FUNDAMENTALS OF THE ANALYTIC NETWORK PROCESS – DEPENDENCE AND FEEDBACK IN DECISION-MAKING WITH A SINGLE NETWORK

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Abstract

The Analytic Network Process (ANP) is a multicriteria theory of measurement used to derive relative priority scales of absolute numbers from individual judgments (or from actual measurements normalized to a relative form) that also belong to a fundamental scale of absolute numbers. These judgments represent the relative influence, of one of two elements over the other in a pairwise comparison process on a third element in the system, with respect to an underlying control criterion. Through its supermatrix, whose entries are themselves matrices of column priorities, the ANP synthesizes the outcome of dependence and feedback within and between clusters of elements. The Analytic Hierarchy Process (AHP) with its independence assumptions on upper levels from lower levels and the independence of the elements in a level is a special case of the ANP. The ANP is an essential tool for articulating our understanding of a decision problem. One had to overcome the limitation of linear hierarchic structures and their mathematical consequences. This part on the ANP summarizes and illustrates the basic concepts of the ANP and shows how informed intuitive judgments can lead to real life answers that are matched by actual measurements in the real world (for example, relative dollar values) as illustrated in market share examples that rely on judgments and not on numerical data.

Keywords: Analytic Network Process (ANP), decisions with feedback, interdependence, market share, paired-comparisons, supermatrix, limit priorities

1. Introduction (Saaty 2004a and 2004b)

The Analytic Network Process (ANP) provides a general framework to deal with decisions without making assumptions about the independence of higher-level elements from lower level elements and about the independence of the elements within a level as in a hierarchy. In fact the ANP uses a network without the need to specify levels. As in the AHP, dominance or the relative importance of influence is a central concept. In the ANP, one provides a judgment from the fundamental scale of the AHP by answering two kinds of question with regard to strength of dominance: 1) Given a criterion which of two elements is more dominant with respect to that criterion, 2) Which of two elements influences a third element more with respect to a criterion? In

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order that all such influences be considered with respect to the same criterion so they would be meaningful to synthesize, it is essential that the same criterion be used to make all the comparisons. Such a criterion is called a control criterion. A control criterion is an important way to focus thinking to answer the question of dominance, thus first decomposing a complex problem with a variety of influences and then pulling it back together by using the weights of these influences. Synthesis has the requirement that scales can be added and multiplied to deal with dependence and feedback using judgments about importance and preference along with likelihood as in presidential elections. Real data and statistics representing probabilities and likelihood can also be used in relative form instead of making pairwise comparisons in the ANP as they are in the AHP.

With regard to understanding the ANP, we would like to encourage the reader by telling him or her that there are numerous elaborately worked out examples of the ANP (numbered in the hundreds) mostly developed by managers, executives, industrial engineers, mature students, and others in the US and abroad (among which are Brazil, Chile, the Czech Republic, Germany, India, Indonesia, Italy, Korea, Poland, Russia, Spain, and Turkey) who have studied and mastered its underlying concepts. About 100 of these applications have been summarized in a book called the *Encyclicon* indicating the use of cycles in contrast with my other book the *Hierarchon* of which I am co-editor, which has more than five hundred examples of hierarchic applications. In the next paper following this one, examples of fully worked out decisions with benefits, opportunities, costs and risks will be illustrated.

The idea of influence is central in decision-making. It is a general term applicable in the physical world (e.g. gravitational pull) in biology (giving birth or dying) in psychology (loving and hating) in politics (persuading, negotiating and opposing) and in every conceivable domain of the world in which we live and the society in which we participate. Influence is a force that produces change, to make order or to create chaos. In process thinking, change is known to be the most fundamental process in nature. As time changes so do all things change subject to the influences that exist from instant to another instant. Anything that exists influences the behavior of other things or influences the state of the environment near and sometimes far away from where it is. When we make a decision we need to look at all the potential influences and not simply the influences from top to bottom or bottom to top as in a hierarchy. Influences spread as in a network or even more generally as in a manifold. Conscious existence is a function of time and is generally recognized in different acts of consciousness, although we try to think about it as a continuous historical event. It is more meaningful for us to understand decisions mathematically in the discrete form of the spread of influence rather than as continuous processes, although the necessary continuous mathematics is in a way easier to develop as generalization of the discrete case.

Most decisions are analyzed in terms of what is important to a person or a group and what is seen as preferred in making a choice. But when we allow feedback, what is likely to turn out as a result of all the influences is what one really would like to know. The resulting priorities enable one to take the necessary actions and make the investments in resources. One would also like to ensure through sensitivity analysis, not only that the most preferred outcome will take place but also that it remain stable to perturbing forces that may occur after it is implemented. Thus the ANP is a useful tool for prediction and for representing a variety of competitors with their explicitly known and implicitly assumed interactions and the relative strengths with which they wield their influence in making a decision. It is also useful in conflict resolution where there can be many opposing influences.

The difference between a hierarchy and a network is illustrated in Figure 1. A hierarchy has a goal or a source node or cluster. It also has a sink node or cluster known in probability theory as an absorbing state that represents the alternatives of the decision. It is a linear top down structure with no feedback from lower to higher levels. However, it does have a loop at the bottom level to indicate that each alternative in that level only depends on itself and thus the elements are considered to be independent from each other. That is the case for any cluster or collection of elements that influences another group (by convention an arrow is directed towards it as in a hierarchy) but is not influenced by any other group; such a cluster is known as. A cluster of elements also has a loop if its elements were to depend on each other resulting in dependence known as inner dependence. Unlike a hierarchy, a network spreads out in all directions and its clusters of elements are not arranged in a particular order. In addition, a network allows influence to be transmitted from a cluster to another one (outer dependence) and back either directly from the second cluster or by transiting through intermediate clusters along a path which sometimes can return to the original cluster forming a cycle. The alternatives cluster of a network may or may not have feedback to other clusters. Figure 2 characterizes the clusters of a system and their connections in greater detail. A system may be generated from a hierarchy by increasing its connections gradually, so that pairs of components are connected as desired and some components have an inner dependence loop.

In Figure 2 no arrow feeds into a source component, no arrow leaves a sink component, and arrows feed into and leave a transient component. A recurrent component falls on a cycle. Loops as in C_2, C_4 and C_5 feed back into the component itself. Each priority vector is derived and introduced in the appropriate position as a column vector in a supermatrix of impacts (with respect to one control criterion) displayed as in Figure 3.

Having been exposed to the AHP, the reader knows that criteria must be weighted. The weights cannot be meaningfully obtained by simply assigning numbers to them but need to be compared with an objective (or multiple objectives) in mind. Comparisons are not only mathematically necessary, but they are our heritage from our biology. Comparisons require judgments. Judgments are associated with feelings, feelings with intensities, intensities with numbers, numbers with a

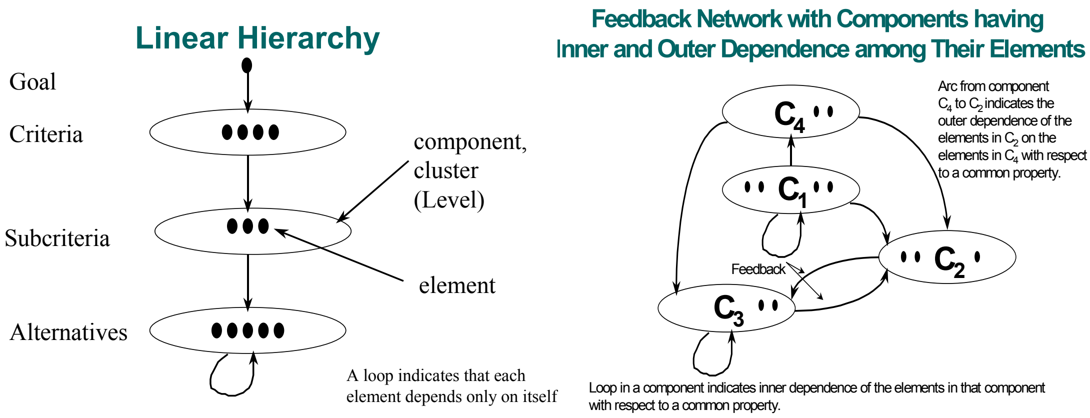


Figure 1 How a hierarchy compares to a network

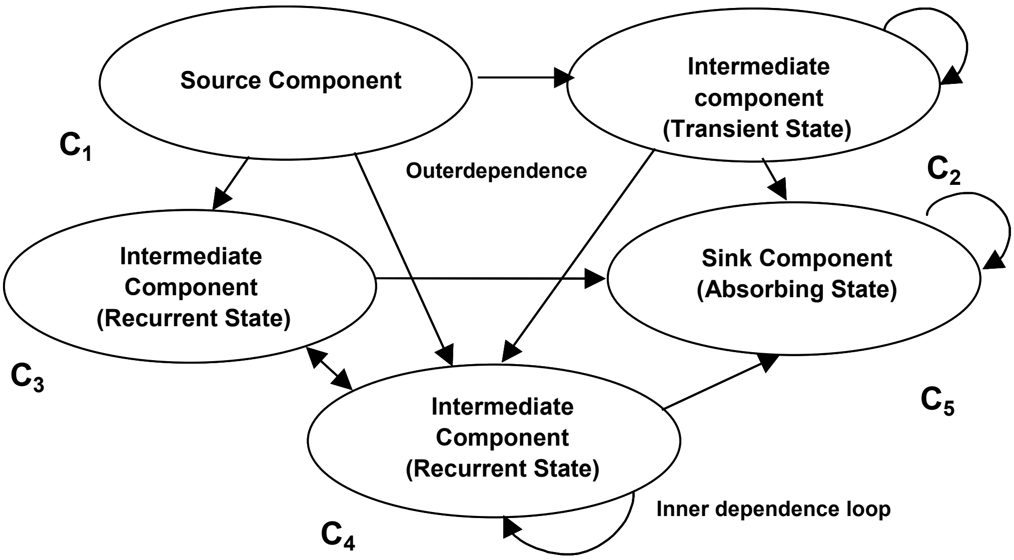


Figure 2 Connections in a network

fundamental scale, and a set of judgments represented by a fundamental scale, with priorities. The fundamental scale that represents dominance of one element over another is an absolute scale and the priorities derived from it are normalized or idealized to again yield an absolute scale. Judgments are usually inconsistent. A modicum of inconsistency is a very useful fact because it indicates that our mind has the ability to learn new things that improve and even change our understanding. But large inconsistency can indicate lack of coherent understanding that may lead to a wrong decision. To capture priorities from inconsistent judgments, the transitivity of influence is an essential consideration. It turns out that the principal right eigenvector of the matrix is necessary for capturing transitivities such as A dominates B by an amount x and B dominates C by and amount y, therefore A dominates C by an amount xy, and is the only way to represent priorities. This kind of representation needs to be validated to be credible. It has frequently been validated in the AHP and we have numerous market-share examples (illustrated below) and other full network examples to validate the ANP.

The basic question before us is how to synthesize all possible transitivities to represent overall priorities. A very useful theorem from the path matrix of a graph of a network is that the number of paths of length k between a pair of vertices is obtained from that matrix raised to the kth power. Here we obtain not the number of paths but the influence along paths of a certain length by raising our influence matrix (called the supermatrix below) to powers. We then use Cesaro summability to determine the overall priorities for all the transitivities of different lengths.

2. The Supermatrix of a Feedback System

Assume that we have a system of N clusters or components, whereby the elements in each component interact or have an impact on or are themselves influenced by some or all of the elements of that component or of another component with respect to a property governing the interactions of the entire system, such as energy or capital or political influence. Assume that component h, denoted by C_h, h = 1, ..., N, has n_h elements, that we denote by $e_{h_1}, e_{h_2}, \ldots, e_{h_{n_h}}$. A priority vector derived from paired comparisons in the usual way represents the impact of a given set of elements in a component on another element in the system. When an element has no influence on another element, its influence priority is **assigned** (not derived) as zero.

The priority vectors derived from pairwise comparison matrices are each entered as a part of some column of a supermatrix. The supermatrix represents the influence priority of an element on the left of the matrix on an element at the top of the matrix. A supermatrix along with an example of one of its general entry *i, j* block are shown in Figure 3. The component C_i alongside the supermatrix includes all the priority vectors derived for nodes that are "parent" nodes in the C_i cluster. Figure 4 gives the supermatrix of a hierarchy along with its supermatrix. The entry in the last row and column of the supermatrix of a hierarchy is the identity matrix *I*.

Figure 5 shows a holarchy whose bottom level is connected to its top level of criteria along with its supermatrix. Note the difference between the two in the last entry of the top and bottom rows.

3. Why Stochasticity of the Supermatrix Is Necessary

Interaction in the supermatrix may be measured according to several different criteria. To display and relate the criteria, we need a separate control hierarchy that includes the criteria and their priorities (see examples below). For each criterion, a different supermatrix of impacts is developed, and in terms of that criterion the components are compared according to their relative impact (or absence of impact) on each other component at the top of the supermatrix, thus developing priorities to weight the block matrices of eigenvector columns under that component in the supermatrix. The resulting stochastic matrix is known as the weighted supermatrix.

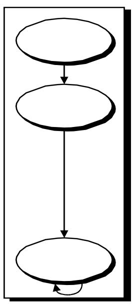
As we shall see below, it needs to be stochastic to derive meaningful limiting priorities.

Before taking the limit, the supermatrix must first be reduced to a matrix, each of whose columns sums to unity, resulting in what is known as a column stochastic matrix (see below why). In general, a supermatrix is not stochastic. This is because its columns are made up of several eigenvectors whose entries in normalized form sum to one and hence that column sums to the number of nonzero eigenvectors. In order to transform it to a stochastic matrix we need to compare its clusters, according to their impact on each other with respect to the general control criterion we have been considering and thus

$$W = \begin{array}{cc} \begin{array}{c} C_1 \\ \\ \\ C_2 \\ \\ \vdots \\ C_m \end{array} & \begin{array}{c} e_{11} \\ e_{12} \\ \vdots \\ e_{1n_1} \\ e_{21} \\ e_{22} \\ \vdots \\ e_{2n_2} \\ \vdots \\ e_{m1} \\ e_{m2} \\ \vdots \\ e_{mn_n} \end{array} \end{array} \begin{bmatrix} W_{11} & W_{12} & \cdots & W_{1m} \\ W_{21} & W_{22} & \cdots & W_{2m} \\ & & \vdots & \\ W_{m1} & W_{m2} & \cdots & W_{mm} \end{bmatrix}$$

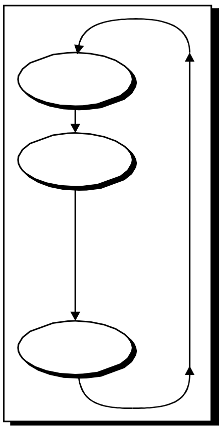
$$W_{ij} = \begin{bmatrix} w_{i_1 j_1} & w_{i_1 j_2} & \cdots & w_{i_1 j_{n_j}} \\ w_{i_2 j_1} & w_{i_2 j_2} & \cdots & w_{i_2 j_{n_j}} \\ & & \vdots & \\ w_{i_{n_i} j_1} & w_{i_{n_i} j_2} & \cdots & w_{i_{n_i} j_{n_j}} \end{bmatrix}$$

Figure 3 The supermatrix of a network and detail of a matrix in it



$$W = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ W_{21} & 0 & 0 & \dots & 0 & 0 \\ 0 & W_{32} & 0 & \dots & 0 & 0 \\ & & \ddots & & \vdots & \\ & \dots & & W_{n-1,n-2} & \dots & \\ 0 & 0 & 0 & \dots & W_{n,n-1} & I \end{bmatrix}$$

Figure 4 The structure and supermatrix of a hierarchy



$$W = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & W_{1,n} \\ W_{21} & 0 & 0 & \cdots & 0 & 0 \\ 0 & W_{32} & 0 & \cdots & 0 & 0 \\ & & \ddots & & \vdots & \\ & \cdots & & W_{n-1,n-2} & \cdots & \\ 0 & 0 & 0 & \cdots & W_{n,n-1} & 0 \end{bmatrix}$$

Figure 5 The structure and supermatrix of a holarchy

must do it several times for a decision problem, once for each control criterion, and for that criterion several matrices are needed. Each one is used to compare the influence of all the clusters on a given cluster to which they are connected. This yields an eigenvector of influence of all the clusters on each cluster. Such a vector would have zero entries when there is no influence. The priority of a component of such an eigenvector is used to weight all the elements in the block of the supermatrix that corresponds to the elements of both the influencing and the influenced cluster. The result is a stochastic supermatrix. This is not a forced way to make the matrix stochastic. It is natural. Why? Because the elements are compared among themselves and one needs information about the importance of the clusters to which they belong, to determine their relative overall weight among all the elements in the other clusters. Here is an example of why it is necessary to weight the priorities of the elements by those of their clusters: If one shouts into a room, Ladies and Gentlemen, the president, everyone is alerted and somewhat awed to expect to see the president of the United States because he is in the news so often. But if the announcement is then followed by, “of the garbage collection association”, the priority immediately drops according to the importance of the group to which that president belongs. We cannot avoid such a consideration.

The supermatrix of a hierarchy given above is already column stochastic and its clusters have equal weights. As a result, all the blocks of the matrix are multiplied by the same number. Thus the clusters do not have to be weighted. Its limit matrix shown in Figure 6 has a form whose first entry in the bottom row is the well-known hierarchic composition principle. In this case, the limit supermatrix is obtained by raising W to powers, but only in this case any kth power ($k \geq n-1$) is sufficient to derive the principle of hierarchic composition in its (k, 1) position.

If the supermatrix is stochastic, the limiting priorities depend on its reducibility, primitivity,

$$W^k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ W_{n,n-1}W_{n-1,n-2}\cdots W_{32}W_{21} & W_{n,n-1}W_{n-1,n-2}\cdots W_{32} & \cdots W_{n,n-1}W_{n-1,n-2} & W_{n,n-1} & I \end{bmatrix}$$

Figure 6 Limit matrix corresponding to hierarchic composition

and cyclicity, with four cases to consider (see Table 1 below). Both acyclic cases are illustrated here. A matrix is reducible if on a permutation of rows and columns it can be put in the form

$$\begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix}$$

where B_1 and B_3 are square submatrices. Otherwise A is irreducible or non-decomposable. It is clear that the supermatrix of a hierarchy is reducible. Its principal eigenvalue $\lambda_{\max}$ is a multiple eigenvalue. A matrix is primitive if some power of it is positive. Otherwise it is called imprimitive. A matrix has to be reducible in order for its powers to cycle that is best illustrated by the following example of successive powers of a matrix and how they shift in an orderly cyclic way the nonzero entries from one power to the next:

$$W = \begin{bmatrix} 0 & W_{12} & 0 \\ 0 & 0 & W_{23} \\ W_{31} & 0 & 0 \end{bmatrix}; \quad W^2 = \begin{bmatrix} 0 & 0 & W_{12}W_{23} \\ W_{23}W_{31} & 0 & 0 \\ 0 & W_{31}W_{12} & 0 \end{bmatrix}$$

$$W^3 = \begin{bmatrix} W_{12}W_{23}W_{31} & 0 & 0 \\ 0 & W_{23}W_{31}W_{12} & 0 \\ 0 & 0 & W_{31}W_{12}W_{23} \end{bmatrix}$$

$$W^{3k} = \begin{bmatrix} (W_{12}W_{23}W_{31})^k & 0 & 0 \\ 0 & (W_{23}W_{31}W_{12})^k & 0 \\ 0 & 0 & (W_{31}W_{12}W_{23})^k \end{bmatrix}$$

$$W^{3k+1} = \begin{bmatrix} 0 & (W_{12}W_{23}W_{31})^k W_{12} & 0 \\ 0 & 0 & (W_{23}W_{31}W_{12})^k W_{23} \\ (W_{31}W_{12}W_{23})^k W_{31} & 0 & 0 \end{bmatrix}$$

$$W^{3k+2} = \begin{bmatrix} 0 & 0 & (W_{12}W_{23}W_{31})^k W_{12}W_{23} \\ (W_{23}W_{31}W_{12})^k W_{23}W_{31} & 0 & 0 \\ 0 & (W_{31}W_{12}W_{23})^k W_{31}W_{12} & 0 \end{bmatrix}$$

Table 1 Characterization of W^{∞} in terms of eigenvalue multiplicity

	Acyclic	Cyclic
Irreducible	$\lambda_{\max}$= 1 is a simple root	C other eigenvalues with modulus = 1 (they occur in conjugate pairs)
Reducible	$\lambda_{\max}$= 1 is a multiple root	C other eigenvalues with modulus = 1 (they occur in conjugate pairs)

Let W be the stochastic matrix for which we wish to obtain $f(W) = W^{\infty}$. We have

$$\max \sum_{j=1}^{n} a_{ij} \geq \sum_{j=1}^{n} a_{ij} \frac{w_j}{w_i} = \lambda_{\max} \quad \text{for max } w_i$$

$$\min \sum_{j=1}^{n} a_{ij} \leq \sum_{j=1}^{n} a_{ij} \frac{w_j}{w_i} = \lambda_{\max} \quad \text{for min } w_i$$

For a row stochastic matrix

$$1 = \min \sum_{j=1}^{n} a_{ij} \leq \lambda_{\max} \leq \max \sum_{j=1}^{n} a_{ij} = 1, \text{thus } \lambda_{\max} = 1.$$

What we must do now is find a way to derive priorities for these four cases. We will consider the two cases when $\lambda_{\max} = 1$ is simple and then again when it is a multiple root.

The following is well known in algebra (Horn and Johnson 1985). According to J.J. Sylvester one can represent an entire function of a (diagonalizable) matrix W whose characteristic roots are distinct as:

$$f(W) = \sum_{i=1}^{n} f(\lambda_i) Z(\lambda_i),$$

where

$$Z(\lambda_i) = \frac{\prod_{j \neq i} (\lambda_j I - W)}{\prod_{j \neq i} (\lambda_j - \lambda_i)}$$

The $Z(\lambda_i)$ can be shown to be complete orthogonal idempotent matrices of W; that is, they have the properties

$$\sum_{i=1}^{k} Z(\lambda_i) = I, \; Z(\lambda_i) Z(\lambda_j) = 0,$$

$$i \neq j, \; Z^2(\lambda_i) = Z(\lambda_i),$$

where I and 0 are the identity and null matrices, respectively. Thus for example if one raises a matrix to arbitrarily large powers, it is enough to raise its eigenvalues to these powers and form the above sum involving polynomials in W. Because the eigenvalues of a stochastic matrix are all less than one, when raised to powers they vanish except when they are equal to one or are complex conjugate roots of one. Because here the eigenvalues are assumed to be distinct, we have the simplest case to deal with, that is $\lambda_{\max} = 1$ is a simple eigenvalue. Formally, because the right hand side is a polynomial in W multiplying both sides by W^{∞} each term on the right would be a constant multiplied by W^{∞} and the final outcome is also a constant multiplied by W^{∞}. Because we are only interested in the relative values of the entries in W^{∞} we can ignore the constant and simply raise W to very large powers which the computer program *Superdecisions* does in this case of distinct eigenvalues.

Next we consider the case where $\lambda_{\max}=1$ is a multiple eigenvalue. For that case we have what is known as the confluent form of Sylvester's theorem:

$$f(W)=\sum_{j=1}^{k}T(\lambda_i)=\sum_{i=1}^{k}\frac{1}{(m_i-1)!}\frac{\mathrm{d}^{m_i-1}}{\mathrm{d}\lambda^{m_i-1}}\left.\left[f(\lambda)(\lambda I-W)^{-1}\frac{\prod_{i=1}^{n}(\lambda-\lambda_i)}{\prod_{i=m_{i+1}}^{n}(\lambda-\lambda_i)}\right]\right|_{\lambda-\lambda_1}$$

where k is the number of distinct roots and m_i is the multiplicity of the root λ_i. However, as we show below, this too tells us that to obtain the limit priorities it is sufficient to raise W to arbitrarily large power to obtain a satisfactory decimal approximation to W^{∞}.

The only possible nonzero survivors as we raise the matrix to powers are those $\lambda's$ that are equal to one or are roots of one (Saaty 2004b). If the multiplicity of the largest real eigenvalue $\lambda_{\max}=1$ is n_1, then we have

$$W^{\infty}=n_1\left.\frac{\frac{\mathrm{d}^{(n_1-1)}}{\mathrm{d}\lambda^{(n_1-1)}}\left[(\lambda I-W)^{-1}\Delta(\lambda)\right]}{\Delta^{(n_1)}(\lambda)}\right|_{\lambda=1}$$

where one takes derivatives of the characteristic polynomial of the matrix W, and $\Delta(\lambda)=\det(\lambda I-W)=\lambda^n+p_1\lambda^{n-1}+...+p_n$. Also $(\lambda I-W)^{-1}=F(\lambda)/\Delta(\lambda)$ and

$$\begin{aligned}F(\lambda)=W^{n-1}&+(\lambda+p_1)W^{n-2}\\&+(\lambda^2+p_1\lambda+p_2)W^{n-3}+\cdots\\&+(\lambda^{n-1}+p_1\lambda^{n-2}+\cdots+p_{n-1})I\end{aligned}$$

is the adjoint of $(\lambda I-W)$.

Now the right side is a polynomial in W. Again, if we multiply both sides by W^{∞}, we would have on the right a constant multiplied by W^{∞} which means that we can obtain W^{∞} by raising W to large powers.

For the cases of roots of one when $\lambda_{\max}=1$ is a simple or a multiple root let us again formally see what happens to our polynomial expressions on the right in both of Sylvester's formulas as we now multiply both on the left and on the right first by $\left(W^c\right)^{\infty}$ obtaining one equation and then again by $\left(W^{c+1}\right)^{\infty}$ obtaining another and so on c times, finally multiplying both sides by $\left(W^{c+c-1}\right)^{\infty}$. We then sum these equations and take their average on both sides. The left side of each of the equations reduces to W^{∞} and the average is W^{∞}/c. On the right side the sum for each eigenvalue that is a root of unity is simply a constant times the sum

$$\left(W^c\right)^{\infty}+\left(W^{c+1}\right)^{\infty}+\cdots+\left(W^{c+c-1}\right)^{\infty}.$$

Also, because this sum is common to all the eigenvalues, it factors out and their different constants sum to a new constant multiplied by (1/c). This is true whether one is a simple or a multiple eigenvalue because the same process applies to accumulating its constants. In the very end we simply have

$$\frac{1}{c}\left[\left(W^{c}\right)^{\infty}+\left(W^{c+1}\right)^{\infty}+\cdots+\left(W^{c+c-1}\right)^{\infty}\right]=$$
$$\frac{1}{c}\left(1+W+\cdots+W^{c-1}\right)\left(W^{c}\right)^{\infty} \quad c\geq 2$$

which amounts to averaging over a cycle of length c obtained in raising W to infinite power. The cyclicity c can be determined, among others, by noting the return of the form of the matrix of powers of W to the original form of blocks of zero in W.

Caution: Some interesting things can happen in the limit supermatrix when it is reducible. For example if we have multiple goals in a hierarchy that are not connected to a higher goal, that is if we have multiple sources, we may have several limit vectors for the alternatives and these must be synthesized somehow to give a unique answer. To do that, the sources need to be connected to a higher goal and prioritized with respect to it. Otherwise, the outcome would not be unique and we would obtain nothing that is meaningful in a cooperative decision (but may be useful in a non-cooperative problem where the goals for example, are different ways of facing an opponent). It is significant to note that a hierarchy always has a single source node (the goal) and a single sink cluster (the alternatives), yet its supermatrix is reducible. Only when the supermatrix is irreducible and thus its graph is strongly connected with a path from any node or cluster to any other node or cluster that the columns of the supermatrix would be identical. It is rare that the supermatrix of a decision problem is irreducible. If the source clusters do not have sufficient interaction to serve as a single source, one could take the average of the alternatives relating to the several sources as if they are equally important to obtain a single overall outcome.

4. Why Dominance Gives Rise to the Principal Eigenvector-Cesaro Summability

In the field of decision-making, the concept of priority is quintessential and how priorities are derived influences the choices one makes. Priorities should be unique and not one of many possibilities, they must also capture the dominance of the order expressed in the judgments of the pairwise comparison matrix. The idea of a priority vector has much less validity for an arbitrary positive reciprocal matrix than for a consistent and a near consistent matrix. A matrix is near consistent if it is a small perturbation of a consistent matrix. The custom is to look for a vector $w = (w_1, \ldots, w_n)$ such that the matrix $W = (w_i / w_j)$ is "close" to $A = (a_{ij})$ by minimizing a metric. Metric closeness to the numerical values of the a_{ij} by itself says little about the numerical precision with which one element dominates another directly as in the matrix itself and indirectly through other elements as represented by the powers of the matrix. We now show that with the idea of dominance, the **principal eigenvector,** known to be unique to within a positive multiplicative constant (thus defining a ratio scale), and made unique through normalization, is the **only plausible candidate for representing priorities derived from a positive reciprocal near consistent**

pairwise comparison matrix.

Let a_{ij} be the relative dominance of A_i over A_j in the paired comparisons process. Let the matrix corresponding to the reciprocal pairwise relation be denoted by (a_{ij}). The relative dominance of A_i over A_j along paths of length *k* is given by

$$\frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}}$$

where $a_{ij}^{(k)}$ is the (*i, j*) entry of the *k*th power of the matrix (a_{ij}).

A consistent matrix *A* of order *n* satisfies the relation $a_{ij} = a_{ik} / a_{jk}$ for all $i, j,$ $k = 1,...,n.$ Thus $A^m = n^{m-1} A$. Note that a consistent matrix is reciprocal with $a_{ji} = 1/a_{ij}$. Because a consistent matrix is always of the form $A = (w_i / w_j)$, we immediately have on using $e = (1,...,1)^{\mathrm{T}}$ with T indicating the transpose vector (all other vectors are column vectors):

$$\lim_{k\to\infty} \frac{\sum_{m=1}^{k} A^m e}{\sum_{m=1}^{k} e^{\mathrm{T}} A^m e} = \lim_{h\to\infty} \frac{A^k e}{e^{\mathrm{T}} A^k e} = cw$$

where, because *A* has rank one, *n* is its principal eigenvalue and $w = (w_1,...,w_n)$ is its corresponding principal right eigenvector and c is a positive constant.

For an inconsistent matrix, the sum of all the dominances along paths of length 1, 2, and so on has a limit determined as a Cesaro sum. That limit is the principal eigenvector of the matrix of judgments. The total dominance $w(A_i)$, of alternative *i* over all other alternatives along paths of all lengths is given by the infinite series

$$w(A_i) = \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}}$$

whose sum is the Cesaro sum

$$\lim_{M\to\infty} \frac{1}{M} \sum_{k=1}^{M} \frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}}.$$

Why? Note that the sums of different sets with *k* numbers in each, determines their ranks according to their total value. The average of each sum is obtained by dividing by *k*. The averages give the same ranks because they only differ by the same constant from the original sums. Often the sum of an infinite series of numbers is infinite but if we form the average, that average as *k* tends to infinity may converge. In that case it converges to the same limit as that of the *k*th term of the infinite sum. Thus taking the limit of the averages gives us a meaningful ranking of the objects. This is a profound observation proved by the Italian Mathematician Ernesto Cesaro (1859-1906).

Cesaro Summability: Let us prove that if a sequence of numbers converges then the sequence of arithmetic means formed from that sequence also converges to the same limit as the sequence.

Proof. Let s_n denote the nth term of the sequence and let

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n}, \text{ if } \lim_{n\to\infty} \sigma_n = S,$$

then *S* is called the Cesaro sum of s_n.

Let $t_n = s_n - S, \tau_n = \sigma_n - S,$

and thus

$$\tau_n = \frac{t_1 + ... + t_n}{n}.$$

We prove that $\tau_n \to 0$ as $n \to \infty$. Choose $a > 0$, so that each $|t_n| < a$. Given $\varepsilon > 0$, choose N so that for $n > N, |t_n| < \varepsilon$.
Now for $n > N$,

$$|\tau_n| \le \frac{|t_1| + ... + |t_N|}{n} + \frac{|t_{N+1}| + ... + |t_n|}{n} < \frac{Na}{n} + \varepsilon.$$

Since ε is arbitrary, it follows that $\lim_{n\to\infty} |\tau_n| = 0$ and $\sigma_n \to S$.

Cesaro' summability ensures that

$$w(A_i) = \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}} = \lim_{M\to\infty} \frac{1}{M} \sum_{k=1}^{M} \frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}}$$

$$= \lim_{k\to\infty} \frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}}.$$

This approach to the idea of derived overall dominance is a variant of the well-known theorem of Oskar Perron for positive matrices in which it is demonstrated that the limit converges to the principal right eigenvector of the matrix. Thus a reciprocal pairwise comparisons reciprocal matrix $A = (a_{ij})$, satisfies the system of homogeneous equations

$$\sum_{j=1}^{n} a_{ij} w_j = \lambda_{\max} w_i,\ i = 1, ..., n,$$

where $\lambda_{\max}$ is the principal eigenvalue of the matrix A and w is its corresponding principal right eigenvector.

5. The Control Hierarchy

Although in this part we only illustrate the use of a single control criterion and a single decision network and supermatrix in the next section, the idea of a hierarchy of control criteria for thinking about the spread of influence is essential for decision-making. What is a control hierarchy? It is a hierarchy with criteria, called control criteria that serve as a basis for making pairwise comparisons about influence. Examples are: economic influence, social influence and environmental influence and so on. For each of these control criteria, one obtains priorities from a limit supermatrix and then combines the several sets of priorities by weighting them by the priorities of the control criteria to obtain an overall outcome.

Analysis of priorities in a system can be thought of in terms of a control hierarchy with dependence among its bottom-level subsystem arranged as a network. Dependence can occur within the clusters and between them. A control network can replace a control hierarchy at the top with dependence among its clusters. More generally, one can have a cascading set of control networks the outcome of one network is used to synthesize the outcomes of what it controls. For obvious reasons relating to the complexity of exposition, apart from a control hierarchy, we will not discuss such complex control structures here. A control hierarchy can also be involved in the network itself with feedback involved from the criteria to the elements of the network and back to the criteria to modify their influence. This kind of closed-circuit interaction between the operating parts and the criteria that drive the parts is likely to be prevalent in the brain.

A component or cluster in the ANP is a collection of elements whose function derives from the synergy of their interaction and hence has a higher-order function not found in any single element. A component is like the audio or visual component of a television set or like an arm or a leg, consisting of muscle and bone, in the human body. The clusters of the system should generally be synergistically different from the elements themselves. Otherwise they would be a mechanical collection with no intrinsic meaning.

The criteria in the control hierarchy that are used for comparing the components are usually the major parent criteria whose subcriteria are used to compare the elements in the components. Thus the criteria for comparison of the components need to be more general than those of the elements because of the greater functional complexity of the components. Sometimes for convenience, interactions of both components and elements are examined in terms of the same criteria in the control hierarchy. Although one does that to economize on the effort spent, it is more meaningful to compare the clusters with respect to control criteria and to compare the elements with respect to subcriteria of the control criteria. Otherwise the process can lead to asking difficult questions in making the paired comparisons.

The control hierarchy, critical for ANP analysis, provides overriding criteria for comparing each type of interaction that is intended by the network representation. There are two types of control criteria (subcriteria). A control criterion may be directly connected to the structure as the goal of a hierarchy if the structure is in fact a hierarchy. In this case the control criterion is called a comparison-"linking" criterion. Otherwise a control criterion does not connect directly to the structure but "induces" comparisons in a network. In that case the control criterion is called a comparison-“inducing” criterion. Note that the structure is the same, but how we think in terms of criteria is different.

An example of dependence between the elements in a component which corresponds to a loop within the component is the input-output of materials among industries. The electric industry supplies electricity to other industries including itself. But it depends more on the coal industry than on its own electricity for operation and also more on the steel industry for its turbines.

To summarize, a control hierarchy is a hierarchy of criteria and subcriteria that help us think about the spread of influence. Priorities are derived for the control criteria with benefits, or opportunities or costs or risks in mind. It is sometimes easier to use the criteria to compare the components of a system, and the subcriteria to compare the elements in the components. The generic question is; given an element in any component, how much more does a given element of a pair influence that element with respect to a control subcriterion (criterion)? The same kind of question is asked about the comparison of components. The weights of the components are used to weight the blocks of the supermatrix corresponding to the component being influenced. The limiting priorities in each supermatrix are weighted by the priority of the corresponding subcriterion and the results are synthesized for all the subcriteria. If it should happen that an element or a component has no input, a zero is entered in the corresponding priority vector.

In each block of the supermatrix, a column is either a normalized eigenvector with possibly some zero entries, or all of its elements are equal to zero. In either case it is weighted by the priority of the corresponding cluster on the left. If it is zero, that column of the supermatrix must be normalized after weighting by the cluster's weights. This operation is equivalent to assigning a zero value to the cluster on the left when weighting a column of a block with zero entries and then renormalizing the weights of the remaining clusters.

6. Three Market Share Examples – A Way to Validate the ANP

People who work in decision making often overlook the fact that the human mind is a composite of at least two parts, thinking and feeling and that our attempt to separate these two is not always to our advantage. In addition, what we think about is a matter of taste which belongs to the domain of feeling and thus we cannot separate the subjects of thinking and feeling. We deceive ourselves in assuming that rational logical thinking is divorced from feeling. Every time we think, our feelings and intuition are hidden behind as a coach of our thinking process because they provide the underlying meaning and the intensity or emphasis we place on what we think about. In attempting to suppress feelings we lose our real understanding of the world around us. Subjectivity and objectivity cannot be separated with the surgical knife of "logical thinking". How to combine the two is by transforming feelings into judgment expressed numerically within elaborate and carefully thought out structures. We have a large number of examples where users of the ANP have obtained surprisingly close answers to what is known in the real world. Many of these examples are done in a little over an hour in class as in short two and a half day courses on the ANP in Sao Paulo, Prague, Jakarta, Hawaii and others places. When properly laid out within a structure and judgments are carefully used, intuition, which usually is not very reliable, turns out to be a powerful and accurate part of the working of the nervous system.

6.1 Market Share in the Hamburger Industry (1996)

The Encyclopedia Americana says that intuition is a way of knowing directly excluding inference, discursive reasoning, logic and the employment of symbols and ideas. It is also a direct acquaintance with oneself that cannot be put into words, or a similar sensitivity to the thoughts and feelings of others. The dictionary defines a hunch as a feeling based on intuition. One reason that science does not trust intuition is because hunches that are discrete instances of intuition can often be wrong. Intuition represents a special type of thought whose separate links run more or less imperceptibly through the consciousness, making it possible to perceive truth (the result of thought) with utmost clarity. Descartes wrote about intuition, “Thanks to its simplicity it is more reliable than deduction itself”. Until the AHP/ANP there has been no formal mathematical way to lay down exhaustively and as best as one can all the factors relating to a problem and establish intensities of hunches in paired comparisons to put all the relevant intuition scientifically together and discover what intuition really says. Intuition drives reason and has in it the meaning behind what reason works on. The ANP combines intuition and judgment with reason. It asks that one do one’s best to lay down all the factors and all the numerically expressed relations among these factors. It works very well. We now have dozens, literally dozens of examples whereby students and other people who know their problem can determine the relative market share of several companies without knowing or using numerical data about them but only judgments. The reader needs to see it happen to believe it because it is done all in class in about an hour without having anticipated the subject from the instructor at all.

This example applies ANP to the problem of predicting the market share for the big three companies in the hamburger fast food industry: McDonald’s, Burger King and Wendy’s. These three firms are very competitive and offer a similar menu of hamburgers and other food items. To attract new customers and to retain their own, they have to compete by setting reasonable prices, making quality hamburgers, and promoting support of the community by sponsoring charity events and other public services.

The ANP model consists of clusters of elements connected by their dependence to one another. A cluster therefore allows one to think about grouping elements that share a set of attributes. The marketing mix is an example of a cluster whose elements are: price, product, promotion and location. The basic requirement when identifying clusters and their elements is that the elements are similar.

For this simple network model, we consider a single control criterion: economic influence. Figure 7 below shows the connections between clusters; a cluster is connected to another cluster when at least one element in it is connected to at least two elements in another cluster. The elements themselves are not shown in this figure. Except for the customer group cluster, inner dependence exists for all other clusters. In that case, the connections between elements are in the same cluster.

Structure

The structure of the model is described by its clusters and elements, and by the connection between them. These connections indicate the flow of influence between the elements. For example with respect to promotion is nutrition more or less important than packaging, and if so, by how much. In other words, given a limited budget, the company has to prioritize spending on promoting one message over others. The importance of this comparison is the basis for connecting Promotion (in the *Marketing Mix* cluster) to elements in the *Contemporary Issues* cluster (packaging, nutrition, waste disposal and recycling). The reverse connection is also important because management is aware of themes in the

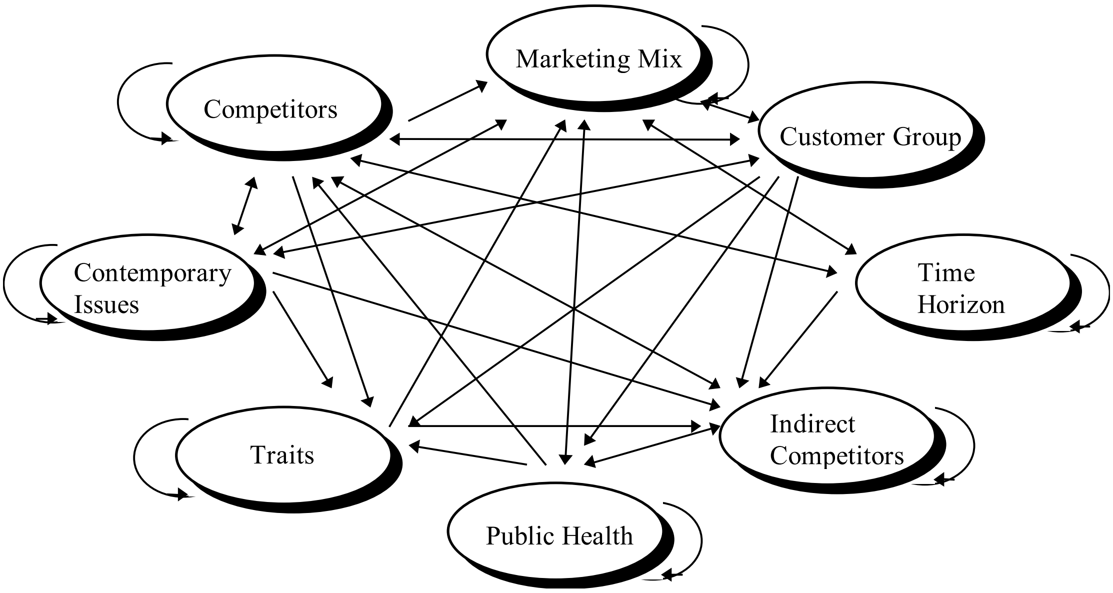


Figure 7 Overall goal: market share of competitor group

contemporary cluster influence elements in the *marketing mix* differently. For example, using more costly materials that can be recycled may raise prices more than the promotion of this fact to the public may bring in new business. Through this process of analyzing dependencies, the prevailing understanding of the marketplace is mapped out in the ANP model.

Direct Competitors Cluster

The big three companies - McDonald's, Burger King and Wendy's - are elements in the *Competitors* cluster. Each is a significant competitor with the other competitors warranting continuous monitoring and quick responses. Pairwise comparisons allow us to evaluate the importance of the two competitors with respect to their influence on each cluster's market share.

Indirect Competitors Cluster

These are companies offering alternatives to hamburgers but competing for the same overall customer base. They include: subs (Miami, Subway, corner sandwich), fried chicken (KFC, BC), Pizza (Pizza Hut), Mexican (Taco Bell), Chinese, Steak (Ponderosa), and Diners (Full service and formal). Companies in this cluster compete indirectly against the big three by offering customers alternate foods and tastes and like the direct competitors, also compete and influence one another for market share.

Customer Cluster

Four basic consumer groups considered include: the white collar professional, the blue collar laborer, the student and the family. These segments help us evaluate the influence other elements may have on each of them. For example, price may influence each segment differently. Students and families on a tight budget may be more concerned with price but the working individual may instead be more concerned with convenience. This is the only cluster without inner dependence as customer segments are perceived not to influence one another.

Marketing Mix Cluster

Price in this model refers to the average price of the typical product, for example, the price of a Big Mac. However, the model could be extended to include other products and their prices. The typical product is that which the company sells the most. For location, we consider the number of established outlets, and for promotion we consider specially packaged lunch deals that would usually be more expensive when bought separately.

Contemporary Concerns Cluster

This cluster includes issues the public is aware of about the fast food industry. For example, CNN raised questions about the nutritional value of fast foods in a news report. Also, environmental groups pressure companies into paying more attention to preserving Nature and the environment by practicing recycling, properly disposing of waste, and by not over-packaging their product. These factors help raise the cost and change the routine of doing business but may also attract more customers.

Public Health Concerns Cluster

Periodic outbreak of meat contamination always serves to create panic as to the safety of the meat supply channels and the adequacy of regulation and inspection. Consumers have also become highly sensitized to other evidence of hygiene related to for example, the site (clean tables, floors) and the personnel (uniforms, hats, gloves, and the handling of money together with food).

Traits Cluster

The elements of this cluster consist of attributes customers may use or recall when judging one eatery over another. They are 1) the speed of service, 2) available seating and parking, 3) whether there is a delivery service and 4) the presence or absence of a drive through facility.

Time Horizon Cluster

This cluster makes managers think about short and medium term measures to improve market share by connecting other elements to this cluster.

Paired Comparisons

In making paired comparisons of homogeneous elements, ratios are estimated by using a 1 to 9 fundamental scale of absolute numbers to compare two alternatives with respect to an attribute, with the smaller or lesser alternative as the unit for that attribute. To estimate the larger one as a multiple of that unit, one assigns to it an absolute number from the fundamental scale. This process is done for every pair. Rather than assigning two numbers w_i and w_j and forming the ratio w_i/w_j, we assign a single number from 1 to 9 to represent the ratio $(w_i / w_j)/1$. The absolute number from the scale is an approximation to the ratio w_i / w_j . The derived scale gives us w_i and w_j . This is the central point in the relative measurement approach of the AHP.

Paired comparisons are needed for all the connections in the model. For example, Burger King is connected to elements in the *Customer Group* cluster. There would be a set of numerical judgments and the derived priority weights from these judgments, represented in the reciprocal matrix shown in Table 2 below.

The judgments in the first row of this reciprocal matrix say that in considering the

Table 2 Pairwise judgments of the customer group for Burger King

Burger King	White Collar	Blue Collar	Student	Family	Priorities
White Collar	1	4	5	1/3	0.299
Blue Collar	1/4	1	4	1/3	0.138
Student	1/5	1/4	1	1/7	0.051
Family	3	3	7	1	0.512

market share of Burger King, White Collar workers are four times more important than Blue Collar workers, White Collar workers are 5 times more important than Students but only a third as important as Family. The derived priorities in the last column are computed by raising the reciprocal matrix to arbitrarily large powers and then normalizing their row sums. Each priority vector's entries sum to one and are placed in their appropriate location in the supermatrix. This vector will be placed in the Burger King column in the rows labeled White collar, Blue collar, etc.

6.2 The Supermatrix

A supermatrix is a two-dimensional matrix of elements by elements. The priority vectors from the paired comparisons appear in the appropriate column of the supermatrix. In the supermatrix of Table 3, the sum of each column corresponds to the number of comparison sets. If Burger King only had two comparison sets, then the column under Burger King (BURG) would sum to 2 because each priority vector sums to 1. In Table 6 we give priorities, derived from paired comparisons, for the clusters as they impact each cluster according to market share. All the numbers in the *i,j* block of Table 3 which correspond to the influence of the C_i cluster on the left on the C_j cluster at the top are multiplied by the weight of the cluster C_i. For example, the 9 numbers in the upper left hand corner of the matrix in the (Competitors, Competitors) component that contains nodes McDonald's, Burger King and Wendy's are multiplied by the first number in the cluster matrix, 0.169. Applying the cluster matrix numbers to their respective blocks in the unweighted supermatrix yields the weighted matrix that is column stochastic, shown in Table 4. Raising this matrix to powers gives the limiting matrix shown in Table 5 that represents all possible interactions in the system.

The predicted relative market share is obtained in the column corresponding to the clusters of direct and indirect competitors from the limiting supermatrix. The predictions for direct competitors and the actual market shares as appeared in the Market Share Reporter (Darney and Reddy 1992) are shown in Table 7. We also found for the indirect competitor the following result in Table 8.

In addition to the priorities for the Competitors, limiting priorities for each node in the model are obtained as shown in Table 9.

This limit supermatrix predicts the market share for three fast-food restaurant chains. Based on the outcome, companies should be able to improve on the dominant factors in the

Table 3 The initial supermatrix for the hamburger model

Initial	MCDO	BURG	WEND	WHIT	BLUE	STUD	FAMI	PRIC	PROD	LOCA	DEAL	NUTR	RECY	WAST	OVER	PERS
MCDONAL	0.0000	0.9000	0.8330	0.7560	0.7350	0.7350	0.6990	0.6960	0.7050	0.7050	0.6910	0.7170	0.7700	0.7310	0.6910	0.7420
BURGER K	0.5000	0.0000	0.1670	0.1880	0.2070	0.2070	0.2370	0.2290	0.2110	0.2110	0.2180	0.2050	0.1620	0.1880	0.2180	0.1830
WENDY'S	0.5000	0.1000	0.0000	0.0560	0.0580	0.0580	0.0640	0.0750	0.0840	0.0840	0.0910	0.0780	0.0680	0.0810	0.0910	0.0750
WHITE CO	0.3080	0.2990	0.1420	0.0000	0.0000	0.0000	0.0000	0.0840	0.2570	0.1720	0.1090	0.2390	0.1680	0.2000	0.2000	0.2760
BLUE COL	0.1010	0.1380	0.2760	0.0000	0.0000	0.0000	0.0000	0.1470	0.1980	0.3170	0.1890	0.2810	0.1980	0.2000	0.2000	0.1180
STUDENTS	0.0540	0.0510	0.0600	0.0000	0.0000	0.0000	0.0000	0.4760	0.0940	0.3890	0.3510	0.1400	0.3950	0.4000	0.4000	0.1180
FAMILIES	0.5370	0.5120	0.5220	0.0000	0.0000	0.0000	0.0000	0.2930	0.4510	0.1220	0.3510	0.3400	0.2390	0.2000	0.2000	0.4880
PRICE	0.3440	0.2600	0.3010	0.1930	0.2370	0.2150	0.1970	0.0000	1.0000	0.0000	0.2500	0.6670	0.7500	0.7500	0.5000	0.0000
PRODUCT	0.1920	0.2600	0.3680	0.4520	0.1370	0.1060	0.1580	0.2500	0.0000	0.0000	0.5000	0.3330	0.2500	0.2500	0.5000	1.0000
LOCATION	0.3900	0.3810	0.2000	0.1430	0.0790	0.1890	0.0900	0.2500	0.0000	0.0000	0.2500	0.0000	0.0000	0.0000	0.0000	0.0000
DEALS	0.0740	0.0990	0.1310	0.2120	0.5470	0.4910	0.5540	0.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
NUTRITIO	0.6260	0.6850	0.6740	0.6380	0.5620	0.4830	0.5990	0.2500	1.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
RECYCLIN	0.1990	0.1070	0.1920	0.1720	0.1990	0.2760	0.1800	0.2500	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000	0.3330	0.0000
WASTE DI	0.0640	0.0570	0.0650	0.0640	0.1400	0.1410	0.1330	0.2500	0.0000	0.0000	0.0000	0.0000	0.3330	0.0000	0.6670	0.0000
OVER PAC	0.1110	0.1500	0.0690	0.1260	0.0990	0.1010	0.0880	0.2500	0.0000	0.0000	0.0000	0.0000	0.6670	0.5000	0.0000	0.0000
PERSONNE	0.4930	0.5280	0.5280	0.4930	0.4930	0.5280	0.5280	0.3330	0.2680	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
FOOD HYG	0.3110	0.3330	0.3330	0.3110	0.3110	0.3330	0.3330	0.3330	0.6150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000
SITE HYG	0.1960	0.1400	0.1400	0.1960	0.1960	0.1400	0.1400	0.3330	0.1170	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000
SPEED OF	0.6740	0.6960	0.5940	0.3490	0.4540	0.4290	0.2110	0.2000	1.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.5000	0.0000
SEATING	0.2260	0.2290	0.2490	0.1640	0.2040	0.2580	0.2160	0.2000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
PARKING	0.1010	0.0750	0.1570	0.1410	0.0910	0.0870	0.1660	0.2000	0.0000	0.2400	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
DELIVERY	0.0000	0.0000	0.0000	0.1070	0.1950	0.1560	0.1170	0.2000	0.0000	0.5500	0.0000	0.0000	0.0000	0.0000	0.2500	0.0000
DRIVE TH	0.0000	0.0000	0.0000	0.2400	0.0560	0.0700	0.2890	0.2000	0.0000	0.2100	0.0000	0.0000	0.0000	0.0000	0.2500	0.0000
SUBS	0.2560	0.1310	0.2800	0.0570	0.2260	0.2360	0.0680	0.0540	0.1080	0.0950	0.0600	0.0800	0.1430	0.1430	0.1430	0.1430

Initial	FOOD	SITE	SPEE	SEAT	PARK	DELI	DRIV	SUBS	CHIC	PIZZ	MEXI	CHIN	STEA	DINE	SHOR	MEDI
MCDONAL	0.7310	0.6670	0.6250	0.0000	0.0000	0.0000	0.5400	0.7050	0.7010	0.6740	0.6250	0.6740	0.6250	0.6140	0.6370	0.6960
BURGER K	0.1880	0.2220	0.2380	0.0000	0.0000	0.0000	0.2970	0.2110	0.1930	0.2260	0.2380	0.2260	0.2380	0.2680	0.2580	0.2290
WENDY'S	0.0810	0.1110	0.1360	0.0000	0.0000	0.0000	0.1630	0.0840	0.1060	0.1010	0.1360	0.1010	0.1360	0.1170	0.1050	0.0750
WHITE CO	0.2660	0.2930	0.4000	0.0000	0.0000	0.2500	0.2980	0.1670	0.2450	0.1110	0.2000	0.2820	0.3070	0.2890	0.0000	0.0000
BLUE COL	0.0910	0.0940	0.2000	0.0000	0.0000	0.2500	0.2100	0.3330	0.1760	0.2720	0.2000	0.1490	0.1190	0.1290	0.0000	0.0000
STUDENTS	0.0910	0.0940	0.2000	0.0000	0.0000	0.2500	0.2460	0.3330	0.2810	0.3620	0.4000	0.0750	0.0690	0.0710	0.0000	0.0000
FAMILIES	0.5520	0.5180	0.2000	0.0000	0.0000	0.2500	0.2460	0.1670	0.2980	0.2550	0.2000	0.4930	0.5050	0.5100	0.0000	0.0000
PRICE	0.0000	0.0000	0.0000	0.0000	0.0000	0.1590	0.1250	0.3500	0.2050	0.1680	0.1520	0.2400	0.2120	0.1630	0.2500	0.1960
PRODUCT	1.0000	1.0000	1.0000	0.2000	0.2500	0.0750	0.1250	0.2160	0.1880	0.1980	0.1170	0.1160	0.1930	0.4900	0.2500	0.3110
LOCATION	0.0000	0.0000	0.0000	0.8000	0.7500	0.5670	0.7500	0.1120	0.1440	0.2390	0.2900	0.1850	0.1430	0.2310	0.0000	0.0000
DEALS	0.0000	0.0000	0.0000	0.0000	0.0000	0.1990	0.0000	0.3220	0.4630	0.3950	0.4410	0.4580	0.4520	0.1160	0.5000	0.4930
NUTRITIO	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5190	0.5000	0.5710	0.5710	0.5000	0.5710	0.5710	0.0000	0.0000
RECYCLIN	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1500	0.1670	0.1430	0.1430	0.1670	0.1430	0.1430	0.0000	0.0000
WASTE DI	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1940	0.1670	0.1430	0.1430	0.1670	0.1430	0.1430	0.0000	0.0000
OVER PAC	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1370	0.1670	0.1430	0.1430	0.1670	0.1430	0.1430	0.0000	0.0000
PERSONNE	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3270	0.3330	0.3330	0.3330	0.3330	0.3330	0.3330	0.0000	0.0000
FOOD HYG	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.4130	0.3330	0.3330	0.3330	0.3330	0.3330	0.3330	0.0000	0.0000
SITE HYG	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2600	0.3330	0.3330	0.3330	0.3330	0.3330	0.3330	0.0000	0.0000
SPEED OF	0.0000	0.0000	0.0000	0.1670	0.0000	0.1270	0.2500	0.2770	0.3620	0.3260	0.3520	0.2840	0.2730	0.2730	0.0000	0.0000
SEATING	0.0000	0.7500	0.0000	0.0000	0.0000	0.2230	0.2500	0.2470	0.2270	0.1180	0.1330	0.2480	0.2730	0.2730	0.0000	0.0000
PARKING	0.0000	0.0000	0.0000	0.4040	0.0000	0.4880	0.2500	0.1920	0.1370	0.1520	0.1630	0.1860	0.2730	0.2730	0.0000	0.0000
DELIVERY	0.0000	0.1250	0.0000	0.1910	0.5000	0.0000	0.2500	0.1270	0.1370	0.2670	0.1890	0.1810	0.0910	0.0910	0.0000	0.0000
DRIVE TH	0.0000	0.1250	0.0000	0.2390	0.5000	0.1620	0.0000	0.1570	0.1370	0.1370	0.1630	0.1010	0.0910	0.0910	0.0000	0.0000
SUBS	0.2830	0.1430	0.1430	0.0000	0.0000	0.0000	0.1430	0.2870	0.1670	0.1290	0.1640	0.0880	0.0640	0.0750	0.1430	0.1430

Table 4 The weighted supermatrix for the hamburger model

Weighted	MCDO	BURG	WEND	WHIT	BLUE	STUD	FAMI	PRIC	PROD	LOCA	DEAL	NUTR	RECY	WAST	OVER	PERS	FOOD	SITE	SPEE	SEAT	PARK	DELI	DRIV	SUBS	CHIC	PIZZ	MEXI	CHIN	STEA	DINE	SHOR	MEDI
MCDONAL	0.0000	0.1521	0.1408	0.1512	0.1470	0.1470	0.1398	0.1051	0.1065	0.1065	0.1043	0.1592	0.1709	0.1623	0.1534	0.1848	0.1820	0.1661	0.1575	0.0000	0.0000	0.0000	0.1361	0.1361	0.1353	0.1301	0.1206	0.1301	0.1206	0.1185	0.2491	0.2721
BURGER K	0.0845	0.0000	0.0282	0.0376	0.0414	0.0414	0.0474	0.0346	0.0319	0.0319	0.0329	0.0455	0.0360	0.0417	0.0484	0.0456	0.0468	0.0553	0.0600	0.0000	0.0000	0.0000	0.0748	0.0407	0.0372	0.0436	0.0459	0.0436	0.0459	0.0517	0.1009	0.0895
WENDY'S	0.0845	0.0169	0.0000	0.0112	0.0116	0.0116	0.0128	0.0113	0.0127	0.0127	0.0137	0.0173	0.0151	0.0180	0.0202	0.0187	0.0202	0.0276	0.0343	0.0000	0.0000	0.0000	0.0411	0.0162	0.0205	0.0195	0.0262	0.0195	0.0262	0.0226	0.0411	0.0293
WHITE CO	0.0573	0.0556	0.0264	0.0000	0.0000	0.0000	0.0000	0.0151	0.0463	0.0310	0.0196	0.0531	0.0373	0.0444	0.0444	0.0483	0.0466	0.0513	0.1008	0.0000	0.0000	0.0630	0.0751	0.0297	0.0436	0.0198	0.0356	0.0502	0.0546	0.0514	0.0000	0.0000
BLUE COL	0.0188	0.0257	0.0513	0.0000	0.0000	0.0000	0.0000	0.0265	0.0356	0.0571	0.0340	0.0624	0.0440	0.0444	0.0444	0.0207	0.0159	0.0165	0.0504	0.0000	0.0000	0.0630	0.0529	0.0593	0.0313	0.0484	0.0356	0.0265	0.0212	0.0230	0.0000	0.0000
STUDENTS	0.0100	0.0095	0.0112	0.0000	0.0000	0.0000	0.0000	0.0857	0.0169	0.0700	0.0632	0.0311	0.0877	0.0888	0.0888	0.0207	0.0159	0.0165	0.0504	0.0000	0.0000	0.0630	0.0620	0.0593	0.0500	0.0644	0.0712	0.0134	0.0123	0.0126	0.0000	0.0000
FAMILIES	0.0999	0.0952	0.0971	0.0000	0.0000	0.0000	0.0000	0.0527	0.0812	0.0220	0.0632	0.0755	0.0531	0.0444	0.0444	0.0854	0.0966	0.0906	0.0504	0.0000	0.0000	0.0630	0.0620	0.0297	0.0530	0.0454	0.0356	0.0878	0.0899	0.0908	0.0000	0.0000
PRICE	0.0478	0.0361	0.0418	0.0349	0.0429	0.0389	0.0357	0.0000	0.1620	0.0000	0.0405	0.1341	0.1508	0.1508	0.1005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0347	0.0273	0.0392	0.0230	0.0188	0.0170	0.0269	0.0237	0.0183	0.0488	0.0382
PRODUCT	0.0267	0.0361	0.0512	0.0818	0.0248	0.0192	0.0286	0.0405	0.0000	0.0000	0.0810	0.0669	0.0503	0.0503	0.1005	0.1570	0.1570	0.1570	0.2180	0.0436	0.0545	0.0164	0.0273	0.0242	0.0211	0.0222	0.0131	0.0130	0.0216	0.0549	0.0488	0.0606
LOCATION	0.0542	0.0530	0.0278	0.0259	0.0143	0.0342	0.0163	0.0405	0.0000	0.0000	0.0405	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1744	0.1635	0.1236	0.1635	0.0125	0.0161	0.0268	0.0325	0.0207	0.0160	0.0259	0.0000	0.0000
DEALS	0.0103	0.0138	0.0182	0.0384	0.0990	0.0889	0.1003	0.0810	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0434	0.0000	0.0361	0.0519	0.0442	0.0494	0.0513	0.0506	0.0130	0.0975	0.0961
NUTRITIO	0.0645	0.0706	0.0694	0.0721	0.0635	0.0546	0.0677	0.0243	0.0970	0.0000	0.0970	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0477	0.0460	0.0525	0.0525	0.0460	0.0525	0.0525	0.0000	0.0000
RECYCLIN	0.0205	0.0110	0.0198	0.0194	0.0225	0.0312	0.0203	0.0243	0.0000	0.0000	0.0000	0.0000	0.0000	0.0635	0.0423	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0138	0.0154	0.0132	0.0132	0.0154	0.0132	0.0132	0.0000	0.0000
WASTE DI	0.0066	0.0059	0.0067	0.0072	0.0158	0.0159	0.0150	0.0243	0.0000	0.0000	0.0000	0.0000	0.0423	0.0000	0.0847	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0178	0.0154	0.0132	0.0132	0.0154	0.0132	0.0132	0.0000	0.0000
OVER PAC	0.0114	0.0155	0.0071	0.0142	0.0112	0.0114	0.0099	0.0243	0.0000	0.0000	0.0000	0.0000	0.0847	0.0635	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0126	0.0154	0.0132	0.0132	0.0154	0.0132	0.0132	0.0000	0.0000
PERSONNE	0.0823	0.0882	0.0882	0.0804	0.0804	0.0861	0.0861	0.0566	0.0456	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0713	0.0726	0.0726	0.0726	0.0726	0.0726	0.0726	0.0000	0.0000
FOOD HYG	0.0519	0.0556	0.0556	0.0507	0.0507	0.0543	0.0543	0.0566	0.1046	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0900	0.0726	0.0726	0.0726	0.0726	0.0726	0.0726	0.0000	0.0000
SITE HYG	0.0327	0.0234	0.0234	0.0319	0.0319	0.0228	0.0228	0.0566	0.0199	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0567	0.0726	0.0726	0.0726	0.0726	0.0726	0.0726	0.0000	0.0000
SPEED OF	0.0499	0.0515	0.0440	0.0394	0.0513	0.0485	0.0238	0.0142	0.0710	0.0000	0.0710	0.0000	0.0000	0.0000	0.0505	0.0000	0.0000	0.0000	0.0000	0.0182	0.0000	0.0138	0.0273	0.0230	0.0300	0.0271	0.0292	0.0236	0.0227	0.0227	0.0000	0.0000
SEATING	0.0167	0.0169	0.0184	0.0185	0.0231	0.0292	0.0244	0.0142	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0660	0.0000	0.0000	0.0000	0.0243	0.0273	0.0205	0.0188	0.0098	0.0110	0.0206	0.0227	0.0227	0.0000	0.0000
PARKING	0.0075	0.0055	0.0116	0.0159	0.0103	0.0098	0.0188	0.0142	0.0000	0.0170	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0440	0.0000	0.0532	0.0273	0.0159	0.0114	0.0126	0.0135	0.0154	0.0227	0.0227	0.0000	0.0000
DELIVERY	0.0000	0.0000	0.0000	0.0121	0.0220	0.0176	0.0132	0.0142	0.0000	0.0391	0.0000	0.0000	0.0000	0.0000	0.0253	0.0000	0.0000	0.0110	0.0000	0.0208	0.0545	0.0000	0.0273	0.0105	0.0114	0.0222	0.0157	0.0150	0.0076	0.0076	0.0000	0.0000
DRIVE TH	0.0000	0.0000	0.0000	0.0271	0.0063	0.0079	0.0327	0.0142	0.0000	0.0149	0.0000	0.0000	0.0000	0.0000	0.0253	0.0000	0.0000	0.0110	0.0000	0.0261	0.0545	0.0177	0.0000	0.0130	0.0114	0.0114	0.0135	0.0084	0.0076	0.0076	0.0000	0.0000
SUBS	0.0415	0.0212	0.0454	0.0131	0.0518	0.0540	0.0156	0.0057	0.0114	0.0101	0.0064	0.0102	0.0182	0.0182	0.0182	0.0160	0.0317	0.0160	0.0242	0.0000	0.0000	0.0000	0.0242	0.0359	0.0209	0.0161	0.0205	0.0110	0.0080	0.0094	0.0395	0.0395

Table 5 The synthesized or limiting global supermatrix for the hamburger model

Global	MCDO	BURG	WEND	WHIT	BLUE	STUD	FAMI	PRIC	PROD	LOCA	DEAL	NUTR	RECY	WAST	OVER	PERS	FOOD	SITE	SPEE	SEAT	PARK	DELI	DRIV	SUBS	CHIC	PIZZ	MEXI	CHIN	STEA	DINE	SHOR	MEDI
MCDONAL	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298	0.1298
BURGER K	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478
WENDY'S	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266
WHITE CO	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414	0.0414
BLUE COL	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312	0.0312
STUDENTS	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323
FAMILIES	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654	0.0654
PRICE	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447	0.0447
PRODUCT	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647	0.0647
LOCATION	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386
DEALS	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281	0.0281
NUTRITIO	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435
RECYCLIN	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118
WASTE DI	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083
OVER PAC	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093
PERSONNE	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601	0.0601
FOOD HYG	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485	0.0485
SITE HYG	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326	0.0326
SPEED OF	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302
SEATING	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145
PARKING	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119	0.0119
DELIVERY	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122	0.0122
DRIVE TH	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113
SUBS	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230	0.0230

Table 6 Cluster weights with respect to economic impact control criterion of the hamburger model

	COMPETI-TORS	CUSTOMER GROUPS	MARKET-ING MIX	CONTEMP-ORARY ISSUES	PUBLIC HEALTH	TRAITS	INDIRECT COMPETI-TORS	TIME HORIZON
COMPETITORS	0.169	0.200	0.151	0.222	0.249	0.252	0.193	0.454
CUSTOMER GROUP	0.186	0.000	0.180	0.222	0.175	0.252	0.178	0.000
MARKETING MIX	0.139	0.181	0.162	0.201	0.157	0.218	0.112	0.226
CONTEMP-ORARY	0.103	0.113	0.097	0.127	0.000	0.000	0.092	0.000
PUBLIC HEALTH	0.167	0.163	0.170	0.000	0.220	0.000	0.218	0.000
TRAITS	0.074	0.113	0.071	0.101	0.088	0.109	0.083	0.000
INDIRECT COMPETITORS	0.162	0.229	0.106	0.127	0.112	0.169	0.125	0.320
TIME HORIZON	0.000	0.000	0.064	0.000	0.000	0.000	0.000	0.000

Table 7 Predicted and actual market shares for direct competitors

Company	Market Predicted %	Share Actual %
McDonald's	62.9	58.2
Burger King	23.9	28.6
Wendy's	13.2	13.2

Table 8 Predicted and actual market shares for indirect competitors

Company	Market Predicted %	Share Actual %
PIZZA	33.7	37.0
CHICKEN	26.0	28.4
MEXICAN	15.2	22.8
SUBS	25.0	11.7

Table 9 Priorities obtained from limit supermatrix

		Priorities from Limit Matrix	Priorities Normalized by Cluster
1 Alternatives	1 McDonald's	0.1749	0.5549
	2 Burger King	0.0883	0.2801
	3 Wendy's	0.0520	0.1650
2 Advertising	1 Creativity	0.0727	0.2071
	2 Promotion	0.0878	0.2501
	3 Frequency	0.1905	0.5427
3 Quality of food	1 Nutrition	0.0087	0.2825
	2 Taste	0.0076	0.2468
	3 Portion	0.0145	0.4708
4 Other	1 Price	0.0462	0.1523
	2 Location	0.0681	0.2245
	3 Service	0.0091	0.0300
	4 Speed	0.0248	0.0818
	5 Cleanliness	0.0271	0.0894
	6 Menu Item	0.0474	0.1563
	7 Take-out	0.0210	0.0692
	8 Reputation	0.0596	0.1965

model to gain market share over their competitors. Sensitivity analysis can be performed to plan various strategies depending on market responses.

Market share for the fast food restaurants as determined by the supermatrix model is as follows:

McDonald's 55.9%
Burger King 28.4%
Wendy's 15.6%

Normalized industry statistics for these restaurant chains in terms of sales in dollars reported March 1993 (published in the Market Share Reporter 1994) reflect market share as follows:

McDonald's 61.4%
Burger King 25.1%
Wendy's 13.5%

For the market of the top 15 restaurant chain industries, early 1994 statistics reported the following:

McDonald's 32.3%
Burger King 13.2%
Wendy's 7.1%
Indirect 47.4%

These figures show that McDonald's has nearly one-third the share of the entire fast-food market.

6.3 AIRLINE EXAMPLE (2001)

James Nagy did the following study of the market share of eight US airlines. Nowhere did he use numerical data, but only his knowledge of the airlines and how good each is relative to the others on the factors mentioned below. Note that in three of the clusters there is an inner dependence loop which indicates that the elements in that cluster depend on each other with respect to market share. Figure 8 shows the clusters and their inner and outer dependence connections. When the priorities of the airlines in the limit supermatrix he obtained the outcome of the model that is then compared with the actual dollar market share of each airline normalized to sum to one. The two are amazingly close. See the results in Table 10.

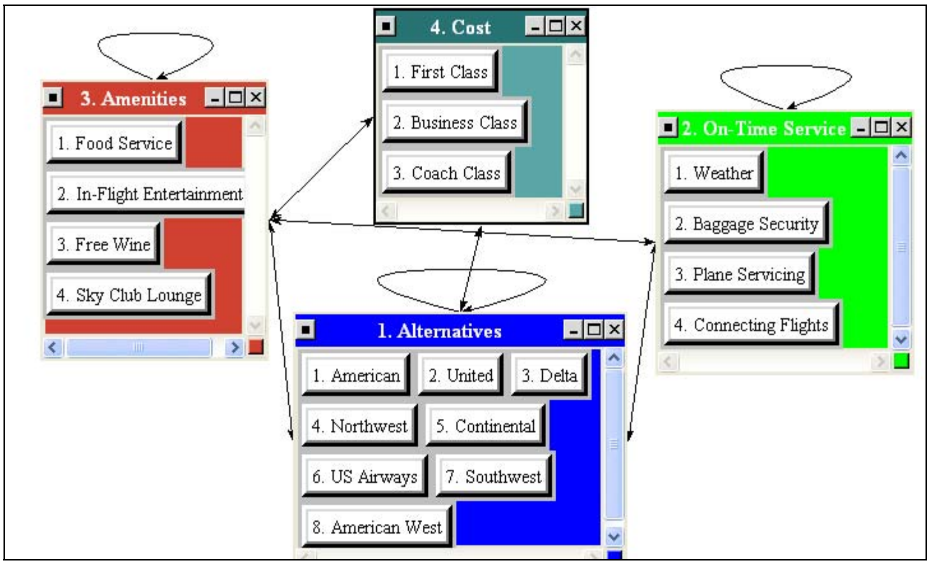


Figure 8 Airline model from the ANP super decisions software

Table 10 Market share of airlines, actual and predicted

	Actual (yr 2000)	Model Estimate
American	23.9	24.0
United	18.7	19.7
Delta	18.0	18.0
Northwest	11.4	12.4
Continental	9.3	10.0
US Airways	7.5	7.1
Southwest	5.9	6.4
American West	4.4	2.9

Nagy writes:

"I initially chose the airline industry for the assignment because I was a frequent traveler. My study group at Katz helped me make the comparisons between airlines that I did not have first hand experience as a passenger. Otherwise, I used my personal experience and perception of consumer sentiment towards the airlines to make the comparison. I was equally surprised at the results. In fact, I initially questioned how they could be so close. I would like to see the results of a study using today's consumer perception. A lot has changed in the industry since the 9/11 tragedy in the year 2001. You could divide the class up into 4 to 5 small groups and let them do the comparisons as individual groups and compare the results."

Cellular Phone Carriers (Done by my graduate students from Germany: Anabel Hengelmann and Andreas Neuhierl and from Chile, Fernandez Rodriguez, March 2004)

All the companies that were compared in the analysis represented in Figure 9 are cellular network operators. They run and maintain a wireless network all over Germany. Comparable companies in the US are hard to find as most cellular operators in the US, such as Verizon or AT&T also offer internet services and regular phone technology. If one made a separate company out of Verizon Wireless, this would resemble one of the companies that are compared. I believe there are actually no companies that operate a nation-wide network. All the four companies in the analysis do in Germany, which is of course much easier as Germany is much smaller than the United States of America.

While the American market had been deregulated long before the German market, deregulation in the early 90's was the driving factor that really triggered the development in the market out of which these companies actually evolved. It is also logical that the development of more competitive markets occurred first in the wireless market as there are fewer property rights involved than when building a wire network.

There are only these four companies competing in the German market, which makes the market relatively easy to oversee. If there were many local small companies it would be much harder to draw a line. The results are shown in Table 11.

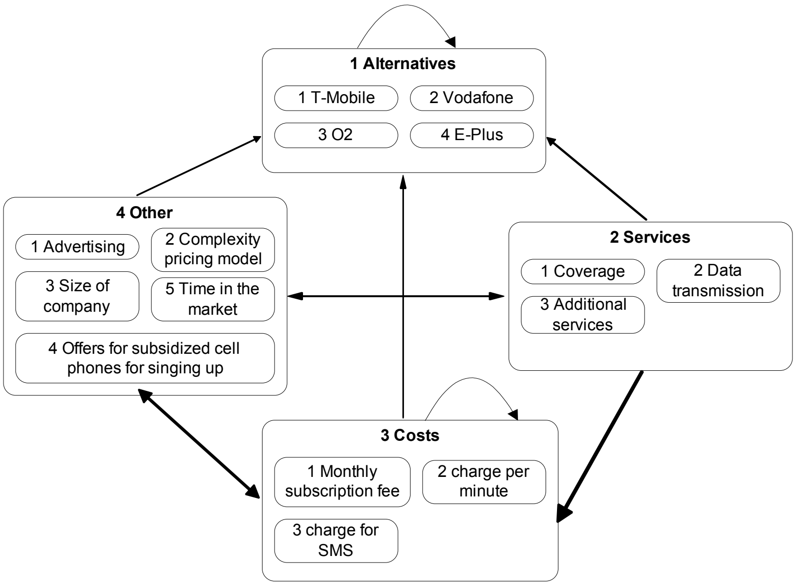


Figure 9 Cellular phones

Table 11 Actual and predicted relative market share of cell-phone providers

	Actual	Predicted
T-Mobile	42.5%	42.17%
Vodafone	38.5%	38.05%
E-Plus	11.8%	12.17%
O2	7.2%	7.61%

Source: http://www.t-mobile.de/downloads/company/roadshows/strategie.pdf

T-Mobile: T-Mobile was started as a daughter of the German Telekom, the former monopolist and was the first operating a wireless network in Germany. The German Telekom is now also a private company but used to be owned by the Federal Republic. A year ago, T-Mobile bought Voicestream and could establish a position in the American market and actually have stores in Pittsburgh.

Vodafone: Vodafone is actually a British company, but bought Mannesmann Mobilfunk in 2001. Mannesmann Mobilfunk was a German company that started operating a wireless network with T-Mobile and was actually the first private telecommunications company in Germany. With the acquisition of Mannesmann Mobilfunk, Vodafone was able to gain a very strong position in the German market.

O2: O2 used to be called Viag Interkom and was actually started as a privatization project of the state of Bavaria. Today it is completely privately owned. While they used to buy communication slots from T-Mobile and Vodafone when they originally started their network, they are now fully operating their own network and do not have to rely on the other companies in the market.

E-Plus: E-Plus is the German entity of the Japanese telecom giant NTTDOCOMO. E-Plus was the third player in the German market and was the first to operate on a new standard (GSM 1800). GSM 1800 allows more cellular users per cellular transmission station but has less sending performance and thus requires the placement of more transmission stations. This was a drawback in the early development of the companies as it took them longer to build up a fully covering network. O2 is also operating on this standard, yet had learned out of E-Plus' experience and was able to build up its network comparably quickly.

Andreas Neuhierl wrote the following:

"I started with the easiest part in this example which is choosing the alternatives. As there are only four competitors this was taken care of pretty quickly.

After that I started to try to identify a number of factors that influence the market share of a company. I had no knowledge of the actual numbers, only a rough idea that the providers would fall into two groups just by the age of the company. I did not think directly of market share as this was a little too abstract for me, rather I thought of what makes me decide on a certain provider. So technically I used myself and my judgment as a proxy to predict what other people would do, as I measured market share by subscribers.

Usually in Germany cellular contracts run for at least two years so before deciding who you are going to sign up with, you do quite a bit of research on what's better, what do I like more. I've been using a cell phone for about 7 years now and I have been a customer of three of the four providers. I've gone through the "research experience" of choosing a cellular provider three times now and I basically just used the criteria I always look for when making a decision. Advertising is certainly not a factor that I actively look for when deciding on a provider, but I suppose that I am at least subconsciously biased by it, so I accounted for that.

When doing the pairwise comparisons, I only used what I had learned from deciding on a provider for myself. I believe that I have a pretty good idea of most of the factors as I had last decided on a provider in October of 2003.

NOTE: While I built the model at home over the weekend, I did the pairwise comparisons in class and actually obtained the chart showing the actual market share during the break in class.

Then I just had the software calculate the numbers. I then wrote up the report and did not even have any numbers of the actual market share in there, just the numbers obtained from the SuperDecisions software and the chart. That's what I submitted. To that point I knew that the model was really good and by looking at the chart I could tell that my numbers were really close, but I could not quantify the difference between the actual and the predicted values. After I submitted the model, I did not change anything at all, had Rozann not asked me to include a table, I probably would have never found out the difference between the relative actual and the predicted numbers. I had to get the numbers from a different source as my first source only had the chart. So there was virtually no chance that I could have adjusted the model afterwards. In addition it could be verified if I did. Rozann has the first version that I submitted and the second version in which I included the table. So if there were a difference in my numbers, Rozann would have seen it." Rozann is this author's wife.

7. Conclusions

The outcomes of the market share examples show us that in the system of input-process-output, with the structure and judgments as input, synthesis as throughput we obtain overall priorities as output. The relative numerical values of this output correspond very closely to actual relative values measured by money. This is an extension of validating the AHP measurement process using a single paired comparison matrix (area, weights, distances, amount of protein in foods, drink consumption, electric appliance energy consumption, relative brightness of chairs), or a hierarchy (currency relative rates, chess championships about who will win and by how many games (Saaty and Vargas 2000), the superbowl (Saaty and Turner 1996) and who will win, presidential elections since 1976 (Saaty 1992)), or a holarchy (percent of increase of GNP and time of recovery of an economy 1992 and 2001 (Blair et al. 2002, Saaty and Vargas 2000)). Even more sophisticated models using networks involving benefits, opportunities, costs and risks to predict the proportion of people to vote for and against digging for oil in Alaska. This more complex approach to decision-making will be the subject of the next part of our presentation. It should further enhance our confidence in the validity of the method and its uses. Such confidence should strengthen our trust in applying it to design strategies for the future and the cause–effect relationship between actions and outcomes.

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Thomas L. Saaty holds the Chair of University Professor, Katz Graduate School of Business, University of Pittsburgh, Pittsburgh, PA, and obtained his Ph.D. in mathematics from Yale University. Before that he was a professor at the Wharton School of the University of Pennsylvania for ten years. Prior to that he spent seven years at the Arms Control and Disarmament Agency in the State Department in Washington, DC, that carried out the arms reduction negotiations with the Soviets in Geneva. His current research interests include decision-making, planning, conflict resolution and synthesis in the brain. As a result of his search for an effective means to deal with weapons tradeoffs at the Disarmament Agency and, more generally, with decision-making and resource allocation, Professor Saaty developed The Analytic Hierarchy Process (AHP) and its generalization to dependence and feedback, the Analytic Network Process (ANP). He is co-developer of the software Expert Choice and of the software Super Decisions for decisions with dependence and feedback. He has authored or co-authored twelve books on the AHP/ANP. Professor Saaty has also written a number of other books that embrace a variety of topics, including *Modern Nonlinear Equations, Nonlinear Mathematics, Graph Theory, The Four Color Problem, Behavioral Mathematics, Queuing Theory, Optimization in Integers, Embracing the Future* and *The Brain: Unraveling the Mystery of How It Works*. His most recent book is *Creative Thinking, Problem Solving & Decision Making*. The book is a rich collection of ideas, incorporating research by a growing body of researchers and practitioners, profiles of creative people, projects and products, theory, philosophy, physics and metaphysics…all explained with a liberal dose of humor. He has published more than 300 refereed articles in a wide variety of professional journals. He has been on the editorial boards of Mathematical Reviews, Operations Research, Naval Research Logistics Quarterly, Mathematical and Computer Modeling, Socio-Economic Planning Sciences, Applied Mathematics Letters, and several others. He also served as consultant to many corporations and governments.