Further input-to-state stability subtleties for discrete-time systems

Further Input-to-State Stability Subtleties for Discrete-time Systems

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Abstract: This paper considers input-to-state stability (ISS) analysis of discrete-time systems using continuous Lyapunov functions. The contributions are as follows. Firstly, the existence of a continuous Lyapunov function is related to inherent input-to-state stability on compact sets with respect to both inner and outer perturbations. If the Lyapunov function is $\mathcal{K}_\infty$-continuous, this result applies to unbounded sets as well. Secondly, continuous control Lyapunov functions are employed to construct input-to-state stabilizing control laws for discrete-time systems subject to bounded perturbations. The goal is to design a receding horizon control scheme that allows the optimization of the ISS gain along a closed-loop trajectory.

Keywords: Discrete-time, Stability, Input-to-state stability, Lyapunov methods, Predictive control.

1. INTRODUCTION

This paper focuses on the design of robust stabilizing control laws in general and the design of robust model predictive control (MPC) laws in particular. A main motivation for this research is that nominally stabilizing (MPC) controllers might have no robustness properties with respect to disturbances. This aspect was for the first time shown in (Grimm et al., 2004), where it was indicated that asymptotically stable MPC closed-loop systems may have zero robustness in the presence of arbitrary small perturbations. This undesired phenomenon was revealed in (Grimm et al., 2004) by showing that an asymptotically stable MPC closed-loop system is not *robustly asymptotically stable* for arbitrary small perturbations. More recently, in (Lazar et al., 2009) the same phenomenon was exposed for globally asymptotically stable (GAS) discrete-time systems in terms of a lack of input-to-state stability (ISS) (Jiang and Wang, 2001) to *arbitrarily small* inputs. The conclusion drawn in (Lazar et al., 2009) is that GAS discrete-time systems which admit a discontinuous Lyapunov function are not necessarily inherently ISS, not even locally. As such, this observation issued a valid warning for nominally stabilizing MPC schemes, as in the case of nonlinear or hybrid systems the MPC candidate Lyapunov function is typically a discontinuous function.

To deal with the phenomenon of non-robustness, it would be useful to establish sufficient conditions under which nominally stable systems are inherently ISS. A conjecture that is frequently employed in the MPC literature is that the existence of a continuous Lyapunov function is sufficient for inherent ISS. The first contribution of this paper is to provide a formal statement of this conjecture along with a complete proof. To this end we will introduce a property called $\mathcal{K}$-continuity, which generalizes Hölder continuity on compact sets, and a property called $\mathcal{K}_\infty$-continuity, which generalizes global Hölder continuity. It is proven that continuity on a compact set is equivalent with $\mathcal{K}$-continuity and that a stronger type of global uniform continuity is equivalent with $\mathcal{K}_\infty$-continuity. These results enable us to establish that every discrete-time system that admits a continuous Lyapunov function is inherently ISS on a robustly positively invariant compact set, with respect to both inner and outer perturbations. The inclusion of inner perturbations (e.g., measurement noise or estimation error) is particularly relevant for MPC, as most of the ISS results in this framework are limited to outer perturbations (e.g., additive disturbances). A previous article that considered nominal robustness of MPC in terms of both inner and outer perturbations is (Messina et al., 2005), where it was established that existence of a continuous Lyapunov function is equivalent with robust GAS (RGAS) and semiglobal practical asymptotic stability (SPAS). Also, therein it was established that RGAS and SPAS are equivalent with attenuated ISS and integral ISS, respectively. As most robust stability results in MPC make use of the ISS framework, see, e.g., (Limon et al., 2006; Magni et al., 2006; Lazar et al., 2008; Lazar and Heemels, 2009), and integral ISS does not necessarily imply ISS (Angeli et al., 2000), in this work we focus on establishing inherent ISS with respect to both inner and outer perturbations. In this context it is worth to mention the article (Roset et al., 2008), where a connection was established between ISS to outer perturbations and ISS to inner perturbations for general constrained discrete-time systems.

The second contribution of the paper deals with the design of stabilizing MPC schemes that explicitly use a pre-defined continuous control Lyapunov function (CLF). The results established in the first part of the paper are used to show that inherent ISS is guaranteed for the resulting closed-loop system. Moreover, the recently introduced notion of *optimized ISS* (Lazar and Heemels, 2008) is employed to improve the disturbance rejection properties of the controller.

2. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$ and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \{k \in \mathbb{Z} \mid k \in \Pi\}$. For a sequence $\mathbf{w} := \{w(l)\}_{l\in\mathbb{Z}_+}$ with $w(l) \in \mathbb{R}^n$, $l \in \mathbb{Z}_+$, let $\|\mathbf{w}\| := \sup\{\|w(l)\| \mid l \in \mathbb{Z}_+\}$ and let $\mathbf{w}_{[k]} := \{w(l)\}_{l\in\mathbb{Z}_{[0,k]}}$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\mathrm{int}(\mathcal{S})$ the interior, by $\partial\mathcal{S}$ the boundary and by $\mathrm{cl}(\mathcal{S})$ the closure of $\mathcal{S}$. For two arbitrary sets $\mathcal{S} \subseteq \mathbb{R}^n$ and $\mathcal{P} \subseteq \mathbb{R}^n$, let $\mathcal{S} \oplus \mathcal{P} := \{x + y \mid x \in \mathcal{S}, y \in \mathcal{P}\}$ denote their Minkowski sum and let $\mathcal{S} \sim \mathcal{P} := \{x \in \mathbb{R}^n \mid x + \mathcal{P} \subseteq \mathcal{S}\}$ denote their Pontryagin difference. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A polytope is a closed and bounded polyhedron. Let $\|\cdot\|$ denote an arbitrary p-norm. For a matrix $Z \in \mathbb{R}^{m\times n}$ let $\|Z\| := \sup_{x\neq 0} \frac{\|Zx\|}{\|x\|}$ denote its corresponding induced matrix norm.

A real valued scalar function φ with $\varphi(\varepsilon) > 0$ for all $\varepsilon \neq 0$ is called a positive function. Let $c_1 \in \mathbb{R}_{>0}$. A function $\varphi : \mathbb{R}_{[0,c_1]} \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\lim_{s\to\infty} \varphi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

Fact 1. Let $c_1 \in \mathbb{R}_{>0}$, $\varphi_1 \in \mathcal{K}$, $\varphi_1 : \mathbb{R}_{[0,c_1]} \to \mathbb{R}_+$ and let $\varphi_2 \in \mathcal{K}_\infty$. Then $\varphi_1^{-1} : \mathbb{R}_{[0,c_2]} \to \mathbb{R}_{[0,c_1]}$ with $c_2 = \varphi_1(c_1)$ is a $\mathcal{K}$-function and $\varphi_2^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{K}_\infty$-function. □

3. UNIFORM CONTINUITY ON COMPACT SETS

Definition 2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called uniformly continuous on $\mathbb{X} \subseteq \mathbb{R}^n$ (or shortly, UC($\mathbb{X}$)) if there exists a positive function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $\varepsilon \in \mathbb{R}_{>0}$ and all $(x, y) \in \mathbb{X}^2 := \mathbb{X} \times \mathbb{X}$ with $\|x - y\| \leq \delta(\varepsilon)$ it holds that $|f(x) - f(y)| \leq \varepsilon$. If f is UC($\mathbb{R}^n$), then f is called globally uniformly continuous (GUC). If f is GUC and moreover, $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$, f is called unbounded GUC. □

Definition 3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called Hölder continuous on $\mathbb{X} \subseteq \mathbb{R}^n$ (or shortly, HC($\mathbb{X}$)) if there exist $a \in \mathbb{R}_{>0}$, $\alpha \in \mathbb{R}_{>0}$ such that $|f(x) - f(y)| \leq a\|x - y\|^\alpha$ for all $(x, y) \in \mathbb{X}^2$. If f is HC($\mathbb{R}^n$), then f is called globally Hölder continuous (GHC). If $\alpha = 1$, then f is called Lipschitz continuous. □

Fact 4. A function $f : \mathbb{X} \to \mathbb{R}$ that is UC($\mathbb{X}$) is continuous on $\mathbb{X}$, for $\mathbb{X} \subseteq \mathbb{R}^n$. □

Fact 5. Heine-Cantor Theorem. Let $\mathbb{X} \subset \mathbb{R}^n$ be a compact set and let $f : \mathbb{X} \to \mathbb{R}$ denote a continuous function on $\mathbb{X}$. Then f is UC($\mathbb{X}$). □

Fact 6. Let $A, B \subset \mathbb{R}^n$ be arbitrary compact sets and let $f : A \to \mathbb{R}$ denote a continuous function on A. Also, let $f(A) := \{f(x) \mid x \in A\}$. Then $f(A)$ and $A \oplus B$ are compact sets. □

Fact 7. Let $\mathbb{X} \subset \mathbb{R}^n$ be a compact set and let $f : \mathbb{X} \to \mathbb{R}$ denote a continuous function on $\mathbb{X}$. Then f attains its minimum and maximum on $\mathbb{X}$. □

Let $\mathbb{X} \subset \mathbb{R}^n$ be a compact set and let $f : \mathbb{X} \to \mathbb{R}$ be a continuous function on $\mathbb{X}$. Let $M_x := \sup_{x\in\mathbb{X}} \|x\|$ and $M_f := \sup_{x\in\mathbb{X}} |f(x)|$, where the supremum is an attainable maximum by continuity of the norm and f, respectively, and Fact 7.

Definition 8. A function $f : \mathbb{X} \to \mathbb{R}$ is called $\mathcal{K}$-continuous on $\mathbb{X} \subset \mathbb{R}^n$ (or shortly, KC($\mathbb{X}$)) if there exists $\varphi : \mathbb{R}_{[0,c_1]} \to \mathbb{R}_+$, for some $c_1 \in \mathbb{R}_{\geq 2M_x}$, such that $\varphi \in \mathcal{K}$ and $|f(x) - f(y)| \leq \varphi(\|x - y\|)$ for all $(x, y) \in \mathbb{X}^2$. □

Definition 9. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called $\mathcal{K}_\infty$-continuous on $\mathbb{R}^n$ (or alternatively, globally $\mathcal{K}_\infty$-continuous (GKC)) if there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi \in \mathcal{K}_\infty$ and $|f(x) - f(y)| \leq \varphi(\|x - y\|)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. □

Notice that the set of HC (GHC) functions is a subset of KC (GKC) functions.

Lemma 10. Let $\mathbb{X} \subset \mathbb{R}^n$ be a compact set. A function $f : \mathbb{X} \to \mathbb{R}$ is UC($\mathbb{X}$) *if and only if* it is $\mathcal{K}$-continuous on $\mathbb{X}$.

The proof of Lemma 10 is given in Appendix A.

Corollary 11. Let $\mathbb{X} \subset \mathbb{R}^n$ be a compact set. A function $f : \mathbb{X} \to \mathbb{R}$ is continuous on $\mathbb{X}$ *if and only if* it is $\mathcal{K}$-continuous on $\mathbb{X}$.

Proof. The claim follows from Lemma 10 in combination with Fact 4 and Fact 5. □

Lemma 12. A function $f : \mathbb{R}^n \to \mathbb{R}$ is unbounded globally uniformly continuous *if and only if* it is $\mathcal{K}_\infty$-continuous.

The proof of Lemma 12 is given in Appendix B.

Corollary 13. Every $\mathcal{K}_\infty$-continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is globally uniformly continuous.

Proof. The claim follows from Lemma 12 and the fact that every unbounded GUC function is a GUC function. □

4. INHERENT INPUT-TO-STATE STABILITY

Consider the discrete-time nominal system

$$x(k + 1) = \Phi(x(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

and its perturbed counterpart

$$x(k + 1) = \Psi(x(k), e(k), d(k)), \quad k \in \mathbb{Z}_+, \tag{2}$$

where $x : \mathbb{Z}_+ \to \mathbb{R}^n$ is the state trajectory, $e : \mathbb{Z}_+ \to \mathbb{R}^n$ is an unknown *inner* perturbation trajectory, $d : \mathbb{Z}_+ \to \mathbb{R}^n$ is an unknown *outer* perturbation trajectory and $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $\Psi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear maps with $\Psi(x, 0, 0) := \Phi(x)$ for all $x \in \mathbb{R}^n$ and $\Phi(0) = 0$. The reason for distinguishing between inner and outer perturbations will be made clear later in the section. For ease of notation we will use x, e and d, respectively, to also denote a vector in $\mathbb{R}^n$. Let $\mathbb{X}$, $\mathbb{E}$ and $\mathbb{D}$ denote subsets of $\mathbb{R}^n$ that contain the origin in their interior.

Definition 14. A set $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \mathrm{int}(\mathcal{P})$ is called a *positively invariant (PI) set* for system (1) if for all $x \in \mathcal{P}$ it holds that $\Phi(x) \in \mathcal{P}$. A set $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \mathrm{int}(\mathcal{P})$ is called a *robustly positively invariant (RPI) set* for system (2) and $(\mathbb{E}, \mathbb{D})$, or shortly, RPI($\mathbb{E}, \mathbb{D}$), if for all $x \in \mathcal{P}$ it holds that $\Psi(x, e, d) \in \mathcal{P}$ for all $(e, d) \in \mathbb{E} \times \mathbb{D}$. □

Definition 15. We call system (1) asymptotically stable in $\mathbb{X}$, or shortly *AS($\mathbb{X}$)*, if there exists a $\mathcal{KL}$-function β such that for each $x(0) \in \mathbb{X}$ it holds that $\|x(k)\| \leq \beta(\|x(0)\|, k)$, $\forall k \in \mathbb{Z}_+$. We call system (1) GAS if it is AS($\mathbb{R}^n$). □

Definition 16. We call system (2) *input-to-state stable in* $\mathbb{X}$ *for inputs in* $\mathbb{E}$ *and* $\mathbb{D}$, or shortly *ISS($\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$)*, if there exist a $\mathcal{KL}$-function β and $\mathcal{K}$-functions γ_1, γ_2 such that, for each $x(0) \in \mathbb{X}$, all $\mathbf{e} = \{e(l)\}_{l\in\mathbb{Z}_+}$ with $e(l) \in \mathbb{E}$ for all $l \in \mathbb{Z}_+$ and all $\mathbf{d} = \{d(l)\}_{l\in\mathbb{Z}_+}$ with $d(l) \in \mathbb{D}$ for all $l \in \mathbb{Z}_+$, it holds that the corresponding state trajectory of (2) satisfies

$$\|x(k)\| \leq \beta(\|x(0)\|, k) + \gamma_1(\|\mathbf{e}_{[k-1]}\|) + \gamma_2(\|\mathbf{d}_{[k-1]}\|)$$

for all $k \in \mathbb{Z}_{[1,\infty)}$. The system (2) is *globally ISS* if it is ISS($\mathbb{R}^n$, $\mathbb{R}^n$, $\mathbb{R}^n$). □

Throughout this article we will employ the following sufficient conditions for analyzing ISS.

Theorem 17. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty, \sigma_1, \sigma_2 \in \mathcal{K}, \mathbb{X} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathbb{X})$. Let $V : \mathbb{X} \to \mathbb{R}_+$ be a function with $V(0) = 0$ and consider the following inequalities:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{3a}$$

$$V(\Psi(x, e, d)) - V(x) \leq -\alpha_3(\|x\|) + \sigma_1(\|e\|) + \sigma_2(\|d\|). \tag{3b}$$

(i) If $\mathbb{X}$ is a RPI($\mathbb{E}$,$\mathbb{D}$) set for system (2) and inequalities (3) hold for all $x \in \mathbb{X}$, $e \in \mathbb{E}$ and all $d \in \mathbb{D}$, then system (2) is *ISS($\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$)*. If inequalities (3) hold for all $(x, e, d) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, then system (2) is *globally ISS*.

(ii) If $\mathbb{X}$ is a PI set for system (1) and inequalities (3) hold for all $x \in \mathbb{X}$ ($x \in \mathbb{R}^n$), $e \in \mathbb{E} = \{0\}$ and $d \in \mathbb{D} = \{0\}$, then system (1) is AS($\mathbb{X}$) (GAS).

Definition 18. A function $V : \mathbb{X} \to \mathbb{R}_+$ that satisfies the hypothesis of Theorem 17-(i) for some sets $\mathbb{E}, \mathbb{D}$ is called an *ISS Lyapunov function on* $\mathbb{X}$ for system (2), or shortly, an ISS($\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$) Lyapunov function. An ISS($\mathbb{R}^n$,$\mathbb{R}^n$,$\mathbb{R}^n$) Lyapunov function is called a *global ISS Lyapunov function*. □

Definition 19. A function $V : \mathbb{X} \to \mathbb{R}_+$ that satisfies the hypothesis of Theorem 17-(ii) is called a *Lyapunov function on* $\mathbb{X}$ for system (2). A Lyapunov function on $\mathbb{R}^n$ is called a *global Lyapunov function*. □

The interested reader is referred to (Jiang and Wang, 2001; Lazar et al., 2008) for a proof of Theorem 17. Notice that in contrast to the continuous-time case, in discrete-time the above sufficient conditions for ISS (GAS) only require the continuity of the system dynamics and the (ISS) Lyapunov function at $x = 0$, as indicated in (Lazar et al., 2006, 2008). However, in what follows we will focus on continuous Lyapunov functions. The interested reader is referred to (Lazar et al., 2009) for ISS subtleties for discrete-time systems regarding *discontinuous* Lyapunov functions.

The next two theorems consider the case when the perturbed system (2) satisfies $\Psi(x, e, d) := \Phi(x+e)+d$ for all $(x, e, d) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, which exposes the difference between inner and outer perturbations. The next result relates existence of a continuous Lyapunov function to inherent ISS for system (1).

Theorem 20. Let $\mathbb{X}$, $\mathbb{E}$ and $\mathbb{D}$ be compact subsets of $\mathbb{R}^n$ with the origin in their interior. Suppose that $\mathbb{X}$ is an RPI($\mathbb{E}, \mathbb{D}$) set for system (2). Furthermore, suppose that system (1) admits a continuous Lyapunov function on $\mathbb{X} \oplus \mathbb{E}$. Then, system (2) is ISS($\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$).

Proof. The hypothesis implies that there exists a continuous function $V : \mathbb{X} \oplus \mathbb{E} \to \mathbb{R}_+$ that satisfies (3a) for all $x \in \mathbb{X} \oplus \mathbb{E}$. Thus, it satisfies (3a) for all $x \in \mathbb{X}$ as well. Next, we prove that V satisfies (3b) for all $(x, e, d) \in \mathbb{X} \times \mathbb{E} \times \mathbb{D}$. Let $\hat{x} := x + e$. As V is a Lyapunov function on $\mathbb{X} \oplus \mathbb{E}$ for system (1), by Definition 19 it follows that

$$V(\Phi(\hat{x})) - V(\hat{x}) + \alpha_3(\|\hat{x}\|) \leq 0, \quad \forall \hat{x} \in \mathbb{X} \oplus \mathbb{E}. \tag{4}$$

Since $\mathbb{X}$ is a RPI($\mathbb{E}, \mathbb{D}$) set for system (2), from Corollary 11, Fact 6, the reverse triangle inequality and using $\hat{x} = x + e$ we also have that there exist $\varphi_1, \varphi_2 \in \mathcal{K}$ such that

$$|V(\hat{x}) - V(x)| \leq \varphi_1(\|e\|), \tag{5a}$$

$$|V(\Phi(\hat{x}) + d) - V(\Phi(\hat{x}))| \leq \varphi_1(\|d\|), \tag{5b}$$

$$|\alpha_3(\|\hat{x}\|) - \alpha_3(\|x\|)| \leq \varphi_2(\|e\|), \tag{5c}$$

for all $(x, e, d) \in \mathbb{X} \times \mathbb{E} \times \mathbb{D}$. Then, using the fact that $a - b \leq |a - b| = |b - a|$ for any $a, b \in \mathbb{R}$ and adding (5b) and (4) yield

$$V(\Phi(\hat{x}) + d) - V(\hat{x}) + \alpha_3(\|\hat{x}\|) - \varphi_1(\|d\|) \leq 0$$

for all $\hat{x} \in \mathbb{X} \oplus \mathbb{E}$ and all $d \in \mathbb{D}$. Adding and subtracting $V(x)$ and $\alpha_3(\|x\|)$ in the above inequality and using (5a) and (5c), respectively, along with the fact that $a - b \leq |a - b| = |b - a|$ for any $a, b \in \mathbb{R}$, yield

$$V(\Phi(x+e)+d) - V(x) \leq -\alpha_3(\|x\|) + \sum_{i=1}^{2} \varphi_i(\|e\|) + \varphi_1(\|d\|),$$

for all $(x, e, d) \in \mathbb{X} \times \mathbb{E} \times \mathbb{D}$. Letting $\sigma_1 := \sum_{i=1}^{2} \varphi_i \in \mathcal{K}$ and $\sigma_2 := \varphi_2 \in \mathcal{K}$ yields that V satisfies (3b) for all $(x, e, d) \in \mathbb{X} \times \mathbb{E} \times \mathbb{D}$. Hence, the claim follows from Theorem 17-(i). □

A global correspondent of Theorem 20 is stated next.

Theorem 21. Suppose that system (1) admits a $\mathcal{K}_\infty$-continuous global Lyapunov function that satisfies (3b) for all $x \in \mathbb{R}^n$, $e \in \mathbb{E} = \{0\}$ and $d \in \mathbb{D} = \{0\}$ with a $\mathcal{K}_\infty$-continuous $\alpha_3 \in \mathcal{K}_\infty$. Then, system (2) is globally ISS.

Proof. The claim follows *via* the reasoning used in the proof of Theorem 20, in combination with Definition 9. □

Consider next the discrete-time nominal system with a control input

$$x(k + 1) = \phi(x(k), u(x(k))), \quad k \in \mathbb{Z}_+, \tag{6}$$

and its perturbed counterpart

$$x(k + 1) = \phi(x(k), u(x(k) + e(k))) + d(k), \quad k \in \mathbb{Z}_+, \tag{7}$$

where $u : \mathbb{R}^n \to \mathbb{R}^m$ is a state-feedback control law and $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear map with $\phi(0, 0) = 0$. For ease of notation we will use u to also denote a vector in $\mathbb{R}^m$. Let $\mathbb{U}$ be a subset of $\mathbb{R}^m$ with $0 \in \text{int}(\mathbb{U})$.

Definition 22. Let $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ be compact sets. A map $\phi : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^n$ is called uniformly $\mathcal{K}$-continuous on $\mathbb{X}$ if there exists $\varphi : \mathbb{R}_{[0,c_1]} \to \mathbb{R}_+$, for some $c_1 \in \mathbb{R}_{\geq 2M_x}$, such that $\varphi \in \mathcal{K}$ and $\|\phi(x, u) - \phi(y, u)\| \leq \varphi(\|x - y\|)$ for all $u \in \mathbb{U}$ and all $(x, y) \in \mathbb{X}^2$. A map $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is called uniformly $\mathcal{K}_\infty$-continuous if there exists $\varphi \in \mathcal{K}_\infty$ such that $\|\phi(x, u) - \phi(y, u)\| \leq \varphi(\|x - y\|)$ for all $u \in \mathbb{R}^m$ and all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. □

Definition 23. Let $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$. Suppose that $u : \mathbb{X} \to \mathbb{U}$ is a known map with $u(0) = 0$. A set $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathcal{P})$ is called a PI set for system (6) if for all $x \in \mathcal{P}$ it holds that $\phi(x, u(x)) \in \mathcal{P}$. A set $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathcal{P})$ is called a RPI($\mathbb{E}, \mathbb{D}$) set for system (7) if for all $x \in \mathcal{P}$ it holds that $\phi(x, u(x + e)) + d \in \mathcal{P}$ for all $(e, d) \in \mathbb{E} \times \mathbb{D}$. □

Theorem 24. Let $\mathbb{X}$, $\mathbb{U}$, $\mathbb{E}$ and $\mathbb{D}$ be compact sets with the origin in their interior and let $u : \mathbb{X} \oplus \mathbb{E} \to \mathbb{U}$ be a known map with $u(0) = 0$. Suppose that $\mathbb{X}$ is a RPI($\mathbb{E}, \mathbb{D}$) set for system (7). Furthermore, suppose that system (6) admits a continuous Lyapunov function on $\mathbb{X} \oplus \mathbb{E}$ and the map ϕ is uniformly $\mathcal{K}$-continuous on $\mathbb{X} \oplus \mathbb{E}$. Then, the system (7) is ISS($\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$).

Proof. Let $\hat{x} := x + e$. Observe that for any $x \in \mathbb{X}$ and $e \in \mathbb{E}$, $\hat{x} \in \mathbb{X} \oplus \mathbb{E}$ and thus, $\phi(\hat{x}, u(\hat{x})) \in \mathbb{X} \oplus \mathbb{E}$, since system (6) admits a Lyapunov function on $\mathbb{X} \oplus \mathbb{E}$. This further implies implies that there exists a continuous function $V : \mathbb{X}\oplus\mathbb{E} \to \mathbb{R}_+$ such that

$$V(\phi(\hat{x}, u(\hat{x}))) - V(\hat{x}) \leq -\alpha_3(\|\hat{x}\|), \quad \forall \hat{x} \in \mathbb{X} \oplus \mathbb{E}. \quad (8)$$

From Corollary 11 and Definition 22 it follows that there exist $\varphi_1, \varphi_2 \in \mathcal{K}$ such that:

$$\begin{aligned} &|V(\phi(\hat{x}, u(\hat{x}))) - V(\phi(x, u(\hat{x})))| \\ &\qquad \leq \varphi_1(\|\phi(\hat{x}, u(\hat{x})) - \phi(x, u(\hat{x}))\|) \\ &\qquad \leq \varphi_1(\varphi_2(\|e\|)) \end{aligned}$$

for all $(x, e) \in \mathbb{X} \times \mathbb{E}$. Adding and subtracting $V(\phi(x, u(\hat{x}))$ in (8) and using the above inequality along with the fact that $a - b \leq |a - b| = |b - a|$ for any $a, b \in \mathbb{R}$, yield

$$V(\phi(x, u(\hat{x}))) - V(\hat{x}) \leq -\alpha_3(\|\hat{x}\|) + \varphi_1(\varphi_2(\|e\|)).$$

Then, the claim follows *via* the reasoning used in the proof of Theorem 20, by considering a perturbed system (2) with $\Psi(x, e, d) := \phi(x, u(x + e)) + d$ for all $(x, e, d) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. □

Theorem 25. Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a known map with $u(0) = 0$. Suppose that system (6) admits a $\mathcal{K}_\infty$-continuous global Lyapunov function that satisfies (3b) for all $x \in \mathbb{R}^n$, $e \in \mathbb{E} = \{0\}$ and $d \in \mathbb{D} = \{0\}$ with a $\mathcal{K}_\infty$-continuous $\alpha_3 \in \mathcal{K}_\infty$. Furthermore, suppose that ϕ is a uniformly $\mathcal{K}_\infty$-continuous map. Then, system (7) is globally ISS.

Proof. The claim follows *via* the reasoning used in the proof of Theorem 24 in combination with the reasoning employed in the proof of Theorem 21 and Definition 22. □

Remark 26. The local inherent ISS results of Theorem 20 and Theorem 24 are obtained using a similar reasoning as the one employed in (Messina et al., 2005) to establish local inherent robustness (RGAS and SPAS) from a global continuous Lyapunov function. The ISS results established in the above-mentioned theorems require the existence of a suitable RPI set, which is an additional requirement compared to (Messina et al., 2005). Similarly to the results in (Messina et al., 2005), the result of Theorem 20, which applies to systems without a control input, does not require continuity of the system dynamics and the result of Theorem 24, which applies to systems with a control input, does not require continuity of the state-feedback control law. The same holds for the ISS results of Theorem 21 and Theorem 25, respectively, which do not a have a correspondent in (Messina et al., 2005). Notice that if $\mathbb{E} = \{0\}$, then the continuity assumptions on ϕ can be removed in Theorem 24 and Theorem 25. □

5. OPTIMIZED INPUT-TO-STATE STABILITY

This section illustrates how a continuous control Lyapunov function (CLF) (Kellett and Teel, 2004) can be employed to design a stabilizing MPC scheme that is inherently input-to-state stabilizing. As such, let us formally define a CLF for system (6).

Definition 27. Let $V : \mathbb{X} \to \mathbb{R}_+$ be a candidate Lyapunov function on $\mathbb{X} \subseteq \mathbb{R}^n$, i.e., a function that satisfies (3a) for all $x \in \mathbb{X}$. V is called a control Lyapunov function on $\mathbb{X}$ for system (6) if there exists a map $u : \mathbb{X} \to \mathbb{U}$ with $u(0) = 0$ such that $\mathbb{X}$ is a PI set for system (6) and

$$V(\phi(x, u(x))) - V(x) \leq -\alpha_3(\|x\|), \quad \forall x \in \mathbb{X}.$$

A control Lyapunov function on $\mathbb{R}^n$ is called a global control Lyapunov function. □

Moreover, as it was recently pointed out in (Lazar and Heemels, 2008), besides guaranteeing inherent ISS, it would be desirable to optimize the ISS gain of the closed-loop system, i.e., the gain of the functions $\gamma_1, \gamma_2 \in \mathcal{K}$. In what follows we briefly recall some of the results in (Lazar and Heemels, 2008), which consider outer perturbations only, and provide some new insights and extensions. For clarity of exposition we will treat the case of outer perturbations separately from the case of inner perturbations, as there is a crucial difference between the two cases.

Let us begin with the case of outer perturbations. To optimize disturbance attenuation for the closed-loop system, at each time instant $k \in \mathbb{Z}_+$ and for a given $x(k) \in \mathbb{X}$, it would be desirable to *simultaneously* compute a control action $u(x(k)) \in \mathbb{U}$ that satisfies

$$\begin{aligned} (i) \quad & V(\phi(x(k), u(x(k))) + d) - V(x(k)) + \alpha_3(\|x(k)\|) \\ & \qquad - \sigma_2(\|d\|) \leq 0, \ \forall d \in \mathbb{D} \end{aligned} \quad (9)$$

for $\sigma_2(s) := \eta_2(k)s^\delta$, $\delta \in \mathbb{R}_{>0}$, $\eta_2(k) \in \mathbb{R}_{>0}$ and *(ii)* minimize $\eta_2(k)$.

Next, we recall a solution to this problem that was given in (Lazar and Heemels, 2008). Let $\mathbb{D}$ be a polytope and let d^o, $o \in \mathbb{Z}_{[1,O]}$, be the vertices of $\mathbb{D}$. Next, consider a finite set of simplices $S_1, \ldots, S_M$ with each simplex S_i equal to the convex hull of the origin and a subset of the vertices of $\mathbb{D}$, and such that $\cup_{i=1}^M S_i = \mathbb{D}$, $\mathrm{int}(S_i) \cap \mathrm{int}(S_j) = \emptyset$ for $i \neq j$, $\mathrm{int}(S_i) \neq \emptyset$ for all i. More precisely, $D_i = \mathrm{Co}\{0, d^{o_{i,1}}, \ldots, d^{o_{i,l}}\}$ and $\{d^{o_{i,1}}, \ldots, d^{o_{i,l}}\} \subset \{d^1, \ldots, d^O\}$ (i.e., $\{o_{i,1}, \ldots, o_{i,l}\} \subset \{1, \ldots, O\}$) with $d^{o_{i,1}}, \ldots, d^{o_{i,l}}$ linearly independent. For each simplex S_i we define the matrix $D_i := [d^{o_{i,1}} \ \ldots \ d^{o_{i,l}}] \in \mathbb{R}^{l\times l}$, which is invertible. Let $\lambda_o(k)$, $k \in \mathbb{Z}_+$, be optimization variables associated with each vertex d^o. *In what follows, when the time dependency is irrelevant, it will be omitted for brevity of presentation.* Consider the following set of inequalities depending on u and $\{\lambda_o\}_{o\in\mathbb{Z}_{[1,O]}}$:

$$V(\phi(x, u(x))) - V(x) + \alpha_3(\|x\|) \leq 0, \quad (10a)$$

$$V(\phi(x, u(x)) + d^o) - V(x) + \alpha_3(\|x\|) - \lambda_o \leq 0, \quad \forall o \in \mathbb{Z}_{[1,O]}. \quad (10b)$$

Theorem 28. (Lazar and Heemels, 2008). Let V be a convex function. Suppose that for some $\alpha_3 \in \mathcal{K}_\infty$ and $x \in \mathbb{R}^n$ there exist $u(x) \in \mathbb{R}^m$ and $\{\lambda_o(k)\}_{o\in\mathbb{Z}_{[1,O]}} \in (\mathbb{R}_+)^O$ such that (10a) and (10b) hold. Then (9) holds for the same $u(x)$, with $\sigma_2(s) := \eta_2 s$ and

$$\eta_2 := \max_{i=1,\ldots,M} \|\bar{\lambda}_i D_i^{-1}\|, \quad (11)$$

where $\bar{\lambda}_i := [\lambda_{o_{i,1}} \ \ldots \ \lambda_{o_{i,l}}] \in \mathbb{R}^{1\times l}$, $i = 1, \ldots, M$.

Let $\bar{\lambda} := [\lambda_1, \ldots, \lambda_O]^\top$ and let $\bar{J}(\bar{\lambda}) : \mathbb{R}^O \to \mathbb{R}_+$ be a function that satisfies $\alpha_4(\|\bar{\lambda}\|) \leq \bar{J}(\bar{\lambda}) \leq \alpha_5(\|\bar{\lambda}\|)$ for some $\alpha_4, \alpha_5 \in \mathcal{K}_\infty$; for example, $\bar{J}(\bar{\lambda}) := \max_{i=1,\ldots,M} \|\bar{\lambda}_i D_i^{-1}\|$. Let $N \in \mathbb{Z}_{\geq 1}$, let $J : \mathbb{R}^n \times \mathbb{R}^m \times \ldots \times \mathbb{R}^m \to \mathbb{R}_+$ denote an arbitrary cost function that is zero at zero and define $\mathbf{u} := \{u_l\}_{l\in\mathbb{Z}_{[1,N]}}$.

Problem 29. Let $\alpha_3 \in \mathcal{K}_\infty$, $J(\cdot)$, $\bar{J}(\cdot)$ and $V : \mathbb{X} \to \mathbb{R}_+$ be given. At time $k \in \mathbb{Z}_+$ let the state $x(k)$ be known and minimize the cost $J + \bar{J}$ over $\mathbf{u}(k), \bar{\lambda}(k)$, subject to

$$\mathbf{u}(k) \in \mathbb{U}^N,\ \bar{\lambda}(k) \in (\mathbb{R}_+)^O,\ \phi(x(k), \mathbf{u}_1(k)) \in \mathbb{X} \sim \mathbb{D}, \tag{12a}$$

$$V(\phi(x(k), \mathbf{u}_1(k))) - V(x(k)) + \alpha_3(\|x(k)\|) \le 0, \tag{12b}$$

$$V(\phi(x(k), \mathbf{u}_1(k)) + d^o) - V(x(k)) + \alpha_3(\|x(k)\|) - \lambda_o(k) \le 0,\ \forall o \in \mathbb{Z}_{[1,O]}. \tag{12c}$$

Set

$$u(x(k)) = \mathbf{u}_1^{\text{feas}}(k). \tag{13}$$

□

In the above problem $\mathbf{u}_1^{\text{feas}}(k)$ denotes a control law that selects the first element of an arbitrary feasible sequence of inputs $\mathbf{u}(k) \in \mathbb{U}^N$ for all $k \in \mathbb{Z}_+$. Moreover, because J, and (12) likewise, is a function of $x(k)$, $\mathbf{u}_1^{\text{feas}}(k)$ is a function of $x(k)$ as well.

Theorem 30. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, a continuous and convex CLF $V : \mathbb{X} \to \mathbb{R}_+$, a cost $J(\cdot)$ and a cost $\bar{J}(\cdot)$ be given. Suppose that Problem 29 is feasible for all $x \in \mathbb{X}$ and the map ϕ is uniformly $\mathcal{K}$-continuous on $\mathbb{X}$. Then the closed-loop system (7)-(13) is ISS($\mathbb{X}$, $\{0\}$, $\mathbb{D}$).

Proof. For any $k \in \mathbb{Z}_+$ and $x(k) \in \mathbb{X}$ it holds that $x(k+1) := \phi(x(k), u(x(k))) + d(k) \in \mathbb{X}$ for all $d(k) \in \mathbb{D}$ by (12a). Hence, Problem 29 is recursively feasible for all $x \in \mathbb{X}$. Then, the result follows from (12b) and Theorem 24 with $\mathbb{E} = \{0\}$. □

It is important to observe that Problem 29, although it inherently guarantees a constant ISS gain, *it provides freedom to optimize the ISS gain of the closed-loop system, by minimizing the variables* $\lambda_1(k), \ldots, \lambda_O(k)$ *via the cost* $\bar{J}(\cdot)$. As such, in reality the gain $\eta_2(k)$ of the function $\sigma_2(\cdot)$ can be much smaller for $k \ge k_0$, for some $k_0 \in \mathbb{Z}_+$, depending on the actual state trajectory.

Next, consider the case of inner perturbations. The goal is now to *simultaneously* compute a control action $u(\hat{x}(k)) \in \mathbb{U}$ at time $k \in \mathbb{Z}_+$ that satisfies

$$(i)\quad V(\phi(x(k), u(x(k) + e))) - V(x(k)) + \alpha_3(\|x(k)\|) - \sigma_1(\|e\|) \le 0,\ \forall e \in \mathbb{E} \tag{14}$$

for $\sigma_1(s) := \eta_1(k)s^\delta$, $\delta \in \mathbb{R}_{>0}$, $\eta_1(k) \in \mathbb{R}_{>0}$ and *(ii)* minimize $\eta_1(k)$. As done for outer perturbations, let $\mathbb{E}$ be a polytope and let e^w, $w = 1, ..., W$, be the vertices of $\mathbb{E}$. Next, consider a finite set of simplices $S_1, \ldots, S_M$ with each simplex S_i equal to the convex hull of a subset of the vertices of $\mathbb{E}$ and the origin, and such that $\cup_{i=1}^M S_i = \mathbb{E}$, $\text{int}(S_i) \cap \text{int}(S_j) = \emptyset$ for $i \ne j$, $\text{int}(S_i) \ne \emptyset$ for all i. More precisely, $S_i = \text{Co}\{0, e^{w_{i,1}}, \ldots, e^{w_{i,l}}\}$ and

$$\{e^{w_{i,1}}, \ldots, e^{w_{i,l}}\} \subseteq \{e^1, \ldots, e^W\}$$

(i.e., $\{w_{i,1}, \ldots, w_{i,l}\} \subseteq \{1, \ldots, W\}$) with $e^{w_{i,1}}, \ldots, e^{w_{i,l}}$ linearly independent. For each simplex S_i we define the matrix $E_i := [e^{w_{i,1}}\ \ldots\ e^{w_{i,l}}] \in \mathbb{R}^{l \times l}$, which is invertible. Let $\lambda_w \in \mathbb{R}_+$ be variables associated with each vertex e^w.

Next, suppose that x is known. Notice that the assumption that x is known is only used here to show how one can transform (14) into a finite dimensional problem. The dependence on x will be removed later, leading to a main stability result and an MPC algorithm that only use the perturbed state $\hat{x}$ for feedback, see Problem 32. Consider the following set of constraints:

$$V(\phi(\hat{x}, u(\hat{x}))) - V(x) + \alpha_3(\|x\|) \le 0, \tag{15a}$$

$$V(\phi(\hat{x} - e^w, u(\hat{x}))) - V(x) + \alpha_3(\|x\|) - \lambda_w \le 0,\quad \forall w \in \mathbb{Z}_{[1,W]}. \tag{15b}$$

Theorem 31. Let $\alpha_3 \in \mathcal{K}_\infty$, let V be a convex function and let ϕ be an affine map of x for all u. If for $x \in \mathbb{R}^n$ and $\hat{x} = x + e \in \mathbb{R}^n$ there exist $u(\hat{x}) \in \mathbb{R}^m$ and $\{\lambda_w\}_{w \in \mathbb{Z}_{[1,W]}} \in (\mathbb{R}_+)^W$ such that (15a) and (15b) hold, then (14) holds for the same $u(\hat{x})$, with $\sigma_1(s) := \eta_1 s$ and

$$\eta_1 := \max_{i=1,\ldots,M} \|\bar{\lambda}_i E_i^{-1}\|, \tag{16}$$

where $\bar{\lambda}_i := [\lambda_{w_{i,1}}\ \ldots\ \lambda_{w_{i,l}}] \in \mathbb{R}^{1 \times l}$ and $\|\cdot\|$ is the corresponding induced matrix norm.

The proof of the above theorem, which is similar, *mutatis mutandis*, to the proof of Theorem 28 is omitted due to space limitations.

Based on the result of Theorem 31 we are now able to formulate a finite dimensional optimization problem that results in closed-loop ISS with respect to inner perturbation $e(k)$ and moreover, in optimization of the closed-loop trajectory-dependent ISS gain. This will be achieved only based on knowledge of the perturbed state $\hat{x}(k)$ and the set $\mathbb{E}$.

Let $\bar{\lambda} := [\lambda_1, \ldots, \lambda_W]^\top$ and let $\bar{J}$ be defined similarly as done for outer perturbations. Define next:

$$V_{\min}(\hat{x}(k)) := \min_{x \in \text{Co}(\{\hat{x}(k) - e^1, \ldots, \hat{x}(k) - e^W\})} V(x) \tag{17}$$

and

$$\alpha_{3,\max}(\hat{x}(k)) := \max_{x \in \text{Co}(\{\hat{x}(k) - e^1, \ldots, \hat{x}(k) - e^W\})} \alpha_3(\|x\|). \tag{18}$$

Problem 32. Let $\alpha_3 \in \mathcal{K}_\infty$, $J(\cdot)$, $\bar{J}(\cdot)$ and $V : \mathbb{X} \to \mathbb{R}_+$ be given. At time $k \in \mathbb{Z}_+$ let the perturbed state $\hat{x}(k)$ be known and minimize the cost $J + \bar{J}$ over $\mathbf{u}(k)$, $\bar{\lambda}(k)$, subject to

$$\mathbf{u}(k) \in \mathbb{U}^N,\ \bar{\lambda}(k) \in (\mathbb{R}_+)^W, \tag{19a}$$

$$\phi(x, \mathbf{u}_1(k)) \in \mathbb{X},\ \forall x \in \text{Co}(\{\hat{x}(k) - e^1, \ldots, \hat{x}(k) - e^W\}), \tag{19b}$$

$$V(\phi(\hat{x}(k), \mathbf{u}_1(k))) - V_{\min}(\hat{x}(k)) + \alpha_{3,\max}(\hat{x}(k)) \le 0, \tag{19c}$$

$$V(\phi(\hat{x}(k) - e^w, \mathbf{u}_1(k))) - V_{\min}(\hat{x}(k)) + \alpha_{3,\max}(\hat{x}(k)) - \lambda_w(k) \le 0, \forall w \in \mathbb{Z}_{[1,W]}. \tag{19d}$$

Set

$$u(\hat{x}(k)) = \mathbf{u}_1^{\text{feas}}(k). \tag{20}$$

□

In the above problem $\mathbf{u}_1^{\text{feas}}(k)$ denotes a control law that selects the first element of an arbitrary feasible sequence of inputs $\mathbf{u}(k) \in \mathbb{U}^N$ for all $k \in \mathbb{Z}_+$. Moreover, because J, and (19) likewise, is a function of $\hat{x}(k)$, $\mathbf{u}_1^{\text{feas}}(k)$ is a function of $\hat{x}(k)$ as well.

Theorem 33. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, a continuous and convex CLF $V : \mathbb{X} \oplus \mathbb{E} \to \mathbb{R}_+$, a cost $J(\cdot)$ and a cost $\bar{J}(\cdot)$ be given. Suppose that Problem 32 is feasible for all $\hat{x} \in \mathbb{X} \oplus \mathbb{E}$ and ϕ is an affine map of x for all u. Then the closed-loop system (7)-(20) is ISS($\mathbb{X}$, $\mathbb{E}$, $\{0\}$).

Proof. For any $k \in \mathbb{Z}_+$ and $\hat{x}(k) \in \mathbb{X} \oplus \mathbb{E}$ it holds that $x(k) \in \text{Co}(\{\hat{x}(k) - e^1, \ldots, \hat{x}(k) - e^W\})$. Hence, it follows that $x(k+1) = \phi(x(k), u(\hat{x}(k))) \in \mathbb{X}$ for all $e(k) \in \mathbb{E}$ by (19b). Hence, Problem 32 is recursively feasible for all $\hat{x} \in \mathbb{X} \oplus \mathbb{E}$. Then, the result follows from (19c) and Theorem 24 with $\mathbb{D} = \{0\}$. □

Notice that a more restrictive condition is imposed on the system dynamics ϕ for inner perturbations, i.e., ϕ should be an affine map in x for all u, than for outer perturbations, i.e., ϕ

should be a uniformly $\mathcal{K}$-continuous map. Under the stronger condition, the proposed MPC schemes can be combined in one algorithm that yields optimized ISS($\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$).

6. CONCLUSIONS

Input-to-state stability analysis of discrete-time systems using continuous Lyapunov functions was considered. Firstly, the existence of a continuous Lyapunov function was related to inherent input-to-state stability on compact sets with respect to both inner and outer perturbations. For $\mathcal{K}_\infty$-continuous Lyapunov functions it was shown that this result applies to unbounded sets as well. Secondly, continuous control Lyapunov functions were employed to construct input-to-state stabilizing control laws for discrete-time systems subject to bounded perturbations.

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Appendix A. PROOF OF LEMMA 10

Let us begin with the *only if* part. As f is UC($\mathbb{X}$), without any loss of generality we can take $\delta : \mathbb{R}_{(0,2M_f]} \to \mathbb{R}_{>0}$ to be a positive, non-decreasing function. Let $\delta^* := \delta(2M_f) > 0$. Next, let

$$\bar{\delta}(\varepsilon) := \begin{cases} \delta(\varepsilon), & \varepsilon \in \mathbb{R}_{(0,2M_f]}, \\ \delta^* + \varepsilon - 2M_f, & \varepsilon \in \mathbb{R}_{(2M_f,2(M_x+M_f)-\delta^*]}. \end{cases}$$

Notice that $\delta^* \leq 2M_x$ and, if $\delta^* = 2M_x$, then $\bar{\delta}(\varepsilon) = \delta(\varepsilon)$ for all $\varepsilon \in \mathbb{R}_{(0,2(M_x+M_f)-\delta^*]}$. Also, $\bar{\delta}(2(M_x+M_f)-\delta^*) = 2M_x$. Observe that the function $\bar{\delta} : \mathbb{R}_{(0,2(M_x+M_f)-\delta^*]} \to \mathbb{R}_{(0,2M_x]}$ is non-decreasing and it extends the domain of $\delta(\varepsilon)$. Next, we prove that there exists $\rho : \mathbb{R}_{[0,2(M_x+M_f)-\delta^*]} \to \mathbb{R}_+$ such that $\rho \in \mathcal{K}$ and $\rho(\varepsilon) \leq \delta(\varepsilon)$ for all $\varepsilon \in \mathbb{R}_{(0,2(M_x+M_f)-\delta^*]}$. Define

$$s_k := \inf_{\varepsilon \in \mathbb{R}_{>0}} \{\varepsilon \mid \bar{\delta}(\varepsilon) \geq 2M_x 0.5^k\}, \quad \forall k \in \mathbb{Z}_+. \tag{A.1}$$

Then, define

$$\rho(\varepsilon) := M_x \left(0.5^k + \frac{0.5^k}{s_k - s_{k+1}}(\varepsilon - s_{k+1})\right), \tag{A.2}$$

for all $\varepsilon \in \mathbb{R}_{[s_{k+1},s_k)}$ and all $k \in \mathbb{Z}_+$, $\rho(0) := 0$. Observe that $\lim_{k\to\infty} s_k = 0$, which implies that ρ is continuous at zero. As $\lim_{\varepsilon\downarrow s_k} \rho(\varepsilon) = \lim_{\varepsilon\uparrow s_k} \rho(\varepsilon) = M_x 0.5^{k-1}$ for all $k \in \mathbb{Z}_{\geq 1}$, $\rho(s_0) = 2M_x$ and $s_0 = 2(M_x + M_f) - \delta^*$, it follows that ρ is continuous on $\mathbb{R}_{[0,2(M_x+M_f)-\delta^*]}$. Next, observing that $\rho(s_k) = M_x 0.5^{k-1} = 2M_x 0.5^k = 2\rho(s_{k+1})$ for all $k \in \mathbb{Z}_+$ yields that ρ is strictly increasing. Hence, the constructed function $\rho : \mathbb{R}_{[0,2(M_x+M_f)-\delta^*]} \to \mathbb{R}_+$ is of class $\mathcal{K}$ and, from (A.1) and (A.2) it follows that $\rho(\varepsilon) \leq \delta(\varepsilon)$ for all $\varepsilon \in \mathbb{R}_{(0,2(M_x+M_f)-\delta^*]}$. As $\rho(2(M_x + M_f) - \delta^*) = 2M_x$, by Fact 1 it holds that $\rho^{-1} : \mathbb{R}_{[0,2M_x]} \to \mathbb{R}_{[0,2(M_x+M_f)-\delta^*]}$ is a $\mathcal{K}$-function. As $\|x - y\| \leq 2M_x$ for any $x, y \in \mathbb{X}^2$, we can define $q := \|x - y\|$ and $w := \rho^{-1}(q)$. Since f is UC($\mathbb{X}$) it follows that for all $(x, y) \in \mathbb{X}^2$,

$$\|x - y\| = q = \rho(w) \leq \delta(w)$$
$$\Rightarrow |f(x) - f(y)| \leq w = \rho^{-1}(\|x - y\|).$$

Observing that $\rho^{-1} : \mathbb{R}_{[0,2M_x]} \to \mathbb{R}_{[0,2(M_x+M_f)-\delta^*]}$ is a $\mathcal{K}$-function completes the *only if* part of the proof.

The *if* part of the proof proceeds as follows. Let $\varepsilon > 0$ and take $\delta(\varepsilon) := \varphi^{-1}(\min\{\varepsilon, 2M_f\})$. Then, by Definition 8, for all $(x, y) \in \mathbb{X}^2$ with $\|x - y\| \leq \delta(\varepsilon)$ it holds that

$$|f(x) - f(y)| \leq \varphi(\|x - y\|) \leq \varphi(\delta(\varepsilon)) \leq \min\{\varepsilon, 2M_f\} \leq \varepsilon,$$

which completes the proof.

Appendix B. PROOF OF LEMMA 12

The claim for the *only if* part follows *mutatis mutandis* by applying the reasoning used in the proof of Lemma 10. The difference is that $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$ and as such, it suffices to construct a $\mathcal{K}_\infty$ function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(\varepsilon) \leq \delta(\varepsilon)$ for all $\varepsilon > 0$. It is straightforward to verify that the function

$$\rho(\varepsilon) := 0.5^k + \frac{0.5^k}{s_k - s_{k+1}}(\varepsilon - s_{k+1}),\ \forall \varepsilon \in \mathbb{R}_{[s_{k+1},s_k)}, \forall k \in \mathbb{Z}$$

and $\rho(0) := 0$, where

$$s_k := \inf_{\varepsilon \in \mathbb{R}_{>0}} \{\varepsilon \mid \delta(\varepsilon) \geq 0.5^k\}, \quad \forall k \in \mathbb{Z}$$

satisfies the desired properties. Similarly, for the *if* part it suffices to observe that $\varphi^{-1} \in \mathcal{K}_\infty$.