l} = \frac{\mu \mathcal{F}}{l} \text{ (T)} \tag{1.6}$$

where the permeability of free space is

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{(H/m)}; \tag{1.7}$$

$\mu = \mu_r \mu_0$ is the permeability, and $\mu_r = \mu/\mu_0$ is the relative permeability (i.e., relative to free space). For free space, insulators, and nonmagnetic conductors, $\mu_r = 1$. For diamagnetics such as copper (Cu), lead (Pb), silver (Ag), and gold (Au), $\mu_r \approx 1 - 10^{-5} \approx 1$. However, for ferromagnetic materials such as iron (Fe), cobalt (Co), nickel (Ni), and their alloys, $\mu_r > 1$ and it can be as high as 100 000. The permeability is the measure of the ability of a material to conduct magnetic flux. It describes how easily a material can be magnetized. For a large value of μ_r, a small current i produces a large flux density B. The magnetic flux takes the path of the highest permeability.

For ferromagnetic materials, the relationship between B and H is nonlinear because the relative permeability μ_r depends on the magnetic field intensity H. Figure 1.1 shows simplified plots of the magnetic flux density B as a function of the magnetic field intensity H for air core inductors and for ferromagnetic core inductors. The straight line describes the air core inductor and has a slope μ_0 for all values of H. These inductors are linear. The piecewise linear approximation corresponds to the ferromagnetic core inductors, where B_s is the saturation flux density and $H_s = B_s/(\mu_r \mu_0)$ is the magnetic field intensity corresponding to B_s. At low values of the magnetic flux density $B < B_s$, the relative permeability μ_r is high and the slope of the B–H curve $\mu_r \mu_0$ is also high. For $B > B_s$, the core saturates and $\mu_r = 1$, reducing the slope of the B–H curve to μ_0.

The total peak magnetic flux density B_{pk}, which in general consists of both the dc component B_{DC} and the amplitude of the ac component B_m, should be lower than the saturation flux density B_s of a magnetic core at the highest operating temperature T_{max}:

$$B_{pk} = B_{DC} + B_m \leq B_s. \tag{1.8}$$

The dc component of the magnetic flux density is caused by the dc component of the inductor current I_L:

$$B_{DC} = \frac{\mu_r \mu_0 N I_L}{l_c}. \tag{1.9}$$

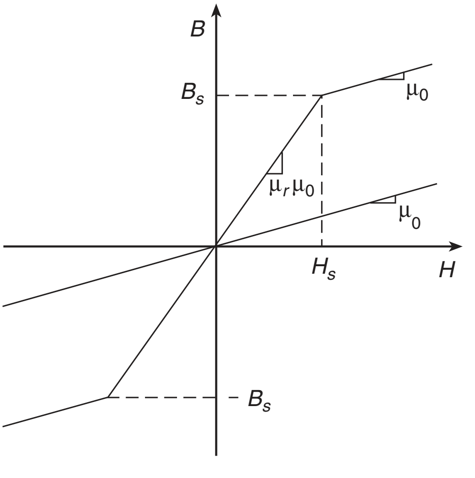

Figure 1.1 Simplified plots of magnetic flux density B as a function of magnetic field intensity H for air core inductors (straight line) and ferromagnetic core inductors (piecewise linear approximation).

The ac component of the magnetic flux density is caused by the ac component of the inductor current with its amplitude I_m:

$$B_m = \frac{\mu_r \mu_0 N I_m}{l_c}. \tag{1.10}$$

Hence,

$$B_{pk} = \frac{\mu_r \mu_0 N I_L}{l_c} + \frac{\mu_r \mu_0 N I_m}{l_c} = \frac{\mu_r \mu_0 N (I_L + I_m)}{l_c} \le B_s. \tag{1.11}$$

The saturation flux density B_s decreases with temperature. For ferrites, B_s may decrease by a factor of 2 as the temperature increases from 20°C to 90°C.

1.2.5 Magnetic Flux Linkage

The *magnetic flux linkage* is the sum of the flux enclosed by each turn of wire wound around the core. The magnetic flux linkage is the magnetic flux linking N turns and is described by

$$\lambda = N\phi = NA_cB = NA_c\mu H = \frac{\mu A_c N^2 i}{l_c} = Li \ \text{(V s)}. \tag{1.12}$$

This equation is analogous to Ohm's law $v = Ri$ and the equation for the capacitor charge $Q = Cv$. The unit of the flux linkage is the weber-turn. For sinusoidal waveforms, the relationship among the amplitudes is

$$\lambda_m = N\phi_m = NA_cB_m = NA_c\mu H_m = \frac{\mu_r \mu_0 A_c N^2 I_m}{l_c}. \tag{1.13}$$

The change in the magnetic linkage can be expressed as

$$\Delta\lambda = \int_{t_1}^{t_2} v dt = \lambda(t_2) - \lambda(t_1). \tag{1.14}$$

1.3 Magnetic Circuits

1.3.1 Reluctance

The *reluctance* $\mathcal{R}$ is the resistance of the core to the flow of the magnetic flux ϕ. It opposes the magnetic flux flow, similarly to the way the resistance opposes the electric current flow. An element of a magnetic circuit can be called a *reluctor*. The concept of reluctance is illustrated in Figure 1.2. The reluctance of a basic magnetic circuit element is given by

$$\mathcal{R} = \frac{1}{\mathcal{P}} = \frac{l_c}{\mu A_c} \left(\frac{\text{A} \cdot \text{turns}}{\text{Wb}}\right) = \frac{l_c}{\mu A_c} \left(\frac{\text{turns}}{\text{H}}\right), \tag{1.15}$$

where A_c is the cross-sectional area of the core (i.e., the area through which the magnetic flux flows) and l_c is the mean magnetic path length (MPL), which is the mean length of the closed path that the magnetic flux flows around the magnetic circuit. The reluctance is directly proportional the length of the magnetic path l_c and is inversely proportional to the cross-sectional area A_c through which the magnetic flux ϕ flows. The *permeance* of a basic magnetic circuit element is

$$\mathcal{P} = \frac{1}{\mathcal{R}} = \frac{\mu A_c}{l_c} \left(\frac{\text{Wb}}{\text{A} \cdot \text{turns}}\right) = \frac{\mu A_c}{l_c} \left(\frac{\text{H}}{\text{turns}}\right). \tag{1.16}$$

When the number of turns $N = 1$, $L = \mathcal{P}$. The reluctance is the magnetic resistance because it opposes the establishment and the flow of a magnetic flux ϕ in a material. A poor conductor of the magnetic flux has a high reluctance and a low permeance. The magnetic Ohm's law is expressed as

$$\phi = \frac{\mathcal{F}}{\mathcal{R}} = \mathcal{P}\mathcal{F} = \frac{\mu A_c N i}{l_c} = \frac{\mu_{rc} \mu_0 A_c N i}{l_c} \ \text{(Wb)}. \tag{1.17}$$

Magnetic flux always takes the path with the highest permeability μ.

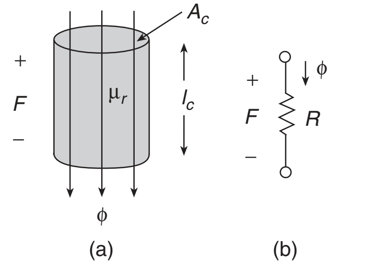

Figure 1.2 Reluctance. (a) Basic magnetic circuit element conducting magnetic flux ϕ. (b) Equivalent magnetic circuit.

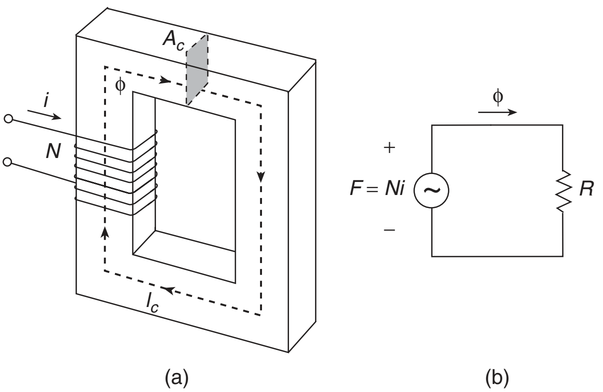

Figure 1.3 Magnetic circuit. (a) An inductor composed of a core and a winding. (b) Equivalent magnetic circuit.

In general, the magnetic circuit is the space in which the magnetic flux flows around the coil(s). Figure 1.3 shows an example of a magnetic circuit. The reluctance in magnetic circuits is analogous to the resistance R in electric circuits. Likewise, the permeance in magnetic circuits is analogous to the conductance in electric circuits. Therefore, magnetic circuits described by the equation $\phi = \mathcal{F}/\mathcal{R}$ can be solved in a similar manner as electric circuits described by Ohm's law $I = V/R = GV = (\sigma A/l)V$, where ϕ, $\mathcal{F}$, $\mathcal{R}$, $\mathcal{P}$, B, λ, and σ correspond to I, V, R, G, J, Q, and μ, respectively. For example, the reluctances can be connected in series or in parallel. In addition, the reluctance $\mathcal{R} = l_c/\mu A_c$ is analogous to electric resistance $R = l/\sigma A$, and flux density $B = \phi/A_c$ is analogous to current density $J = I/A$. Table 1.1 lists analogous magnetic and electric quantities.

1.3.2 Magnetic Kirchhoff's Voltage Law

Physical structures made of magnetic devices, such as inductors and transformers, can be analyzed just like electric circuits. The magnetic law, analogous to Kirchhoff's voltage law, states that the sum of the magnetomotive forces $\mathcal{F}_k$ and the magnetic potential differences $\mathcal{R}_k\phi_k$ around the closed magnetic loop is zero:

$$\sum_{k=1}^{n} \mathcal{F}_k - \sum_{k=1}^{m} \mathcal{R}_k\phi_k = 0. \tag{1.18}$$

For instance, an inductor with a simple core having an air gap as illustrated in Figure 1.4 is given by

$$Ni = \mathcal{F} = \mathcal{F}_c + \mathcal{F}_g = \phi(\mathcal{R}_c + \mathcal{R}_g) \tag{1.19}$$

Table 1.1 Analogy between magnetic and electric quantities

Magnetic quantity	Electric quantity
$\mathcal{F} = Ni$	V
ϕ	I
H	E
B	J
$\mathcal{P}$	G
λ	Q
μ	σ
L	C
$\phi = \frac{\mathcal{F}}{\mathcal{R}}$	$I = \frac{V}{R}$
$B = \frac{\phi}{A}$	$J = \frac{I}{A}$
$H = \frac{\mathcal{F}}{l} = \frac{Ni}{l}$	$E = \frac{V}{l}$
$\mathcal{R} = \frac{l}{\mu A}$	$R = \frac{l}{\sigma A}$
$B = \mu H$	$D = \epsilon E$
$\lambda = Li$	$Q = Cv$
$w_m = \frac{1}{2}\mu H^2$	$w_e = \frac{1}{2}\epsilon E^2$
$W_m = \frac{1}{2}Li^2$	$W_e = \frac{1}{2}Cv^2$

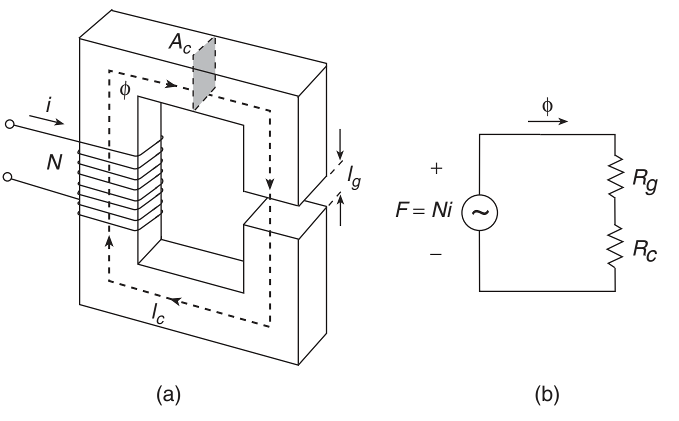

Figure 1.4 Magnetic circuit illustrating the magnetic Kirchhoff's voltage law. (a) An inductor composed of a core with air gap and a winding. (b) Equivalent magnetic circuit.

where the reluctance of the core is

$$\mathcal{R}_c = \frac{l_c}{\mu_{rc}\mu_0 A_c}, \tag{1.20}$$

the reluctance of the air gap is

$$\mathcal{R}_g = \frac{l_g}{\mu_0 A_c}, \tag{1.21}$$

and it assumed that $\phi_c = \phi_g = \phi$. The reluctance of the air gap $\mathcal{R}_g$ is much higher than the reluctance of the core $\mathcal{R}_c$.

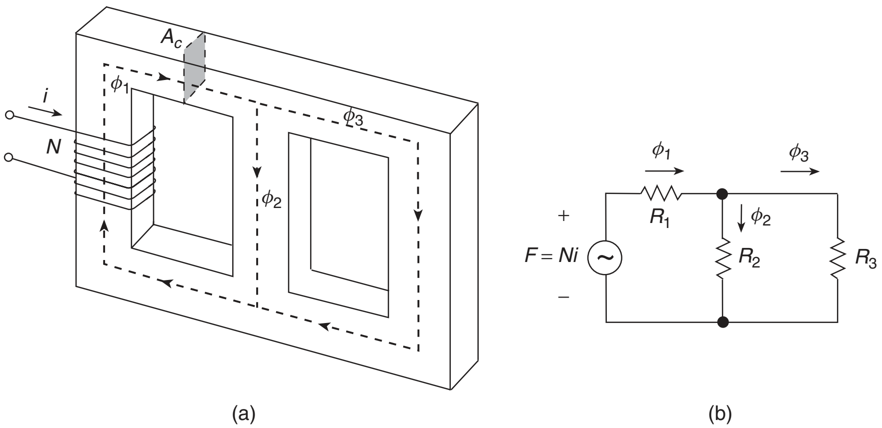

Figure 1.5 Magnetic circuit illustrating the continuity of the magnetic flux. (a) An inductor composed of a core and a winding. (b) Equivalent magnetic circuit.

1.3.3 Magnetic Flux Continuity

The continuity of the magnetic flux law states that the net magnetic flux through any closed surface is always zero,

$$\phi = \oiint_A BdA = 0, \tag{1.22}$$

or the net magnetic flux entering the node is zero,

$$\sum_{k=1}^{n} \phi_k = \sum_{k=1}^{n} A_k B_k = 0. \tag{1.23}$$

This law is analogous to Kirchhoff's current law introduced by Gauss and can be called Kirchhoff's flux law. Figure 1.5 illustrates the continuity of the magnetic flux law. For example, when three core legs meet at a node,

$$\phi_1 = \phi_2 + \phi_3, \tag{1.24}$$

which can be expressed by

$$\frac{\mathcal{F}_1}{\mathcal{R}_1} = \frac{\mathcal{F}_2}{\mathcal{R}_2} + \frac{\mathcal{F}_3}{\mathcal{R}_3}. \tag{1.25}$$

If all three legs of the core have windings, then we have

$$\frac{N_1 i_1}{\mathcal{R}_1} = \frac{N_2 i_2}{\mathcal{R}_2} + \frac{N_3 i_3}{\mathcal{R}_3}. \tag{1.26}$$

Usually, most of the magnetic flux is confined inside an inductor, e.g., for an inductor with a toroidal core. The flux outside an inductor is called the *leakage flux*.

1.4 Magnetic Laws

1.4.1 Ampère's Law

Ampère's law (1826) states that a time-varying current $i(t)$ induces a time-varying magnetic field $H(t)$. When a conductor (such as an inductor) carries a time-varying current $i(t)$, a magnetic field $H(t)$ is induced. In a conductor, the induced magnetic field may be due to the conductor's own ac current or the ac current of other adjacent conductors. The integral form of Ampère's circuital law, or simply Ampère's law, states that the closed line integral of the magnetic field intensity **H** around a closed path C is equal to the total current i_{enc} passing through the interior of the closed path bounding the surface S:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int \int_S \mathbf{J} \cdot d\mathbf{S} = \sum_{n=1}^{N} i_n = i_1 + i_2 + \cdots + i_N = i_{enc}, \tag{1.27}$$

where $d\mathbf{l}$ is the vector length element pointing in the direction of the Amperian path. The current i_{enc} enclosed by the path C is given by the integral of the normal component **J** over the surface S. The surface integral of the current density **J** is equal to the conduction current I flowing through the surface S. For good conductors, the displacement current can be ignored. For an inductor with N turns, Ampère's law is

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = Ni. \tag{1.28}$$

Ampère's law in its discrete form can be expressed as

$$\sum_{k=1}^{n} H_k l_k = \sum_{k=1}^{m} N_k i_k. \tag{1.29}$$

For example, Ampère's law for an inductor with an air gap is

$$H_c l_c + H_g l_g = Ni. \tag{1.30}$$

If the current density J is uniform and perpendicular to the surface S,

$$HC = SJ. \tag{1.31}$$

The current density J in winding conductors of magnetic components used in power electronics is usually in the range of 0.1 to 10 A/mm^2. The displacement current is neglected in (1.27). Ampère's law constitutes one of Maxwell's equations in integral form.

Example 1.1

An infinitely long round straight wire of radius r_o carries current $i = I_m \cos \omega t$ in steady state at low frequencies. Determine the waveforms of the magnetic field intensity $H(r, t)$ inside and outside the wire.

Solution: At low frequencies, the skin effect can be neglected and the current is uniformly distributed over the cross section of the wire, as shown in Figure 1.6.

The magnetic field intensity inside the wire. The current in the conductor induces a concentric magnetic field inside and outside the conductor. The current flowing through the area enclosed by the cylindrical shell of radius r at low frequencies is given by

$$i_{enc} = I_{m(enc)} \cos \omega t \tag{1.32}$$

where $I_{m(enc)}$ is the amplitude of the current enclosed by the shell. Hence, the amplitude of the current density at a radius r is

$$J_m(r) = \frac{I_{m(enc)}}{\pi r^2}, \quad \text{for} \quad r \le r_o, \tag{1.33}$$

and the amplitude of the current density at the wire surface $r = r_o$ is

$$J_m(r_o) = \frac{I_m}{\pi r_o^2}. \tag{1.34}$$

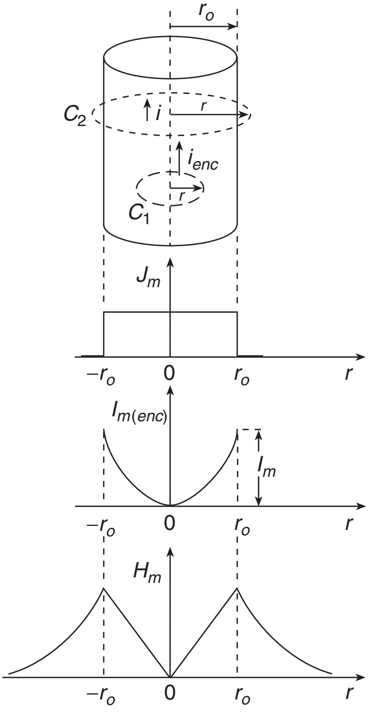

Figure 1.6 Cross section of an infinitely long round straight wire carrying a uniform current i and amplitudes of current density J_m, enclosed current $I_{m(enc)}$, and magnetic field intensity H_m as a function of the distance from the wire center r at low frequencies (i.e., when the skin effect can be neglected).

The current density is uniform at low frequencies (where the skin effect can be neglected), i.e., $J_m(r) = J_m(r_o)$, yielding the amplitude of the enclosed current

$$I_{m(enc)} = I_m \left(\frac{r}{r_o}\right)^2, \quad \text{for} \quad r \leq r_o. \tag{1.35}$$

From Ampère's law,

$$I_{m(enc)} = \oint_{C_1} \mathbf{H} \cdot d\mathbf{l} = H_m(r) \oint_{C_1} dl = 2\pi r H_m(r), \quad \text{for} \quad r \leq r_o, \tag{1.36}$$

where $C_1 = 2\pi r$ with $r \leq r_o$. Figure 1.6 shows a plot of $I_{m(enc)}$ as a function of distance from the conductor center r. Equating the right-hand sides of (1.35) and (1.36), we obtain the amplitude of the magnetic field intensity inside the wire for low frequencies

$$H_m(r) = I_m \left(\frac{r}{r_o}\right)^2 \frac{1}{2\pi r} = I_m \frac{r}{2\pi r_o^2}, \quad \text{for} \quad r \leq r_o. \tag{1.37}$$

Figure 1.6 shows a plot of the amplitude of the magnetic field intensity H_m as a function of r. The amplitude of the magnetic field intensity H_m is zero at the wire center because the enclosed current is zero. The waveform of the magnetic field inside the wire at low frequencies is given by

$$H(r,t) = I_m \frac{r}{2\pi r_o^2} \cos \omega t, \quad \text{for} \quad r \leq r_o. \tag{1.38}$$

Thus, the amplitude of the magnetic field intensity H_m inside the wire at radius r is determined solely by the amplitude of the current inside the radius r.

The magnetic field intensity outside the wire. The entire current $i = I_m \cos \omega t$ is enclosed by a path of radius $r \geq r_o$. From Ampère's law, the amplitude of the entire current i is

$$I_m = \oint_{C_2} \mathbf{H} \cdot d\mathbf{l} = H_m(r) \oint_{C_2} dl = 2\pi r H_m(r), \quad \text{for} \quad r \geq r_o, \tag{1.39}$$

where $C_2 = 2\pi r$ with $r \geq r_o$. The amplitude of the magnetic field intensity outside the conductor at any frequency is given by the expression

$$H_m(r) = \frac{I_m}{2\pi r}, \quad \text{for} \quad r \geq r_o, \tag{1.40}$$

and the waveform of this field is

$$H(r,t) = \frac{I_m}{2\pi r} \cos \omega t, \quad \text{for} \quad r \geq r_o. \tag{1.41}$$

The amplitude of the magnetic field intensity increases linearly with r inside the wire from 0 to $H_m(r_o) = I_m/(2\pi r_o)$ at low frequencies. The amplitude of the magnetic field intensity is inversely proportional to r outside the wire at any frequency.

1.4.2 Faraday's Law

A time-varying current produces a magnetic field, and a time-varying magnetic field can produce an electric current. In 1820, Hans Christian Oersted showed that a conductor carrying a current produces a magnetic field, which can affect a compass needle. Ampère measured that this magnetic field intensity is linearly related to the current which produces it. In 1831, Michael Faraday discovered that a current can be produced by an alternating magnetic field and that a time-varying magnetic field can induce a voltage, or an electromotive force, in an adjacent circuit. This voltage is proportional to the rate of change of magnetic flux linkage λ, or magnetic flux ϕ, or the current i, producing the magnetic field.

Faraday's law (1831) states that a time-varying magnetic flux $\phi(t)$ passing through a closed loop, such as an inductor turn, generates a voltage $v(t)$ in the loop and for a linear inductor it is expressed by

$$v(t) = \frac{d\lambda}{dt} = \frac{d(N\phi)}{dt} = N\frac{d\phi}{dt} = N\frac{d}{dt}\left(\frac{Ni}{\mathcal{R}}\right) = \frac{N^2}{\mathcal{R}}\frac{di}{dt} = L\frac{di}{dt}$$

$$= NA\frac{dB}{dt} = NA\mu\frac{dH}{dt} = \frac{\mu A N^2}{l}\frac{di}{dt}. \tag{1.42}$$

The voltage v is proportional to the rate of change of current i. This voltage, in turn, may produce a current in the circuit. The inductance L relates the induced voltage v to the current i. The voltage v across the terminals of an inductor L is proportional to the time rate of change of the current i in the inductor. If the inductor current is constant, the voltage across an ideal inductor is zero. The inductor behaves as a short circuit for a dc current. The inductor current cannot change instantaneously.

For sinusoidal waveforms, the derivative d/dt can be replaced by $j\omega$ and Faraday's law in phasor form can be expressed as

$$\mathbf{V_{Lm}} = j\omega\boldsymbol{\lambda}_{\mathbf{m}}. \tag{1.43}$$

For nonlinear, time-varying inductors, the relationships are

$$\lambda(t) = L(t)i(t) \tag{1.44}$$

and

$$v(t) = L(t)\frac{di(t)}{dt} + i(t)\frac{dL(t)}{dt}. \tag{1.45}$$

The impedance of lossless inductive components in terms of phasors of sinusoidal inductor current $\mathbf{I_{Lm}}$ and voltage $\mathbf{V_m} = j\omega\boldsymbol{\lambda}_{\mathbf{m}}$ is

$$Z = \frac{\mathbf{V_m}}{\mathbf{I_{Lm}}} = \frac{j\omega\lambda}{I_m} = j\omega L. \tag{1.46}$$

The impedance of lossy inductive components in terms of phasors is

$$Z = \frac{\mathbf{V_m}}{\mathbf{I_m}} = \frac{j\omega\lambda_m}{I_m} = R + j\omega L. \tag{1.47}$$

Since

$$vdt = L\left(\frac{di}{dt}\right)dt = Ldi, \tag{1.48}$$

the current in an inductor is given by

$$i(t) = \frac{1}{L}\int_0^t vdt + i(0) = \frac{1}{\omega L}\int_0^{\omega t} vd(\omega t) + i(0). \tag{1.49}$$

1.4.3 Lenz's Law

Lenz's law (1834) states that the voltage $v(t)$ induced by an applied time-varying magnetic flux $\phi_a(t)$ has a direction that induces current $i_E(t)$ in the closed loop, which in turn induces a magnetic flux $\phi_i(t)$ that tends to oppose the change in the applied flux $\phi_a(t)$, as illustrated in Figure 1.7. The direction of the induced current is always such that it produces a magnetic field that opposes the change in the original flux. If $\phi_a(t)$ increases, the induced current produces an opposing flux $\phi_i(t)$. If $\phi_a(t)$ decreases, the induced current produces an aiding flux $\phi_i(t)$. The induced currents in the closed loops are called *eddy currents*. Eddy currents occur when a conductor is subjected to time-varying magnetic fields. In accordance with Lenz's law, the eddy currents produce their own magnetic fields to oppose the original field. The effects of eddy currents on winding conductors and magnetic cores are: nonuniform current distribution, increased effective resistance, increased power loss, and reduced internal inductance. If the resistivity of a conductor was zero (as in a perfect conductor), eddy-current loops would be generated with such a magnitude and phase to exactly cancel the applied magnetic field. A perfect conductor would oppose any change in externally applied magnetic field. Circulating eddy currents would be induced to oppose any buildup of magnetic field in the conductor.

1.4.4 Ohms's Law

The point form of Ohm's law (1827) for conducting materials is

$$\mathbf{E} = \rho\mathbf{J} = \frac{\mathbf{J}}{\sigma}, \tag{1.50}$$

where ρ is the resistivity and $\sigma = 1/\rho$ is the conductivity of a material.

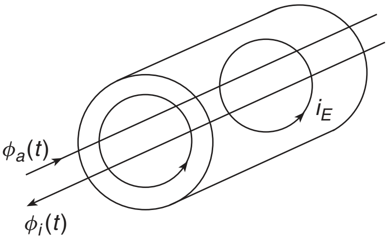

Figure 1.7 Illustration of Lenz's law generating eddy currents. The applied time-varying magnetic flux $\phi_a(t)$ induces eddy current $i_E(t)$, which in turn generates induced flux $\phi_i(t)$ that opposes changes in the applied flux $\phi_a(t)$.

1.4.5 Maxwell's Equations

Maxwell's equations (1865) govern electromagnetic waves. They couple electric fields, magnetic fields, and current densities. Maxwell's equations in differential (point) form in the time domain at any point in space and at any time are given by

$$\nabla \times \mathcal{H} = \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t} \quad \text{(Ampère's law)}, \tag{1.51}$$

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} = -\mu \frac{\partial \mathcal{H}}{\partial t} \quad \text{(Faraday's law)}, \tag{1.52}$$

$$\nabla \cdot \mathcal{D} = \rho_v \quad \text{(Gauss's law)}, \tag{1.53}$$

$$\nabla \cdot \mathcal{B} = 0 \quad \text{(flux continuity law)}. \tag{1.54}$$

where $\partial \mathcal{D}/\partial t$ is the displacement current density. The current density $\mathcal{J}$ and the volume charge density ρ_v are the sources of electromagnetic fields $\mathcal{H}$, $\mathcal{B} = \mu\mathcal{H}$, $\mathcal{E}$, and $\mathcal{D} = \epsilon\mathcal{E}$, where μ is the permeability and ϵ is the permittivity. They are related by the charge or current conservation equation

$$\nabla \cdot \mathcal{J} + \frac{\partial \rho_v}{\partial t} = 0. \tag{1.55}$$

Script letters are used to designate instantaneous field quantities, which are functions of position and time, e.g., $\mathcal{E}(x, y, z, t)$.

The phasor technique is a useful mathematical tool for solving problems in linear systems that involve periodic sinusoidal or nonsinusoidal waveforms in steady state. A periodic nonsinusoidal waveform, such as the rectangular wave, can be expanded into a Fourier series of sinusoidal components, which is a superposition of harmonic sinusoids. If the excitation is a sinusoidal function of time, the steady-state waveforms described in the time domain can be represented by phasors and integro-differential equations become linear equations with no sinusoidal functions, which are easy to solve. Differentiation in the time domain is equivalent to multiplication by $j\omega$ in the phasor domain, and integration in the time domain is equivalent to division by $j\omega$ in the phasor domain. The solutions in the phasor domain can be converted back into the time domain.

The electric field intensity for the one-dimensional case in the time domain is given by

$$\mathcal{E}(x, t) = E_m(0)e^{-\frac{x}{\delta_w}} \cos\left(\omega t - \frac{x}{\delta_w} + \phi_o\right) = \mathrm{Re}\{\mathbf{E}(x)e^{j\omega t}\}, \tag{1.56}$$

where δ_w is the skin depth and the phasor of the electric field intensity is

$$\mathbf{E}(x) = E_m(0)e^{-\frac{x}{\delta_w}} e^{-j\frac{x}{\delta_w}} e^{j\phi_o}. \tag{1.57}$$

Similarly, the magnetic field intensity is

$$\mathcal{H}(x, t) = H_m(0)e^{-\frac{x}{\delta_w}} \cos\left(\omega t - \frac{x}{\delta_w} + \theta_o\right) = \mathrm{Re}\{\mathbf{H}(x)e^{j\omega t}\}, \tag{1.58}$$

where the phasor of the magnetic field intensity is

$$\mathbf{H}(x) = H_m(0)e^{-\frac{x}{\delta_w}} e^{-j\frac{x}{\delta_w}} e^{j\theta_o}. \tag{1.59}$$

Substituting the electric and magnetic field intensities into Maxwell's equation in the time domain, we obtain

$$\nabla \times \mathrm{Re}\{\mathbf{E}(x)e^{j\omega t}\} = -\frac{\partial}{\partial t}\mathrm{Re}\{\mu\mathbf{H}(x)e^{j\omega t}\}, \tag{1.60}$$

which becomes

$$\mathrm{Re}\{\nabla \times \mathbf{E}(x)e^{j\omega t}\} = \mathrm{Re}\{-j\omega\mu\mathbf{H}(x)e^{j\omega t}\}. \tag{1.61}$$

Thus, $\frac{\partial}{\partial t}$ in Maxwell's equations in the time domain can be replaced by $j\omega$ to obtain Maxwell's equations for sinusoidal field waveforms in phasor form:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} = -j\omega\mathbf{B}, \tag{1.62}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D} = (\sigma + j\omega\epsilon)\mathbf{E}, \tag{1.63}$$

$$\nabla \cdot \mathbf{D} = \rho_v, \tag{1.64}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{1.65}$$

The constitutive equations for linear and isotropic materials are:

$$\mathbf{B} = \mu\mathbf{H}, \tag{1.66}$$

$$\mathbf{D} = \epsilon\mathbf{H}, \tag{1.67}$$

$$\mathbf{J} = \sigma\mathbf{E}. \tag{1.68}$$

1.4.6 Maxwell's Equations for Good Conductors

In general, Maxwell's equation in phasor form, which is the differential form of Ampère's equation, together with Ohm's law ($\mathbf{J} = \sigma\mathbf{E}$), is given by

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D} = \sigma\mathbf{E} + j\omega\epsilon\mathbf{E} = (\sigma + j\omega\epsilon)\mathbf{E}, \tag{1.69}$$

where σ is the conductivity. For good conductors, conduction current density $\mathbf{J} = \sigma\mathbf{E}$ dominates displacement current density $j\omega\epsilon\mathbf{E}$, and the following inequality is satisfied:

$$\sigma \gg \omega\epsilon. \tag{1.70}$$

For copper, this inequality is satisfied for frequencies up to 10^{16} Hz. Maxwell's equation for good conductors becomes

$$\nabla \times \mathbf{H} \approx \mathbf{J}. \tag{1.71}$$

Maxwell's equation in phasor form, which is the differential form of Faraday's law, is expressed as

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}. \tag{1.72}$$

Using Ohm's law $\mathbf{E} = \mathbf{J}/\sigma$, we obtain

$$\nabla \times \frac{\mathbf{J}}{\sigma} = -j\omega\mu\mathbf{H}, \tag{1.73}$$

producing another form of Maxwell's equation,

$$\nabla \times \mathbf{J} = -j\omega\mu\sigma\mathbf{H}. \tag{1.74}$$

Assuming that σ and μ are homogeneous, taking the curl on both sides of the above equation, and substituting into Maxwell's equation,

$$\nabla \times (\nabla \times \mathbf{J}) = -j\omega\mu\sigma\nabla \times \mathbf{H} = -j\omega\mu\sigma\mathbf{J}. \tag{1.75}$$

Expanding the left-hand side,

$$\nabla(\nabla \cdot \mathbf{J}) - \nabla^2\mathbf{J} = -j\omega\mu\sigma\mathbf{J}, \tag{1.76}$$

where the principle of conservation of charge states that charge can be neither created nor destroyed, and its point form is expressed by $\nabla \cdot \mathbf{J} = 0$. It is a point form of Kirchhoff's current law. The conduction current density $\mathbf{J}$ in good conductors must satisfy the second-order partial differential equation

$$\nabla^2\mathbf{J} = j\omega\mu\sigma\mathbf{J}. \tag{1.77}$$

For good conductors,

$$\nabla \cdot (\nabla \times \mathbf{H}) = (\sigma + j\omega\epsilon)(\nabla \cdot \mathbf{E}) = 0. \tag{1.78}$$

Hence, Maxwell's equation for good conductors becomes

$$\nabla \cdot \mathbf{D} = \rho_v = 0. \tag{1.79}$$

In good conductors, mobile electrons drift through a lattice of positive ions encountering frequent collisions. On average, the net charge over a large volume (compared with atomic dimensions) is zero even though some of the charges are moving and causing current flow. The net movement velocity or drift velocity is proportional to the electric field intensity.

1.4.7 Poynting Vector

The *instantaneous Poynting vector* (1884) at a given point is expressed by

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}} \quad (\mathrm{W/m^2}). \tag{1.80}$$

The Poynting vector represents the direction and density of power flow at any point, i.e., it is the rate at which energy flows through a unit surface area perpendicular to direction of wave propagation. The direction of $\boldsymbol{\mathcal{S}}$ is normal to both $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$. The cross product $\mathbf{E} \times \mathbf{H}$ points in the direction of power flow. The vector $\boldsymbol{\mathcal{S}}$ represents an instantaneous surface power density, i.e., *an instantaneous power per unit area*. Since $\boldsymbol{\mathcal{E}}$ is measured in V/m and $\boldsymbol{\mathcal{H}}$ in A/m, $\boldsymbol{\mathcal{S}}$ is measured in (V/m) $\times$ (A/m) $=$ VA/m^2 $=$ W/m^2.

For time-harmonic fields, the *complex Poynting vector* is

$$\mathbf{S}_c = \mathbf{E} \times \mathbf{H}^* \quad (\mathrm{W/m^2}). \tag{1.81}$$

The time-averaged power density (i.e., averaged over one period of the sinusoidal excitation) is given by the *time-averaged Poynting vector*

$$\mathbf{S}_{av} = \frac{1}{2}\mathrm{Re}\{\mathbf{E} \times \mathbf{H}^*\} \quad (\mathrm{W/m^2}). \tag{1.82}$$

The amount of time-averaged power passing through a surface S is

$$P_{av} = \int_S \mathbf{S}_{av} \cdot d\mathbf{s} = \frac{1}{2}\ \mathrm{Re}\left\{\int_s \left(\mathrm{E} \times \mathrm{H}^*\right) \bullet ds\right\} \quad (\mathrm{W}). \tag{1.83}$$

For a linear, isotropic, and time-invariant medium, the Poynting theorem is

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = -\int_V \mathbf{J} \cdot \mathbf{E} dV - \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right) dV. \tag{1.84}$$

For sinusoidal field waveforms,

$$\begin{aligned}\oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} &= -\int_V [\mathbf{E} \cdot \mathbf{J}^* + j\omega(\mathbf{H}^* \cdot \mathbf{B} + \mathbf{E}^* \cdot \mathbf{D}]dV \\ &= -\frac{1}{2}\int_V \rho J^2 dV - j\omega \int_V \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right) dV \\ &= -\int_V p_d dV - \frac{\partial}{d\partial}\int_V (w_m + w_e) dV,\end{aligned} \tag{1.85}$$

where an asterisk * indicates a complex conjugate quantity, $\mathbf{E} = \rho\mathbf{J}$ is point Ohm's law, $w_m = \frac{1}{2}\mu H^2$ is the point magnetic energy density, and $w_e = \frac{1}{2}\epsilon E^2$ is the point electric energy density. The first term on the right-hand side of (1.85) represents the ohmic power dissipated as heat in the volume V as a result of the flow of conduction current density $\mathbf{J} = \sigma\mathbf{E}$ due to the presence of the electric field $\mathbf{E}$. This power exits the volume V through its surface S. The second and third terms represent the time rate of change (decrease) of the magnetic and electric energies stored in the magnetic and electric fields, respectively. The principle of conservation of energy states in this case that the total power flow out of a closed surface S at any instant is equal to the sum of the ohmic power dissipated within the enclosed volume V and the rates of decrease of the stored magnetic and electric energies.

In steady state, the complex power flowing into a volume V surrounded by a closed surface S is given by

$$P = \frac{1}{2}\oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} = P_D + 2j\omega(W_m + W_e) \quad \text{(W)}, \tag{1.86}$$

where the time-averaged real power dissipated in the volume V is given by Joule's law as

$$P_D = \frac{1}{2}\int_V \mathbf{E} \cdot \mathbf{J}^* dV = \frac{1}{2}\int_V \rho|\mathbf{J}|^2 dV = \frac{1}{2}\int_V \sigma|\mathbf{E}|^2 dV. \tag{1.87}$$

$w_m = \mu H^2/2$ (J/m^3) is the magnetic energy density stored in the magnetic field in the volume V, $w_e = \epsilon E^2/2$ (J/m^3) is the electric energy density stored in the electric field in the volume V, $\mathbf{B} \cdot \mathbf{H}/2$ is the magnetic energy density, $\mathbf{D} \cdot \mathbf{E}/2$ is the electric energy density, and $p_D = \rho J^2$ (W/m^3) is the ohmic power loss density.

1.4.8 Joule's Law

The current density in a conductor in the time domain in steady state for the one-dimensional case is described by

$$\mathrm{Re}\{\mathbf{J}(x)e^{j\omega t}\} = J(x,t) = J_m(0)e^{-\frac{x}{\delta_w}} \cos\left(\omega t - \frac{x}{\delta_w} + \phi_o\right), \tag{1.88}$$

where δ_w is the skin depth and ϕ_o is the initial phase. It is assumed that the current amplitude varies only in the x direction. From Ohm's law,

$$E(x,t) = \rho J(x,t) = \rho J_m(0)e^{-\frac{x}{\delta_w}} \cos\left(\omega t - \frac{x}{\delta_w} + \phi_o\right), \tag{1.89}$$

where $E_m(0) = \rho J_m(0)$. Assuming that ρ is a real number, the phase shift between $J(x,t)$ and $E(x,t)$ is zero. The instantaneous power density at a point is given by

$$\begin{aligned} p(x,t) = J(x,t)E(x,t) &= J_m(0)E_m(0)e^{-\frac{2x}{\delta_w}} \cos^2\left(\omega t - \frac{x}{\delta_w} + \phi_o\right) \\ &= \rho J_m^2(0)e^{-\frac{2x}{\delta_w}} \cos^2\left(\omega t - \frac{x}{\delta_w} + \phi_o\right) \\ &= \frac{J_m(0)E_m(0)}{2}e^{-\frac{2x}{\delta_w}} + \frac{J_m(0)E_m(0)}{2}e^{-\frac{2x}{\delta_w}} \cos 2\left(\omega t - \frac{x}{\delta_w} + \phi_o\right) \\ &= \frac{\rho J_m^2(0)}{2}e^{-\frac{2x}{\delta_w}} + \frac{\rho J_m^2(0)}{2}e^{-\frac{2x}{\delta_w}} \cos 2\left(\omega t - \frac{x}{\delta_w} + \phi_o\right) \\ &= p_D(x) + p_D(x) \cos 2\left(\omega t - \frac{x}{\delta_w} + \phi_o\right), \end{aligned} \tag{1.90}$$

where $\cos^2 z = \frac{1}{2} + \frac{1}{2}\cos 2z$. The first term in the above equation represents the time-averaged real power density dissipated in a conductor at a point, and the second term represents the ac component of the instantaneous real power density dissipated in a conductor as heat at a point. The time-averaged real power density dissipated in a conductor at a point is

$$p_D(x) = \frac{1}{T}\int_0^T p(x,t)dt = \frac{1}{2\pi}\int_0^{2\pi} p(x,\omega t)d(\omega t) = \frac{J_m(0)E_m(0)}{2}e^{-\frac{2x}{\delta_w}} = \frac{\rho J_m^2(0)}{2}e^{-\frac{2x}{\delta_w}}, \tag{1.91}$$

where T is the period. The total time-averaged power dissipated as heat in a conductor of volume V is

$$P_D = \int\int\int_V p_D(x)dV = \frac{1}{2}\int\int\int_V J_m(0)E_m(0)e^{-\frac{2x}{\delta_w}}dxdydz$$
$$= \frac{1}{2}\int\int\int_V \rho J_m^2(0)e^{-\frac{2x}{\delta_w}}dxdydz. \tag{1.92}$$

When electromagnetic fields are sinusoidal, phasors are described in space as follows: $\mathbf{H}(\mathbf{r}) = \mathbf{H}(x, y, z)$, $\mathbf{E}(\mathbf{r}) = \mathbf{E}(x, y, z)$, and $\mathbf{J}(\mathbf{r}) = \mathbf{J}(x, y, z)$. The point (local) power density is

$$p(\mathbf{r}, t) = \text{Re}\{\mathbf{J}(\mathbf{r}, t\} \cdot \text{Re}\{\mathbf{E}(\mathbf{r}, t\} = \frac{1}{4}\left[\mathbf{J}(\mathbf{r}, t) + \mathbf{J}^*(\mathbf{r}, t)\right]\left[\mathbf{E}(\mathbf{r}, t) + \mathbf{E}^*(\mathbf{r}, t)\right]$$
$$= \frac{1}{4}\left[\mathbf{J}(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})e^{2j\omega t} + \mathbf{J}^*(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) + \mathbf{J}^*(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r})e^{-2j\omega t}\right]$$
$$= \frac{1}{2}\text{Re}\left[\mathbf{J}(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})e^{2j\omega t}\right]. \tag{1.93}$$

The time-averaged real power density dissipated in a conductor at a point $\mathbf{r}$ is

$$p_D(\mathbf{r}) = \frac{1}{T}\int_0^T p(\mathbf{r}, t)dt = \frac{1}{2}\text{Re}\left[\mathbf{J}(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r})\right]. \tag{1.94}$$

The time-averaged power dissipated as heat in the conductor of volume V is given by

$$P_D = \int\int\int_V p_D(\mathbf{r})dV = \frac{1}{2}\text{Re}\int\int\int_V \mathbf{J}(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r})dV$$
$$= \frac{1}{2}\int\int\int_V \rho\mathbf{J}(\mathbf{r}) \cdot \mathbf{J}^*(\mathbf{r})dV = \frac{1}{2}\int\int\int_V \rho|\mathbf{J}(\mathbf{r})|^2 dV. \tag{1.95}$$

The current density in phasor form for the one-dimensional case is given by

$$\mathbf{J}(x) = J_m(0)e^{-\frac{x}{\delta_w}}e^{-j\frac{x}{\delta_w}}e^{j\phi_o} = J_m(x)e^{j(\phi_o - \frac{x}{\delta_w})} \tag{1.96}$$

where the amplitude is

$$J_m(x) = J_m(0)e^{-\frac{x}{\delta_w}}. \tag{1.97}$$

The time-averaged point power density for sinusoidal waveforms is given by point Joule's law in phasor form:

$$P_D(x) = \frac{1}{2}\text{Re}(\mathbf{J} \cdot \mathbf{E}^*) = \frac{1}{2}\rho\mathbf{J} \cdot \mathbf{J}^* = \frac{1}{2}\rho|J(x)|^2 = \frac{1}{2}\rho J_m^2(0)e^{-\frac{2x}{\delta_w}}, \tag{1.98}$$

where δ_w is the skin depth. For periodic waveforms, the time-averaged real power dissipated in a conductor of volume V and resistivity ρ due to conversion of electromagnetic energy to thermal energy (heat) is given by Joule's law in phasor form:

$$P_D = \frac{1}{2}\text{Re}\int\int\int_V \mathbf{J} \cdot \mathbf{E}^* dV = \frac{1}{2}\int\int\int_V \rho\mathbf{J} \cdot \mathbf{J}^* dV = \frac{1}{2}\int\int\int_V \rho|J|^2 dV, \tag{1.99}$$

where J and E are the amplitudes of the current density and the electric field intensity, respectively. The time-averaged power loss density P_v is defined as the total time-averaged power loss P_D per unit volume,

$$P_v = \frac{P_D}{V}, \tag{1.100}$$

where V is the volume carrying the current.

Since $\mathbf{B} = \mu\mathbf{H}$, the point (local) magnetic energy density for sinusoidal waveforms is given by

$$w_m(x) = \frac{1}{2}\mathbf{B} \cdot \mathbf{H}^* = \frac{1}{2}\mu\mathbf{H} \cdot \mathbf{H}^* = \frac{1}{2}\mu|H(x)|^2. \tag{1.101}$$

The total magnetic energy stored in inductor L is given by

$$W_m = \int ivdt = \int iL\frac{di}{dt}dt = L\int_0^{I_m} idi = \frac{1}{2}LI_m^2 = \frac{1}{2}\frac{N^2}{R} = \frac{F_m^2}{2R} = \frac{1}{2}\frac{\mu N^2 S}{l}\left(\frac{Bl}{\mu N}\right)^2$$
$$= \frac{1}{2}\int\int\int_V \mu|H(x)|^2 dV, \tag{1.102}$$

where V is the volume of the interior of the inductor and $v = Ldi/dt$. The magnetic energy density w_m is defined as the magnetic energy W_m per unit volume,

$$w_m = \frac{W_m}{V}. \tag{1.103}$$

The time-averaged local magnetic energy density is given by

$$w_m(x) = \frac{1}{2}\mathbf{B}\cdot\mathbf{H}^* = \frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H}^* = \frac{1}{2}\mu|H(x)|^2 \left(\frac{\mathrm{J}}{\mathrm{m}^3}\right). \tag{1.104}$$

The total time-averaged magnetic energy is

$$W_m = \frac{1}{2}\int\int\int_V \mathbf{B(x)}\cdot\mathbf{H}^*(x)dV = \frac{1}{2}\int\int\int_V \mu|H(x)|^2 dV \ \ (\mathrm{J}). \tag{1.105}$$

1.5 Eddy Currents

Figure 1.8 illustrates eddy current induced by a time-varying magnetic field. Eddy currents circulate in closed paths. In a conductor, the induced magnetic field may be caused by the conductor's own ac current or by the ac current of other adjacent conductors. According to Lenz's law, the magnetic field induces eddy currents, which generate magnetic field that opposes the original magnetic field. According to Ampère's law,

$$\Delta \times \mathbf{H} = \mathbf{J}_a + \mathbf{J}_e, \tag{1.106}$$

where $\mathbf{J}_a$ is the applied current and $\mathbf{J}_e$ is the eddy current. When the applied current $\mathbf{J}_a$ is zero and the magnetic field is generated by adjacent conductors, we have

$$\Delta \times \mathbf{H} = \mathbf{J}_e. \tag{1.107}$$

The eddy-current density can be described by

$$\mathbf{J}_e = \sigma\mathbf{E} = \frac{\mathbf{E}}{\rho}. \tag{1.108}$$

For sinusoidal waveforms, the phasor of the eddy-current density is given by

$$\mathbf{J}_e = -j\omega\sigma\mathbf{A}, \tag{1.109}$$

where $\mathbf{A}$ is the phasor of the magnetic vector potential.

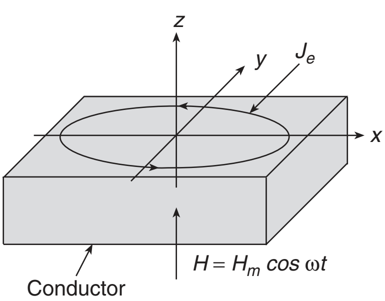

Figure 1.8 Eddy current.

1.6 Core Saturation

For an inductor with a magnetic core of cross-sectional area A_c and a saturation magnetic flux density B_s, the magnetic flux at which the magnetic core begins to saturate is

$$\phi_s = A_c B_s. \tag{1.110}$$

Hence, the magnetic flux linkage at which the magnetic core begins to saturate is given by

$$\lambda_s = N\phi_s = NA_c B_s = LI_{m(max)}. \tag{1.111}$$

Thus,

$$N_{max} A_c B_{pk} = LI_{m(max)}, \tag{1.112}$$

yielding the maximum number of turns

$$N_{max} = \frac{LI_{m(max)}}{A_c B_{max}}. \tag{1.113}$$

According to (1.2), the magnetic field intensity H is proportional to $\mathcal{F} = Ni$. Therefore, there is a maximum amplitude of the inductor current $I_{m(max)}$ at which the core saturates. Figure 1.9 shows plots of B as a function of H and i. Since

$$B_s = \mu H_s = \mu \frac{NI_{m(max)}}{l_c}, \tag{1.114}$$

to avoid core saturation, the *ampere-turn limit* is given by

$$N_{max} I_{m(max)} = \frac{B_s l_c}{\mu_{rc}\mu_0} = B_s A_c \mathcal{R} = \frac{B_s l_c}{\mu_{rc}\mu_0}. \tag{1.115}$$

From Faraday's law, $d\lambda = v_L(t)dt$. Hence, the general relationship between the inductor voltage and the flux linkage is given by

$$\lambda(t) = \int_0^t v_L(t)dt + \lambda(0) = \frac{1}{\omega}\int_0^{\omega t} v_L(\omega t)d(\omega t) + \lambda(0). \tag{1.116}$$

For a transformer,

$$(N_1 i_1 + N_2 i_2 + \cdots)_{max} \le B_s A_c R = \frac{B_s l_c}{\mu_{rc}\mu_0}.$$

1.6.1 Core Saturation for Sinusoidal Inductor Voltage

Consider an inductor with a magnetic core of saturation flux density B_s. Figure 1.10 shows sinusoidal waveforms of the inductor voltage v_L and the magnetic flux linkage λ. The dc components of these waveforms are assumed to be zero. The inductor voltage is given by

$$v_L = V_{Lm} \sin \omega t. \tag{1.117}$$

The magnetic flux linkage is

$$\begin{aligned}\lambda(t) &= \frac{1}{\omega}\int_0^{\omega t} v_L(\omega t)d(\omega t) + \lambda(0) = \frac{1}{\omega}\int_0^{\omega t} V_{Lm}\sin \omega t d(\omega t) + \lambda(0)\\ &= \frac{V_{Lm}}{\omega}(1 - \cos \omega t) + \lambda(0) = \frac{V_{Lm}}{\omega} - \frac{V_{Lm}}{\omega}\cos \omega t + \lambda(0).\end{aligned} \tag{1.118}$$

Thus, the peak-to-peak value of the magnetic flux linkage is

$$\Delta\lambda = \lambda(\pi) - \lambda(0) = \frac{2V_{Lm}}{\omega} = N\phi = NA_c B_m < NA_c B_s. \tag{1.119}$$

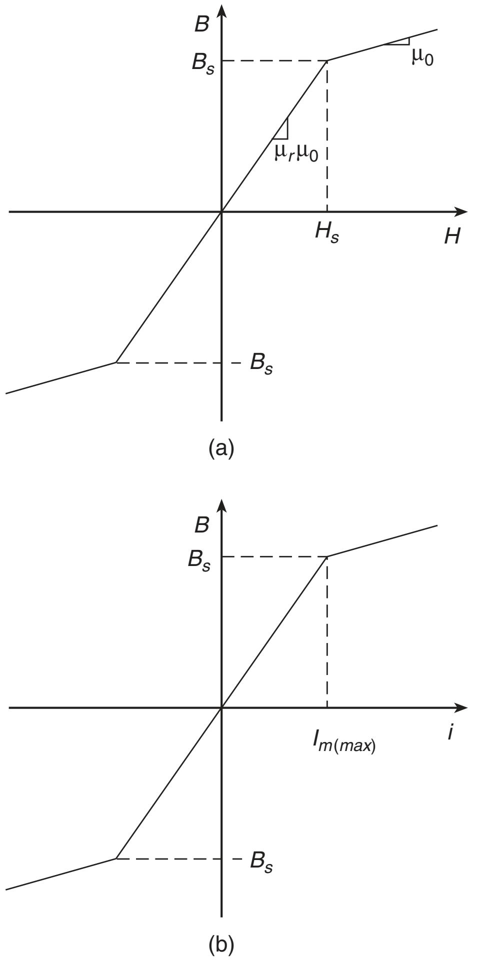

Figure 1.9 (a) Magnetic flux density B as a function of magnetic field intensity H. (b) Magnetic flux density B as a function of inductor current i at a fixed number of turns N.

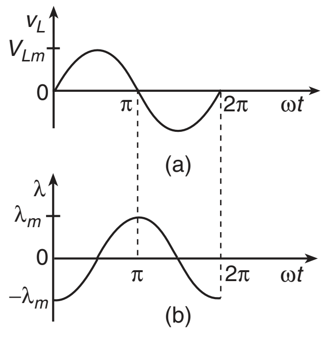

Figure 1.10 Waveforms of the inductor voltage and the magnetic flux linkage for sinusoidal inductor voltage. (a) Waveform of the inductor voltage v_L. (b) Waveform of the magnetic flux linkage λ.

The initial value of the flux linkage is

$$\lambda(0) = -\frac{\Delta\lambda}{2} = -\frac{V_{Lm}}{\omega}. \tag{1.120}$$

The steady-state waveform of the magnetic flux linkage is given by

$$\lambda(t) = -\frac{V_{Lm}}{\omega}\cos\omega t = -\lambda_m \cos\omega t, \tag{1.121}$$

where the amplitude of the flux linkage is

$$\lambda_m = \frac{V_{Lm}}{\omega}. \tag{1.122}$$

Thus, the amplitude of the magnetic flux linkage λ_m increases as the frequency f decreases. The minimum frequency f_{min} occurs when the amplitude of the magnetic flux linkage λ_m reaches the saturation value λ_s:

$$\lambda_m = \lambda_s = \frac{V_{Lm(max)}}{\omega_{min}}. \tag{1.123}$$

The lowest frequency at which the inductor can operate without saturating the core is given by

$$f_{min} = \frac{V_{Lm(max)}}{2\pi\lambda_s} = \frac{V_{Lm(max)}}{2\pi NA_cB_s} = \frac{\sqrt{2}V_{Lrms(max)}}{2\pi NA_cB_s} = \frac{V_{Lrms(max)}}{K_fNA_cB_s}, \tag{1.124}$$

where the waveform factor for a sinusoidal inductor voltage is

$$K_f = \frac{2\pi}{\sqrt{2}} = 4.44. \tag{1.125}$$

The minimum frequency f_{min} decreases as N increases, A_c increases, B_s increases, and $V_{Lm(max)}$ decreases. As the temperature increases, B_s decreases. For ferrite cores, B_s may decrease by a factor of 2 as T increases from room temperature to 100°C.

Another method to derive the minimum frequency is as follows. Assume that the initial condition is $\lambda(0) = -\lambda_s$. The magnetic flux linkage at core saturation is given by

$$\lambda_s = \frac{1}{\omega_{min}}\int_0^{\pi} v_L d(\omega t) + \lambda(0) = \frac{1}{\omega_{min}}\int_0^{\pi} V_{Lm}\sin\omega t d(\omega t) - \lambda_s = \frac{2V_{Lm}}{\omega_{min}} - \lambda_s, \tag{1.126}$$

resulting in

$$\lambda_s = \frac{\Delta\lambda_{max}}{2} = \frac{V_{Lm(max)}}{\omega_{min}}. \tag{1.127}$$

Hence, the lowest frequency at which the inductor can operate without saturating the core is given by

$$f_{min} = \frac{V_{Lm(max)}}{2\pi\lambda_s} = \frac{V_{Lm(max)}}{2\pi NA_cB_s}. \tag{1.128}$$

The maximum root mean square (rms) value of the sinusoidal voltage across an inductor is

$$V_{Lrms(max)} = \frac{\omega NA_cB_s}{\sqrt{2}} = \frac{2\pi fNA_cB_s}{\sqrt{2}} = K_f fNA_cB_s = 4.44fNA_cB_s. \tag{1.129}$$

1.6.2 Core Saturation for Square Wave Inductor Voltage

If the inductor voltage waveform is a square wave $\pm V$, the magnetic flux linkage is a symmetrical triangular wave, as shown in Figure 1.11. For the first half of the cycle,

$$v_L = V, \quad \text{for } 0 \le t \le \frac{T}{2}, \tag{1.130}$$

and

$$\lambda(t) = \int_0^t v_L(t)dt + \lambda(0) = \int_0^t Vdt + \lambda(0) = Vt + \lambda(0), \quad \text{for } 0 \le t \le \frac{T}{2}. \tag{1.131}$$

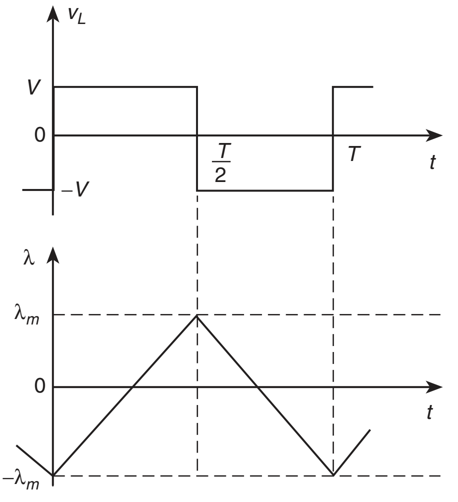

Figure 1.11 Waveforms of the inductor voltage and the magnetic flux linkage for a square wave inductor voltage.

The flux linkage at $t = T/2$ is

$$\lambda\left(\frac{T}{2}\right) = \frac{VT}{2} + \lambda(0). \tag{1.132}$$

For the second half of the cycle,

$$v_L = -V, \quad \text{for } \frac{T}{2} \le t \le T, \tag{1.133}$$

and

$$\begin{aligned}\lambda(t) &= \int_{T/2}^{t} v_L(t)dt + \lambda\left(\frac{T}{2}\right) = \int_{T/2}^{t} (-V)dt + \lambda\left(\frac{T}{2}\right) \\ &= -V\left(t - \frac{T}{2}\right) + \lambda\left(\frac{T}{2}\right), \quad \text{for } \frac{T}{2} \le t \le T.\end{aligned} \tag{1.134}$$

Hence, the peak-to-peak value of the magnetic flux linkage is

$$\Delta\lambda = \lambda\left(\frac{T}{2}\right) - \lambda(0) = \frac{VT}{2} + \lambda(0) - \lambda(0) = \frac{VT}{2} = \frac{V}{2f}, \tag{1.135}$$

$$-\lambda_m = \lambda(0) = -\frac{\Delta\lambda}{2} = -\frac{VT}{4} = -\frac{V}{4f}, \tag{1.136}$$

and

$$\lambda_m = \lambda\left(\frac{T}{2}\right) = \frac{\Delta\lambda}{2} = \frac{VT}{4} = \frac{V}{4f}. \tag{1.137}$$

The steady-state waveform of the magnetic linkage is

$$\lambda(t) = Vt - \frac{V}{4f}, \quad \text{for } 0 \le t \le \frac{T}{2}, \tag{1.138}$$

and

$$\lambda(t) = -V\left(t - \frac{T}{2}\right) + \frac{V}{4f}, \quad \text{for } \frac{T}{2} \le t \le T. \tag{1.139}$$

The rms value of the square wave inductor voltage is obtained as

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v_L^2 dt} = \sqrt{\frac{1}{T}\int_0^T V^2 dt} = V. \tag{1.140}$$

For core saturation,

$$\lambda_s = \frac{\Delta\lambda_m}{2} = \frac{V_{max}}{4f_{min}} = N\phi_s = NA_cB_s, \tag{1.141}$$

where $V_{max} = V_{Lrms(max)}$ for the square wave inductor current. The minimum frequency at which the core can be operated without saturation is given by

$$f_{min} = \frac{V_{max}}{4\lambda_s} = \frac{V_{max}}{4NA_cB_s} = \frac{V_{max}}{K_fNA_cB_s} \tag{1.142}$$

where the waveform coefficient of the square inductor voltage is

$$K_f = 4. \tag{1.143}$$

The maximum peak voltage of the square inductor voltage at the operating frequency f is

$$V_{rms} = V_{max} = 4fNA_cB_s. \tag{1.144}$$

In general, the minimum core cross-sectional area is given by

$$A_c = \frac{V_{Lrms}}{K_ff_{min}NB_{pk}} = \frac{V_{Lrms}}{K_ff_{min}N(B_{DC}+B_m)}, \tag{1.145}$$

where

$$B_{pk} = B_{DC} + B_m \le B_s, \quad \text{for} \quad T \le T_{max}, \tag{1.146}$$

and K_f is the waveform coefficient of the inductor voltage. The peak value of the flux density B_{pk} must be lower than B_s at the maximum operating temperature T_{max} to avoid core saturation. The amplitude of the ac component of the flux density B_m must be limited to avoid core saturation or to reduce core loss. As the amplitude of the ac component of the flux density B_m increases, the core loss also increases.

The saturation flux density B_s limits the maximum amplitude of the magnetic field intensity

$$H_s = H_{m(max)} = \frac{B_s}{\mu_{rc}\mu_0} = \frac{NI_{Lm(max)}}{l_c}. \tag{1.147}$$

The maximum amplitude of the current in the winding at which the core saturates is

$$I_{m(max)} = \frac{l_cB_s}{N\mu_{rc}\mu_0}. \tag{1.148}$$

As the amplitude of the inductor current I_{Lm} increases, the amplitude of the magnetic field H_m also increases. To avoid core saturation,

$$NI_{Lm(max)} < \frac{B_sl_c}{\mu_{rc}\mu_0}. \tag{1.149}$$

When a core with an air gap is used, both amplitudes H_m and I_{Lm} can be increased to

$$H_{m(max)} = \frac{B_s}{\mu_{re}\mu_0} \tag{1.150}$$

and

$$I_{Lm(max)} = \frac{l_cB_s}{N\mu_{re}\mu_0}, \tag{1.151}$$

where μ_{re} is the core effective relative permeability.

1.6.3 Core Saturation for Rectangular Wave Inductor Voltage

Consider the situation where the inductor voltage waveform is a rectangular wave whose high level is V_H and low level is $-V_L$, as depicted in Figure 1.12. The magnetic flux linkage is an asymmetrical triangular wave, as shown in Figure 1.11. For the first part of the cycle,

$$v_L = V_H, \quad \text{for} \quad 0 \le t \le DT, \tag{1.152}$$

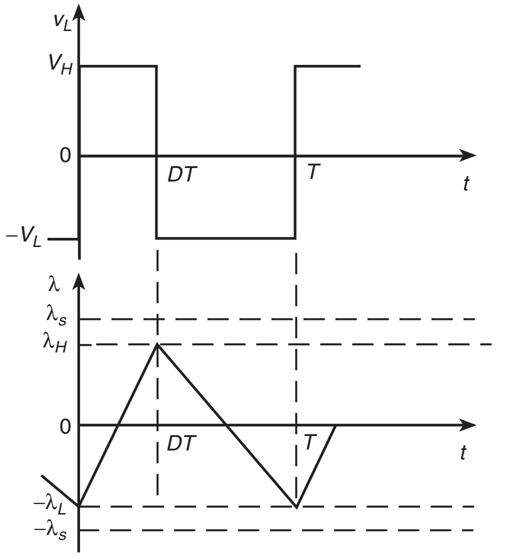

Figure 1.12 Waveforms of the inductor voltage and the magnetic flux linkage for a rectangular wave inductor voltage.

and

$$\lambda(t) = \int_0^t v_L(t)dt + \lambda(0) = \int_0^t V_H dt + \lambda(0) = V_H t + \lambda(0), \quad \text{for} \;\; 0 \le t \le DT, \tag{1.153}$$

where D is the duty cycle. The flux linkage at $t = DT$ is given by

$$\lambda(DT) = V_H DT + \lambda(0). \tag{1.154}$$

For the second part of the cycle,

$$v_L = -V_L, \quad \text{for} \;\; DT \le t \le T, \tag{1.155}$$

and

$$\begin{aligned}\lambda(t) &= \int_{DT}^t v_L(t)dt + \lambda(DT) = \int_{DT}^t (-V_L)dt + \lambda(DT) \\ &= -V_L(t - DT) + \lambda(DT), \quad \text{for} \;\; DT \le t \le T.\end{aligned} \tag{1.156}$$

Hence, the peak-to-peak value of the magnetic flux linkage is

$$\Delta\lambda = \lambda(DT) - \lambda(0) = V_H DT + \lambda(0) - \lambda(0) = V_H DT = \frac{DV_H}{f}. \tag{1.157}$$

The rms value of the inductor voltage is

$$V_{Lrms} = \sqrt{\frac{1}{T}\int_0^T v_L^2 dt} = \sqrt{\frac{1}{T}\left(\int_0^{DT} V_H^2 dt + \int_{DT}^T V_L^2 dt\right)} = \sqrt{DV_H^2 + (1 - D)V_L^2}. \tag{1.158}$$

Using the volt-second balance law, we obtain

$$V_H DT = V_L(1 - D)T \tag{1.159}$$

yielding

$$\frac{V_L}{V_H} = \frac{D}{1 - D}. \tag{1.160}$$

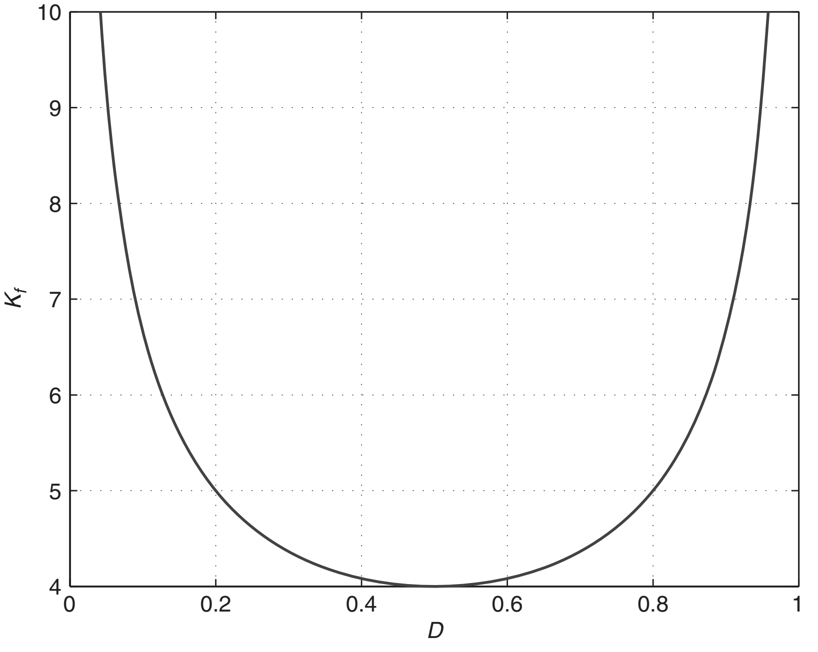

Figure 1.13 Waveform coefficient K_f as a function of duty cycle D.

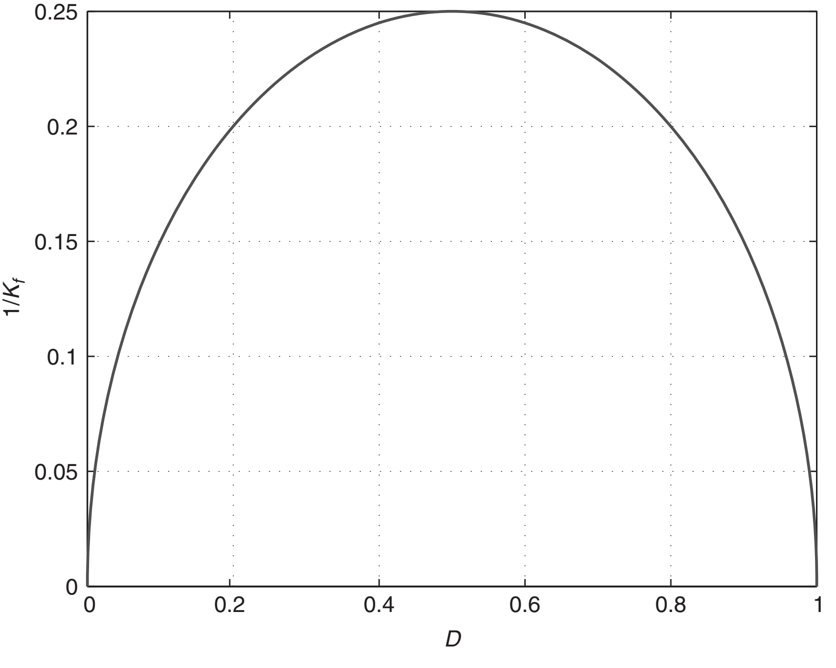

Figure 1.14 Coefficient $1/K_f$ as a function of duty cycle D.

Therefore,

$$V_{Lrms} = V_H \sqrt{\frac{D}{1-D}} = V_L \sqrt{\frac{1-D}{D}}. \tag{1.161}$$

The flux linkage at the beginning of core saturation is

$$\lambda_s = \frac{\Delta\lambda_{max}}{2} = \frac{DV_H}{2f_{min}} = N\phi_s = NA_c B_s. \tag{1.162}$$

Hence, the minimum operating frequency is

$$f_{min} = \frac{DV_H}{2NA_c B_s} = \frac{V_{Lrms}}{NA_c B_s}\frac{\sqrt{D(1-D)}}{2} = \frac{V_{Lrms}}{K_f NA_c B_s}, \tag{1.163}$$

where the waveform coefficient is

$$K_f = \frac{2}{\sqrt{D(1-D)}}. \tag{1.164}$$

The minimum cross-sectional area is given by

$$A_c = \frac{V_{Lrms}}{K_{fmax} f B_s}. \tag{1.165}$$

Figure 1.13 shows a plot of K_f as a function of the duty cycle D; like many of the figures in this book, it was created in MATLAB (see Appendix B). The minimum value of K_f occurs at $D = 0.5$. The lowest value of $f_{\min}$ occurs at $D = 0.5$. Figure 1.14 depicts $1/K_f$ as a function of D. The core cross-sectional area A_c is proportional to $1/K_f$. The maximum value of A_c occurs at $D = 0.5$.

1.7 Volt-Second Balance

For periodic waveforms in steady state,

$$\lambda(T) - \lambda(0) = \int_0^T v_L(t)dt = 0. \tag{1.166}$$

This equation is called a *volt-second balance*, which states that the area enclosed by the waveform v_L above zero must be equal to the area enclosed by the waveform v_L below zero for steady state. The volt-second balance can be expressed by

$$\int_0^{t_o} v_L(t)dt = -\int_{t_o}^T v_L(t)dt. \tag{1.167}$$

1.8 Inductance

1.8.1 Definitions of Inductance

A coil is generally formed by winding a wire on a cylindrical former, called a bobbin. The inductance depends on (1) winding geometry, (2) core geometry, (3) permeability of the core material, and (4) frequency. There are several methods to determine the inductance.

Magnetic Flux Linkage Method

The *inductance* (or *self-inductance*) for linear inductors is defined as the ratio of the total magnetic flux linkage λ to the time-varying (ac) current i, producing the flux linkage

$$L = \frac{\lambda}{i}. \tag{1.168}$$

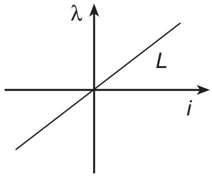

Figure 1.15 Magnetic flux linkage λ as a function of current i producing the flux linkage for linear inductors.

The inductance of a linear inductor is a proportionality constant in the expression $\lambda = Li$. In general, an ac current flowing through a conductor produces the magnetic linkage inside the conductor (an internal magnetic linkage) λ_{int} and outside the conductor (an external magnetic linkage) λ_{ext}. Therefore, the inductance is defined as

$$L = \frac{\lambda}{i} = \frac{\lambda_{int} + \lambda_{ext}}{i}. \tag{1.169}$$

A single conductor carrying an ac current i is linked by its own magnetic flux. For linear inductors, the flux linkage λ is proportional to current i, resulting in $\lambda = Li$. The inductance L is the slope of the λ–i characteristic, as illustrated in Figure 1.15. This characteristic is analogous to the resistor characteristic $v = Ri$ or the capacitor characteristic $Q = Cv$. A circuit that is designed to have a self-inductance is called an inductor. An inductor has a self-inductance of 1 H if a current of 1 A produces a flux linkage of 1 V·s (or 1 Wb·turn).

A change in the current flowing through the inductor produces an induced electromotive force, called an electromotance, or voltage

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \frac{\lambda}{dt} = L\frac{di}{dt}. \tag{1.170}$$

An inductor has a self-inductance of 1 H if the current flowing through it changes at a rate of 1 A/s when the voltage difference between its terminals is 1 V. The inductance L is a function of the number of turns N, core permeability μ_{rc}, core geometry, and frequency f.

The inductance can be defined as

$$L = \frac{\lambda}{i} = \frac{1}{i}\int\int_S \mathbf{B} \cdot d\mathbf{S}. \tag{1.171}$$

The magnetic field produced by a current-carrying conductor links itself. The associated inductance is called self-inductance. In some cases, the magnetic flux links only a part of the current and the inductance is defined as

$$L = \frac{1}{i}\int\int_S \frac{i_{enclosed}}{i} d\phi. \tag{1.172}$$

The total inductance of a conductor is made up of two components: an external inductance L_{ext} and an internal inductance L_{int}:

$$L = L_{ext} + L_{int}. \tag{1.173}$$

The external inductance L_{ext} is due to the magnetic energy stored in the magnetic field outside the conductor. This inductance is usually independent of frequency. The internal inductance L_{int} is due to the magnetic energy stored in the internal magnetic field inside the conductor. This inductance depends on frequency because the magnetic field intensity H distribution inside the conductor is a function of frequency due to the skin effect. The internal inductance usually decreases with frequency.

The voltage across the inductance is

$$v_L = \frac{d\lambda}{dt} = N\frac{d\phi}{dt} = N\frac{d\phi}{di_L}\frac{di_L}{dt} = L\frac{di_L}{dt}. \tag{1.174}$$

The self-inductance L relates the voltage induced in an inductor v_L to the time-varying current i_L flowing through the same inductor.

Reluctance Method

The inductance of an inductor can be determined using the core reluctance $\mathcal{R}$ or the core permeance $\mathcal{P}$:

$$L = \frac{N^2}{\mathcal{R}} = \mathcal{P}N^2 = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c}. \tag{1.175}$$

If $N = 1$, $L = \mathcal{P} = 1/\mathcal{R}$.

Magnetic Energy Method

Inductance may be equivalently defined using magnetic energy,

$$W_m = \frac{1}{2}LI_m^2 = \frac{1}{2}\int_V (\mathbf{B}\cdot\mathbf{H}^*)dV, \tag{1.176}$$

yielding the inductance

$$L = \frac{2W_m}{I_m^2} = \frac{1}{I_m^2}\int_V (\mathbf{B}\cdot\mathbf{H}^*)dV, \tag{1.177}$$

where I_m is the amplitude of the current flowing in the closed path and W_m is the energy stored in the magnetic field produced by the current flowing through the inductor,

$$W_m = \frac{1}{2\mu}\int\int\int_V B^2 dV. \tag{1.178}$$

Small-Signal Inductance

The small-signal (or incremental) inductance of a nonlinear inductor is defined as the ratio of the infinitesimal change in the flux linkage to the infinitesimal change in the current producing it at a given operating point $Q(I_{DC}, \lambda_{DC})$:

$$L = \left.\frac{d\lambda}{di}\right|_Q. \tag{1.179}$$

Inductors with ferrous cores are nonlinear because the permeability depends on the applied magnetic field H. Figure 1.16 shows a plot of the magnetic flux linkage λ as a function of current i for nonlinear inductors. At low values of current, the core is not saturated and the relative permeability is high, resulting in a high slope of the λ–i curve and a large inductance L_1. When the core saturates, the

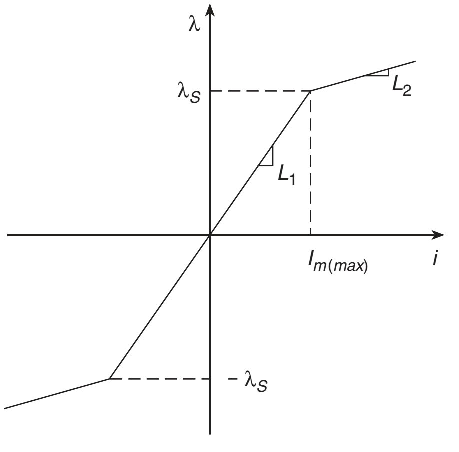

Figure 1.16 Magnetic flux linkage λ as a function of current i producing the flux linkage for nonlinear inductors.

relative permeability μ_{rc} becomes equal to 1, the slope of the $\lambda - i$ curve decreases and the inductance decreases to a lower value L_2.

Vector Magnetic Potential Method

The inductance can be determined using the vector magnetic potential **A**:

$$L = \frac{1}{I^2} \int\int\int_V \mathbf{A} \cdot \mathbf{J} dV. \tag{1.180}$$

The vector magnetic potential is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int\int\int_V \frac{\mathbf{J}(\mathbf{r})}{R} dV. \tag{1.181}$$

Hence, the inductance is given by

$$L = \frac{1}{I^2} \int\int\int_V \left[\frac{\mu}{4\pi} \int\int\int_V \frac{\mathbf{J}(\mathbf{r})}{R} dV \right] \cdot \mathbf{J}(\mathbf{r}) dV. \tag{1.182}$$

Example 1.2

An inductor is wound on a CC core (see Figure 2.9) whose cross-sectional area is 2 cm × 2 cm, $l_c = 16$ cm, the core window is 3 cm × 3 cm, and $\mu_{rc} = 100$. The core has no air gap. There is a magnetic flux in the core ϕ_c and a leakage flux ϕ_l in the air. The inductor has 10 turns. Estimate the inductance using the reluctance method.

Solution: The total magnetic flux is

$$\phi = \phi_c + \phi_l. \tag{1.183}$$

The inductance is

$$L = \frac{N\phi}{i} = N^2 \left(\frac{1}{R_c} + \frac{1}{R_l} \right) = N^2 \left(\frac{\mu_{rc}\mu_0 A_c}{l_c} + \frac{\mu_0 A_l}{l_l} \right). \tag{1.184}$$

Let $l_l = l_c/2$ and $A_l = 4A_c$. In this case, the inductance is given by

$$L = N^2 \left(\frac{\mu_{rc}\mu_0 A_c}{l_c} + \frac{8\mu_0 A_c}{l_c} \right) = N^2 \left(\frac{\mu_0 A_c}{l_c} \right) (\mu_{rc} + 8)$$

$$= 10^2 \left(\frac{4\pi \times 10^{-7} \times 4 \times 10^{-4}}{16 \times 10^{-2}} \right) (100 + 8) = 33.929\,\mu\text{H}. \tag{1.185}$$

The inductance is increased by 8% due to the leakage magnetic flux.

1.8.2 Inductance of Solenoid

Neglecting the end effects, the magnetic flux density inside a long solenoid is uniform and it is given by

$$B = \frac{\mu N I}{l_c}. \tag{1.186}$$

The magnetic flux inside is

$$\phi = A_c B = \frac{\mu N I A_c}{l_c} = \frac{\pi \mu N I r^2}{l_c}. \tag{1.187}$$

The flux linkage is

$$\lambda = N\phi = \frac{\mu N^2 I A_c}{l_c} = \frac{\pi \mu N^2 I r^2}{l_c}. \tag{1.188}$$

The inductance of a long solenoid (theoretically infinitely long) with a core and without an air gap at low frequencies is

$$L_\infty = \frac{\lambda}{I} = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c} = \frac{\pi \mu_{rc}\mu_0 r^2 N^2}{l_c} = \frac{N^2}{l_c/(\mu A_c)} = \frac{N^2}{\mathcal{R}} \tag{1.189}$$

where $A_c = \pi r^2$ is the core cross-sectional area, r is the mean coil radius, l_c is the mean core length, μ_{rc} is the relative permeability of the core, and N is the total number of turns. The inductance L is proportional to the square of the number of turns N^2 and the cross-sectional area A_c, and is inversely proportional to its length l_c. More precisely, the inductance L is proportional to the ratio of the core cross-sectional area to the magnetic path length A_c/l_c.

The inductance of a short solenoid is smaller than that of an infinitely long round solenoid. As r/l_c increases, L/L_∞ decreases, where L_∞ is the inductance of an infinitely long solenoid. For example, $K = L/L_\infty = 0.85$ for $r/l_c = 0.2$, $K = 0.74$ for $r/l_c = 0.4$, $K = 0.53$ for $r/l_c = 1$, $K = 0.2$ for $r/l_c = 5$, and $K = 0.12$ for $r/l_c = 10$. A first-order approximation is

$$K = \frac{L}{L_\infty} \approx \frac{1}{1 + 0.9\dfrac{r}{l_c}}, \tag{1.190}$$

resulting in $L \approx KL_\infty = L_\infty/(1 + 0.9r/l_c)$. The inductance of a round single-layer solenoid of a finite length l_c can be approximated by Wheeler's formula [38], which is correct to within 1% for $r/l_c < 1.25$ (or $l_c/(2r) > 0.4$):

$$L = \frac{L_\infty}{1 + 0.9\dfrac{r}{l_c}} = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c\left(1 + 0.9\dfrac{r}{l_c}\right)} = \frac{\pi \mu_{rc}\mu_0 r^2 N^2}{l_c\left(1 + 0.9\dfrac{r}{l_c}\right)} = \frac{\pi \mu_{rc}\mu_0 r^2 N^2}{l_c + 0.9r} \quad \text{(H)}$$

$$= \frac{0.4\pi^2 \mu_{rc} r^2 N^2}{l_c + 0.9r} \quad (\mu\text{H}), \quad \text{for} \quad \frac{r}{l_c} < 1.25. \tag{1.191}$$

Figure 1.17 shows a plot of L/L_∞ as a function of r/l_c. As the ratio of the external diameter to the internal diameter decreases, the inductance also decreases.

The inductance of a multi-layer solenoid is given by

$$L = \frac{0.8\mu\pi r^2 N^2}{l_c + 0.9r + b}, \tag{1.192}$$

where b is the thickness of all layers (or coil build) and r is the average radius of the winding.

A more accurate equation for the inductance of a multi-layer inductor is

$$L = \frac{\mu\pi r^2 N^2}{l_c} \frac{1}{1 + 0.9\dfrac{r}{l_c} + 0.32\dfrac{b}{r} + 0.84\dfrac{b}{l_c}}. \tag{1.193}$$

The inductance predicted by this equation is within 2% of the exact value.

Example 1.3

An air core solenoid has $N = 20$, $l_c = 15$ cm, and $r = 3$ cm. Find the inductance.

Solution: The inductance of the solenoid is

$$L = \frac{\pi \mu_{rc}\mu_0 r^2 N^2}{l_c\left(1 + 0.9\dfrac{r}{l_c}\right)} = \frac{\pi \times 1 \times 4\pi \times 10^{-7} \times (3 \times 10^{-2})^2 \times 20^2}{15 \times 10^{-2}\left(1 + 0.9 \times \dfrac{3 \times 10^{-2}}{15 \times 10^{-2}}\right)} = 8.0295\,\mu\text{H}. \tag{1.194}$$

Note that the inductance calculated in this example is about 15% less than the inductance calculated for a very long inductor because $K = 0.8475$.

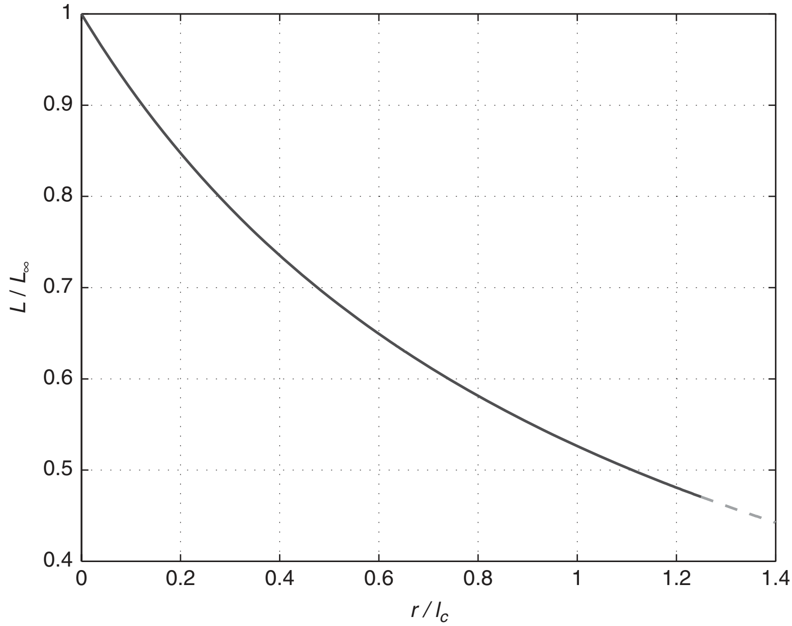

Figure 1.17 Plot of L/L_∞ as a function of r/l_c.

1.8.3 Inductance of Inductor with Toroidal Core

An idealized toroid can be thought as a finite-length solenoid bent around to close on itself to form a doughnut shape. A toroidal inductor with a rectangular cross section is shown in Figure 1.18. The dimensions of the magnetic core are: a is the inner radius, b is the outer radius, and h is the toroid height. The toroid is symmetrical about its axis. Thus,

$$dl = rd\varphi. \tag{1.195}$$

Applying Ampère's law,

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_0^{2\pi} Brd\varphi = Br\int_0^{2\pi} d\varphi = 2\pi rB. \tag{1.196}$$

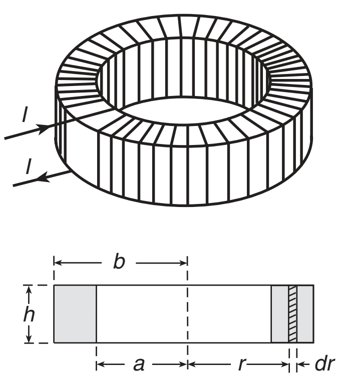

Figure 1.18 Toroidal inductor.

Since the path of integration encircles a total current NI, we obtain

$$2\pi rB = \mu NI. \tag{1.197}$$

Hence, the magnetic flux density inside the toroidal core is given by

$$B(r) = \frac{\mu NI}{2\pi r}, \quad \text{for } \ a \le r \le b. \tag{1.198}$$

Since $dS = hdr$, the magnetic flux inside the toroidal core is

$$\phi = \int\int_S B(r)dS = \int_a^b \int_0^h \left(\frac{\mu NI}{2\pi r}\right)(hdr) = \frac{\mu NIh}{2\pi}\int_a^b \frac{dr}{r} = \int_S \left(\frac{\mu NI}{2\pi r}\right)(hdr)$$
$$= \frac{\mu NIh}{2\pi}\ln\left(\frac{b}{a}\right) \tag{1.199}$$

where S is the surface bounded by the path C. The flux linkage of the toroidal inductor is

$$\lambda = N\phi = \frac{\mu hN^2 I}{2\pi}\ln\left(\frac{b}{a}\right) \tag{1.200}$$

resulting in the inductance of a toroidal coil

$$L = \frac{\lambda}{I} = \frac{\mu_{rc}\mu_0 hN^2}{2\pi}\ln\left(\frac{b}{a}\right). \tag{1.201}$$

The inductance of a toroidal coil with a round cross section can be described by the expression for the inductance of a long solenoid

$$L = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c} = \frac{\mu_{rc}\mu_0 A_c N^2}{2\pi R} \tag{1.202}$$

where $R = (a+b)/2$ is the mean radius of the core, $l_c = 2\pi R = \pi(a+b)$, and $A_c = \pi(b-a)^2/4$ is the cross-sectional area of the core. Hence,

$$L = \frac{\mu_{rc}\mu_0 N^2(b-a)^2}{4(a+b)}. \tag{1.203}$$

Example 1.4

An inductor is wound on a toroidal core where $\mu_{rc} = 150$, $h = 1$ cm, $a = 4$ cm, and $b = 5$ cm. The inductor has 20 turns. Find the inductance.

Solution: The inductance is

$$L = \frac{\mu_{rc}\mu_0 hN^2}{2\pi}\ln\left(\frac{b}{a}\right) = \frac{150 \times 4\pi \times 10^{-7} \times 10^{-2} \times 20^2}{2\pi}\ln\left(\frac{5}{4}\right) = 26.777\,\mu\text{H}. \tag{1.204}$$

1.8.4 Inductance of Inductor with Pot Core

The geometry of an inductor with a pot core is very complex and the inductance of these inductors can be determined only approximately. The core cross-sectional area of the pot core is approximately equal to the cross-sectional area of the center post,

$$A_c = \frac{\pi d^2}{4}, \tag{1.205}$$

where d is the diameter of the center post. The average diameter of the mean magnetic path is given by

$$D_{av} = \frac{D_i + D_o}{2} \tag{1.206}$$

where D_i is the inner diameter of the outer core area and D_o is the outer diameter of the outer core area. The mean magnetic path length is given by

$$l_c = 2D_{av} + 4H = D_i + D_o + 4H \tag{1.207}$$

where H is the height of the core halve. The inductance of an inductor with a pot core can be approximated by

$$L = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c} = \frac{\pi \mu_{rc}\mu_0 d^2 N^2}{4(D_i + D_o + 4H)}. \tag{1.208}$$

1.8.5 Air Gap

The overall reluctance $\mathcal{R}$ can be controlled by an air gap in the core. Therefore, the magnetic flux ϕ, flux density B, and inductance L can be controlled by the length of the air gap l_g. Air gaps can be bulk or distributed. In a gapped core, a small section of the magnetic flux path is replaced by a nonmagnetic medium, such as air or nylon. It is often filled with a spacer. The air gap length l_g is usually twice the spacer thickness. Some cores have pre-fabricated air gaps. Standard values of l_g are 0.5, 0.6, 0.7, ..., 5 mm. Adding an air gap in a core is equivalent to adding a large gap reluctance in series with the core reluctance (i.e., a series reluctor). As a result, the magnitude of the magnetic flux ϕ_m at a fixed value of NI_m is reduced. This effect is analogous to adding a series resistor in an electric circuit to reduce the magnitude of the current at a fixed source voltage.

Figure 1.19(a) illustrates an inductor whose core has an air gap. An equivalent magnetic circuit of an inductor with an air gap is shown in Figure 1.19(b). The inductance of a coil with a magnetic core having an air gap at low frequencies is expressed as

$$L = \frac{N^2}{\mathcal{R}_g + \mathcal{R}_c} = \frac{N^2}{\dfrac{l_g}{\mu_0 A_c} + \dfrac{l_c}{\mu_{rc}\mu_0 A_c}} = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c + \mu_{rc} l_g} = \frac{\mu_0 A_c N^2}{l_g + \dfrac{l_c}{\mu_{rc}}}$$
$$= \frac{\mu_{rc}\mu_0 A_c N^2}{l_c\left(1 + \dfrac{\mu_{rc} l_g}{l_c}\right)} = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c F_g}, \tag{1.209}$$

where the reluctance of the air gap is

$$\mathcal{R}_g = \frac{l_g}{\mu_0 A_c}, \tag{1.210}$$

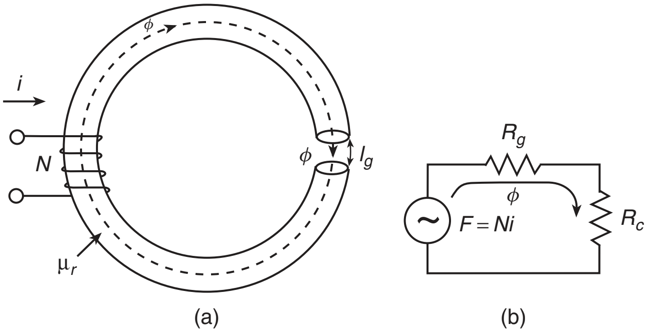

Figure 1.19 Inductor with an air gap. (a) Inductor. (b) Magnetic circuit of an inductor with an air gap.

the reluctance of the core is

$$\mathcal{R}_c = \frac{l_c - l_g}{\mu_{rc}\mu_0 A_c} \approx \frac{l_c}{\mu_{rc}\mu_0 A_c}, \tag{1.211}$$

the overall reluctance is

$$\mathcal{R} = \mathcal{R}_c + \mathcal{R}_g = \frac{l_c}{\mu_{rc}\mu_0 A_c} + \frac{l_g}{\mu_0 A_c} = \frac{l_c}{\mu_{rc}\mu_0 A_c}\left(1 + \frac{\mu_{rc} l_g}{l_c}\right) = F_g \mathcal{R}_c, \tag{1.212}$$

the *air gap factor* is

$$F_g = \frac{\mathcal{R}}{\mathcal{R}_c} = \frac{\mathcal{R}_c + \mathcal{R}_g}{\mathcal{R}_c} = 1 + \frac{\mathcal{R}_g}{\mathcal{R}_c} = 1 + \frac{\mu_{rc} l_g}{l_c}, \tag{1.213}$$

and the effective relative permeability of a core with an air gap is

$$\mu_{re} = \frac{\mu_{rc}}{1 + \dfrac{\mu_{rc} l_g}{l_c}} = \frac{\mu_{rc}}{F_g}. \tag{1.214}$$

The air gap causes a considerable decrease in the effective relative permeability. However, it produces a more stable effective permeability and reluctance, resulting in a more predictable and stable inductance. For example, inductors used in resonant circuits should be predictable and stable. Usually, at least 95% of the inductance comes from the air gap. The length of the air gap is given by

$$l_g = \frac{\mu_0 A_c N^2}{L} - \frac{l_c}{\mu_{rc}}. \tag{1.215}$$

The number of turns of an inductor whose core has an air gap is given by

$$N = \sqrt{\frac{L\left(l_g + \dfrac{l_c}{\mu_{rc}}\right)}{\mu_0 A_c}}. \tag{1.216}$$

For $l_g \gg l_c/\mu_{rc}$, $\mathcal{R}_g \gg \mathcal{R}_c$, and

$$L \approx \frac{\mu_0 A_c N^2}{l_g} = \frac{N^2}{\mathcal{R}_g}. \tag{1.217}$$

Therefore, the inductance of an inductor with a core air gap is a function of the air gap length l_g and is almost independent of the core relative permeability μ_{rc}. The number of turns is

$$N \approx \sqrt{\frac{L l_g}{\mu_0 A_c}}, \quad \text{for} \quad l_g \gg \frac{l_c}{\mu_{rc}}. \tag{1.218}$$

The core permeability varies with temperature and flux level. Inductors that carry dc currents and have dc magnetic flux require long air gaps to avoid saturation.

The relationships for the inductor with an air gap can be written as

$$\mathcal{F} = Ni = H_c l_c + H_g l_g = \frac{B_c l_c}{\mu_{rc}\mu_0} + \frac{B_g l_g}{\mu_0} = \frac{\phi_c l_c}{A_c \mu_{rc}\mu_0} + \frac{\phi_g l_g}{A_a \mu_0} = \mathcal{R}_c \phi_c + \mathcal{R}_g \phi_g. \tag{1.219}$$

For $\mathcal{R}_g \gg \mathcal{R}_c$, $\phi \approx Ni/R_g$. Neglecting the flux fringing effect, $B_g = B_c$. Hence,

$$Ni = B_c\left(\frac{l_c}{\mu_{rc}\mu_0} + \frac{l_g}{\mu_0}\right). \tag{1.220}$$

The magnetic flux density in the core with an air gap is given by

$$B_c = \frac{\mu_0 Ni}{l_g + \dfrac{l_c}{\mu_{rc}}}. \tag{1.221}$$

Hence, the maximum flux density in the core with an air gap, which is caused by the dc component of the inductor current I_L and the amplitude of the ac component of the inductor current I_m, is expressed by

$$B_{c(pk)} = B_{DC} + B_m = \frac{\mu_0 N (I_L + I_m)}{l_g + \dfrac{l_c}{\mu_{rc}}} \leq B_s, \quad \text{for} \quad T \leq T_{max}. \tag{1.222}$$

The magnetic flux density and the magnetic field intensity in the core are

$$B_c = \frac{\phi_c}{A_c} \tag{1.223}$$

and

$$H_c = \frac{B_c}{\mu_{rc}\mu_0}. \tag{1.224}$$

Assuming a uniform magnetic flux density in the air gap and neglecting the fringing effect, the magnetic flux, magnetic flux density, and magnetic field intensity in the air gap are

$$\phi_g = \phi_c = A_c B_c = A_g B_g, \tag{1.225}$$

$$B_g = \frac{A_c}{A_g} B_c \approx B_c, \tag{1.226}$$

and

$$H_g = \frac{B_g}{\mu_0} = \frac{B_c}{\mu_0} = \mu_{rc} H_c. \tag{1.227}$$

Gap losses consist of winding loss, core loss, and hardware loss (e.g., power loss in clamps or bolts).

The maximum MMF is

$$\mathcal{F}_{max} = N_{max} I_{Lmax} = \phi(\mathcal{R}_g + \mathcal{R}_c) = B_{pk} A_c (\mathcal{R}_g + \mathcal{R}_c) \approx B_{pk} A_c \mathcal{R}_g = \frac{B_{pk} l_g}{\mu_0}, \tag{1.228}$$

where $\mathcal{R}_g = l_g/(\mu_0 A_c)$. To avoid core saturation, the maximum number of turns is given by

$$N_{max} = \frac{B_{pk} l_c}{\mu_0 I_{Lmax}}. \tag{1.229}$$

As the air gap length l_c increases, NI_m can be increased and the core loss decreases. However, the number of turns N must be incressed to achieve a specified inductance L. Increasing the number of turns increases the winding loss. In addition, the leakage inductance increases and the air gap radiates a larger amount of electromagnetic interference (EMI).

The behavior of an inductor with an air gap is similar to an amplifier with negative feedback,

$$A_f = \frac{A}{1 + \beta A} = \frac{\mu_{rc}}{1 + \mu_{rc}\dfrac{l_g}{l_c}}. \tag{1.230}$$

Thus, μ_{rc} is analogous to A and l_g/l_c is analogous to β.

Example 1.5

A PQ4220 Magnetics core [48] has $\mu_{rc} = 2500$, $l_c = 4.63\,\text{cm}$, and $A_c = 1.19\,\text{cm}^2$. The inductor wound on this core has $N = 10$ turns. The required inductance should be $L = 55.6\,\mu\text{H}$. Find the length of the air gap l_g.

Solution: The length of the air gap in the core is

$$l_g = \frac{\mu_0 A_c N^2}{L} - \frac{l_c}{\mu_{rc}} = \frac{4\pi \times 10^{-7} \times 1.19 \times 10^{-4} \times 10^2}{55.6 \times 10^{-6}} - \frac{4.63 \times 10^{-2}}{2500}$$
$$= (0.2689564 - 0.01852) \times 10^{-3} = 0.2504\,\text{mm}. \tag{1.231}$$

1.8.6 Fringing Flux

A fringing flux is present around the air gap whenever the core is excited, as shown in Figure 1.20. Figure 1.21 depicts fringing flux in an inductor with an EE core (see Figure 2.14) and air gap. The magnetic flux lines bulge outward because the magnetic lines repel each other when passing through nonmagnetic material. As a result, the cross-sectional area of the magnetic field is increased and the flux density is decreased. Typically, 10% is added to the air gap cross-sectional area. This effect is called the *fringing flux*. The percentage of the fringing flux in the total magnetic flux increases as the air-gap length l_g increases. The maximum increase in the radius of the magnetic flux due to the fringing effect is approximately equal to the length of the air gap length l_g. Figure 1.22 shows a magnetic equivalent circuit for the inductor with an air gap and fringing flux. The fringing permeance is shunting the gap permeance.

Due to the continuity of magnetic flux, the magnetic flux in the core ϕ_c is equal to the sum of the magnetic flux in the air gap ϕ_g and the fringing flux ϕ_f:

$$\phi_c = \phi_g + \phi_f. \tag{1.232}$$

The permeance of the core is

$$\mathcal{P}_c = \frac{1}{\mathcal{R}_c} = \frac{\mu_{rc}\mu_0 A_c}{l_c}. \tag{1.233}$$

The permeance of the air gap is

$$\mathcal{P}_g = \frac{1}{\mathcal{R}_g} = \frac{\mu_0 A_c}{l_g}. \tag{1.234}$$

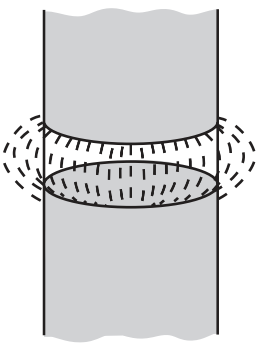

Figure 1.20 Fringing magnetic flux in an air gap.

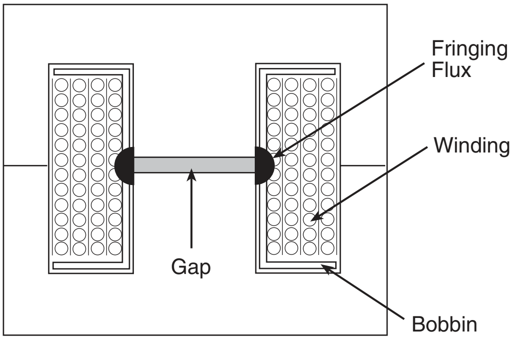

Figure 1.21 Fringing magnetic flux in an inductor with gapped pot core.

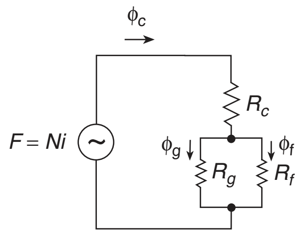

Figure 1.22 Magnetic equivalent circuit of an inductor with an air gap and fringing magnetic flux.

The permeance of the fringing area is

$$\mathcal{P}_f = \frac{1}{\mathcal{R}_f} = \frac{\mu_0 A_f}{l_f} \tag{1.235}$$

where A_f is the fringing area and l_f is the magnetic path length in the fringing area. Assuming that $A_g = A_c$, the total reluctance is given by

$$\begin{aligned}\mathcal{R} &= \mathcal{R}_c + \mathcal{R}_g || \mathcal{R}_f = \mathcal{R}_c + \frac{\mathcal{R}_g \mathcal{R}_f}{\mathcal{R}_g + \mathcal{R}_f} = \frac{l_c}{\mu_{rc}\mu_0 A_c} + \frac{\dfrac{l_g}{\mu_0 A_g} \times \dfrac{l_f}{\mu_0 A_f}}{\dfrac{l_g}{\mu_0 A_g} + \dfrac{l_f}{\mu_0 A_f}} \\ &= \frac{l_c}{\mu_{rc}\mu_0 A_c} + \frac{l_g l_f}{l_g \mu_0 A_f + l_f \mu_0 A_g} = \frac{l_c}{\mu_{rc}\mu_0 A_c}\left(1 + \frac{\mu_{rc} A_c}{l_c}\,\frac{l_g l_f}{l_f A_g + l_g A_f}\right) \\ &= \frac{l_c}{\mu_{rc}\mu_0 A_c}\left(1 + \frac{\mu_{rc} l_g}{l_c}\,\frac{1}{1 + \dfrac{l_g A_f}{l_f A_g}}\right).\end{aligned} \tag{1.236}$$

Hence, the inductance of an inductor with an air gap and a fringing flux is given by

$$L_f = \frac{N^2}{\mathcal{R}} = N^2 \Bigg/ \left[\frac{l_c}{\mu_{rc}\mu_0 A_c}\left(1 + \frac{\mu_{rc} l_g}{l_c}\,\frac{1}{1 + \dfrac{l_g A_f}{l_f A_g}}\right)\right]. \tag{1.237}$$

Neglecting the permeance of the core, the total permeance of the air gap an the fringing area is

$$\mathcal{P} = \mathcal{P}_g + \mathcal{P}_f = \frac{\mu_0 A_c}{l_g} + \frac{\mu_0 A_f}{l_f} = \frac{\mu_0 A_c}{l_g}\left(1 + \frac{A_f l_g}{A_c l_f}\right) = \frac{\mu_0 A_c F_f}{l_g} = F_f \mathcal{P}_g. \tag{1.238}$$

Thus, $\mathcal{R} = \mathcal{R}_g / F_f$. The inductance of the inductor with an air gap and the fringing flux is

$$L_f = \mathcal{P} N^2 = \frac{\mu_0 A_c N^2}{l_g} + \frac{\mu_0 A_f N^2}{l_f} = \frac{\mu_0 A_c N^2}{l_g}\left(1 + \frac{A_f l_g}{A_c l_f}\right) = \frac{\mu_0 A_c N^2 F_f}{l_g} = F_f L, \tag{1.239}$$

where the *fringing factor* is defined as the ratio of the inductance with an air gap and with the fringing effect L_f to the ideal inductance with an air gap and with no fringing effect,

$$F_f = \frac{L_f}{L} = 1 + \frac{A_f l_g}{A_c l_f}. \tag{1.240}$$

Thus, the fringing effect increases the inductance. The number of turns required to obtain a desired inductance is

$$N = \sqrt{\frac{l_g L}{\mu_0 A_c F_f}}. \tag{1.241}$$

If the air gap is enclosed by the winding, the fringing flux is reduced, lowering the value of F_f. However, the inductor losses increase by as much as 5 times. To reduce these losses, the winding should be moved away from the air gap by a distance equal to two to three times the air gap length l_g. Short distributed air gaps significantly reduce the fringing flux and power losses. Cores with a large relative permeability require long air gaps, which increase the fringing flux.

Consider a round core with a single air gap. The cross-sectional area of the core with diameter D_c is given by

$$A_c = \frac{\pi}{4} D_c^2. \tag{1.242}$$

It is difficult to determine the cross-sectional area A_f and the mean path length l_f of the magnetic flux. Assume that the outer diameter of the fringing magnetic flux is $D_f = D_c + 2l_g$ and the mean magnetic path length of the fringing flux is $l_f = 2l_g$. The cross-sectional area of the fringing flux is

$$A_f = \frac{\pi}{4}(D_c + 2l_g)^2 - \frac{\pi}{4} D_c^2 = \pi l_g (D_c + l_g). \tag{1.243}$$

Hence, the fringing factor is

$$F_f = 1 + \frac{A_f l_g}{A_c l_f} = 1 + 2l_g \left(\frac{1}{D_c} + \frac{l_g}{D_c^2} \right). \tag{1.244}$$

The permeance of the air gap is

$$\mathcal{P}_g = \frac{\mu_0 A_c}{l_g} = \frac{\pi \mu_0 D_c^2}{4 l_g}. \tag{1.245}$$

The permeance of the fringing area is

$$\mathcal{P}_f = \frac{\mu_0 A_f}{l_f} = \frac{\pi \mu_0 (D_c + l_g)}{2}. \tag{1.246}$$

The total permeance of the air gap and the fringing area is

$$\mathcal{P} = \mathcal{P}_g + \mathcal{P}_f = \frac{\mu_0 A_c}{l_g} + \frac{\mu_0 A_f}{l_f} = \frac{\pi \mu_0 D_c^2}{4 l_g} + \frac{\pi \mu_0 (D_c + l_g)}{2}. \tag{1.247}$$

The inductance is

$$L_f = \mathcal{P} N^2 = \left[\frac{\pi \mu_0 D_c^2}{4 l_g} + \frac{\pi \mu_0 (D_c + l_g)}{2} \right] N^2 = \frac{\pi \mu_0 D_c^2 N^2}{4 l_g} \left[1 + \frac{2 l_g (D_c + l_g)}{D_c^2} \right] = F_f L, \tag{1.248}$$

where

$$F_f = 1 + \frac{2 l_g (D_c + l_g)}{D_c^2} \approx 1 + \frac{2 l_g}{D_c} \quad \text{for} \quad l_g \ll D_c. \tag{1.249}$$

Example 1.6

An inductor with an air gap has a round core with $D_c = 10\,\text{mm}$ and $l_g = 1\,\text{mm}$. Find F_f and N_f/N.

Solution: The fringing factor is

$$F_f = \frac{L_f}{L} = 1 + \frac{2 l_g (D_c + l_g)}{D_c^2} = 1 + \frac{2 \times 1 \times (10 + 1)}{10^2} = 1.22. \tag{1.250}$$

The ratio of the turns is

$$\frac{N_f}{N} = \frac{1}{\sqrt{F_f}} = \frac{1}{\sqrt{1.22}} = 0.9054. \tag{1.251}$$

If we use the approximate equation for F_f, we have

$$F_f = \frac{L_f}{L} = 1 + \frac{2 l_g}{D_c} = 1 + \frac{2 \times 1}{10} = 1.2 \tag{1.252}$$

and

$$\frac{N_f}{N} = \frac{1}{\sqrt{F_f}} = \frac{1}{\sqrt{1.2}} = 0.9123. \tag{1.253}$$

Consider a core with a single rectangular air gap. The dimensions of the air gap are a and b. The cross-sectional area of the air gap is

$$A_c = ab. \tag{1.254}$$

Assume that the outer dimensions of the fringing magnetic flux are $A = a + 2l_g$ and $B = b + 2l_g$, and the mean magnetic path length of the fringing flux is $l_f = 2l_g$. The cross-sectional area of the fringing flux is

$$A_f = (a + 2l_g)(b + 2l_g) - ab = 2l_g(a + b) + 4l_g. \tag{1.255}$$

The fringing factor is

$$F_f = \frac{L_f}{L} = 1 + \frac{l_g(a + b + 2l_g)}{ab}. \tag{1.256}$$

Example 1.7

An inductor with a single rectangular air gap has $a = 10$ mm, $b = 20$ mm, and $l_g = 1$ mm. Find F_f and N_f/N.

Solution: The fringing factor is

$$F_f = \frac{L_f}{L} = 1 + \frac{l_g(a + b + 2l_g)}{ab} = 1 + \frac{1 \times (10 + 20 + 2 \times 1)}{10 \times 20} = 1.16. \tag{1.257}$$

The ratio of the turns is

$$\frac{N_f}{N} = \frac{1}{\sqrt{F_f}} = \frac{1}{\sqrt{1.16}} = 0.9285. \tag{1.258}$$

The fringing factor given in [5], [10] is described by

$$F_f = 1 + \frac{al_g}{\sqrt{A_c}} \ln\left(\frac{2w}{l_g}\right) \approx 1 + \frac{l_g}{\sqrt{A_c}} \ln\left(\frac{2w}{l_g}\right) \tag{1.259}$$

where w is the width of the core window, a lies between 0.85 and 0.95 for round cores and between 1 and 1.1 for rectangular cores. Typical values of the fringing factor F_f are between 1.1 and 1.4. The fringing flux reduces the total reluctance of the magnetic path $\mathcal{R}$, and therefore it increases the inductance L. The inductance is increased due to the fringing effect and is given by

$$L_f = F_f L = \left[1 + \frac{al_g}{\sqrt{A_c}} \ln\left(\frac{2w}{l_g}\right)\right] L = \left[1 + \frac{al_g}{\sqrt{A_c}} \ln\left(\frac{2w}{l_g}\right)\right] \frac{\mu_{rc}\mu_0 A_c N^2}{l_c(1 + \mu_{rc} l_g)}$$
$$= \frac{\mu_0 A_c N^2 F_f}{l_g + \dfrac{l_c}{\mu_{rc}}}. \tag{1.260}$$

Therefore, the number of turns N to obtain a required inductance L of an inductor with an air gap and the fringing effect should be reduced to

$$N_f = \sqrt{\frac{L_f\left(l_g + \dfrac{l_c}{\mu_{rc}}\right)}{\mu_0 A_c F_f}} = \frac{N}{\sqrt{F_f}}. \tag{1.261}$$

Fringing flux generates eddy currents, which cause hot spots in both the core and the winding, resulting in power losses. The winding, banding, and clips should be kept away from the fringing flux to reduce power losses. A distributed air gap along the magnetic path reduces winding loss, as compared to the winding loss due to a single air gap.

1.8.7 Inductance of Strip Transmission Line

Consider a strip transmission line, where d is the distance between the conductors, l is the length of the strip, and w is the width of the strip. The magnetic field intensity between conducting parallel plates is

$$H = \frac{Il}{w}, \tag{1.262}$$

resulting in the flux linkage

$$\lambda = \int\int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^l \int_0^d \frac{\mu I}{w} dxdz = \frac{\mu Ild}{w}. \tag{1.263}$$

Hence, the inductance of the strip transmission line is given by

$$L = \frac{\lambda}{I} = \frac{\mu dl}{w}. \tag{1.264}$$

1.8.8 Inductance of Coaxial Cable

The inductance of a coaxial cable of inner radius a, outer radius b, and length l_w is given by

$$L = \frac{\mu l_w}{2\pi} \ln\left(\frac{b}{a}\right). \tag{1.265}$$

1.8.9 Inductance of Two-Wire Transmission Line

The inductance of a two-wire transmission line of round conductor of radius a, distance between the conductor centers d, and length l_w is

$$\begin{aligned} L &= \frac{\mu l_w}{\pi} \cosh^{-1}\left(\frac{d}{2a}\right) = \frac{\mu l_w}{\pi} \ln\left[\frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right] \\ &\approx \frac{\mu l_w}{\pi} \ln\left(\frac{d}{a}\right), \quad \text{for} \quad \left(\frac{d}{2a}\right)^2 \gg 1, \end{aligned} \tag{1.266}$$

where $\cosh^{-1} x \approx \ln(2x)$ for $x \ll 1$.

1.9 Inductance Factor

Equation (1.189) for the inductance can be written as

$$L = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c} = A_L N^2. \tag{1.267}$$

The *specific inductance* of a core, also called the *core inductance factor*, is defined as the inductance per single turn,

$$A_L = \frac{L}{N^2} = \frac{\mu_{rc}\mu_0 A_c}{l_c} = \frac{1}{\mathcal{R}} = \mathcal{P} \left(\frac{\mathrm{H}}{t^2}\right). \tag{1.268}$$

Each core of different materials, shapes, and sizes will have a unique value of A_L, some of which are not easy to predict analytically, especially for complex core shapes. Core manufacturers give the values of A_L in data sheets.

The specific inductance (or the inductance index) A_L is usually specified in henries per turn, in millihenries per 1000 turns, or in microhenries per 100 turns for cores without and with air gaps. If the specific inductance A_L is expressed in henries per turn, the number of turns is given by

$$N = \sqrt{\frac{L(\mathrm{H})}{A_L}}. \tag{1.269}$$

If the specific inductance $A_{L(1000)}$ is expressed in millihenries per 1000 turns, the inductance is given by

$$L = \frac{A_{L(1000)} N^2}{(1000)^2} \text{ (mH)} \tag{1.270}$$

and the number of turns is

$$N = 1000\sqrt{\frac{L(\mathrm{mH})}{A_{L(1000)}}}. \tag{1.271}$$

For most ferrite cores, the specific inductance $A_{L(100)}$ is specified in microhenries per 100 turns. In this case, the inductance is given by

$$L = \frac{A_{L(100)} N^2}{(100)^2} \text{ (μH)}. \tag{1.272}$$

To compute the required number of turns N in order to achieve a desired inductance L in microhenries, the following formula can be used for ferrite cores:

$$N = 100\sqrt{\frac{L(\mu\mathrm{H})}{A_{L(100)}}}. \tag{1.273}$$

Common values of $A_{L(100)}$ are 16, 25, 40, 63, 100, 250, 400, and so on.

Air core inductors are linear devices because the relationship $B = \mu_0 H$ is linear. In general, inductors with magnetic cores are nonlinear devices as the relationship between B and H is nonlinear. For $B < B_s$, inductors can be modeled as linear devices.

Example 1.8

The relative permeability of the Ferroxcube ferrite magnetic core material is $\mu_{rc} = 1800$. The toroidal core made of this material has inner diameter $d = 13.1$ mm, external diameter $D = 23.7$ mm, and height $h = 7.5$ mm. Find the specific inductance of this core. What is the inductance of the inductor with this core if the number of turns is $N = 10$?

Solution: The magnetic path length is

$$l_c = \pi \frac{d + D}{2} = \pi \frac{13.1 + 23.7}{2} = 57.805\,\mathrm{mm} \tag{1.274}$$

and the cross-sectional area of the core is

$$A_c = h\frac{(D - d)}{2} = 7.5 \times 10^{-3} \times \frac{(23.7 - 13.1) \times 10^{-3}}{2} = 39.75 \times 10^{-6}\,\mathrm{m}^2. \tag{1.275}$$

Hence, the specific inductance of the core is

$$A_L = \frac{\mu_{rc}\mu_0 A_c}{l_c} = \frac{1800 \times 4\pi \times 10^{-7} \times 39.75 \times 10^{-6}}{57.805 \times 10^{-3}} = 1.5554\,\mu\mathrm{H/turn}\,. \tag{1.276}$$

The inductance at $N = 10$ is

$$L = N^2 A_L = 10^2 \times 1.5554 \times 10^{-6} = 155.54\,\mu\mathrm{H}. \tag{1.277}$$

1.10 Magnetic Energy

The instantaneous power of an inductor is

$$p(t) = i_L(t)v_L(t) = i_L\left(L\frac{di_L}{dt}\right) = Li_L\frac{di_L}{dt}. \tag{1.278}$$

Power is the time rate of change of energy $P = W/\Delta t$. The instantaneous *magnetic energy* stored in the magnetic field of an inductor without an air gap is given by

$$\begin{aligned} w_m(t) &= \int_0^t p(t)dt = \int_0^t i_L v_L dt = \int_0^t i_L L\frac{di_L}{dt}dt = L\int_0^{i_L} i_L di_L = \frac{1}{2}Li_L^2 = \frac{1}{2}\lambda i_L = \frac{\lambda^2}{2L} \\ &= \frac{1}{2}\frac{N^2}{\mathcal{R}}i_L^2 = \frac{1}{2}\frac{N^2}{\dfrac{l_c}{\mu_{rc}\mu_0 A_c}}\left(\frac{Hl_c}{N}\right)^2 = \frac{1}{2}\mu_{rc}\mu_0 H^2 A_c l_c \\ &= \frac{B^2 l_c A_c}{2\mu_{rc}\mu_0} = \frac{B^2 V_c}{2\mu_{rc}\mu_0} \text{ (J)} \end{aligned} \tag{1.279}$$

where $V_c = l_c A_c$ is the core volume, $v_L = Ldi_L/dt$, $i_L = \lambda/L$, $L = N^2/\mathcal{R}$, and $H = B/\mu$. The magnetic energy is proportional to the core volume V_c and the flux density B, and it is inversely proportional to the core relative permeability μ_{rc}.

The *magnetic energy density* is

$$w_m = \frac{W_m}{V_c} = \frac{B^2}{2\mu_{rc}\mu_0} = \frac{1}{2}\mu_{rc}\mu_0 H^2 = \frac{1}{2}\mu H^2 \left(\frac{\text{J}}{\text{m}^3}\right). \tag{1.280}$$

For an inductor with an air gap, the energy stored in the gap is

$$W_g = \frac{B^2 l_g A_g}{2\mu_0} \approx \frac{B^2 l_g A_c}{2\mu_0} \tag{1.281}$$

where $A_g \approx A_c$. The energy stored in the core is

$$W_c = \frac{B^2 l_c A_c}{2\mu_{rc}\mu_0}. \tag{1.282}$$

The total energy stored in an inductor with an air gap is equal to the sum of the energy stored in the gap W_g and the energy stored in the core W_c:

$$W_m = W_g + W_c = \frac{B^2 A_c}{2\mu_0}\left(l_g + \frac{l_c}{\mu_{rc}}\right). \tag{1.283}$$

For $l_g \gg l_c/\mu_{rc}$, almost all the inductor energy is stored in the air gap:

$$W_m \approx W_g = \frac{B^2 l_g A_g}{2\mu_0} \approx \frac{B^2 l_g A_c}{2\mu_0}. \tag{1.284}$$

The maximum energy that can be stored in an inductor is limited by the core saturation flux density B_s, the core volume V_c, and the core relative permeability μ_{rc}. The maximum energy stored in an inductor with a core without an air gap is given by

$$W_{c(max)} = \frac{B_s^2 l_c A_c}{2\mu_{rc}\mu_0} = \frac{B_s^2 V_c}{2\mu_{rc}\mu_0}. \tag{1.285}$$

The maximum energy that can be stored in an inductor with a core with an air gap is

$$W_{g(max)} = \frac{B_s^2 l_g A_c}{2\mu_0} = \frac{B_s^2 V_g}{2\mu_0} \tag{1.286}$$

where $V_g = l_g A_c$ is the air gap volume. The ratio of the two energies is

$$\frac{W_{g(max)}}{W_{c(max)}} = \frac{l_g}{l_c}\mu_{rc}. \tag{1.287}$$

A winding represents a series combination of an inductance and a frequency-dependent resistance. The quality factor of an inductor at a given frequency f is defined as

$$Q_{Lo} = \frac{\omega L}{r_L}, \tag{1.288}$$

where r_L is the equivalent series resistance at frequency f.

Example 1.9

A Ferroxcube 528T500-4C4 ferrite magnetic core has $A_c = 1.17\,\text{cm}^2$, $l_c = 8.49\,\text{cm}$, and $\mu_{rc} = 125$. (a) Determine the maximum magnetic energy that can be stored in the inductor with this core. (b) Determine the maximum magnetic energy that can be stored in the inductor whose core has an air gap $l_g = 0.5\,\text{mm}$. (c) Find the ratio of the maximum magnetic energies.

Solution: The saturation flux density B_s for ferrite cores is $B_s = 0.3\,\text{T}$ at room temperature. At $T = 100^\circ\text{C}$, the saturation flux density B_s for ferrite cores decreases by a factor of 2. Thus,

$$B_s = \frac{0.3}{2} = 0.15\,\text{T}. \tag{1.289}$$

The maximum magnetic energy that can be stored in the inductor is

$$W_{m(max)} = \frac{B_s^2 l_c A_c}{2\mu_{rc}\mu_0} = \frac{0.15^2 \times 8.49 \times 10^{-2} \times 1.17 \times 10^{-4}}{2 \times 125 \times 4\pi \times 10^{-7}} = 0.711\,\text{mJ}. \tag{1.290}$$

The maximum magnetic energy that can be stored in the inductor whose core contains an air gap is

$$W_{g(max)} = \frac{B_s^2 l_g A_c}{2\mu_0} = \frac{0.15^2 \times 0.5 \times 10^{-3} \times 1.17 \times 10^{-4}}{2 \times 4\pi \times 10^{-7}} = 0.5237\,\text{mJ}. \tag{1.291}$$

Hence,

$$\frac{W_{g(max)}}{W_{m(max)}} = \frac{0.5237}{0.711} = 0.7362. \tag{1.292}$$

1.11 Self-Resonant Frequency

The distributed capacitance between the winding turns acts like a shunt capacitance, conducting a high-frequency current. This capacitance is called a *stray capacitance* or a *self-capacitance C_s* [32], [33]. It depends on the winding geometry, the proximity of turns, core, and shield, and the permittivity of the dielectric insulator, in which the winding wire is coated. A single, well-spaced secondary winding is recommended to reduce the interwinding capacitance. The core should be insulated to reduce the capacitance between the winding and the core. The inductance and the self-capacitance form a parallel resonant circuit, having a fundamental (parallel) self-resonant frequency

$$f_r = \frac{1}{2\pi\sqrt{LC_s}}. \tag{1.293}$$

Below this frequency, the inductor impedance is inductive. Above the self-resonant frequency f_r, the inductor impedance is capacitive. Therefore, the useful operating frequency range of an inductor is usually from dc to $0.9 f_r$.

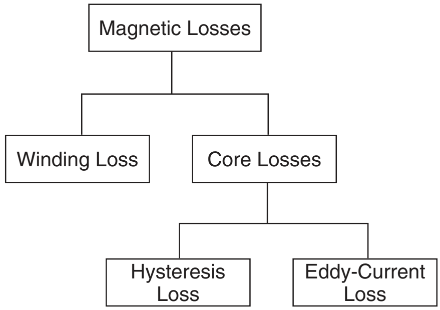

Figure 1.23 Classification of power losses in magnetic components.

1.12 Classification of Power Losses in Magnetic Components

Figure 1.23 shows a classification of power losses in magnetic components. These losses can be categorized into winding (or copper) loss P_{Rw} and core losses P_C. In turn, core losses can be divided into hysteresis loss P_H and eddy-current loss P_E:

$$P_C = P_H + P_E. \tag{1.294}$$

Hence, the total inductor power loss P_L is given by

$$P_L = P_{Rw} + P_C = P_{Rw} + P_H + P_E. \tag{1.295}$$

There are two kinds of eddy-current losses: the skin-effect loss and the proximity-effect loss. Both these effects cause current crowding. Eddy-current losses are magnetically induced losses.

1.13 Noninductive Coils

In some applications, it is desired to have a noninductive coil. Precision resistors are usually noninductive. For example, current probes require noninductive resistors. A noninductive coil is usually made using closely spaced, parallel windings, called the *bifilar winding*, as illustrated in Figure 1.24(a). Therefore, every coil turn has an adjacent turn, which carries current in the opposite direction. The magnetic fields generated by adjacent turns cancel each other, as shown in Figure 1.24(b). As a result, the coil does not store magnetic flux and presents no inductance.

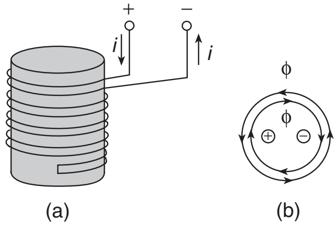

Figure 1.24 Noninductive coil. (a) Bifilar winding (b) Magnetic flux cancellation.

1.14 Summary

- Magnetic fields can be categorized as self, proximity, and fringing magnetic fields.
- The instantaneous field vector is a function of position and time.
- The phasor field vector is a function of position only.
- The magnetomotive force $\mathcal{F} = Ni$ is a source in magnetic circuits.
- A changing current in an inductor produces a changing magnetic flux, which induces voltage between the terminals of the inductor.
- Ampère's law states that the line integral of $\mathbf{H}$ around a closed contour C is equal to the current traversing the surface bounded by the contour.
- Faraday's law states that an ac voltage is induced in a coil, which contains a time-varying magnetic flux, regardless of the source of flux.
- According to Faraday's law, the voltage induced in the inductor is proportional to the number of turns N and the time rate of change of the magnetic flux ϕ.
- The magnetic flux always takes the path with the highest permeability μ.
- The Poynting vector represents the direction and the density of power flow.
- The self-inductance of a wire-wound inductor depends on its geometry and is proportional to the square of the number of turns N.
- The reluctance is directly proportional to the length of the magnetic path l_c and inversely proportional to the core cross-sectional area A_c through which the magnetic flux ϕ flows.
- The inductance is proportional to the ratio of the core cross-sectional area to the magnetic path length A_c/l_c.
- The inductance of an inductor with a ferromagnetic core is μ_{rc} times higher than that of an air core inductor. An inductor has an ampere-turn maximum value of $(NI_m)_{max}$ limited by the core saturation flux density B_s.
- An air gap is used to prevent core saturation and to make the inductance almost independent of μ_{rc}, yielding good inductance repeatability.
- The air gap contains nearly all of the magnetic field energy.
- An air gap in the core increases the energy storage capability of an inductor or a transformer.
- The core relative permeability μ_{rc} varies considerably with temperature and current. Therefore, it is desirable to maintain $\mathcal{R}_c \ll \mathcal{R}_g$ to achieve a stable inductance.
- The inductance of an inductor with an air gap is lower than the inductance of an inductor without an air gap.
- Whenever the core is excited, the fringing flux is present around the air gap, increasing the inductance and causing power losses. The fringing field decreases substantially within one air-gap length l_g of the edge of the core.
- Fringing flux represents a larger percentage of the total flux for larger gaps.
- Fringing flux and inductor losses can be reduced by dividing a large air gap into several shorter air gaps.
- The fringing flux reduces the total reluctance $\mathcal{R}$ and increases the inductance L. Therefore, the number of turns should be reduced if the exact value of the inductance is required.
- The winding should be moved away from the air gap by a distance of twice the air gap length.

- The thickness of the shield foils should be low compared to the skin depth. As the distance between the shields decreases, the inductance also decreases.
- The self-resonant frequency of an inductor is the resonant frequency of the resonant circuit formed by the inductance and stray capacitance.
- Power losses in inductors and transformers consist of winding and core losses.
- Core losses consist of hysteresis loss and eddy-current loss.
- The turns should be evenly spaced to achieve consistent inductance and reduce leakage inductance.

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1.16 Review Questions

1.1. What is the magnetomotive force?

1.2. What is the magnetic field intensity?

1.3. What is the magnetic flux density?

1.4. What is the magnetic linkage?

1.5. Define magnetic susceptibility.

1.6. Define relative permeability.

1.7. What is the reluctance of an inductor?

1.8. What is the magnetic circuit? Give an example.

1.9. Can magnetic field exist in a good conductor?

1.10. State Ampère's circuital law.

1.11. State Faraday's law.

1.12. State Lenz's law.

1.13. What is Joule's law?

1.14. What is the point form of Ohm's law?

1.15. Give Maxwell's equations.

1.16. Give Maxwell's equations for good conductors.

1.17. What is core saturation?

1.18. Define the inductance of a linear inductor.

1.19. Define the inductance of a nonlinear inductor.

1.20. Give an expression for the current of a nonlinear inductor.

1.21. What is the core inductance factor?

1.22. How is the inductance of a coil related to its number of turns?

1.23. What is the effect of an air gap on the inductance?

1.24. What is the fringing factor?

1.25. What is the effect of an air gap on core saturation?

1.26. Where is the magnetic energy stored in an inductor with an air gap?

1.27. Is the magnetic field intensity in the air gap higher or lower than that in the core?

1.28. Is the magnetic flux density in the air gap higher or lower than that in the core?

1.29. What is the volt-second balance?

1.30. Give expressions for magnetic energy in terms of H and B.

1.31. What are the mechanisms of power losses in magnetic components?

1.32. What are winding losses?

1.33. What is hysteresis loss?

1.34. What is eddy-current loss?

1.35. What are the effects of eddy currents on winding conductors and magnetic cores?

1.36. What is the self-resonant frequency?

1.37. What is the difference between fringing flux and leakage flux?

1.38. The line integral of the magnetic field intensity H over a closed contour is zero. What is the net current flowing through the surface enclosed by the contour?

1.17 Problems

1.1. A current flows in the inner conductor of a long coaxial cable and returns through the outer conductor. What is the magnetic field intensity in the region outside the coaxial cable? Explain why.

1.2. Sketch the shape of the magnetic field around a current-carrying conductor and show how the direction of the field is related to the direction of the current in the conductor.

1.3. A toroidal inductor has $N = 20$ turns, inner radius $a = 1\,\text{cm}$, outer radius $b = 2\,\text{cm}$, and height $h = 1\,\text{cm}$. The core relative permeability is $\mu_r = 100$. Find the inductance.

1.4. An inductor has $N = 300$ turns, current $I = 0.1\,\text{A}$, and $B = 0.5\,\text{T}$. The cross-sectional area $A_c = 4\,\text{cm}^2$, and the length $l_c = 15\,\text{cm}$. Find the magnetic field intensity, the magnetic flux, and the flux linkage.

1.5. An inductor has $\mu_r = 800$, $N = 700$, $\phi = 0.4\,\text{mWb}$, $l_w = 22\,\text{cm}$, and $A_c = 4 \times 10^{-4}\,\text{m}^2$. Find the current I.

1.6. An inductor has $L = 100\,\mu\text{H}$, $l_c = 2.5\,\text{cm}$, and $A_c = 2\,\text{cm}^2$. Find the number of turns N for (a) $\mu_{rc} = 1$, (b) $\mu_{rc} = 25$ and no air gap, (c) $\mu_{rc} = 25$ and $l_g = 3\,\text{mm}$, and (d) $\mu_{rc} = 2500$ and $l_g = 3\,\text{mm}$.

1.7. A core has $A_L = 30\,\mu\text{H}/100\,\text{turns}$. Find N to make an inductor of $L = 1\,\mu\text{H}$.

1.8. A toroidal core inductor has: $N = 500$, $\mu_r = 200$, $A_c = 4\,\text{cm}^2$, $r = 2\,\text{cm}$, $I_m = 1\,\text{A}$, $f = 10\,\text{MHz}$, $\rho_w = \rho_{Cu} = 1.724 \times 10^{-6}\,\Omega\text{cm}$, and $\rho_c = 10^5\,\Omega\text{m}$. Find L, A_L, $\mathcal{R}$, H_m, B_m, ϕ_m, and λ_m.

1.9. A toroidal core of $\mu_{rc} = 3000$ has a mean radius $R = 80\,\text{mm}$ and a circular cross section with radius $b = 25\,\text{mm}$. The core has an air gap $l_g = 3\,\text{mm}$ and a current I flows in a 500-turn winding to produce a magnetic flux of $10^{-4}\,\text{Wb}$. Neglect the leakage flux. (a) Determine the reluctances of the air gap and the core. (b) Find B_g and H_g in the air gap and B_c and H_c in the core. (c) Find the required current I.

1.10. An inductor has $N = 100$, $A_c = 1\,\text{cm}^2$, $B_s = 0.3\,\text{T}$, and $v_L = V_{Lm}\cos\omega t = 10\cos\omega t$ (V). Find $\lambda(t)$ and f_{min}.

1.11. Derive an expression for the internal and external inductances of a two-wire transmission line consisting of two parallel round conductors of radius a, which carry currents I in opposite directions. The axis-to-axis distance between the two conductors is $d \gg a$.