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Research Article

Furstenberg Families and Sensitivity

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We introduce and study some concepts of sensitivity via Furstenberg families. A dynamical system (X, f) is $\mathcal{F}$-sensitive if there exists a positive ε such that for every $x \in X$ and every open neighborhood U of x there exists $y \in U$ such that the pair (x, y) is not $\mathcal{F}$-ε-asymptotic; that is, the time set $\{n : d(f^n(x), f^n(y)) > \varepsilon\}$ belongs to $\mathcal{F}$, where $\mathcal{F}$ is a Furstenberg family. A dynamical system (X, f) is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive if there is a positive ε such that every $x \in X$ is a limit of points $y \in X$ such that the pair (x, y) is $\mathcal{F}_1$-proximal but not $\mathcal{F}_2$-ε-asymptotic; that is, the time set $\{n : d(f^n(x), f^n(y)) < \delta\}$ belongs to $\mathcal{F}_1$ for any positive δ but the time set $\{n : d(f^n(x), f^n(y)) > \varepsilon\}$ belongs to $\mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are Furstenberg families.

1. Introduction

Throughout this paper a topological dynamical system (TDS) is a pair (X, f), where X is a compact metric space with a metric d and $f : X \to X$ is a continuous surjective map. Let $\mathbb{Z}_+$ be the set of nonnegative integers.

The phrase—sensitive dependence on initial condition—was first used by Ruelle [1], to indicate some exponential rate of divergence of orbits of nearby points. Following the work by Guckenheimer [2], Auslander and Yorke [3], Devaney [4], a TDS (X, f) is called *sensitive* if there exists a positive ε such that for every $x \in X$ and every open neighborhood U of x, there exist $y \in U$ and $n \in \mathbb{Z}_+$ with $d(f^n(x), f^n(y)) > \varepsilon$; that is, there exists a positive ε' such that in any opene (= open and nonempty) set there are two distinct points whose trajectories are apart from ε' (at least one moment).

Recently, several authors studied the sensitive property (cf. Abraham et al. [5], Akin and Kolyada [6]). The following proposition holds according to [6].

Proposition 1.1. *Let (X, f) be a TDS. The following conditions are equivalent.*

(1) (X, f) *is sensitive.*

(2) *There exists a positive ε such that for every $x \in X$ and every open neighborhood U of x there exists $y \in U$ with $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon$.*

(3) *There exists a positive ε such that in any opene set $U \subset X$ there exist $x, y \in U$ and $n \in \mathbb{Z}_+$ with $d(f^n(x), f^n(y)) > \varepsilon$.*

(4) *There exists a positive ε such that in any opene set $U \subset X$ there exist $x, y \in U$ with $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon$.*

From Proposition 1.1, we know that a TDS (X, f) is sensitive if and only if there exists a positive ε such that in any opene set there are two distinct points whose trajectories are infinitely many times apart at least of ε.

Some authors introduced concepts which link the Li-Yorke versions of chaos with the sensitivity in the recent years. Blanchard et al. [7] introduced the concept of spatiotemporal chaos. A TDS (X, f) is called *spatiotemporally chaotic* if every $x \in X$ is a limit of points $y \in X$ such that the pair (x, y) is proximal but not asymptotic; that is, the pair (x, y) is a Li-Yorke scrambled pair [8]. That is

$$\liminf_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0, \qquad \limsup_{n\to\infty} d\big(f^n(x), f^n(y)\big) > 0. \tag{1.1}$$

Akin and Kolyada [6] introduced the concept of Li-Yorke sensitivity. A TDS (X, f) is called *Li-Yorke sensitive* if there is a positive ε such that every $x \in X$ is a limit of points $y \in X$ such that the pair (x, y) is proximal but not ε-asymptotic. That is,

$$\liminf_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0, \qquad \limsup_{n\to\infty} d\big(f^n(x), f^n(y)\big) > \varepsilon. \tag{1.2}$$

We see that Li-Yorke sensitivity clearly implies spatiotemporal chaos, but the latter property is strictly weaker (see [6]).

Let $J \subset \mathbb{Z}_+$. The upper density of J is

$$\overline{\mu}(J) = \limsup_{n\to\infty} \frac{\sharp(J \cap \{0, 1, \ldots, n-1\})}{n}, \tag{1.3}$$

where $\sharp$ denotes the cardinality of the set. The lower density of J is

$$\underline{\mu}(J) = \liminf_{n\to\infty} \frac{\sharp(J \cap \{0, 1, \ldots, n-1\})}{n}. \tag{1.4}$$

A pair $(x, y) \in X \times X$ is *distributively scrambled pair* [9] if there is positive ε such that $\underline{\mu}(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \varepsilon\} = 0$, that is, $\overline{\mu}(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \geq \varepsilon\} = 1$, and $\overline{\mu}(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\}) = 1$ for any positive δ.

Let $\mathbb{Z}_+$ be the set of nonnegative integers, and let $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is called a Furstenberg family [10] if it is hereditary upwards; that is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$.

In the past few years, some authors [11–14] investigated proximity, mixing, and chaos via Furstenberg family. In [13], $\mathcal{F}$-scrambled pair was defined via a Furstenberg family $\mathcal{F}$. A pair (x, y) is called *$\mathcal{F}$-scrambled pair* if there is positive ε such that $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \geq \varepsilon\} \in \mathcal{F}$, and $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}$ for any positive δ. In [14], *$(\mathcal{F}_1, \mathcal{F}_2)$-scrambled pair* was defined via Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$. A pair (x, y) is called *$(\mathcal{F}_1, \mathcal{F}_2)$-scrambled pair* if there is positive ε such that $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}_1$ for any positive δ, and $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \geq \varepsilon\} \in \mathcal{F}_2$.

In this paper we investigate the sensitivity from the viewpoint of Furstenberg families.

A dynamical system (X, f) is *$\mathcal{F}$-sensitive* if there exists a positive ε such that for every $x \in X$ and every open neighborhood U of x there exists $y \in U$ such that $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\}$ belongs to $\mathcal{F}$, where $\mathcal{F}$ is a Furstenberg family.

A dynamical system (X, f) is *$(\mathcal{F}_1, \mathcal{F}_2)$-sensitive* if there is a positive ε such that every $x \in X$ is a limit of points $y \in X$ such that $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\}$ belongs to $\mathcal{F}_1$ for any positive δ but $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\}$ belongs to $\mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are Furstenberg families.

In Section 2, some basic notions related to Furstenberg families are introduced. In Section 3, we introduce and study the concept of $\mathcal{F}$-sensitivity. In Section 4, the notion of $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity is introduced and investigated, and the sensitivity of symbolic dynamics in the sense Furstenberg families is discussed finally.

2. Preliminary

In this section, we introduce some basic notions related Furstenberg families (for details see [10]). For a Furstenberg family $\mathcal{F}$, the dual family is

$$k\mathcal{F} = \{F \in \mathcal{P} : F \cap F' \neq \emptyset,\ \forall F' \in \mathcal{F}\} = \{F \in \mathcal{P} : \mathbb{Z}_+ \setminus F \notin \mathcal{F}\}. \tag{2.1}$$

Clearly, if $\mathcal{F}$ is a Furstenberg family then so is $k\mathcal{F}$. Let $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}_+$. It is easy to see that $k\mathcal{P} = \emptyset, k\emptyset = \mathcal{P}$. Clearly, $k(k\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}_1 \subset \mathcal{F}_2$ implies $k\mathcal{F}_2 \subset k\mathcal{F}_1$. Let $\mathcal{B}$ be the family of all infinite subsets of $\mathbb{Z}_+$. It is easy to see that $\mathcal{B}$ is a Furstenberg family and $k\mathcal{B}$ is the family of all cofinite subsets.

A Furstenberg family $\mathcal{F}$ is proper if it is a proper subset of $\mathcal{P}$. It is easy to see that a Furstenberg family $\mathcal{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Any subset $\mathcal{A}$ of $\mathcal{P}$ can generate a Furstenberg family $[\mathcal{A}] = \{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}$.

A Furstenberg family $\mathcal{F}$ is countably generated [10, 13] if there exists a countable subset $\mathcal{A}$ of $\mathcal{P}$ such that $[\mathcal{A}] = \mathcal{F}$. Clearly, $k\mathcal{B}$ is a countably generated proper family.

For Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$, let $\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \cap F_2 : F_1 \in \mathcal{F}, F_2 \in \mathcal{F}_2\}$. A Furstenberg family $\mathcal{F}$ is full if it is proper and $\mathcal{F} \cdot k\mathcal{F} \subset \mathcal{B}$. It is easy to see that a Furstenberg family $\mathcal{F}$ is full if and only if $k\mathcal{B} \cdot \mathcal{F} \subset \mathcal{F}$. Clearly, $k\mathcal{B}$ and $\mathcal{B}$ are full. Clearly, if $\mathcal{F}$ is full then $k\mathcal{B} \subset \mathcal{F}$. A Furstenberg family $\mathcal{F}$ is a filterdual if $\mathcal{F}$ is proper and $\mathrm{k}\mathcal{F} \supset k\mathcal{F} \cdot k\mathcal{F}$.

For every $s \in [0, 1]$, let

$$\overline{M}(s) = \{F \in \mathcal{B} : \overline{\mu}(F) \geq s\}. \tag{2.2}$$

Clearly, $\overline{M}(0) = \mathcal{B}$ and every $\overline{M}(s)$ is a full Furstenberg family (see [13]).

Let (X,f) be a TDS and $U,V\subset X$. We define the meeting time set

$$N(U,V)=\{n\in\mathbb{Z}_+ : f^n(\mathrm{U})\cap V\neq\emptyset\}. \tag{2.3}$$

In particular we have $N(x,V)=\{n\in\mathbb{Z}_+ : f^n(x)\in V\}$ for $x\in X$.

Let $A\subset X$ and $x\in X$. If $N(x,A)\in\mathcal{F}$, x is called an $\mathcal{F}$-attaching point of A. The set of all $\mathcal{F}$-attaching points of A is called the set of $\mathcal{F}$-attaching of A, denoted by $\mathcal{F}(A)$. Clearly,

$$\mathcal{F}(A)=\bigcup_{F\in\mathcal{F}}\bigcap_{n\in F}f^{-n}(A)=\bigcap_{F\in k\mathcal{F}}\bigcup_{n\in F}f^{-n}(A). \tag{2.4}$$

Let $\mathcal{F}\subset\mathcal{B}$ be a Furstenberg family. Recall that a TDS (X,f) is $\mathcal{F}$-transitive if for each pair of opene subsets U and V of X, $N(U,V)\in\mathcal{F}$. (X,f) is $\mathcal{F}$-mixing if $(X\times X, f\times f)$ is $\mathcal{F}$-transitive.

Let (X,f) be a TDS. A Furstenberg family $\mathcal{F}$ is compatible with the system (X,f) [13] if the set of $\mathcal{F}$-attaching of U is a G_δ set of X for each open set U of X.

3. $\mathcal{F}$-Sensitivity

In this section, we introduce and study the concept of $\mathcal{F}$-sensitivity. Let (X,f) be a TDS and $\mathcal{F}$ a Furstenberg family. Suppose that $A\subset X$. Let $[A]_\delta=\{x\in X : d(x,A)<\delta\}$. $\overline{A}$ denotes the closure of A. A subset B of X is called invariant for f if $f(B)\subset B$.

We will use the following relations on X:

$$\Delta=\{(x,x) : x\in X\},\qquad V_\varepsilon=\{(x,y) : d(x,y)<\varepsilon\},$$
$$\overline{V}_\varepsilon=\{(x,y) : d(x,y)\leq\varepsilon\}. \tag{3.1}$$

For any subset $R\subset X\times X$ and any point $x\in X$, we write

$$R(x)=\{y : (x,y)\in R\}. \tag{3.2}$$

We define the sets of $\mathcal{F}$-asymptotic pairs

$$\operatorname{Asym}_\varepsilon(\mathcal{F})=\left\{(x,y) : N\left((x,y),\overline{V}_\varepsilon\right)\in k\mathcal{F}\right\}=k\mathcal{F}\left(\overline{V}_\varepsilon\right),$$
$$\operatorname{Asym}_\varepsilon(\mathcal{F})(x)=\{y : (x,y)\in\operatorname{Asym}_\varepsilon(\mathcal{F})\},$$
$$\operatorname{Asym}(\mathcal{F})=\bigcap_{\varepsilon>0}\operatorname{Asym}_\varepsilon(\mathcal{F}),$$
$$\operatorname{Asym}(\mathcal{F})(x)=\bigcap_{\varepsilon>0}\operatorname{Asym}_\varepsilon(\mathcal{F})(x). \tag{3.3}$$

We say that (X, f) is *weakly $\mathcal{F}$-sensitive* [10] if there is a positive ε—a weakly $\mathcal{F}$-sensitive constant—such that in every opene subset U of X there exist x and y of U such that the pair (x, y) is not $\mathcal{F}$-ε-asymptotic. That is, $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}$, or $N((x, y), X \times X \setminus \overline{V}_\varepsilon) \in \mathcal{F}$.

We say that (X, f) is *$\mathcal{F}$-sensitive* if there exists a positive ε—a $\mathcal{F}$-sensitive constant—such that for every $x \in X$ and every open neighborhood U of x there exists $y \in U$ such that the pair (x, y) is not $\mathcal{F}$-ε-asymptotic.

Theorem 3.1. *Let (X, f) be a TDS. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Furstenberg families. Suppose that $k\mathcal{F}_1 \supset k\mathcal{F}_2 \cdot k\mathcal{F}_2$. If (X, f) is weakly $\mathcal{F}_1$-sensitive, then (X, f)that is $\mathcal{F}_2$-sensitive.*

Proof. If (X, f) is not $\mathcal{F}_2$-sensitive, then for each $\varepsilon > 0$ there exists a $x \in X$ and there exists an open neighborhood U of x such that $N((x, y), X \times X \setminus \overline{V}_\varepsilon) \notin \mathcal{F}_2$ for each $y \in U$. Thus $\mathbb{Z}_+ \setminus N((x, y), \overline{V}_\varepsilon) \notin \mathcal{F}_2$, this implies $N((x, y), \overline{V}_\varepsilon) \in k\mathcal{F}_2$. Since $k\mathcal{F}_1 \supset k\mathcal{F}_2 \cdot k\mathcal{F}_2$, by the triangle inequality we have $N((a, b), \overline{V}_{2\varepsilon}) \in k\mathcal{F}_1$ for any a and b of U. Then $\mathbb{Z}_+ \setminus N((x, y), X \times X \setminus \overline{V}_{2\varepsilon}) \in k\mathcal{F}_1$. So $N((x, y), X \times X \setminus \overline{V}_{2\varepsilon}) \notin \mathcal{F}_1$, this contradicts the (X, f) is weakly $\mathcal{F}_1$-sensitive. □

Corollary 3.2. *Let (X, f) be a TDS and $\mathcal{F}$ a filderdual. The system (X, f) is weakly $\mathcal{F}$-sensitive if and only if it is $\mathcal{F}$-sensitive.*

Lemma 3.3. *Let (X, f) be a TDS. A Furstenberg family $\mathcal{F}$ is compatible with the system (X, f) if and only if the set of $k\mathcal{F}$-attaching of V is an F_σ set of X for each closed subset V of X.*

Proof. Suppose that V is a closed subset of X, then

$$\begin{aligned} x \in k\mathcal{F}(V) &\Longleftrightarrow N(x, V) \in k\mathcal{F} \\ &\Longleftrightarrow \mathbb{Z}_+ \setminus N(x, X \setminus V) \in k\mathcal{F} \\ &\Longleftrightarrow N(x, X \setminus V) \notin \mathcal{F} \\ &\Longleftrightarrow x \notin \mathcal{F}(X \setminus V). \end{aligned} \tag{3.4}$$

Hence, $k\mathcal{F}(V) = X \setminus \mathcal{F}(X \setminus V)$. Thus $\mathcal{F}(X \setminus V)$ is a G_δ set of X, if and only if $k\mathcal{F}(V)$ is an F_σ set of X. □

Lemma 3.4. *Let (X, f) be a TDS and $\mathcal{F}_1$ and $\mathcal{F}_2$ Furstenberg families. Suppose that $\mathcal{F}_2$ is compatible with the system $(X \times X, f \times f)$, and $k\mathcal{F}_1 \supset k\mathcal{F}_2 \cdot k\mathcal{F}_2$. If (X, f) is weakly $\mathcal{F}_1$-sensitive, then there exists a positive ε such that for every $x \in X$, $X \setminus \mathrm{Asym}_\varepsilon(\mathcal{F}_2)(x)$ is a dense G_δ set.*

Proof. Since $\mathcal{F}_2$ is compatible with the system $(X \times X, f \times f)$, then $\mathrm{Asym}_\varepsilon(\mathcal{F}_2)$ is an F_σ set of $X \times X$ by Lemma 3.3. Suppose that $\mathrm{Asym}_\varepsilon(\mathcal{F}_2) = \bigcup_{n=1}^{\infty} C_n$, where every C_n is a closed subset of $X \times X$, then $\mathrm{Asym}_\varepsilon(\mathcal{F}_2)(x) = \bigcup_{n=1}^{\infty} C_n(x)$. Suppose that for each $\varepsilon > 0$ there exists $x \in X$ such that $\mathrm{Asym}_\varepsilon(\mathcal{F}_2)(x)$ is not first category. By Baire theorem there exists an opene subset U of X for some n such that $U \subset C_n(x)$. Hence for each $y \in U$, $N((x, y), \overline{V}_\varepsilon) \in k\mathcal{F}_2$. Since $k\mathcal{F}_1 \supset k\mathcal{F}_2 \cdot k\mathcal{F}_2$, by the triangle inequality we have $N((a, b), \overline{V}_{2\varepsilon}) \in k\mathcal{F}_1$ for any a and b of U. Then $\mathbb{Z}_+ \setminus N((a, b), X \times X \setminus \overline{V}_{2\varepsilon}) \in k\mathcal{F}_1$. So $N((a, b), X \times X \setminus \overline{V}_{2\varepsilon}) \notin \mathcal{F}_1$, this contradicts the (X, f) that is weakly $\mathcal{F}_1$-sensitive. □

The following lemma is proved in [13]. We give another proof here for completeness.

Lemma 3.5. *Let* (X, f) *be a TDS and* $\mathcal{F}$ *a Furstenberg family. If* $k\mathcal{F}$ *is a countably generated proper family, or* $\mathcal{F} = \overline{M}(t), t \in [0,1]$*, then* $\mathcal{F}$ *is compatible with the system* (X, f)*.*

Proof. (1) Let V be a closed subset of X. Suppose that $k\mathcal{F}$ is a proper family countably generated by $\mathcal{A}$, where $\mathcal{A}$ is countable set, then

$$k\mathcal{F}(V) = \bigcup_{F \in k\mathcal{F}} \bigcap_{n \in F} f^{-n}(V) = \bigcup_{F \in \mathcal{A}} \bigcap_{n \in F} f^{-n}(V). \tag{3.5}$$

Hence, $k\mathcal{F}(V)$ is an F_σ set.

(2) Suppose that $\mathcal{F} = \overline{M}(t), t \in [0,1]$. If $t = 0$, then $\mathcal{F} = \mathcal{B}$. Since $k\mathcal{B}$ is a countably generated proper family, the result is true by (1).

Suppose that $t \in (0,1]$. It is easy to see that $k\mathcal{F} = \{F \in \mathcal{B} : \underline{\mu}(F) > 1 - t\}$

$$\begin{aligned}
k\mathcal{F}(V) &= \left\{ x : \liminf_{m \to \infty} \frac{\sharp\{j \in \{1,2,\ldots,m\} : x \in f^{-j}(V)\}}{m} > 1 - t \right\} \\
&= \left\{ x : \exists n \in \mathbb{Z}_+, \quad \forall m > n, \frac{\sharp\{j \in \{1,2,\ldots,m\} : x \in f^{-j}(V)\}}{m} > 1 - t \right\} \\
&= \Bigg\{ x : \exists n \in \mathbb{Z}_+, \quad \forall m > n, \exists l \in \{1,2,\ldots,m\} : \frac{l}{m} > 1 - t, \\
&\qquad \exists (r_1, r_2, \ldots, r_l) : 1 \le r_1 \le \cdots \le r_l \le m \quad \forall i \in \{1,2,\ldots,l\}, f^{r_i}(x) \in V \Bigg\} \\
&= \bigcup_{n=1}^{\infty} \bigcap_{m=n+1}^{\infty} \bigcup_{l \in \Theta_m} \bigcup_{(r_1,\ldots,r_l) \in \Lambda_{l,m}} \bigcap_{i=1}^{l} f^{-r_i}(V),
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
\Theta_m &= \left\{ 1 \le l \le m : \frac{l}{m} > 1 - t \right\}, \\
\Lambda_{l,m} &= \{(r_1, \ldots, r_l) : 1 \le r_1 \le \cdots \le r_l \le m\}.
\end{aligned} \tag{3.7}$$

Hence $k\mathcal{F}(V)$ is an F_σ set. □

By Lemma 3.3, 3.5 holds.

Example 3.6. Let $\mathcal{F}_1 = \{F \in \mathcal{B} : \underline{\mu}(F) > 0.8\}$ and $\mathcal{F}_2 = \{F \in \mathcal{B} : \overline{\mu}(F) \ge 0.4\}$. If (X, f) is weakly $\mathcal{F}_1$-sensitive, then

(1) (X, f) is $\mathcal{F}_2$-sensitive,

(2) there exists a positive ε such that for every $x \in X$, $X \setminus \mathrm{Asym}_\varepsilon(\mathcal{F}_2)(x)$ is a dense G_δ set.

Since $\mathcal{F}_1 = \{F \in \mathcal{B} : \underline{\mu}(F) > 0.8\}$, then $k\mathcal{F}_1 = \{F \in \mathcal{B} : \overline{\mu}(F) \geq 0.2\}$. Since $\mathcal{F}_2 = \{F \in \mathcal{B} : \overline{\mu}(F) \geq 0.4\}$, then $k\mathcal{F}_2 = \{F \in \mathcal{B} : \underline{\mu}(F) > 0.6\}$. For any $F_1, F_2 \in k\mathcal{F}_2$, then

$$\begin{aligned}\overline{\mu}(F_1 \cap F_2) &= 1 - \underline{\mu}(\mathbb{Z}_+ \setminus (F_1 \cap F_2)) \\ &= 1 - \underline{\mu}((\mathbb{Z}_+ \setminus F_1) \cup (\mathbb{Z}_+ \setminus F_2)) \\ &\geq 1 - 0.4 - 0.4 \\ &= 0.2.\end{aligned} \tag{3.8}$$

Hence $k\mathcal{F}_1 \supset k\mathcal{F}_2 \cdot k\mathcal{F}_2$. If (X, f) is weakly $\mathcal{F}_1$-sensitive, then (X, f) is $\mathcal{F}_2$-sensitive by Theorem 3.1. By Lemmas 3.5 and 3.4 if (X, f) is weakly $\mathcal{F}_1$-sensitive, then there exists a positive ε such that for every $x \in X$, $X \setminus \mathrm{Asym}_\varepsilon(\mathcal{F}_2)(x)$ is a dense G_δ set.

The following theorem is based on arguments in Huang and Ye [15]. It is called Huang-Ye equivalences in [6]. We state it here via Furstenberg families.

Theorem 3.7. *Let (X, f) be a TDS. If $\mathcal{F}$ is a filterdual and is compatible with $(X \times X, f \times f)$, then the following statements are equivalent.*

(1) *(X, f) is weakly $\mathcal{F}$-sensitive.*

(2) *There exists a positive ε such that $\mathrm{Asym}_\varepsilon(\mathcal{F})$ is a first category subset of $X \times X$.*

(3) *There exists a positive ε such that for every $x \in X$, $\mathrm{Asym}_\varepsilon(\mathcal{F})(x)$ is a first category subset of X.*

(4) *There exists a positive ε such that for every $x \in X$, $x \in \overline{X \setminus \mathrm{Asym}_\varepsilon(\mathcal{F})(x)}$.*

(5) *There exists a positive ε such that*

$$\mathcal{F}\left(X \times X \setminus \overline{V}_\varepsilon\right) \tag{3.9}$$

is dense in $X \times X$.

(6) *(X, f) is $\mathcal{F}$-sensitive.*

Proof. (1) $\Leftrightarrow$ (6). By Corollary 3.2, it holds.

(2) $\Rightarrow$ (1). If the system is not weakly $\mathcal{F}$-sensitive then for every $\varepsilon > 0$, there exists an opene subset U of X such that $N((x, y), X \times X \setminus \overline{V}_\varepsilon) \notin \mathcal{F}$ for each $(x, y) \in U \times U$, that is, $\mathbb{Z}_+ \setminus N((x, y), \overline{V}_\varepsilon) \notin \mathcal{F}$. Then $N((x, y), \overline{V}_\varepsilon) \in k\mathcal{F}$, this implies $(x, y) \in \mathrm{Asym}_\varepsilon(\mathcal{F})$. Hence, $U \times U \subset \mathrm{Asym}_\varepsilon(\mathcal{F})$. So $\mathrm{Asym}_\varepsilon(\mathcal{F})$ is not of first category.

(3) $\Rightarrow$ (2). By Lemma 3.3, we know that $\mathrm{Asym}_\varepsilon(\mathcal{F}) = k\mathcal{F}(\overline{V_\varepsilon})$ is an F_σ set. Suppose that $\mathrm{Asym}_\varepsilon(\mathcal{F}) = \bigcup_{i=1}^{\infty} C_i$, where C_i is a closed subset of $X \times X$. Then $\mathrm{Asym}_\varepsilon(\mathcal{F})(x) = \bigcup_{i=1}^{\infty} C_i(x)$. If $\mathrm{Asym}_\varepsilon(\mathcal{F})$ is not first category then by the Baire category theorem some C_i has nonempty interior. If $U \times V \subset C_i$ and $x \in U$, then $V \subset C_i(x)$. So $\mathrm{Asym}_\varepsilon(\mathcal{F})(x)$ is not first category.

(1) $\Rightarrow$ (3). By Lemma 3.4, it holds.

Thus, we have proved that (1)–(3) are equivalent.

(4) $\Rightarrow$ (1). If (X, f) is not weakly $\mathcal{F}$-sensitive, then for any $\varepsilon > 0$ there exists an opene subset $U \subset X$ such that $U \times U \subset \mathrm{Asym}_\varepsilon(\mathcal{F})$. Let $x \in U$. Then $x \in U \subset \mathrm{Asym}_\varepsilon(\mathcal{F})(x)$, this implies $x \notin \overline{X \setminus \mathrm{Asym}_\varepsilon(\mathcal{F})(x)}$.

(3) $\Rightarrow$ (4). If there exists a positive ε such that for every $x \in X$, $\text{Asym}_\varepsilon(\mathcal{F})(x)$ is a first category subset of X, then $X \setminus \text{Asym}_\varepsilon(\mathcal{F})(x)$ is a dense G_δ subset of X. Thus (4) is true.

(2) $\Leftrightarrow$ (5). At first, we note that

$$\mathcal{F}\left(X \times X \setminus \overline{V}_\varepsilon\right) = X \times X \setminus k\mathcal{F}\left(\overline{V}_\varepsilon\right) = X \times X \setminus \text{Asym}_\varepsilon(\mathcal{F}). \tag{3.10}$$

By Baire theorem, (2) $\Leftrightarrow$ (5). □

Theorem 3.8. *Let (X, f) be a TDS. Suppose that (X, f) have two nonempty invariant subsets A and B of X with $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\} > 0$ such that $\bigcup_{i=1}^{\infty} f^{-i}(A)$ and $\bigcup_{i=1}^{\infty} f^{-i}(B)$ are dense subsets of X, then there exists a positive ε such that $k\mathcal{B}(X \times X \setminus \overline{V}_\varepsilon)$ is a dense subset of $X \times X$, and if $\mathcal{F}$ is a full Furstenberg family then (X, f) is weakly $\mathcal{F}$-sensitive.*

Proof. Since $d(A, B) > 0$, there exist positive numbers δ and ε such that $[A]_\delta \times [B]_\delta \subset X \times X \setminus \overline{V}_\varepsilon$. Since $\bigcup_{i=1}^{\infty} f^{-i}(A)$ and $\bigcup_{i=1}^{\infty} f^{-i}(B)$ are dense subsets of X, it is easy to check that so are $k\mathcal{B}([A]_\delta)$ and $k\mathcal{B}([B]_\delta)$. Since $k\mathcal{B}(X \times X \setminus \overline{V}_\varepsilon) \supset k\mathcal{B}([A]_\delta \times [B]_\delta) \supset k\mathcal{B}([A]_\delta) \times k\mathcal{B}([B]_\delta)$, then $k\mathcal{B}(X \times X \setminus \overline{V}_\varepsilon)$ is a dense subset of $X \times X$. Since $\mathcal{F}$ is full then $k\mathcal{B} \subset \mathcal{F}$, this implies that $\mathcal{F}(X \times X \setminus \overline{V}_\varepsilon)$ is a dense subset of $X \times X$. Hence, (X, f) is weakly $\mathcal{F}$-sensitive. □

A map is semiopen if the image of an opene subset contains an opene subset. A factor map $\pi : (X, f) \to (Y, g)$ between dynamical systems is a continuous surjective map $\pi : X \to Y$ such that $g \circ \pi = \pi \circ f$. The weakly $\mathcal{F}$-sensitivity can be lifted up by a semi-open factor map.

Theorem 3.9. *Let (X, f) and (Y, g) be TDS and $\pi : X \to Y$ semi-open factor map. Let $\mathcal{F}$ be a Furstenberg family. If (Y, g) is weakly $\mathcal{F}$-sensitive, so is (X, f).*

Proof. Let ε be a weakly $\mathcal{F}$-sensitive constant for (Y, g). Since π is continuous then there is $\delta > 0$ such that if $d_2(\pi(x), \pi(y)) > \varepsilon$ then $d_1(x, y) > \delta$.

Let U be an opene subset of X. As π is semi-open, $\pi(U)$ contains an opene subset V of Y. Since (Y, g) is weakly $\mathcal{F}$-sensitive, then there exist y_1 and y_2 of V such that $\{n \in \mathbb{Z}_+ : d_2(g^n(y_1), g^n(y_2)) > \varepsilon)\} \in \mathcal{F}$. Let $x_1, x_2 \in U$ with $\pi(x_1) = y_1$ and $\pi(x_2) = y_2$. Then $\{n \in \mathbb{Z}_+ : d_1(f^n(x_1), f^n(x_2)) > \delta\} \in \mathcal{F}$, that is, (X, f) is weakly $\mathcal{F}$-sensitive. □

4. $(\mathcal{F}_1, \mathcal{F}_2)$-Sensitivity

In this section, we introduce and study the notion of $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity which links chaos and sensitivity via a couple Furstenberg families $(\mathcal{F}_1, \mathcal{F}_2)$.

Let (X, f) be a TDS and $F \in \mathcal{B}$. A pair $(x, y) \in X \times X$ is called F-proximal if

$$\liminf_{F \ni n \to \infty} d\left(f^n(x), f^n(y)\right) = 0. \tag{4.1}$$

We denote the set of all F-proximal pairs by P_F.

The following lemma comes from [11].

Lemma 4.1. *Let* (X, f) *be a TDS and* $F = \{t_1 < t_2 < \cdots\} \in \mathcal{B}$*. Then*

$$P_F = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} (f \times f)^{-t_i} V_{1/n}. \tag{4.2}$$

Let $\mathcal{F}$ *be a Furstenberg family. A pair* $(x, y) \in X \times X$ *is called* $\mathcal{F}$*-proximal if* $(x, y) \in \mathcal{F}(V_\varepsilon)$ *for any* $\varepsilon > 0$*. We denote the set of all* $\mathcal{F}$*-proximal pairs by* $P_{\mathcal{F}}$*.*

Note that [12]:

$$\begin{aligned} P_{\mathcal{F}} &= \bigcap_{\varepsilon > 0} \bigcup_{F \in \mathcal{F}} \bigcap_{n \in F} (f \times f)^{-n} (V_\varepsilon) \\ &= \bigcap_{k=1}^{\infty} \bigcap_{F \in k\mathcal{F}} \bigcup_{n \in F} (f \times f)^{-n} (V_{1/k}) \\ &= \bigcap_{k=1}^{\infty} \mathcal{F}(V_{1/k}) \\ &= \bigcap_{F \in k\mathcal{F}} P_F. \end{aligned} \tag{4.3}$$

Suppose that $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are Furstenberg families.*

A TDS (X, f) *is called* $(\mathcal{F}_1, \mathcal{F}_2)$*-spatiotemporally chaotic if every* $x \in X$ *is a limit of points* $y \in X$ *such that the pair* (x, y) *is* $\mathcal{F}_1$*-proximal but not* $\mathcal{F}_2$*-asymptotic. That is, for all* $x \in X$

$$x \in \overline{P_{\mathcal{F}_1}(x) \setminus \mathrm{Asym}(\mathcal{F}_2)(x)}. \tag{4.4}$$

When $\mathcal{F}_1 = \mathcal{F}_2 = \overline{M}(0) = \mathcal{B}$*, it is the usual spatiotemporal chaos.*

A TDS (X, f) *is called* $(\mathcal{F}_1, \mathcal{F}_2)$*-sensitive if there is a positive* ε *such that every* $x \in X$ *is a limit of points* $y \in X$ *such that the pair* (x, y) *is* $\mathcal{F}_1$*-proximal but not* $\mathcal{F}_2$*-*ε*-asymptotic.*

That is, for all $x \in X$

$$x \in \overline{P_{\mathcal{F}_1}(x) \setminus \mathrm{Asym}_\varepsilon(\mathcal{F}_2)(x)} = \overline{P_{\mathcal{F}_1}(x) \cap \mathcal{F}_2(X \times X \setminus \overline{V}_\varepsilon)(x)}. \tag{4.5}$$

When $\mathcal{F}_1 = \mathcal{F}_2 = \overline{M}(0) = \mathcal{B}$*,* (X, f) *is the usual Li-Yorke sensitivity.*

If the pair (x, y) *is* $\overline{M}(1)$*-proximal but not* $\overline{M}(1)$*-*ε*-asymptotic, then* (x, y) *is the usual distributively scrambled pair.*

We will use the following lemmas which comes from [10, 11], respectively.

Lemma 4.2. *Let $\mathcal{F}$ be a full Furstenberg family. If (X,f) is $\mathcal{F}$-mixing, then $P_F(x)$ is a dense G_δ set of X for each $F \in k\mathcal{F}$ and each $x \in X$.*

Lemma 4.3. *Let (X,f) be a TDS and $\mathcal{F}$ a Furstenberg family. (X,f) is $\mathcal{F}$-transitive if and only if for every $F \in k\mathcal{F}$ and every opene subset U of X, $\bigcup\{f^{-t}(U) : t \in F\}$ is an open and dense subset of X (see [10, Proposition 4.1]).*

Theorem 4.4. *Let (X,f) be a TDS. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Furstenberg families. If there exists a positive ε such that $X \setminus \mathrm{Asym}_\varepsilon(\mathcal{F}_2)(x)$ is a dense G_δ set for every $x \in X$, and $P_{\mathcal{F}_1}(x)$ is a dense G_δ set of X for every $x \in X$, then (X,f) is $(\mathcal{F}_1,\mathcal{F}_2)$-sensitive.*

Proof. Since $P_{\mathcal{F}_1}(x) \setminus \mathrm{Asym}_\varepsilon(\mathcal{F}_2)(x)$ is a dense G_δ subset of X. Hence, (X,f) is $(\mathcal{F}_1,\mathcal{F}_2)$-sensitive. □

Theorem 4.5. *Let (X,f) be a nontrivial TDS and $\mathcal{F}$ a full filterdual. Suppose that $k\mathcal{F}$ is countably generated. If (X,f) is $\mathcal{F}$-mixing, then (X,f) is $(\mathcal{F},\mathcal{F})$-sensitive.*

Proof. Suppose that $k\mathcal{F}$ is a proper family countably generated by $\mathcal{A}$, where $\mathcal{A}$ is a countable set. Then $P_{\mathcal{F}}(x) = \bigcap_{F\in k\mathcal{F}} P_F(x) = \bigcap_{F\in\mathcal{A}} P_F(x)$. By Lemmas 4.1 and 4.2, $P_{\mathcal{F}}(x)$ is a dense G_δ set of X. Choose $\varepsilon > 0$ such that $X \times X \setminus \overline{V}_\varepsilon$ is a nonempty open subset of $X \times X$. By Lemma 4.3, $\mathcal{F}(X\times X\setminus\overline{V}_\varepsilon) = \bigcap_{F\in k\mathcal{F}} \bigcup_{n\in F} (f\times \mathrm{f})^{-n}(X\times X\setminus\overline{V}_\varepsilon)$ is a dense G_δ subset of $X\times X$. By Theorem 3.7, (X,f) is $\mathcal{F}$-sensitive. Thus (X,f) is $(\mathcal{F},\mathcal{F})$-sensitive by Theorem 4.4. □

Lemma 4.6. *Let (X,f) be a TDS. Suppose that $\mathcal{F}$ is a full Furstenberg family and is compatible with the system $(X\times X, f\times f)$. If there is a fixed point p of f such that $\bigcup_{i=1}^{\infty} f^{-i}(p)$ is dense subset of X, then $P_{\mathcal{F}}$ is a dense G_δ set of $X\times X$.*

Proof. As $\bigcup_{i=1}^{\infty} f^{-i}(p)$ is dense subset of X, it is easy to check that so is $k\mathcal{B}(\{p\}_\varepsilon)$ for any positive ε. Since $k\mathcal{B}(V_\varepsilon) \supset k\mathcal{B}(\{(p,p)\}_\varepsilon) \supset k\mathcal{B}(\{p\}_\delta)\times k\mathcal{B}(\{p\}_\delta)$ for some positive δ, then $k\mathcal{B}(V_\varepsilon)$ is a dense set of $X\times X$. As $\mathcal{F}$ is full then $k\mathcal{B} \subset \mathcal{F}$, this implies that $\mathcal{F}(V_\varepsilon)$ is a dense subset of $X\times X$. And since $\mathcal{F}$ is compatible with the system $(X\times X, f\times f)$, $\mathcal{F}(V_\varepsilon)$ is a G_δ set of $X\times X$. By $P_{\mathcal{F}} = \bigcap_{k=1}^{\infty}\mathcal{F}(V_{1/k})$, then $P_{\mathcal{F}}$ is a dense G_δ set of $X\times X$. □

5. $(\mathcal{F}_1,\mathcal{F}_2)$-Sensitivity of Symbolic Dynamics (Σ_N,σ)

Finally, as examples we will discuss the $\mathcal{F}$-sensitivity and $(\mathcal{F}_1,\mathcal{F}_2)$-sensitivity of symbolic dynamics.

Let $E = \{1,2,\dots,N\}(N \geq 2)$ with the discrete topology. Let $E_i = E$, for all $i \geq 1$. Let $\Sigma_N = \prod_{i=1}^{\infty} E_i$ with the product topology. Then Σ_N is a compact metric space. Σ_N is called the symbolic space generated by $E = \{1,2,\dots,N\}$. Let $\sigma : \Sigma_N \to \Sigma_N$ be the shift which will be defined as $\sigma(x_1x_2x_3\cdots) = x_2x_3\cdots$ for any $x_1x_2x_3\cdots$ of Σ_N. Then (Σ_N,σ) is called symbolic dynamics. Let $[i_1i_2\cdots i_n] = \{x\in\Sigma_N : x_1 = i_1, x_2 = i_2,\dots,x_n = i_n\}$.

We define a metric d which is compatible with the product topology on Σ_N as follows: for all $x = x_1x_2\dots,\ y = y_1y_2\cdots \in \Sigma_N$,

$$d(x,y) = \begin{cases} 0, & x = y, \\ \dfrac{1}{N^k}, & k = \min\{i : x_i \neq y_i\} - 1. \end{cases} \tag{5.1}$$

Theorem 5.1. *Let $\mathcal{F}$ be a full Furstenberg family. Then (Σ_N, σ) is $\mathcal{F}$-sensitive.*

Proof. Let $p = 111\cdots$ and $q = 222\cdots$. Then p and q are fixed points of σ, and both $\bigcup_{i=1}^{\infty} \sigma^{-i}(p)$ and $\bigcup_{i=1}^{\infty} \sigma^{-i}(q)$ are dense subsets of Σ_N. By Theorem 3.8, (Σ_N, σ) is weakly $\mathcal{F}$-sensitive. Let ε be a weakly $\mathcal{F}$-sensitive constant. Now we show that (Σ_N, σ) is also $\mathcal{F}$-sensitive. For any $x = x_1x_2x_3\cdots$ of Σ_N and for any open neighborhood $[x_1x_2\cdots x_n]$ of x, there exist points $y = y_1y_2y_3\cdots$ and $z = z_1z_2z_3\cdots$ of $[x_1x_2\cdots x_n]$ such that $N((y,z), X \times X \setminus \overline{V_\varepsilon}) \in \mathcal{F}$. Choose $u = u_1u_2u_3\cdots$ of $[x_1x_2\cdots x_n]$ such that $u_t = x_t$ if $y_t = z_t$ otherwise $u_t \neq x_t$. Then $N((x,u), X \times X \setminus \overline{V_\varepsilon}) = N((y,z), X \times X \setminus \overline{V_\varepsilon}) \in \mathcal{F}$, so (Σ_N, σ) is $\mathcal{F}$-sensitive. □

Lemma 5.2. *Suppose that $\mathcal{F}$ is a full Furstenberg family and is compatible with the system $(\Sigma_N \times \Sigma_N, \sigma \times \sigma)$, then $P_{\mathcal{F}}(x)$ is a dense G_δ set of Σ_N for every x of Σ_N.*

Proof. By Lemma 4.6, $P_{\mathcal{F}}(x)$ is a G_δ set of Σ_N for every x of Σ_N. □

Now we show that $P_{\mathcal{F}}(x)$ is dense for every $x = x_1x_2x_3\cdots$ of Σ_N. For any $y = y_1y_2y_3\cdots$ of Σ_N and for any open neighborhood $[y_1y_2\cdots y_n]$ of y. Choose $z = z_1z_2\cdots$ of $[y_1y_2\cdots y_n]$ such that $\sigma^n(x) = \sigma^n(z)$, then $z \in k\mathcal{B}(V_\varepsilon)(x)$ for any positive ε, this implies that $k\mathcal{B}(V_\varepsilon)(x)$ is dense. Since $\mathcal{F}$ is a full then $k\mathcal{B} \subset \mathcal{F}$, so $\mathcal{F}(V_\varepsilon)(x)$ is dense G_δ set of Σ_N. By $P_{\mathcal{F}}(x) = \bigcap_{k=1}^{\infty} \mathcal{F}(V_{1/k})(x)$, then $P_{\mathcal{F}}(x)$ is a dense G_δ set of Σ_N.

Lemma 5.3. *Suppose that $\mathcal{F}$ is a full Furstenberg family and is compatible with the system $(\Sigma_N \times \Sigma_N, \sigma \times \sigma)$, then there exists a positive ε such that for every $x \in \Sigma_N$, $\Sigma_N \setminus \mathrm{Asym}_\varepsilon(\mathcal{F})(x)$ is a dense G_δ set of Σ_N.*

Proof. Let $p = 111\cdots$ and $q = 222\cdots$. Then p and q are fixed points of σ, and both $\bigcup_{i=1}^{\infty} \sigma^{-i}(p)$ and $\bigcup_{i=1}^{\infty} \sigma^{-i}(q)$ are dense subsets of Σ_N. By Theorem 3.8 there exists a positive ε such that $k\mathcal{B}(\Sigma_N \times \Sigma_N \setminus \overline{V_\varepsilon})$ is a dense set of $\Sigma_N \times \Sigma_N$. Now we show for every $x \in X$, $X \setminus \mathrm{Asym}_\varepsilon(\mathcal{F})(x)$ is a dense G_δ set of Σ_N. For any $y = y_1y_2y_3\cdots$ of Σ_N, and for any open neighborhood $[y_1y_2\cdots y_n] \times [x_1x_2\cdots x_n]$ of (y,x) of $\Sigma_N \times \Sigma_N$, there exists (u,v) of $[y_1y_2\cdots y_n] \times [x_1x_2\cdots x_n]$ such that $N((u,v), \Sigma_N \times \Sigma_N \setminus \overline{V_\varepsilon}) \in k\mathcal{B}$. Choose $z = z_1z_2z_3\cdots$ of $[y_1y_2\cdots y_n]$ such that, $z_t = x_t$ if $u_t = v_t$ otherwise $z_t \neq x_t$, when $t > n$. Then $N((u,x), \Sigma_N \times \Sigma_N \setminus \overline{V_\varepsilon}) \in k\mathcal{B}$. Since $\mathcal{F}$ is full, then $k\mathcal{B} \subset \mathcal{F}$, this implies that $N((u,x), \Sigma_N \times \Sigma_N \setminus \overline{V_\varepsilon}) \in \mathcal{F}$. Hence $\mathcal{F}(\Sigma_N \times \Sigma_N \setminus \overline{V_\varepsilon})(x)$ is a dense set of Σ_N. Since $\mathcal{F}$ is compatible with the system $(\Sigma_N \times \Sigma_N, \sigma \times \sigma)$, then $\mathcal{F}(\Sigma_N \times \Sigma_N \setminus \overline{V_\varepsilon})$ is a G_δ set of $\Sigma_N \times \Sigma_N$. Hence $\mathcal{F}(\Sigma_N \times \Sigma_N \setminus \overline{V_\varepsilon})(x)$ is a dense G_δ set of Σ_N. □

By Lemmas 5.2 and 5.3, the following theorem holds.

Theorem 5.4. *Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are full, and are compatible with $(\Sigma_N \times \Sigma_N, \sigma \times \sigma)$, then (Σ_N, σ) is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive. In particular, (Σ_N, σ) is $(\overline{M}(1), \overline{M}(1))$-sensitive.*

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