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FURTHER MONOTONICITY PROPERTIES FOR SPECIALIZED RENEWAL PROCESSES[1]

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Define $Z(t)$ to be the forward recurrence time at t for a renewal process with interarrival time distribution, F, which is assumed to be IMRL (increasing mean residual life). It is shown that $E\phi(Z(t))$ is increasing in $t \geq 0$ for all increasing convex ϕ. An example demonstrates that $Z(t)$ is not necessarily stochastically increasing nor is the renewal function necessarily concave. Both of these properties are known to hold for F DFR (decreasing failure rate).

1. Introduction. It is shown in Brown [3] that if F is IMRL (increasing mean residual life) and $Z(t)$ is the forward recurrence time at t for a renewal process with interarrival time distribution F, then $EZ(t)$ is increasing in $t \geq 0$. In this paper the result is strengthened to $E\phi(Z(t))$ increasing in t for all increasing convex ϕ. An example is given to show that the result cannot be extended to general increasing functions, equivalently that $Z(t)$ need not be stochastically increasing in t. The same example also shows that F IMRL does not imply that the renewal function $M(t)$ is concave, nor that $EA(t)$, the expected renewal age at time t, is increasing. The table below summarizes the monotonicity results of this paper and [3].

	IMRL	DFR
Z(t) **(forward recurrence time)**	$E\phi(Z(t))\uparrow$ for increasing convex ϕ; $Z(t)$ not necessarily stochastically increasing	$Z(t)$ stochastically increasing in $t \geq 0$
A(t) **(renewal age)**	$EA(t)$ not necessarily increasing	$A(t)$ stochastically increasing in $t \geq 0$
M(t) **(renewal function)**	$M(t) - \frac{t}{\mu} \uparrow$ in $t \geq 0$; $M(t)$ not necessarily concave	$M(t)$ concave. The renewal density exists on $(0, \infty)$ and is decreasing

The result $E\phi(Z(t)) \uparrow$ for increasing convex ϕ does not appear to be provable by the methodology employed in [3]. A new approach is followed based on a renewal theory identity (Theorem 1) which may be of independent interest and use.

2. Definitions and preliminaries. A random variable X with c.d.f. F is defined to have an IMRL (increasing mean residual life) distribution on $[0, \infty)$ if $\mu_1 = EX < \infty$, $F(0-) = 0$, $F(0) < 1$, and $E(X - t \mid X > t)$ is increasing in $t \geq 0$. The term increasing (decreasing) is used for monotone non-decreasing (non-increasing). X is defined to have a DFR distribution on $[0, \infty)$ if $F(0-) = 0$, $F(0) < 1$, and $X - t \mid X > t$ (the conditional distribution of $X - t$ given that $X > t$) is stochastically increasing in $t \geq 0$. Lemma 1 below reviews several properties of IMRL and DFR distributions.

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LEMMA 1 (i) F DFR $\Leftrightarrow \log \bar{F}(x)$ *convex* $\Leftrightarrow F$ *is absolutely continuous on* $(0, \infty)$ *and possesses a version of its* pdf, f, *for which the failure rate function* $h(x) = f(x)/\bar{F}(x)$ *is decreasing.*

(ii) F DFR *with finite mean implies* F IMRL; F IMRL *does not imply* F DFR.

(iii) F IMRL $\Leftrightarrow G$ DFR *where* $G(x) = (1/\mu) \int_0^x \bar{F}(y)\, dy$.

(iv) F IMRL $\Leftrightarrow E[\phi(X - t) \mid X > t] \uparrow$ *in* $t \geq 0$ *for all convex increasing* ϕ.

(v) *Mixtures of* DFR *distributions are* DFR. *A mixture of* IMRL *distributions possessing a finite mean is* IMRL.

(vi) *Define* $Z(t)$ *to be the forward recurrence time at* t *for a renewal process with interarrival time distribution* F. *Then* F IMRL (DFR) *implies* $Z(t)$ IMRL (DFR).

PROOF. (i) The fact that a DFR distribution on $[0, \infty)$ is absolutely continuous on $(0, \infty)$ (it can have an atom at $\{0\}$) can be proved by following an argument of Barlow and Proschan [1] page 77. The other implications are straightforward.

(ii) The first part is trivial. The distribution discussed in Section 5 (i) is IMRL but not DFR.

(iii) $E(X - t \mid X > t) = 1/h_G(t)$, where h_G is the hazard function of G.

(iv) Let ϕ be an increasing convex function. Then $\phi(x) = \phi(0+) + \int_0^x \phi'(y)\, dy$ with $\phi' \uparrow$. The expectations $E_F\phi(X)$, $E_F[\phi(X - t) \mid X > t)]$, $E_G[\phi'(X - t) \mid X > t]$ are all well defined with $+\infty$ as a possible value. If $E_F\phi(X) = \infty$ then $E_F[\phi(X - t) \mid X > t] = \infty$ for all t and the result is trivially true. If $E_F\phi(X) < \infty$ then all the above expectations are finite. Integration by parts gives:

$$E_F[\phi(X - t) \mid X > t] = \phi(0+) + E_F(X - t \mid X > t)E_G[\phi'(X - t) \mid X > t] \tag{1}$$

Since G is DFR (iii), ϕ' is increasing, and $E_F(X - t \mid X > t) \uparrow$, the result follows.

(v) The closure of DFR under mixtures is found in Barlow and Proschan [1] page 103. Consider $\bar{F}(t) = \int \bar{F}_\alpha(t)\, dP(\alpha)$, where F_α is IMRL, and P a probability measure. Define $\mu(\alpha) = E_{F_\alpha}X$ and $\bar{G}_\alpha(t) = (1/\mu(\alpha)) \int_t^\infty \bar{F}_\alpha(x)\, dx$. Then $\bar{G}(t) = (1/\mu) \int_t^\infty \bar{F}(x)\, dx = \int \bar{G}_\alpha(t) \cdot ((\mu_\alpha/\mu)\, dP(\alpha))$. Each G_α is DFR by (iii), therefore G is a mixture of DFR's and is therefore DFR. By (iii) F is IMRL.

(vi) The distribution of $Z(t)$ is a mixture of the distributions of $\{X - s \mid X > s, 0 \leq s \leq t\}$. Thus the result for F DFR follows directly from (v). For F IMRL we can apply (v) and obtain the desired result provided that we show that $EZ(t) < \infty$. Since $E(X - t \mid X > t)$ is increasing: $EZ(t) \leq E(X - t \mid X > t) = (1/\bar{F}(t)) \int_t^\infty \bar{F}(x)\, dx \leq (\mu/\bar{F}(t)) < \infty$.

3. Identity. We will be working with two renewal processes: one with interarrival time distribution F, the other with distribution G where $G(t) = (1/\mu) \int_0^t \bar{F}(x)\, dx$. It is important to note that the renewal process with interarrival time distribution G is not the stationary renewal process corresponding to F (which would have its first renewal governed by G and all subsequent renewals governed by F) but rather a renewal process with all interarrival times distributed as G. Define $Z_F(t)(Z_G(t))$ to be the forward recurrence time at t for the renewal process with interarrival time distribution $F(G)$. Define $\bar{F}_{Z_F(t)}^{(x)} = Pr(Z_F(t) > x)$, and $g_{Z_G(t)}^{(x)}$ to be the pdf of $Z_G(t)$ evaluated at x. Note that G is absolutely continuous and thus so is $Z_G(t)$. Define M_F, M_G to be the renewal functions for the two processes. Both start with a renewal epoch at $\{0\}$, so that, for example, $M_F(t) = \sum_{k=0}^\infty F_{(t)}^{(k)}$ rather than $\sum_{k=1}^\infty F_{(t)}^{(k)}$. By Wald's identity $EZ_F(t) = \mu M_F(t) - t$. Since $M_F(t) - t$ is of bounded variation on $[0, t]$ it makes sense to talk about $dEZ_F(y) = \mu\, dM_F(y) - dy$. Note that $dE(Z_F(y))$ has an atom of size $\mu/\bar{F}(0)$ at $y = 0$.

In Theorem 1 below we use the following version of $g_{Z_G(t)}$:

$$g_{Z_G(t)}^{(x)} = \frac{1}{\mu} \int_{0-}^{t} \bar{F}(t - y + x)\, dM_G(y) \tag{2}$$

Thus we regard $g_{Z_G(t)}$ as having the above well defined value for each x rather than as an equivalence class of a.e. equal functions.

THEOREM 1. *For all* $t \geq 0$, $x \geq 0$:

$$\bar{F}^{(x)}_{Z_F(t)} = \int_{0-}^{t} g^{(x)}_{Z_G(t-y)}\, dE(Z_F(y)) = \frac{\mu}{\bar{F}(0)} g^{(x)}_{Z_G(t)} + \int_{0+}^{t} g^{(x)}_{Z_G(t-y)}\, dE(Z_F^{(y)}) \tag{3}$$

PROOF. For $s > 0$ define the following Laplace transforms:

$$\psi_1(s, x) = \int_{t=0}^{\infty} e^{-st} g^{(x)}_{Z_G(t)}\, dt,\ \psi_2(s, x) = \int_{t=0}^{\infty} e^{-st} \bar{F}^{(x)}_{Z_F(t)}\, dt,$$

$$R_x(s) = \int_{t=0}^{\infty} e^{-st} \bar{F}(t + x)\, dt,\ \psi_{M_F}(s) = \int_{t=0}^{\infty} e^{-st}\, dM_F(t),$$

$$\psi_F(s) = \int_{t=0}^{\infty} e^{-st}\, dF(t), \text{ and similarly } \psi_{M_G}(s) \text{ and } \psi_G(s).$$

By (2):

$$\psi_1(s, x) = \mu^{-1} R_x(s) \psi_{M_G}(s) \tag{4}$$

Noting that $\bar{F}^{(x)}_{Z_F(t)} = \int_{0-}^{t} \bar{F}(t - y + x)\, dM_F(y)$ it follows that:

$$\psi_2(s, x) = R_x(s) \psi_{M_F}(s) \tag{5}$$

From (4) and (5) we obtain:

$$\frac{\psi_2(s, x)}{\psi_1(s, x)} = \frac{\mu \psi_{M_F}(s)}{\psi_{M_G}(s)} \tag{6}$$

From (6) we see that ψ_2/ψ_1 is independent of x. This suggests that $\bar{F}_{Z_F(t)}$ is the convolution of $g_{Z_G(\cdot)}$ and a function which does not depend on x. Next:

$$\mu \psi_{M_F}(s) = \mu(1 - \psi_F(s))^{-1} \tag{7}$$

$$\psi_{M_G}(s) = (1 - \psi_G(s))^{-1} = (1 - [(1 - \psi_F(s))/s\mu])^{-1} \tag{8}$$

From (6), (7) and (8):

$$\frac{\psi_2(s, x)}{\psi_1(s, x)} = \mu(1 - \psi_F(s))^{-1} - s^{-1} \tag{9}$$

But $\mu(1 - \psi_F)^{-1}$ is the Laplace transform $\mu \psi_{M_F}(s)$ and s^{-1} is the Laplace transform of $f(t) = t$. Thus $\mu \psi_{M_F}(s)/\psi_{M_G}(s)$ is the Laplace transform of $\mu M_F(t) - t = EZ_F(t)$. The result now follows.

Splitting the range of integration $[0, t]$ into $\{0\}$ and $(0, t]$ and recalling that $dEZ_F(y)$ has an atom of size $\mu/\bar{F}(0)$ at 0, gives the alternative expression in (1). □

4. Monotonicity Result. The main monotonicity result is now derived.

THEOREM 2. *If F is* IMRL *then $E\phi(Z(t))$ is increasing in $t \geq 0$ for all increasing convex functions ϕ.*

PROOF. Since ϕ is an increasing convex function, $\phi(t) = \phi(0+) + \int_0^t \phi'(x)\,dx$ with ϕ' $\uparrow$. It is straightforward to show that the expectations $E(\phi(Z_F(t)))$, $E(\phi'(Z_G(t)))$, $E_G\phi'(X)$ are all well defined and are either finite or equal to $+\infty$ depending on whether $E_F\phi(X)$ is finite or equals $+\infty$.

Start with (3) (the identity of Theorem 1), multiply both sides by $\phi'(x)$ and integrate x from 0 to ∞. The left side, after integration by parts, reduces to $E\phi(Z_F(t)) - \phi(0+)$. It is finite (equals $+\infty$) if and only if $E\phi(X)$ is finite (equals $+\infty$). The right side, after interchanging the order of integration, reduces to $\int_{0-}^t E\phi'(Z_G(t-y))\,dE(Z_F(y))$, and it too is finite (equals $+\infty$) if and only if $E\phi(X)$ is finite (equals $+\infty$). Thus:

$$E\phi(Z_F(t)) = \phi(0+) + \int_{0-}^{t} E\phi'(Z_G(t-y)\,dE(Z_F(y)). \tag{10}$$

Since ϕ' is increasing and G is DFR (Lemma 1 (iii)) it follows from Brown [3], Theorem 3, that $E\phi'(Z_G(t-y))$ is increasing in t for each y. Since $EZ_F(y)$ is increasing (Brown [3], Theorem 2), $\phi' \geq 0$, and $E\phi'(Z_G(t-y))$ is increasing, it follows from (10) that $E\phi(Z_F(t)) \uparrow$. □

5. Examples. (i) In this example F is IMRL, $Z(t)$ is not stochastically increasing, m is not decreasing (and thus M is not concave) and $EA(t)$ is not increasing. The hazard function of F is given by

$$h(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 4 & 1 \leq x < 2 \\ .01 & x \geq 2 \end{cases}$$

For $t \geq 1$, $X - t \mid X > t$ is stochastically increasing and thus increasing in mean. For $0 \leq t \leq 1$, $E(X - t \mid X > t) = 1 + ce^t$ where $c = e^{-1}(99.75e^{-4} - .75) > 0$. Thus F is IMRL.

For $0 < \delta < 1$, $Pr(Z(0) > \delta) = e^{-\delta}$. For $t = 1$, the hazard function, $h_{Z(1)}^{(x)}$, for each $x \in [0, \delta]$ is a weighted average of 1 and 4, with 4 receiving positive weight. Thus $h_{Z(1)}^{(x)} > 1$ for $x \in [0, \delta]$ and consequently $Pr(Z(1) > \delta) < e^{-\delta} = Pr(Z(0) > \delta)$. Therefore $Z(t)$ is not stochastically increasing.

Finally for $s < 1$, $E[A(1+\epsilon) \mid A(1) = s] = s + \epsilon(1-s) + o(\epsilon)$, while $E(A(1+\epsilon) \mid A(1) = 1) = 1 - 3\epsilon + o(\epsilon)$. Thus $E(A(1+\epsilon)) = 1 - e^{-1} - 2\epsilon e^{-1} + o(\epsilon) = EA(1) - 2\epsilon e^{-1} + o(\epsilon)$; thus $(d/dt)\, EA(t) \big|_{t=1^+} = -2e^{-1}$ and $EA(t)$ is not increasing.

(ii) Berman [2], page 429, raised the question of whether F NWU implies that the renewal density is decreasing. In the example below, F is DFRA (and thus NWU) and the renewal density is not decreasing. Recall that F is defined to be NWU if $X - t \mid X > t$ is stochastically greater than X for all $t \geq 0$, and DFRA if $H(t)/t = (-\ln \bar{F}(t))/t$ is decreasing in $t > 0$.

$$F \text{ is defined by the hazard function } h(x) = \begin{cases} 100 & 0 \leq x < 1 \\ 1 & 1 \leq x < 2 \\ 2 & x \geq 2 \end{cases}$$

Clearly F is DFRA (and thus NWU). Define $p(t) = Pr(1 < A(t) < 2)$. Note that $p(2) = p(2-) - Pr(N(2) = 0) = p(2-) - e^{-101}$. Thus $m(2-) = 100(1 - p(2-)) + p(2-)$ while $m(2) = 100(1 - p(2-)) + (p(2-) - e^{-101}) + 2e^{-101} = m(2-) + e^{-101}$. It follows that for some $\epsilon > 0$, $m(t) > m(s)$ for $2 + \epsilon > t > 2 > s > 2 - \epsilon$, and thus m is not decreasing.

(iii) It does not appear that the prospects are good for deriving monotonicity results for renewal processes with IFR interarrival times. Berman [2] points out that the convolution of three exponentials with common parameter does not have an increasing renewal density. Moreover, it is easy to construct absolutely continuous IFR distributions which are arbitrarily close to a degenerate distribution, say for example a degenerate distribution at $\{1\}$. Such a distribution has a renewal function which, for small to moderate t, differs little from that of the degenerate distribution at $\{1\}$.

6. Remark. In Brown [3], Theorem 3, it was shown that F DFR is a sufficient condition for the renewal function to be concave. I conjecture that this condition is also necessary.

Define $\mathscr{A}$ to be the class of distributions on $[0, \infty)$ for which the renewal function, M, is concave. Then $\mathscr{A}$ is closed under geometric compounding, i.e. if $\{X_i, i \geq 1\}$ is i.i.d. with distribution $F \in \mathscr{A}$ and N is independent of $\{X_i\}$ and geometrically distributed with parameter p, then F_p the distribution of $\sum_1^N X_i$ belongs to $\mathscr{A}$. This is true because $M_p(t) - M_p(0) = p(M(t) - M(0))$ where M_p is the renewal function corresponding to F_p. It is not known whether the class of DFR distributions is closed under geometric compounding. If it is not closed then the above conjecture is false.

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