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# FURTHER IMPROVEMENTS OF LOWER BOUNDS FOR THE LEAST COMMON MULTIPLES OF ARITHMETIC PROGRESSIONS

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Abstract. For relatively prime positive integers $u_0$ and $r$, we consider the arithmetic progression $\{u_k := u_0 + kr\}_{k=0}^n$.

Define $L_n := \text{lcm}\{u_0, u_1, ..., u_n\}$ and let $a \geq 2$ be any integer. In this paper, we show that for integers $\alpha, r \geq a$ and $n \geq 2\alpha r$, we have

$$L_n \geq u_0 r^{\alpha+a-2}(r+1)^n.$$

In particular, letting $a = 2$ yields an improvement to the best previous lower bound on $L_n$ (obtained by Hong and Yang) for all but three choices of $\alpha, r \geq 2$.

## 1. Introduction

The search for effective estimates on the least common multiples of finite arithmetic progressions began with the work of Hanson [Han72] and Nair [Nai82], who found, respectively, the upper and lower bounds for $\text{lcm}\{1, \ldots, n\}$.

Inspired by this work, Bateman, Kalb, and Stenger [BKS02] and Farhi [Far05], respectively, sought asymptotics and nontrivial lower bounds for the least common multiples of general arithmetic progressions. Farhi [Far05] obtained several nontrivial bounds and posed a conjecture which was later confirmed by Hong and Feng [HF06]. Additionally, Hong and Feng [HF06] obtained an improved lower bound for sufficiently large arithmetic progressions; this result was recently sharpened further by Hong and Yang [HY08a]. Hong and Yang [HY08b] and Farhi and Kane [FK09] also obtained some related results regarding the least common multiple of a finite number of consecutive integers. The theorem of Farhi and Kane [FK09] was very recently extended to general arithmetic progressions by Hong and Qian [HQ09].

In this article, we study finite arithmetic progressions $\{u_k := u_0 + kr\}_{k=0}^n$ with $u_0, r \geq 1$ integers satisfying $(u_0, r) = 1$. Throughout, we define

$$L_n := \text{lcm}\{u_0, u_1, \ldots, u_n\}$$

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to be the least common multiple of the sequence $\{u_k\}_{k=0}^n$. The following lower bound on $L_n$ was found by Hong and Yang [HY08a].

**Theorem 1.1** ([HY08a]). *Let $\alpha \geq 1$ be an integer. If $n > r^\alpha$, then we have $L_n \geq u_0 r^\alpha (r+1)^n$.*

If $r = 1$, then the content of Theorem 1.1 is the conjecture of Farhi [Far05] proven by Hong and Feng [HF06]. If $\alpha = 1$, then Theorem 1.1 becomes the improved lower bound of Hong and Feng [HF06].

In this paper, we sharpen the lower bound in Theorem 1.1 whenever $\alpha, r \geq 2$. In particular, we prove the following theorem which replaces the exponential condition $n > r^\alpha$ of Theorem 1.1 with a linear condition, $n \geq 2\alpha r$.

**Theorem 1.2.** *Let $a \geq 2$ be any given integer. Then for any integers $\alpha, r \geq a$ and $n \geq 2\alpha r$, we have $L_n \geq u_0 r^{\alpha+a-2}(r+1)^n$.*

Letting $a = 2$, we see that Theorem 1.2 improves upon Theorem 1.1 for all but three choices of $\alpha, r \geq 2$.

The remainder of this paper is organized as follows. In Section 2, we introduce relevant notation and previous results. In Section 3, we prove Theorem 1.2 and, as a corollary, we obtain arbitrarily strong sharpenings of Theorem 1.1 which apply in all but finitely many cases. Then, in Section 4, we discuss when the condition $n > r^\alpha$ is necessary in Theorem 1.1.

## 2. Notation and previous results

For any real numbers $x$ and $y$, we say that $y$ *divides* $x$ if there exists an integer $z$ such that $x = y \cdot z$. If $x$ divides $y$, then we write $y \mid x$. As usual, we let $\lfloor x \rfloor$ denote the largest integer no more than $x$.

Following Hong and Yang [HY08a], we denote, for each integer $0 \leq k \leq n$,

$$C_{n,k} := \frac{u_k \cdots u_n}{(n-k)!}, \quad L_{n,k} := \operatorname{lcm}\{u_k, \ldots, u_n\}.$$

From the latter definition, we have that $L_n = L_{n,0}$.

The following lemma first appeared in [Far05] and was reproven in several sources:

**Lemma 2.1** ([Far05], [Far07], [HF06]). *For any integer $n \geq 1$, $C_{n,0} \mid L_n$.*

From Lemma 2.1, we see immediately that

$$L_{n,k} = A_{n,k} \frac{u_k \cdots u_n}{(n-k)!} = A_{n,k} \cdot C_{n,k} \tag{1}$$

for an integer $A_{n,k} \geq 1$.

Following Hong and Feng [HF06] and Hong and Yang [HY08a], we define, for any $n \geq 1$,

$$k_n := \max\left\{0, \left\lfloor \frac{n-u_0}{r+1} \right\rfloor + 1\right\}. \tag{2}$$

Hong and Feng [HF06] proved the following result.

**Lemma 2.2** ([HF06]). *For all $n \geq 1$ and $0 \leq k \leq n$,*

$$L_n \geq L_{n,k_n} \geq C_{n,k_n} \geq u_0(r+1)^n.$$

## 3. Proof of the main theorem and corollary

We begin with a lemma which is similar to a key step of the proof of Theorem 1.1. The proof of this result closely follows the approach of Hong and Yang [HY08a], but simplifies the analysis.

**Lemma 3.1.** *Let $a \geq 2$ be any given integer. Then for any integers $\alpha, r \geq a$ and $n \geq 2\alpha r$, we have $n - k_n > (\alpha + a - 2)r$.*

*Proof.* If $n \leq u_0$, then by the definition (2) we have $k_n \leq 1$. Since $\alpha, r \geq a \geq 2$ and $n \geq 2\alpha r$, we deduce that $n - k_n \geq n - 1 \geq 2\alpha r - 1 > (\alpha + a - 2)r$.

Now, we suppose that $n > u_0$. In this case, we have

$$k_n = \left\lfloor \frac{n - u_0}{r+1} \right\rfloor + 1;$$

it follows that

$$k_n \leq \frac{n - u_0}{r+1} + 1 \leq \frac{n-1}{r+1} + 1 = \frac{n+r}{r+1}.$$

From this, we then see that

$$n - k_n \geq n - \frac{n+r}{r+1} = \frac{(n-1)r}{r+1} \geq \frac{(2\alpha r - 1)r}{r+1}. \tag{3}$$

However, the assumption $\alpha, r \geq a$ implies that

$$\begin{aligned}(2\alpha r - 1) - (r+1)(\alpha + a - 2) &= (r-1)\alpha - 1 - (r+1)(a-2) \\ &\geq a(r-1) - 1 - (r+1)(a-2) \\ &= 2(r-a) + 1 > 0.\end{aligned} \tag{4}$$

Therefore from (4), we infer that

$$\frac{2\alpha r - 1}{r+1} > \alpha + a - 2. \tag{5}$$

The desired result then follows immediately from (3) and (5). □

From Lemma 3.1, the proof of Theorem 1.2 follows directly via the same argument as in the endgame of the proof of Theorem 1.1. For completeness, we reproduce this elegant argument here.

*Proof of Theorem* 1.2. By hypothesis, we have $\alpha, r \geq a \geq 2$ and $n \geq 2\alpha r$. As a consequence of Lemma 3.1, we therefore obtain that $r^{\alpha+a-2} \mid (n - k_n)!$. Thus, we may express $(n-k_n)!$ in the form $r^{\alpha+a-2} \cdot B_n = (n-k_n)!$, with $B_n \geq 1$ an integer. If we choose $k = k_n$ in (1), we find that

$$r^{\alpha+a-2} \cdot B_n \cdot L_{n,k_n} = A_{n,k_n} \cdot u_{k_n} \cdots u_n.$$

It then follows that $r^{\alpha+a-2} \mid A_{n,k_n}$, since the requirement $(r, u_0) = 1$ implies that $(r, u_k) = 1$ for all $0 \leq k \leq n$. Then, we obtain from (1) and Lemma 2.2 that

$$L_{n,k_n} \geq r^{\alpha+a-2} C_{n,k_n} \geq u_0 r^{\alpha} (r+1)^n.$$

Theorem 1.2 follows. □

As a corollary of Theorem 1.2, we obtain a substantial sharpening of Theorem 1.1.

**Corollary 3.2.** *Fix integers $a \geq 2$ and $\beta \geq 1$. Then, for all but finitely many choices of integers $\alpha, r \geq a$, we have that $L_n \geq u_0 r^{\alpha+\beta+a-2}(r+1)^n$ whenever $n > r^\alpha$.*

*Proof.* By Theorem 1.2, we have $L_n \geq u_0 r^{\alpha+\beta+a-2}(r+1)^n$ whenever $n \geq 2(\alpha+\beta+a-2)r$. If $r^\alpha + 1 \geq 2(\alpha+\beta+a-2)r$, then the condition $n > r^\alpha$ guarantees that $n \geq 2(\alpha+\beta+a-2)r$. Since, for any given integer $\beta \geq 1$, we have $r^\alpha + 1 \geq 2(\alpha+\beta+a-2)r$ for all but finitely many choices of $\alpha, r \geq a$, the result follows immediately. □

The bound of Corollary 3.2 becomes effective even for small $\alpha$ and $r$. For example, the choices of $a = 2$ and $\beta = 1$ in Corollary 3.2 sharpen Theorem 1.1 by a factor of $r$ for all but six choices of $\alpha, r \geq 2$.

## 4. Examples with $L_n < u_0 r^\alpha (r+1)^n$

In their article, Hong and Yang [HY08a] asserted that their condition $n > r^\alpha$ is actually necessary for the bound $L_n > u_0 r^\alpha (r+1)^n$ in Theorem 1.1. This assertion was accompanied by an example,

$$u_0 = r = 2, \quad \alpha = 3, \quad n = 8, \tag{6}$$

in which $L_n = 5040 < 104976 = u_0 r^\alpha (r+1)^n$ (see Remark 3.1 of [HY08a]). This example (6) not only satisfies $r^\alpha = 8 \not< 8 = n$, but also satisfies $2\alpha r = 12 \not< 8 = n$. Unfortunately, (6) does not satisfy the condition $(u_0, r) = 1$, so it does not actually suffice to demonstrate the necessity of the condition $n > r^\alpha$ in Theorem 1.1 when $r = 2$ and $\alpha = 3$.

As $2\alpha r < r^\alpha + 1$ for all but three choices of $\alpha, r \geq 2$, examples with $L_n < u_0 r^\alpha (r+1)^n$ and $n = r^\alpha$ are available for at most three choices of $\alpha, r \geq 2$. A computer search of all $u_0 < n = r^\alpha$ with $(u_0, r) = 1$ in these three cases[1] indicates that there exists only one example with $L_n < u_0 r^\alpha (r+1)^n$, $(u_0, r) = 1$, and $r^\alpha = n$:

$$u_0 = 1, \quad r = \alpha = 2, \quad n = 4,$$

in which $L_4 = \text{lcm}\{1, 3, 5, 7, 9\} = 315 < 324 = 1 \cdot 2^2 (2+1)^4$.

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The footnote on page 1 should read "The first author was partly supported by the National Science Foundation of China, Grant No. 10971145, and by the Program for New Century Excellent Talents in University, Grant No. NCET-06-0785."

Page 3, line -4 should read:

$$L_{n,k_n} \geq r^{\alpha+a-2} C_{n,k_n} \geq u_0 r^{\alpha+a-2}(r+1)^n.$$

[1] We need only consider the cases with $u_0 < n$, as the proof of Lemma 3.1 shows that $\alpha r < n - k_n$ *a priori*—and so the result of Theorem 1.2 holds—whenever $u_0 \geq n$.

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