Research Article

Dorota Kuziak* and Ismael G. Yero

# Further new results on strong resolving partitions for graphs

https://doi.org/10.1515/math-2020-0142
received July 30, 2019; accepted March 12, 2020

**Abstract:** A set $W$ of vertices of a connected graph $G$ strongly resolves two different vertices $x, y \notin W$ if either $d_G(x, W) = d_G(x, y) + d_G(y, W)$ or $d_G(y, W) = d_G(y, x) + d_G(x, W)$, where $d_G(x, W) = \min\{d(x,w): w \in W\}$ and $d(x,w)$ represents the length of a shortest $x - w$ path. An ordered vertex partition $\Pi = \{U_1, U_2,...,U_k\}$ of a graph $G$ is a strong resolving partition for $G$, if every two different vertices of $G$ belonging to the same set of the partition are strongly resolved by some other set of $\Pi$. The minimum cardinality of any strong resolving partition for $G$ is the strong partition dimension of $G$. In this article, we obtain several bounds and closed formulae for the strong partition dimension of some families of graphs and give some realization results relating the strong partition dimension, the strong metric dimension and the order of graphs.

**Keywords:** strong resolving set, strong metric dimension, strong resolving partition, strong partition dimension, strong resolving graph

**MSC 2010:** 05C12, 05C70

## 1 Introduction

Given a connected graph $G$, a vertex $v \in V(G)$ distinguishes two distinct vertices $x, y \in V(G)$, if $d_G(v, x) \neq d_G(v, y)$, where $d_G(v, y)$ represents the length of a shortest $x - y$ path. A set $S \subset V(G)$ is a *metric generator* for $G$, if any pair of vertices of $G$ is distinguished by at least one vertex of $S$. A metric generator of minimum cardinality is called a *metric basis*, and its cardinality the *metric dimension* of $G$, denoted by $\dim(G)$. Such parameter was introduced by Slater in [1], where it was called *locating number* and the metric generators were called *locating sets*. An equivalent terminology was also introduced by Harary and Melter in [2], where metric generators were called *resolving sets*. The terminology of metric generators was first presented in [3], as a more natural way of understanding such structure. This latter terminology is arising from the theory of metric spaces. That is, $(G, d_G)$, where $G = (V, E)$ is a connected graph, $d_G: V \times V \to \mathbf{N}$ is clearly a metric space.

Slater described the usefulness of these ideas in long range aids to navigation [1]. Also, these concepts have some applications in chemistry for representing chemical compounds [4,5], or in problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [6]. Other applications of this concept to navigation of robots in networks and other areas appear in [7–9]. Hence, according to its applicability metric generators have become an interesting and popular topic of investigation in the graph theory. While applications have been continuously appearing, this invariant has also been theoretically studied in a high number of other papers, which we do not mention here due to the

* **Corresponding author: Dorota Kuziak,** Departamento de Estadística e Investigación Operativa, Escuela Politécnica Superior de Algeciras, Universidad de Cádiz, Av. Ramón Puyol s/n, 11202 Algeciras, Spain, e-mail: dorota.kuziak@uca.es
**Ismael G. Yero:** Departamento de Matemáticas, Escuela Politécnica Superior de Algeciras, Universidad de Cádiz, Av. Ramón Puyol s/n, 11202 Algeciras, Spain, e-mail: ismael.gonzalez@uca.es

high amount of them. Moreover, several variations of metric generators including resolving dominating sets [10], independent resolving sets [11], local metric generators [12], strong resolving sets [3], $k$-metric generators [13], edge metric generators [14], mixed metric generators [15], antiresolving sets [16], multiset resolving sets [17], metric colorings [18], resolving partitions [19] and strong resolving partitions [20] have been introduced and studied. The last three cases are remarkable in the sense that they concern with a partition of the vertex set of the graph which uniquely identifies every vertex of the graph. For instance, resolving partitions were introduced in [19] to gain more insight into metric generators for graphs. In a similar manner, strong resolving partitions were introduced in [20] to gain more insight into strong metric generators for graphs. It is now our goal to give a few other new results on the strong partition dimension of graphs.

Strong metric generators in metric spaces or graphs were first described in [3], where the authors presented some applications of this concept to combinatorial search. Metric generators were also studied in [21], where the authors found an interesting connection between the strong metric bases of a graph and the vertex cover number of a related graph which they called “strong resolving graph”. This connection allowed them to prove that finding the strong metric dimension of a graph is NP-hard. Nevertheless, when the problem is restricted to trees, it can be solved in polynomial time. This fact was also noticed in [3]. A vertex $v$ of a graph $G$ *strongly resolves* the two different vertices $x, y$, if either $d_G(x, v) = d_G(x, y) + d_G(y, v)$ or $d_G(y, v) = d_G(y, x) + d_G(x, v)$. A set $S$ of vertices in a connected graph $G$ is a *strong resolving set* for $G$ if every two vertices of $G$ are strongly resolved by some vertex of $S$. A strong resolving set of minimum cardinality is called a *strong metric basis*, and its cardinality the *strong metric dimension* of $G$, which is denoted by $\dim_s(G)$.

Now, given a vertex $x$ and a set $W$ of a graph $G$, the distance between $x$ and $W$ is defined as $d_G(x, W) = \min\{d(x, w): w \in W\}$ (or $d(x, W)$ for short). A set $W$ of vertices of $G$ *strongly resolves* two different vertices $x, y \notin W$ if either $d_G(x, W) = d_G(x, y) + d_G(y, W)$ or $d_G(y, W) = d_G(y, x) + d_G(x, W)$. An ordered vertex partition $\Pi = \{U_1, U_2, \ldots, U_k\}$ of a graph $G$ is a *strong resolving partition* for $G$, if every two distinct vertices of $G$, belonging to the same set of the partition, are strongly resolved by some set of $\Pi$. A strong resolving partition of minimum cardinality is called a *strong partition basis*, and its cardinality the *strong partition dimension* of $G$, denoted by $pd_s(G)$. The strong partition dimension of graphs was introduced in [20] and further studied in [22].

The study of the strong metric dimension of a graph is closely related to the study of the vertex cover number of a related graph, known as strong resolving graph [21]. To introduce such a graph, we need the following terminology. A vertex $u$ of $G$ is *maximally distant* from $v$, if for every vertex $w$ adjacent to $u$, $d_G(v, w) \le d_G(u, v)$. If $u$ is maximally distant from $v$ and $v$ is maximally distant from $u$, then we say that $u$ and $v$ are *mutually maximally distant*. The *boundary* of $G = (V, E)$ is defined as $\partial(G) = \{u \in V$: there exists $v \in V$ such that $u, v$ are mutually maximally distant$\}$. For some basic graph classes, such as complete graphs $K_n$, complete bipartite graphs $K_{r,s}$, cycles $C_n$ and hypercube graphs $Q_k$, the boundary is simply the whole vertex set. It is not difficult to see that this property holds for all 2-antipodal[1] graphs and also for all distance-regular graphs. Notice that the boundary of a tree consists exactly of the set of its leaves. A vertex of a graph is a *simplicial vertex* if the subgraph induced by its neighbors is a complete graph. Given a graph $G$, we denote by $\varepsilon(G)$ the set of simplicial vertices of $G$. If the simplicial vertex has degree one, then it is called an *end vertex*. We denote by $\tau(G)$ the set of end vertices of $G$. Notice that $\varepsilon(G) \subseteq \partial(G)$.

The *strong resolving graph*[2] of a graph $G$ is a graph $G_{SR}$ with vertex set $V(G_{SR}) = \partial(G)$, where two vertices $u,v$ are adjacent in $G_{SR}$ if and only if $u$ and $v$ are mutually maximally distant in $G$. There are some families of graphs for which their strong resolving graphs can be obtained relatively easily. For instance, we emphasize the following cases.

**1** The diameter of $G = (V, E)$ is defined as $D(G) = \max_{u,v \in V}\{d(u, v)\}$. We recall that $G = (V, E)$ is 2-antipodal if for each vertex $x \in V$ there exists exactly one vertex $y \in V$ such that $d_G(x, y) = D(G)$.

**2** In [21], the strong resolving graph $G_{SR}$ was defined over $V(G)$ instead of $\partial(G)$ and those vertices not belonging to $\partial(G)$ were isolated in $G_{SR}$. In this sense, in this work we do not consider such isolated and defined the strong resolving graph over the set $\partial(G)$.

**Remark 1.**

(i) If $\partial(G) = \varepsilon(G)$, then $G_{SR} \cong K_{|\partial(G)|}$. In particular, $(K_n)_{SR} \cong K_n$ and for any tree $T$ with $l(T)$ leaves, $(T)_{SR} \cong K_{l(T)}$.

(ii) For any connected block graph[3] $G$ of order $n$ and $c$ cut vertices, $G_{SR} \cong K_{n-c}$.

(iii) For any 2-antipodal graph $G$ of order $n$, $G_{SR} \cong \cup_{i=1}^{\frac{n}{2}} K_2$. In particular, $(C_{2k})_{SR} \cong \cup_{i=1}^{k} K_2$ and for any hypercube graph $Q_r$, $(Q_r)_{SR} \cong \cup_{i=1}^{2^{r-1}} K_2$.

(iv) For any positive integer $k$, $(C_{2k+1})_{SR} \cong C_{2k+1}$.

(v) For any positive integers $r$, $t$, $(K_{r,t})_{SR} \cong K_r \cup K_t$.

For more information on the strong resolving graph of a graph (as a proper graph operation), we suggest a previous study [23], where several structural properties of such graphs were studied, and the recent work [24] which gives some new results concerning strong resolving graphs.

## 2 Some known results

A natural relationship between the strong metric dimension and the strong partition dimension of graphs was first given in [20].

**Theorem 2.** [20] *For any connected graph G,* $\mathrm{pd_s}(G) \le \dim_s(G) + 1$.

A set $S$ of vertices of $G$ is a *vertex cover* of $G$ if every edge of $G$ is incident with at least one vertex of $S$. The *vertex cover number* of $G$, denoted by $\alpha(G)$, is the smallest cardinality of a vertex cover of $G$. We refer to an $\alpha(G)$ set in a graph $G$ as a vertex cover of cardinality $\alpha(G)$. Oellermann and Peters-Fransen [21] showed that the problem of finding the strong metric dimension of a connected graph $G$ can be transformed to the problem of finding the vertex cover number of $G_{SR}$. That is, for any connected graph $G$,

$$\dim_s(G) = \alpha(G_{SR}). \tag{1}$$

As a consequence of this and Theorem 2, the following result is known from [20].

**Theorem 3.** [20] *For any connected graph G,* $\mathrm{pd_s}(G) \le \alpha(G_{SR}) + 1$.

A *clique* in a graph $G$ is a set of vertices $S$ such that the subgraph induced by $S$ is isomorphic to a complete graph. The maximum cardinality of a clique in a graph $G$ is the *clique number* and is denoted by $\omega(G)$. A set $S$ is an $\omega(G)$ clique if $|S| = \omega(G)$. Since any two maximally distant vertices of a graph $G$ must belong to two different sets in any strong resolving partition for $G$, the following was deduced in [20].

**Theorem 4.** [20] *For any connected graph G,* $\mathrm{pd_s}(G) \ge \omega(G_{SR})$.

## 3 Realization results

Some partial realization results concerning $\dim_s(G)$ and $\mathrm{pd_s}(G)$ were presented in [20]. Specifically, it was proved that for any integers $r$, $t$, $n$ such that $3 \le r \le t \le \frac{n+r-2}{2}$, there exists a connected graph $G$ of order $n$ with $\mathrm{pd_s}(G) = r$ and $\dim_s(G) = t$. In connection with this, the following question was posted in [20].

3 $G$ is a block graph if every biconnected component (also called block) is a clique. Notice that any vertex in a block graph is either a simplicial vertex or a cut vertex.

**Problem.** Is it true that $\dim_s(G) \le \frac{pd_s(G)+n-2}{2}$ for every nontrivial connected graph $G$ of order $n$?

From now on, we center our attention into answering this question in the negative and into giving a more general, although still not complete, realization result for $\dim_s(G)$ and $pd_s(G)$. To this end, we consider the following family of graphs. Let $a$, $b$, $c$ be three integers such that $a \ge 0$ and $b \le c$. We construct a graph $G_{a,b,c}$ as follows. We begin with a complete bipartite graph $K_{b,c}$ with bipartition sets $B$, $C$ such that $|B| = b$ and $|C| = c$. Now if $a > 0$, then we construct the graph $G_{a,b,c}$ by adding a path $P_a$ and joining with an edge one of the leaves of $P_a$ with one vertex of the set $C$. If $a = 0$, then $G_{a,b,c}$ is simply taken as $K_{b,c}$.

**Proposition 5.** *For any three integers $a$, $b$, $c$ such that $a \ge 0$ and $b \le c$,*

$$\dim_s(G_{a,b,c}) = b + c - 2 \text{ and } pd_s(G_{a,b,c}) = \begin{cases} c, & \text{if } b < c, \\ c+1, & \text{if } b = c. \end{cases}$$

**Proof.** Let $B = \{u_1,\dots,u_b\}$ and $C = \{v_1,\dots,v_c\}$ be the bipartition sets of the complete bipartite graph $K_{b,c}$ used to generate $G_{a,b,c}$, and let $P_a = w_1, w_2,\dots,w_a$ if $a > 0$. Assume that the edge $w_a v_c$ has been added to have $G_{a,b,c}$. We observe that any two vertices of $B$ are mutually maximally distant between them and they are not mutually maximally distant with any other vertex of $G_{a,b,c}$. Moreover, any two vertices of $C - \{v_c\} \cup \{w_1\}$ if $a > 0$, or any two vertices of $C$ if $a = 0$, are mutually maximally distant between them and they are not mutually maximally distant with any other vertex of $G_{a,b,c}$. Thus, in any case $(G_{a,b,c})_{SR} \cong K_b \cup K_c$. By Theorem 4 and Remark 1, we obtain that $pd_s(G_{a,b,c}) \ge \omega(K_b \cup K_c) = c$, and by equality (1), $\dim_s(G_{a,b,c}) = b + c - 2$. □

First, we assume $b < c$ and let $\Pi = \{\{u_1, v_1\},\{u_2, v_2\},\dots,\{u_b, v_b\},\{v_{b+1}\},\{v_{b+2}\},\dots,\{v_c\}\}$, if $a = 0$, or $\Pi = \{\{u_1, v_1\},\{u_2, v_2\},\dots,\{u_b, v_b\},\{v_{b+1}\},\{v_{b+2}\},\dots,\{v_c, w_1, w_2,\dots,w_a\}\}$, if $a > 0$. For any $i \in \{1,\dots,b\}$, there exists a vertex $v_j$ with $j \in \{b+1,\dots,c\}$ such that $u_i$ lies in a shortest $v_i - v_j$ path. So, the set $\{v_j\} \in \Pi$ or the set $\{v_c, w_1, w_2,\dots,w_a\} \in \Pi$ strongly resolves the pair $u_i$, $v_i$. Also, any two vertices of the set $\{w_1, w_2,\dots,w_a, v_c\}$ are strongly resolved by any set $\{u_i, v_i\} \in \Pi$ for $i \in \{1,\dots,b\}$. Thus, $\Pi$ is a strong resolving partition for $G_{a,b,c}$, which leads to $pd_s(G_{a,b,c}) \le c$ and the equality follows for the case $b < c$.

Assume now that $b = c$. Since any two vertices in $B$ are mutually maximally distant between them, it follows that they must belong to two different sets in any strong resolving partition for $G_{a,b,c}$. Analogously, any two distinct vertices of $C - \{v_c\} \cup \{w_1\}$ if $a > 0$, or any two vertices of $C$ if $a = 0$, must belong to two different sets in any strong resolving partition for $G_{a,b,c}$. If $pd_s(G_{a,b,c}) = c$, then any set of every strong resolving partition $\Pi'$ for $G_{a,b,c}$ must contain exactly one vertex $u_i \in B$, and exactly one vertex $v_j \in C - \{v_c\} \cup \{w_1\}$, if $a > 0$, or exactly one vertex $v_j \in C$, if $a = 0$. If $a = 0$, then clearly there are two vertices $u_h \in B$ and $v_f \in C$ belonging to the same set of $\Pi'$ such that they have distance 1 to any other different set of $\Pi'$, which is a contradiction, since in such case $u_h$, $v_f$ are not strongly resolved by $\Pi'$. Thus $a > 0$. Now, the vertex $v_c$ must belong to a set $S \in \Pi'$. Let $u_j \in B$ such that also $u_j \in S$. Notice that $d(v_c,W) = 1$ for every $W \in \Pi'$. Hence, in order to strongly resolve the pair $v_c$, $u_j$, there must be a set $S' \in \Pi'$ such that $w_1 \in S'$ and $S' \subset \{w_1, w_2, \dots, w_a\}$. However, this means there exists a vertex $v_l \in C - \{v_c\}$ such that $v_l \in S$ since any two vertices of the set $C - \{v_c\} \cup \{w_1\}$ belong to different sets of $\Pi'$. Thus, the pair $v_c$, $v_l$ satisfies that $d(v_c,W) = d(v_l,W) = 1$ for every set $W \in \Pi' - S$. So, $v_c$, $v_l$ are not strongly resolved by any set of $\Pi'$, a contradiction. As a consequence, $pd_s(G_{a,b,c}) \ge c + 1$.

On the other hand, let $\Pi'' = \{\{u_1, v_1\}, \{u_2, v_2\},\dots,\{u_{c-1}, v_{c-1}\}, \{u_c\}, \{v_c, w_1, w_2,\dots,w_a\}\}$. As in the case $b < c$, it clearly follows that any two vertices $u_i,v_i$ with $i \in \{1,\dots,c-1\}$ are strongly resolved by $\{u_c\}$ and by $\{v_c, w_1, w_2,\dots,w_a\}$. Moreover, any two different vertices in $\{v_c, w_1, w_2,\dots,w_a\}$ are strongly resolved by any other set in $\Pi''$. Thus, $\Pi''$ is a strong resolving partition for $G_{a,b,c}$, which means that $pd_s(G_{a,b,c}) \le c + 1$ and the proof is completed for $b = c$.

As a consequence of the aforementioned proposition, we can assert the following realization result for $pd_s(G)$, $\dim_s(G)$ and the order of $G$. In connection with this, we must recall some realization facts from [20].

For instance, if $t \neq 1$, then the values 2, $t$, $n$ are not realizable, and if $r < t + 1 = n$, then the values $r$, $t$, $n$ are also not realizable. Moreover, constructions for the cases $3 \leq r = t + 1 < n$, $3 \leq r = t \leq n - 3$ and $3 \leq r < t \leq \frac{n+r-2}{2}$ were given in [20]. In this sense, it was remaining the case in which $3 \leq r < t$ together with the condition $n - 2 \geq t > \frac{n+r-2}{2}$. We now give a realization for one part of this interval. In such result, we must assume $r > (n + 4)/5$ (otherwise our new interval for realization is included in the previous mentioned situation), and that $2r - 3 < n - 2$ (otherwise the realization result is completely done).

**Theorem 6.** *For any integers $r$, $t$, $n$ such that $2 \leq r \leq t \leq 2r - 3$, there exists a connected graph $G$ of order $n$ with $\mathrm{pd}_s(G) = r$ and $\dim_s(G) = t$.*

**Proof.** To obtain our result, we just need to consider a graph $G_{a,b,c}$ as described above, where $a = n - t - 2$, $b = t - r + 2$ (with $b < c$) and $c = r$. Clearly, $G_{a,b,c}$ has the order $a + b + c = n - t - 2 + t - r + 2 + r = n$ and satisfies $\dim_s(G_{a,b,c}) = b + c - 2 = t - r + 2 + r - 2 = t$ and $\mathrm{pd}_s(G_{a,b,c}) = c = r$. Since $a \geq 0$ and $1 \leq b < c$, we obtain that $t \leq n - 2$, $r \leq t - 1$ and $t \leq 2r - 3$, which coincides with the condition that $n$, $\mathrm{pd}_s(G)$ and $\dim_s(G)$ satisfy. □

As a consequence of the aforementioned result, in connection with the fact that $t \leq 2r - 3$, one could think into the following problem.

**Problem.** Is it true that $\dim_s(G) \leq 2\mathrm{pd}_s(G) - 3$ for every nontrivial connected graph $G$?

We close this section by noticing that the case $a = 0$ in Proposition 5 leads to the complete bipartite graph $K_{b,c}$. In this sense, the following result can be deduced from Proposition 5.

**Corollary 7.** *For any integers $r$, $t$ with $1 \leq r \leq t$,*

$$\mathrm{pd}_s(K_{r,t}) = \begin{cases} t, & \text{if } r < t, \\ t + 1, & \text{if } r = t. \end{cases}$$

# 4 Corona graphs and join graphs

Let $G$ and $H$ be two graphs of order $n$ and $m$, respectively. The *corona graph* $G \odot H$ is defined as the graph obtained from one copy of $G$ and $n$ copies of $H$, by adding an edge between each vertex from the $i$th copy of $H$ and the $i$th vertex of $G$. We denote by $V(G) = \{v_1, v_2, \ldots, v_n\}$ the set of vertices of $G$, and let $H_i = (V_i, E_i)$ be the copy of $H$ such that $v_i \sim v$ for every $v \in V_i$. The *join graph* $G + H$ is defined as the graph obtained from the disjoint graphs $G$ and $H$ by adding an edge between each vertex of $G$ and each vertex of $H$. Notice that the corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. Figure 1 shows an example of one corona graph.

We next study the strong partition dimension of corona graphs. To this end, we shall use the following observations.

**Claim 8.**
(i) Any two distinct vertices from two different copies of $H$ in any corona graph $G \odot H$ belong to different sets in any strong resolving partition for $G \odot H$.
(ii) Any two distinct vertices from the same set of a strong resolving partition for a corona graph $G \odot H$ belong to the same copy of the graph $H$.
(iii) Any two distinct vertices from a set $A \subset V(H_i)$ of a strong resolving partition for a corona graph $G \odot H$ are strongly resolved by a set $B$ which is a subset of the set $V(H_i)$.

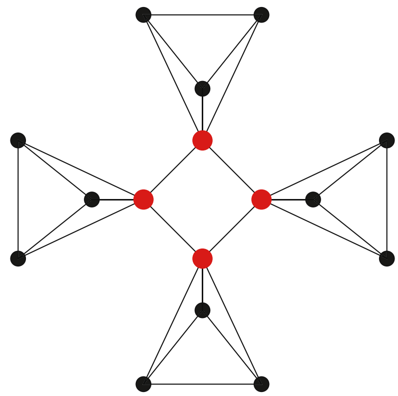

**Figure 1:** The corona graph $C_4 K_3$.

**Proof.** (i) Follows from the fact that any two distinct vertices from two different copies of $H$ are mutually maximally distant between them, and thus they are not resolved by any vertex other than themselves. Consequently, any two distinct vertices from the same set of a strong resolving partition for $G \odot H$ must belong to the same copy of the graph $H$, which is the statement of (ii). Finally, (iii) follows since any two vertices of a same copy of $H$ have the same distance to any other vertex of $G \odot H$ which is not in this copy. □

A direct consequence of the aforementioned result is that the restriction of any strong resolving partition $\Pi$ for a corona graph $G \odot H$, to a copy $H_i$ of $H$ in $G \odot H$ is a strong resolving partition for $H_i$. That is stated in the next corollary.

**Corollary 9.** *For any connected graph $G$ of order $n$ and any connected graph $H$,* $\mathrm{pd}_s(G \odot H) \geq n\mathrm{pd}_s(H)$.

**Proof.** Let $\Pi$ be a strong partition basis for $G \odot H$. For any $i \in \{1,\dots,n\}$, the restriction $\Pi_i$ of $\Pi$ to the copy $H_i$ of $H$ in $G \odot H$ is a strong resolving partition for $H_i$. Since every set in $\Pi_i$ has empty intersection with $V(H_j)$ for every $j \neq i$, it follows that

$$\mathrm{pd}_s(G \odot H) = |\Pi| = \sum_{i=1}^{n} |\Pi_i| \geq \sum_{i=1}^{n} \mathrm{pd}_s(H_i) = n\mathrm{pd}_s(H),$$

and the proof is completed. □

**Theorem 10.** *Let $G$ be a connected graph of order $n$. If $H$ is a complete graph $K_m$ or an edge-less graph $N_m$, then* $\mathrm{pd}_s(G \odot H) = nm$.

**Proof.** It is easy to see that any two distinct vertices of any copy of $H$ (including the case in which the vertices are in the same copy) are mutually maximally distant between them. Thus $(G \odot H)_{SR} \cong K_{nm}$. Therefore, the result follows from Theorems 3 and 4. □

**Theorem 11.** *Let $G$ be a connected graph of order $n$. If $H$ has diameter 2, then* $\mathrm{pd}_s(G \odot H) = n\mathrm{pd}_s(H)$.

**Proof.** Let $\Pi_i = \{A_{i,1},\dots,A_{i,r}\}$ be a strong resolving partition for $H_i$ with $i \in \{1,\dots,n\}$ and $r = \mathrm{pd}_s(H)$. We shall prove that the vertex partition $\Pi = \{A_{1,1} \cup \{v_1\}, A_{1,2},\dots,A_{1,r}, A_{2,1} \cup \{v_2\}, A_{2,2},\dots,A_{2,r},\dots,A_{n,1} \cup \{v_n\}, A_{n,2},\dots,A_{n,r}\}$ is a strong resolving partition for $G \odot H$.

Let $x, y$ be two distinct vertices of $G \odot H$ belonging to the same set of $\Pi$. If $x = v_j$ for some $j \in \{1,\dots,n\}$, then $y \in A_{j,\ell}$, for some $\ell \in \{1,\dots,r\}$, and they are adjacent. Hence, for any $B \in \Pi$ such that $B \cap V(H_j) = \emptyset$, it clearly follows that $d_{G \odot H}(y, B) = d_{G \odot H}(y, x) + d_{G \odot H}(x, B)$. Thus, $x, y$ are strongly resolved by $B$. On the other hand, if $x, y \in A_{j,k}$ for some $j \in \{1,\dots,n\}$ and $k \in \{1,\dots,r\}$, then they are strongly resolved in $H_j$ by some set $A_{j,l}$ with $l \neq k$, which means that without loss of generality, $d_{H_j}(x, A_{j,l}) = d_{H_j}(x, y) + d_{H_j}(y, A_{j,l})$. Since $H$ has diameter 2, $d_{G\odot H}(a, b) = d_{H_j}(a, b)$ for any $a, b \in V(H_j)$, and we have that

$$d_{G\odot H}(x, A_{j,l}) = d_{H_j}(x, A_{j,l}) = d_{H_j}(x, y) + d_{H_j}(y, A_{j,l}) = d_{G\odot H}(x, y) + d_{G\odot H}(y, A_{j,l}).$$

Thus, $x$, $y$ are strongly resolved by $A_{j,l}$. As a consequence, $\Pi$ is a strong resolving partition for $G \odot H$ and we obtain that $\mathrm{pd}_s(G \odot H) \leq n\mathrm{pd}_s(H)$. The equality is finally obtained by using Corollary 9. □

**Theorem 12.** *For any connected graph $G$ of order $n$ and any graph $H$ of diameter at least3,* $\mathrm{pd}_s(G \odot H) = n\mathrm{pd}_s(K_1 + H)$.

**Proof.** By using a similar procedure like that in the proof of the upper bound in Theorem 11, we obtain the upper bound $\mathrm{pd}_s(G \odot H) \leq n\mathrm{pd}_s(K_1 + H)$. Notice that now, the given partition is obtained from a strong resolving partition for $K_1 \odot H$ instead of that for $H$ (like in Theorem 11). □

On the other hand, by Claim 8, any two vertices belonging to two distinct copies of the graph $H$ belong to different sets of any strong resolving partition for $G \odot H$. Also, no vertex $v_i \in V(G)$ influences into strongly resolving a pair of vertices in $V(H_i)$. Moreover, for every $i \in \{1,\ldots,n\}$, any vertex $u \in V(H_i)$ and $v_i$ can be strongly resolved by every set which has empty intersection with $V(H_i)$ in any strong resolving partition for $G \odot H$. Thus, always exists a strong partition basis $\Pi$ for $G \odot H$, such that for every $i \in \{1,\ldots,n\}$, the vertex $v_i$ belongs to some set $A \in \Pi$ and $A \subset V(H_i) \cup \{v_i\}$. Let $\Pi_i$ be the restricted vertex partition of $\Pi$ over the subgraph $H_i'$ induced by $\{v_i\} \cup V(H_i)$. We first note that if there are two distinct vertices $x, y \in V(H_i)$ for some $i \in \{1,\ldots,n\}$ such that they belong to the same set $A_j \in \Pi$, then they are strongly resolved by some set $A_k \subset V(H_i)$, since $x$, $y$ have the same distance to any other vertex not in $V(H_i)$ and also $A_k$ cannot contain vertices from two different copies of $H$ (Claim 8) and, in addition, it cannot contain $v_i$. Also, note that $A_k$ is a set of the partition $\Pi_i$. Thus, $x$,$y$ are strongly resolved by $\Pi_i$ and, as a consequence, $\Pi_i$ is a strong resolving partition for $H_i'$. Therefore, $\mathrm{pd}_s(G \odot H) \geq n\mathrm{pd}_s(K_1 + H)$ and the equality follows.

As a consequence of the aforementioned result, it is natural to think about the relationship that exists between the strong partition dimension of a graph $G$ and that of the join graph $K_1 + G$, for the case in which $G$ has diameter at least 3. To this end, we need the following terminology and notation. Given a graph $G$, two vertices $u$, $v$ are called true or false twins if and only if they have the same closed or open neighborhood, respectively. The complement $G^c$ of $G$ is the graph with vertex set $V(G^c) = V(G)$, such that two vertices $u$, $v$ are adjacent in $G^c$ if and only if $u$, $v$ are not adjacent in $G$. Moreover, we denote by $G^{ct}$, the graph with vertex set $V(G)$ such that two vertices $u$, $v$ are adjacent in $G^{ct}$ if and only if either $u$, $v$ are adjacent in $G^c$ or $u$, $v$ are true twins in $G$. According to these definitions, the following remark is straightforward to observe for the case when $G$ has diameter at least 3. This is based on the fact that $K_1 + G$ has diameter 2, which means that any two nonadjacent vertices are mutually maximally distant in $K_1 + G$, as well as those vertices which are true twins. Moreover, the vertex of $K_1$ is not mutually maximally distant with any other vertex.

**Remark 13.** Let $G$ be any connected graph of diameter at least 3.
(i) If $G$ has true twins, then $(K_1 + G)_{\mathrm{SR}} \cong G^{ct}$.
(ii) If $G$ has no true twins, then $(K_1 + G)_{\mathrm{SR}} \cong G^c$.

A set $S$ is an *independent set* in a graph $G$, if the subgraph induced by $S$ has no edges. The maximum cardinality of any independent set is the *independence number* of $G$ and is denoted by $\beta(G)$. Now, by using the aforementioned remark we derive some conclusions on the strong partition dimension of the join graph $K_1 + G$ for some classes of graphs $G$.

**Proposition 14.** *Let $G$ be any connected graph of order $n$ and diameter at least 3.*
(i) If $G$ has $t$ true twins, then $\beta(G) \leq \mathrm{pd}_s(K_1 + G) \leq n - \omega(G) + t$.

(ii) If $G$ has no true twins, then $\beta(G) \leq \mathrm{pd_s}(K_1 + G) \leq n - \omega(G) + 1$.

**Proof.** The lower bounds follow from Theorem 4 and Remark 13. That is $\mathrm{pd_s}(K_1 + G) \geq \omega((K_1 + G)_{SR}) = \omega(G') \geq \omega(G^c) = \beta(G)$, where $G'$ is either $G^c$ or $G^{ct}$.

To prove the upper bounds, we use Theorem 3 and Remark 13. For the case of (i), let $G^\star$ be the subgraph of $G^{ct}$ induced by the true twin vertices of $G$. Thus, we have

$$\mathrm{pd}_s(K_1 + G) \leq \alpha(G^{ct}) + 1 \leq \alpha(G^c) + 1 + \alpha(G^\star) \leq \alpha(G^c) + t = n - \beta(G^c) + t = n - \omega(G) + t,$$

and for the case (ii) it follows:

$$\mathrm{pd}_s(K_1 + G) \leq \alpha(G^c) + 1 = n - \beta(G^c) + 1 \leq n - \omega(G) + 1,$$

which completes the proof. □

We observe that the bounds given in (ii) are tight, for instance, for the graph $G_{q,r}$ obtained in the following way. We begin with a complete graph $K_q$ with $q \geq 3$. Then, we add $r \geq 2$ pendant vertices to all but one vertex of $K_q$. We observe that $G_{q,r}$ does not have true twins and has the order $(r + 1)(q - 1) + 1$. Also, $\beta(G_{q,r}) = r(q - 1) + 1$ and $\omega(G_{q,r}) = q$. So, the aforementioned result leads to $r(q - 1) + 1 = \beta(G) \leq \mathrm{pd_s}(K_1 + G) \leq n - \omega(G) + 1 = (r + 1)(q - 1) + 1 - q + 1 = r(q - 1) + 1$, from which we deduce that $\mathrm{pd_s}(K_1 + G) = r(q - 1) + 1$ and the bounds are achieved. We next study the strong partition dimension of the join graph $K_1 + G$ for some specific classes of graphs $G$.

**Proposition 15.**

(i) *For any cycle graph $C_n$ with $n \geq 6$, $\mathrm{pd}_s(K_1 + C_n) = \left\lceil \frac{n}{2} \right\rceil$.*

(ii) *For any path graph $P_n$ with $n \geq 5$, $\mathrm{pd}_s(K_1 + P_n) = \left\lceil \frac{n}{2} \right\rceil$.*

**Proof.** (i) We observe that for any three distinct vertices of the cycle $C_n$, at least two of them are at distance 2 and they are mutually maximally distant, since $K_1 + C_n$ has diameter 2. Thus, any set belonging to any strong resolving partition for $K_1 + C_n$ contains at most two vertices of the cycle $C_n$. As a consequence, we obtain that $\mathrm{pd}_s(K_1 + C_n) \geq \left\lceil \frac{n}{2} \right\rceil$. On the other hand, let $u$ be the vertex of $K_1$ and let $v_1, v_2, \ldots, v_n$ be the vertex set of $C_n$. Hence, it is not difficult to observe that the partition $\Pi = \{\{u, v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{n-1}, v_n\}\}$, if $n$ is even, or $\Pi = \{\{u, v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{n-2}, v_{n-1}\}, \{v_n\}\}$, if $n$ is odd, is a strong resolving partition for $K_1 + C_n$. Thus, the equality follows since $|\Pi| = \left\lceil \frac{n}{2} \right\rceil$.

(ii) A similar argument as above gives the result. □

# 5 Circulant graphs

Let $\mathbb{Z}_n$ be the additive group of integers modulo $n$ and let $M \subset \mathbb{Z}_n$, such that, $i \in M$ if and only if $-i \in M$. We can construct a graph $G = (V, E)$ as follows, the vertices of $V$ are the elements of $\mathbb{Z}_n$ and $(i, j)$ is an edge in $E$ if and only if $j - i \in M$. This graph is called a *circulant graph of order n* and is denoted by $CR(n, M)$. With this notation, a cycle graph is $CR(n, \{-1, 1\})$ and the complete graph is $CR(n, \mathbb{Z}_n)$. In order to simplify the notation, we shall use $CR(n, t)$, $0 < t \leq \left\lfloor \frac{n}{2} \right\rfloor$, instead of $CR(n, \{-t, -t + 1, \ldots, -1, 1, 2, \ldots, t\})$. We emphasize that $CR(n, t)$ is a $(2t)$-regular graph.

To study the strong partition dimension of circulant graphs, we require the following concepts presented first in [19]. A set $W$ of vertices of $G$ *resolves* two different vertices $x, y \in V(G)$ if $d_G(x, W) \neq d_G(y, W)$. An ordered vertex partition $\Pi = \{U_1, U_2, \ldots, U_k\}$ of a graph $G$ is a *resolving partition* for $G$ if every two different vertices of $G$ are resolved by some set of $\Pi$. The minimum cardinality of any resolving partition for $G$ is the *partition dimension* of $G$, denoted by $pd(G)$. Clearly every strong resolving set is also a resolving set and, as a consequence of this, for any connected graph $G$ it follows that

$$\mathrm{pd}(G) \le \mathrm{pd}_s(G). \tag{2}$$

The partition dimension of circulant graphs has been studied in several articles. For instance, in [25] the following result has been proved.

**Theorem 16.** [25] *For any circulant graph $CR(n,t)$ with $0 < t < \left\lfloor \frac{n}{2} \right\rfloor$, $\mathrm{pd}(CR(n,t)) \ge t+1$.*

By using inequality (2) and Theorem 16, we are able to give the following bounds for the strong partition dimension of circulant graphs.

**Theorem 17.** *For any circulant graph $CR(n,t)$ with $0 < t < \left\lfloor \frac{n}{2} \right\rfloor$,*

$$t+1 \le \mathrm{pd}_s(CR(n,t)) \le t+2.$$

**Proof.** The lower bound follows by using inequality (2) and Theorem 16. For the upper bound, let $V = \{v_0, v_1, ..., v_{n-1}\}$ be the vertex set of $CR(n,t)$. From now on, in this proof, operations with the subindices of vertices in $V$ are done modulo $n$. Assume that every vertex $v_i$ is adjacent to $v_{i+j}$ for every $j \in \{-t, -t+1, ..., -1, 1, ..., t\}$. We shall prove that the partition $\Pi = \{A_0, A_1, ..., A_{t+1}\}$ such that $A_0 = \{v_0\}$, $A_t = \{v_t, v_{t+1}, \ldots, v_{t+\lceil (n-t)/2 \rceil}\}$ and $A_{t+1} = \{v_{t+\lceil (n-t)/2\rceil+1}, v_{t+\lceil (n-t)/2\rceil+2}, \ldots, v_{n-1}\}$ is a strong resolving partition for $CR(n,t)$.

Notice that $v_t$ is adjacent to every vertex in $\{v_0, v_1, ..., v_{t-1}\}$ and, moreover, for any vertex $v_i \in A_t - \{v_t\}$ there exists at least one vertex $v_j \in \{v_0, v_1, ..., v_{t-1}\}$ such that $v_i, v_j$ are not adjacent. So, for any two distinct vertices $v_i, v_l \in A_t$ there is a vertex $v_j$ with $j \in \{0, ..., t-1\}$, such that either $v_i$ is contained in a shortest $v_l - v_j$ path or $v_l$ is contained in a shortest $v_i - v_j$ path. Thus, $A_j$ strongly resolves the pair $v_i$, $v_l$. Analogously, it can be proved that any two distinct vertices of $A_{t+1}$ are strongly resolved by some set $A_j$ with $j \in \{0, ..., t-1\}$. Therefore, $\Pi$ is a strong resolving partition and the upper bound follows. □

# 6 Bouquet of cycles

Throughout this section, let $\mathcal{B}_{a,b,c}$ be a family of graphs obtained in the following way. Each graph $B \in \mathcal{B}_{a,b,c}$ is a bouquet of $a+b+c$ cycles with $a, b, c \ge 0$ and $a+b+c \ge 2$. That is, $a+b+c$ cycles having a common vertex, where $a$ of them are even cycles (of order at least four), $b$ are odd cycles of order larger than three, and $c$ are cycles of order three. Let $w$ be the common vertex to all cycles of $B \in \mathcal{B}_{a,b,c}$ (see Figure 2 for an example). We shall denote the cycles of order larger than three in $B$ as follows: $C_{r_1}, C_{r_2}, \ldots, C_{r_a}, C_{s_1}, C_{s_2}, \ldots, C_{s_b}$.

We first construct the strong resolving graph of any graph $B \in \mathcal{B}_{a,b,c}$. In this sense, we note the following observations.

- Any two vertices belonging to different cycles of $B$ which are diametral with the vertex $w$ are mutually maximally distant between them.
- Two diametral vertices of a same cycle (different from $w$) are also mutually maximally distant between them in $B$.
- The two vertices of a cycle of order three, different from $w$, are mutually maximally distant between them, as well as, with any other vertex which is diametral with $w$ from every other cycle of $B$.
- The two vertices of an odd cycle which are diametral with $w$ are mutually maximally distant with any other vertex which is diametral with $w$ from every other cycle of $B$, but they are not mutually maximally distant between them.

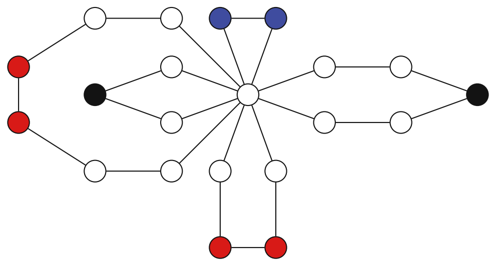

**Figure 2:** A bouquet of five cycles of the family $\mathcal{B}_{2,2,1}$ containing the cycles $C_6$, $C_4$, $C_7$, $C_5$ and $C_3$.

By using the observations above, we are then able to describe the structure of the strong resolving graph of the graph $B \in \mathcal{B}_{a,b,c}$ as follows.

- The set of $a$ vertices of the cycles $C_{r_1}, C_{r_2}, \ldots, C_{r_a}$ which are diametral with $w$ induces a complete graph in $B_{SR}$. We denote such set as $V_a$ (in Figures 2 and 3, the black-colored vertices).
- The set of $2b$ vertices of the cycles $C_{s_1}, C_{s_2}, \ldots, C_{s_b}$ which are diametral with $w$ induces a complete multipartite graph $K_{2,\ldots,2}$ with $b$ bipartition sets each of cardinality two in $B_{SR}$. We denote such set as $V_{2b}$ (in Figures 2 and 3, the red-colored vertices).
- The set of $2c$ vertices of the cycle $C_3$ different from $w$ induces a complete graph in $B_{SR}$. We denote such set as $V_{2c}$ (in Figures 2 and 3, the blue-colored vertices).
- The set of vertices of each odd cycle $C_{s_i}$, $i \in \{1,\ldots,b\}$, which are different from $w$ induces a path of order $s_i - 1$, in $B_{SR}$, whose leaves are the two vertices that are diametral with $w$.
- The set of vertices of each cycle $C_{r_j}$, $j \in \{1,\ldots,a\}$, which are not diametral with $w$ induces a graph isomorphic to the disjoint union of $(r_j - 2)/2$ complete graphs $K_2$ in $B_{SR}$.
- Every three vertices $x$, $y$, $z$ such that $x \in V_a$, $y \in V_{2b}$ and $z \in V_{2c}$ are pairwise adjacent.

Figure 3 shows the strong resolving graph of the graph given in Figure 2.

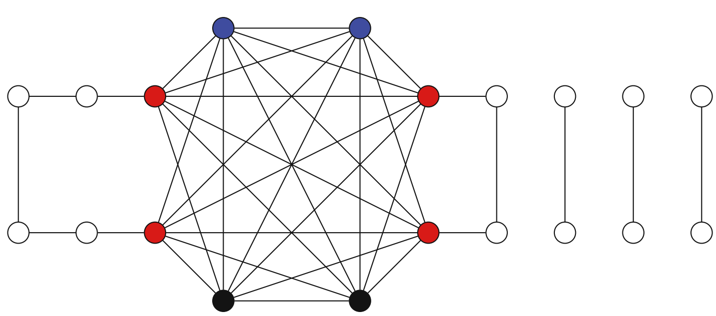

**Figure 3:** The strong resolving graph $B_{SR}$ of the graph given in Figure 2.

We are then able to present upper and lower bounds for the strong partition dimension of a bouquet of cycles.

**Proposition 18.** *For any bouquet of cycles $B \in \mathcal{B}_{a,b,c}$,*

$$a + b + 2c \le pd_s(B) \le a + b + 2c + 2.$$

**Proof.** According to the construction of the strong resolving graph $B_{SR}$, by using Theorem 4, the lower bound follows from the fact that the set $V_a \cup V_{2c}$ together with exactly one vertex from each bipartition set of the subgraph induced by $V_{2b}$ forms a clique in $B_{SR}$.

On the other hand, we consider the following vertex partition, denoted by $\Pi$, of $V(B)$.

- For every even cycle $C_{r_i}$, $i \in \{1,...,a\}$, we create a set $A_i = \{v\}$, where $v$ is the vertex diametral with $w$ in $C_{r_i}$.
- For every odd cycle $C_{s_i}$, $i \in \{1,...,b\}$, we create a set $B_i = \{u, v\}$, where $u, v$ are the vertices diametral with $w$ in $C_{s_i}$.
- For every cycle $C_3^j$ of order three, we create two sets $D_j$ and $F_j$, each of them containing one of the two vertices different from $w$.
- Finally, we create two sets $R$ and $S$ in the following way. If the vertex set of a cycle $C_n$ (even or odd, but not of order three) of $B$ is represented as $V(C_n) = \{w = w_0, w_1,...,w_{n-1}\}$ (with the natural style of adjacency between the vertices), then $R \cap V(C_n) = \{w_0, w_1, \ldots, w_{\lfloor n/2 \rfloor -1}\}$ and $S \cap V(C_n) = \{w_{\lceil n/2 \rceil +1}, w_{\lceil n/2 \rceil +2}, \ldots, w_{n-1}\}$.

We note that the partition $\Pi$ created in this way has cardinality $a + b + 2c + 2$. We need to show now that such $\Pi$ is a strong resolving partition for $B$. Let $x$, $y$ be any two vertices belonging to a same set of the partition. If $x, y \in B_i$ for some $i \in \{1,...,b\}$, then we easily observe that $x$, $y$ are strongly resolved by $R$ and by $S$. Assume now $x, y \in R$ for instance. If $x$, $y$ belong to a same cycle, say $C_{r_i}$ (even), then we note that the set $A_i$ strongly resolves the vertices $x$, $y$. Similar conclusion can be deduced, if $x$, $y$ belong to a cycle $C_{s_j}$ (odd). If $x$ and $y$ belong to two different cycles, then there are two sets, which can be of type $A_i$ or of type $B_i$ (depending on the parity of the cycles to which they belong), that strongly resolve the vertices $x$, $y$. Analogously, it can be deduced that any two vertices of $S$ are strongly resolved by a set of $\Pi$. Therefore, $\Pi$ is a strong resolving partition for $B$, and the upper bound follows. □

## References


[1] P. J. Slater, *Leaves of trees*, Congr. Numer. **14** (1975), 549–559.
[2] F. Harary and R. A. Melter, *On the metric dimension of a graph*, Ars Combin. **2** (1976), 191–195.
[3] A. Sebö and E. Tannier, *On metric generators of graphs*, Math. Oper. Res. **29** (2004), 383–393, DOI: 10.1287/moor.1030.0070.
[4] M. A. Johnson, *Structure–activity maps for visualizing the graph variables arising in drug design*, J. Biopharm. Statist. **3** (1993), 203–236, DOI: 10.1080/10543409308835060.
[5] M. A. Johnson, *Browsable structure–activity datasets*, in R. Carbó-Dorca and P. Mezey, (eds), Advances in Molecular Similarity, JAI Press, Connecticut, 1998, pp. 153–170.
[6] R. A. Melter and I. Tomescu, *Metric bases in digital geometry*, Comput. Gr. Image Process **25** (1984), 113–121, DOI: 10.1016/0734-189X(84)90051-3.
[7] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, *Resolvability in graphs and the metric dimension of a graph*, Discrete Appl. Math. **105** (2000), 99–113, DOI: 10.1016/S0166-218X(00)00198-0.
[8] B. L. Hulme, A. W. Shiver, and P. J. Slater, *A Boolean algebraic analysis of fire protection*, in R. E. Burkard, R. A. Cuninghame-Green, U. Zimmermann, (eds), Algebraic and Combinatorial Methods in Operations Research, North-Holland Mathematics Studies, Elsevier, vol. 95, 1984, pp. 215–227.
[9] S. Khuller, B. Raghavachari, and A. Rosenfeld, *Landmarks in graphs*, Discrete Appl. Math. **70** (1996), 217–229, DOI: 10.1016/0166-218X(95)00106-2.
[10] R. C. Brigham, G. Chartrand, R. D. Dutton, and P. Zhang, *Resolving domination in graphs*, Math. Bohem. **128** (2003), 25–36.
[11] G. Chartrand, C. Poisson, and P. Zhang, *Resolvability and the upper dimension of graphs*, Comput. Math. Appl. **39** (2000), 19–28, DOI: 10.1016/S0898-1221(00)00126-7.
[12] F. Okamoto, B. Phinezyn, and P. Zhang, *The local metric dimension of a graph*, Math. Bohem. **135** (2010), 239–255.
[13] A. Estrada-Moreno, J. A. Rodríguez-Velázquez, and I. G. Yero, *The k-metric dimension of a graph*, Appl. Math. Inf. Sci. **9** (2015), 2829–2840.
[14] A. Kelenc, N. Tratnik, and I.G. Yero, *Uniquely identifying the edges of a graph: the edge metric dimension*, Discrete Appl. Math. **251** (2018), 204–220, DOI: 10.1016/j.dam.2018.05.052.
[15] A. Kelenc, D. Kuziak, A. Taranenko, and I. G. Yero, *Mixed metric dimension of graphs*, Appl. Math. Comput. **314** (2017), 429–438, DOI: 10.1016/j.amc.2017.07.027.
[16] R. Trujillo-Rasúa and I. G. Yero, *k-metric antidimension: a privacy measure for social graphs*, Inform. Sci. **328** (2016), 403–417, DOI: 10.1016/j.ins.2015.08.048.
[17] R. Gil-Pons, Y. Ramírez-Cruz, R. Trujillo-Rasua, and I. G. Yero, *Distance-based vertex identification in graphs: the outer multiset dimension*, Appl. Math. Comput. **363** (2019), 124612, DOI: 10.1016/j.amc.2019.124612.

[18] G. Chartrand, F. Okamoto, and P. Zhang, *The metric chromatic number of a graph*, Australasian J. Combin. **44** (2009), 273–286.

[19] G. Chartrand, E. Salehi, and P. Zhang, *The partition dimension of a graph*, Aequationes Math. **59** (2000), 45–54, DOI: 10.1007/PL00000127.

[20] I. G. Yero, *On the strong partition dimension of graphs*, Electron. J. Combin. **21** (2014), P3.14, DOI: 10.37236/3474.

[21] O. R. Oellermann and J. Peters-Fransen, *The strong metric dimension of graphs and digraphs*, Discrete Appl. Math. **155** (2007), 356–364, DOI: 10.1016/j.dam.2006.06.009.

[22] I. G. Yero, *Strong resolving partitions for strong product graphs and Cartesian product graphs*, Discrete Appl. Math. **202** (2016), 70–78, DOI: 10.1016/j.dam.2015.08.026.

[23] D. Kuziak, M. L. Puertas, J. A. Rodríguez-Velázquez, and I. G. Yero, *Strong resolving graphs: the realization and the characterization problems*, Discrete Appl. Math. **236** (2018), 270–287, DOI: 10.1016/j.dam.2017.11.013.

[24] S. Klavžar and I. G. Yero, *The general position problem and strong resolving graphs*, Open Math. **17** (2019), 1126–1135, DOI: 10.1515/math-2019-0088.

[25] C. Grigorious, S. Stephen, B. Rajan, M. Miller, and P. Manuel, *On the partition dimension of circulant graphs*, Inform. Process. Lett. **114** (2014), 353–356, DOI: 10.1016/j.ipl.2014.02.005.