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Further on Fuzzy Pseudo Near Compactness and *ps-ro* Fuzzy Continuous Functions

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Abstract

Main objective of this paper is to study further properties of fuzzy pseudo near compactness via *ps-ro* closed fuzzy sets, fuzzy nets and fuzzy filterbases. It is shown by an example that *ps-ro* fuzzy continuous and fuzzy continuous functions do not imply each other. Several characterizations of *ps-ro* fuzzy continuous function are obtained in terms of a newly introduced concept of *ps-ro* interior operator, *ps-ro q-nbd* and its graph.

Keywords: Fuzzy pseudo near compactness, fuzzy net, fuzzy filterbase, *ps-ro* interior, *ps-ro q-nbd*.
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1. Introduction

In (Ray & Chettri, 2010), while finding interplay between a fuzzy topological space (fts, for short) (X, τ) and its corresponding strong α-level topology(general) on X, the concept of pseudo regular open(closed) fuzzy sets and *ps-ro* fuzzy topology on X was introduced, members of which are called *ps-ro* open fuzzy sets and their complements are *ps-ro* closed fuzzy sets on (X, τ). In (Ray & Chettri, 2011), in terms of above fuzzy sets, a fuzzy continuous type function called *ps-ro* fuzzy continuous function and a compact type notion called fuzzy pseudo near compactness were introduced and different properties were studied.

In this paper, fuzzy pseudo near compactness has been studied via *ps-ro* closed fuzzy sets, fuzzy nets and fuzzy filterbases. Further, it is shown by an example that *ps-ro* fuzzy continuous and fuzzy continuous functions are independent of each other. An interior-type operator called *ps-ro* interior is introduced and several properties of such functions are studied interms of this operator, *ps-ro q-nbd* and its graph.

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We state a few known definitions and results here that we require subsequently. A fuzzy point x_α is said to q-coincident with a fuzzy set A, denoted by $x_\alpha qA$ if $\alpha + A(x) > 1$. If A and B are not q-coincident, we write $A \not{q} B$. A fuzzy set A is said to be a q-neighbourhood (in short, q-*nbd*.) of a fuzzy point x_α if there is a fuzzy open set B such that $x_\alpha qB \leqslant A$ (Pao-Ming & Ying-Ming, 1980). Let f be a function from a set X into a set Y. Then the following holds:
(i) $f^{-1}(1 - B) = 1 - f^{-1}(B)$, for any fuzzy set B on Y.
(ii) $A_1 \leqslant A_2 \Rightarrow f(A_1) \leqslant f(A_2)$, for any fuzzy sets A_1 and A_2 on X. Also, $B_1 \leqslant B_2 \Rightarrow f^{-1}(B_1) \leqslant f^{-1}(B_2)$, for any fuzzy sets B_1 and B_2 on Y.
(iii) $ff^{-1}(B) \leqslant B$, for any fuzzy set B on Y and the equality holds if f is onto. Also, $f^{-1}f(A) \geqslant A$, for any fuzzy set A on X, equality holds if f is one-to-one (Chang, 1968). For a function $f : X \to Y$, the graph $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$, for each $x \in X$, where X and Y are any sets. Let X, Y be fts and $g : X \to X \times Y$ be the graph of the function $f : X \to Y$. Then if A, B are fuzzy sets on X and Y respectively, $g^{-1}(A \times B) = A \wedge f^{-1}(B)$(Azad, 1981). Let Z, X, Y be fts and $f_1 : Z \to X$ and $f_2 : Z \to Y$ be two functions. Let $f : Z \to X \times Y$ be defined by $f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1 U_2 are fuzzy sets on Z, X, Y respectively such that $f(B) \leqslant U_1 \times U_2$, then $f_1(B) \leqslant U_1$ and $f_2(B) \leqslant U_2$ (Bhattacharyya & Mukherjee, 2000). A function f from a fts (X, τ) to fts (Y, σ) is said to be fuzzy continuous, if $f^{-1}(\mu)$ is fuzzy open on X, for all fuzzy open set μ on Y (Chang, 1968). For a fuzzy set μ in X, the set $\mu^\alpha = \{x \in X : \mu(x) > \alpha\}$ is called the strong α-level set of X. In a fts (X, τ), the family $i_\alpha(\tau) = \{\mu^\alpha : \mu \in \tau\}$ for all $\alpha \in I_1 = [0, 1)$ forms a topology on X called strong α-level topology on X (Lowen, 1976), (Kohli & Prasannan, 2001). A fuzzy open(closed) set μ on a fts (X, τ) is said to be pseudo regular open(closed) fuzzy set if the strong α-level set μ^α is regular open(closed) in $(X, i_\alpha(\tau)), \forall \alpha \in I_1$. The family of all pseudo regular open fuzzy sets form a fuzzy topology on X called *ps-ro* fuzzy topology on X which is coarser than τ. Members of *ps-ro* fuzzy topology are called *ps-ro* open fuzzy sets and their complements are known as *ps-ro* closed fuzzy sets on (X, τ) (Ray & Chettri, 2010). A function f from a fts (X, τ_1) to another fts (Y, τ_2) is pseudo fuzzy *ro* continuous (in short, *ps-ro* fuzzy continuous) if $f^{-1}(U)$ is *ps-ro* open fuzzy set on X for each pseudo regular open fuzzy set U on Y. For a fuzzy set A, $\wedge\{B : A \leqslant B,\ B$ is *ps-ro* closed fuzzy set on $X\}$ is called fuzzy *ps*-closure of A. In a fts (X, τ), a fuzzy set A is said to be a *ps-ro nbd*. of a fuzzy point x_α, if there is a *ps-ro* open fuzzy set B such that $x_\alpha \in B \leqslant A$. In addition, if A is *ps-ro* open fuzzy set, the *ps-ro nbd*. is called *ps-ro* open *nbd*. A fuzzy set A is called *ps-ro* quasi neighborhood or simply *ps-ro q-nbd*. of a fuzzy point x_α, if there is a *ps-ro* open fuzzy set B such that $x_\alpha qB \leqslant A$. In addition, if A is *ps-ro* open, the *ps-ro q-nbd*. is called *ps-ro* open *q-nbd*. Let $\{S_n : n \in D\}$ be a fuzzy net on a fts X. i.e., for each member n of a directed set $(D, \leqslant)$, S_n be a fuzzy set on X. A fuzzy point x_α on X is said to be a fuzzy *ps*-cluster point of the fuzzy net if for every $n \in D$ and every *ps-ro* open *q-nbd*. V of x_α, there exists $m \in D$, with $n \leqslant m$ such that $S_m qV$. A collection $\mathcal{B}$ of fuzzy sets on a fts (X, τ) is said to form a fuzzy filter base in X if for every finite subcollection $\{B_1, B_2, ..., B_n\}$ of $\mathcal{B}$, $\bigwedge_{i=1}^n B_i \neq 0$ (Ray & Chettri, 2011).

2. Fuzzy Pseudo Near Compactness

It is easy to observe, as pseudo regular open fuzzy sets form a base for *ps-ro* fuzzy topology, replacing *ps-ro* open cover by pseudo regular open cover, we may obtain pseudo near compactness.

Definition 2.1. Let x_α be a fuzzy point on a fts X. A fuzzy net $\{S_n : n \in (D, \geqslant)\}$ on X is said to ps-converge to x_α, written as $S_n \xrightarrow{ps} x_\alpha$ if for each *ps-ro* open *q-nbd*. W of x_α, there exists $m \in D$ such that $S_n q W$ for all $n \geqslant m$,$(n \in D)$.

Definition 2.2. Let x_α be a fuzzy point on a fts X. A fuzzy filterbase $\mathcal{B}$ is said to
(i) ps-adhere at x_α written as $x_\alpha \leqslant ps\text{-}ad.\mathcal{B}$ if for each *ps-ro* open *q-nbd*. U of x_α and each $B \in \mathcal{B}$, BqU.
(ii) ps-converge to x_α, written as $\mathcal{B} \xrightarrow{ps} x_\alpha$ if for each *ps-ro* open *q-nbd*. U of x_α, there coresponds some $B \in \mathcal{B}$ such that $B \leqslant U$.

Theorem 2.1. A fts (X, τ) is fuzzy pseudo nearly compact iff every $\{B_\alpha : \alpha \in \Lambda\}$ of *ps-ro* closed fuzzy sets on X with $\wedge_{\alpha\in\Lambda} B_\alpha = 0$, there exist a finite subset Λ_0 of Λ such that $\wedge_{\alpha\in\Lambda_0} B_\alpha = 0$.
Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a *ps-ro* open cover of X. Now, $\wedge_{\alpha\in\Lambda}(1 - U_\alpha) = (1 - \vee_{\alpha\in\Lambda} U_\alpha) = 0$. As $\{1 - U_\alpha : \alpha \in \Lambda\}$ is a collection of *ps-ro* closed fuzzy sets on X, by given condition, there exist a finite subset Λ_0 of Λ such that $\wedge_{\alpha\in\Lambda_0}(1 - U_\alpha) = 0 \Rightarrow 1 - \vee_{\alpha\in\Lambda_0} U_\alpha = 0$. i.e., $1 = \vee_{\alpha\in\Lambda_0} U_\alpha$. So, X is fuzzy pseudo nearly compact.
Conversely, let $\{B_\alpha : \alpha \in \Lambda\}$ be a family of *ps-ro* closed fuzzy sets on X with $\wedge_{\alpha\in\Lambda} B_\alpha = 0$. Then $1 = 1 - \wedge_{\alpha\in\Lambda} B_\alpha \Rightarrow 1 = \vee_{\alpha\in\Lambda}(1 - B_\alpha)$. By given condition there exist a finite subset Λ_0 of Λ such that $1 = \vee_{\alpha\in\Lambda_0}(1 - B_\alpha) \Rightarrow 1 = (1 - \wedge_{\alpha\in\Lambda_0} B_\alpha)$. Hence, $\wedge_{\alpha\in\Lambda_0} B_\alpha \leqslant (\wedge_{\alpha\in\Lambda_0} B_\alpha) \wedge (1 - \wedge_{\alpha\in\Lambda_0} B_\alpha) = 0$. Consequently, $\wedge_{\alpha\in\Lambda_0} B_\alpha = 0$.

Theorem 2.2. For a fuzzy set A on a fts, the following are equivalent:
(a) Every fuzzy net in A has fuzzy ps-cluster point in A.
(b) Every fuzzy net in A has a ps-convergent fuzzy subnet.
(c) Every fuzzy filterbase in A ps-adheres at some fuzzy point in A.
Proof. $(a) \Rightarrow (b)$**:** Let $\{S_n : n \in (D, \geqslant)\}$ be a fuzzy net in A having fuzzy ps-cluster point at $x_\alpha \leqslant A$. Let $Q_{x_\alpha} = \{A : A$ is *ps-ro* open *q-nbd*. of $x_\alpha\}$. For any $B \in Q_{x_\alpha}$, some $n \in D$ can be chosen such that $S_n qB$. Let E denote the set of all ordered pairs (n, B) with the property that $n \in D$, $B \in Q_{x_\alpha}$ and $S_n qB$. Then $(E, \succ)$ is a directed set where $(m, C) \succ (n, B)$ iff $m \geqslant n$ in D and $C \leqslant B$. Then $T : (E \succ) \to (X, \tau)$ given by $T(n, B) = S_n$, is a fuzzy subnet of $\{S_n : n \in (D, \geqslant)\}$. Let V be any *ps-ro* open *q-nbd*. of x_α. Then there exists $n \in D$ such that $(n, V) \in E$ and hence $S_n qV$. Now, for any $(m, U) \succ (n, V)$, $T(m, U) = S_m qU \leqslant V \Rightarrow T(m, U)qV$. Hence, $T \xrightarrow{ps} x_\alpha$.
$(b) \Rightarrow (a)$ If a fuzzy net $\{S_n : n \in (D, \geqslant)\}$ in A does not have any fuzzy ps-cluster point, then there is a *ps-ro* open *q-nbd*. U of X_α and $n \in D$ such that $S_n \not{q} U, \forall\, m \geqslant n$. Then clearly no fuzzy subnet of the fuzzy net can ps-converge to x_α.
$(c) \Rightarrow (a)$ Let $\{S_n : n \in (D, \geqslant)\}$ be a fuzzy net in A. Consider the fuzzy filter base $\mathcal{F} = \{T_n : n \in D\}$ in A, generated by the fuzzy net, where $T_n = \{S_m : m \in (D, \geqslant)$ and $m \geqslant n\}$. By (c), there exist a fuzzy point $a_\alpha \leqslant A \wedge (ps\text{-}ad\mathcal{F})$. Then for each *ps-ro* open *q-nbd*. U of a_α and each $F \in \mathcal{F}$, UqF, i.e., $UqT_n, \forall\, n \in D$. Hence, the given fuzzy net has fuzzy ps-cluster point a_α.
$(a) \Rightarrow (c)$ Let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a fuzzy filterbase in A. For each $\alpha \in \Lambda$, choose a fuzzy point $x_{F_\alpha} \leqslant F_\alpha$, and construct the fuzzy net $S = \{x_{F_\alpha} : F_\alpha \in \mathcal{F}\}$ in A with $(\mathcal{F}, \succ\succ)$ as domain, where for two members $F_\alpha, F_\beta \in \mathcal{F}$, $F_\alpha \succ\succ F_\beta$ iff $F_\alpha \leqslant F_\beta$. By (a), the fuzzy net has a fuzzy ps-cluster

point say $x_t \leqslant A$, where $0 < t \leqslant 1$. Then for any ps-ro open q-nbd. U of x_t and any $F_\alpha \in \mathcal{F}$, there exists $F_\beta \in \mathcal{F}$ such that $F_\beta >> F_\alpha$ and $x_{F_\beta} qU$. Then $F_\beta qU$ and hence $F_\alpha qU$. Thus $\mathcal{F}$ adheres at x_t.

Theorem 2.3. If a fts is fuzzy pseudo nearly compact, then every fuzzy filterbase on X with at most one ps-adherent point is ps-convergent.

Proof. Let $\mathcal{F}$ be a fuzzy filterbase with at most one ps-adherent point in a fuzzy pseudo nearly compact fts X. Then by Theorem (2.2), $\mathcal{F}$ has at least one ps-adherent point. Let x_α be the unique ps-adherent point of $\mathcal{F}$. If $\mathcal{F}$ does not ps-converge to x_α, then there is some ps-ro open q-nbd. U of x_α such that for each $F \in \mathcal{F}$ with $F \leqslant U$, $F \wedge (1-U) \neq 0$. Then $\mathcal{G} = \{F \wedge (1-U) : F \in \mathcal{F}\}$ is a fuzzy filterbase on X and hence has a ps-adherent point y_t(say) in X. Now, $U \not{q} G$, for all $G \in \mathcal{G}$, so that $x_\alpha \neq y_t$. Again, for each ps-ro open q-nbd. V of y_t and each $F \in \mathcal{F}$, $Vq(F \wedge (1-U)) \Rightarrow VqF \Rightarrow y_t$ is a ps-adherent point of $\mathcal{F}$, where $x_\alpha \neq y_t$. This shows that y_t is another ps-adherent point of $\mathcal{F}$, which is not the case.

3. *ps-ro* Fuzzy Continuous Functions

We begin this section by introducing an interior-type operator, called ps-interior operator and observe a few useful properties of that operator.

Definition 3.1. The union of all ps-ro open fuzzy sets, each contained in a fuzzy set A on a fts X is called fuzzy ps-interior of A and is denoted by ps-$int(A)$. So, ps-$int(A) = \vee\{B : B \leqslant A,\ B$ is ps-ro open fuzzy set on $X\}$

Some properties of ps-int operator are furnished below. The proofs are straightforward and hence omitted.

Theorem 3.1. For any fuzzy set A on a fts (X, τ), the following hold:
(a) ps-$int(A)$ is the largest ps-ro open fuzzy set contained in A.
(b) ps-$int(0) = 0$, ps-$int(1) = 1$.
(c) ps-$int(A) \leqslant A$.
(d) A is ps-ro open fuzzy set iff $A = ps$-$int(A)$.
(e) ps-$int(ps$-$int(A)) = ps$-$int(A)$.
(f) ps-$int(A) \leqslant ps$-$int(B), if A \leqslant B$.
(g) ps-$int(A \wedge B) = ps$-$int(A) \wedge ps$-$int(B)$.
(h) ps-$int(A \vee B) \geqslant ps$-$int(A) \vee ps$-$int(B)$.
(i) ps-$int(ps$-$int(A)) = ps$-$int(A)$.
(j) $1 - ps$-$int(A) = ps$-$cl(1 - A)$.
(k) $1 - ps$-$cl(A) = ps$-$int(1 - A)$.

Now, we recapitulate the definition of ps-ro fuzzy continuous functions.

Definition 3.2. A function f from fts (X, τ_1) to fts (Y, τ_2) is pseudo fuzzy ro continuous (in short, ps-ro fuzzy continuous) if $f^{-1}(U)$ is ps-ro open fuzzy set on X for each pseudo regular open fuzzy set U on Y.

The following Example shows that *ps-ro* fuzzy continuity and fuzzy continuity do not imply each other.

Example 3.1. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let A, B and C be fuzzy sets on X defined by $A(a) = 0.2, A(b) = 0.4, A(c) = 0.4$, $B(t) = 0.4, \forall t \in X$ and $C(t) = 0.2, \forall t \in X$. Let D and E be fuzzy sets on Y defined by $D(t) = 0.2, \forall t \in Y$ and $E(x) = 0.6, E(y) = 0.7, E(z) = 0.7$. Clearly, $\tau_1 = \{0, 1, A, B, C\}$ and $\tau_2 = \{0, 1, D, E\}$ are fuzzy topologies on X and Y respectively. In the corresponding topological space $(X, i_\alpha(\tau_1)), \forall \alpha \in I_1 = [0, 1)$, the open sets are $\phi, X, A^\alpha, B^\alpha$ and C^α, where $A^\alpha = \begin{cases} X, & \text{for } \alpha < 0.2 \\ \{b, c\}, & \text{for } 0.2 \leqslant \alpha < 0.4 \\ \phi, & \text{for } \alpha \geqslant 0.4 \end{cases}$, $B^\alpha = \begin{cases} X, & \text{for } \alpha < 0.4 \\ \phi, & \text{for } \alpha \geqslant 0.4 \end{cases}$ and $C^\alpha = \begin{cases} X, & \text{for } \alpha < 0.2 \\ \phi, & \text{for } \alpha \geqslant 0.2 \end{cases}$

For $0.2 \leqslant \alpha < 0.4$, the closed sets are on $(X, i_\alpha(\tau_1))$ are ϕ, X and $\{a\}$. Therefore, $int(cl(A^\alpha)) = X$. So, A^α is not regular open on $(X, i_\alpha(\tau_1))$ and hence, A is not pseudo regular open fuzzy sets on (X, τ_1) for $0.2 \leqslant \alpha < 0.4$. Similarly, it can be seen that $0, 1, B$ and C are pseudo regular open fuzzy set on (X, τ_1). Therefore, *ps-ro* fuzzy topology on X is $\{0, 1, B, C\}$. Again, E is not pseudo regular open fuzzy set for $0.6 \leqslant \alpha < 0.7$ on Y. Therefore, *ps-ro* fuzzy topology on Y is $\{0, 1, D\}$. Now, $ps\text{-}cl(B) = 1 - B$ and $ps\text{-}cl(C) = 1 - B$ where, $(1 - B)(t) = 0.6, \forall t \in X$. Define a function $f : X \to Y$ by $f(a) = x, f(b) = y$ and $f(c) = z$. Then, $f^{-1}(D)(t) = 0.2 = C(t), \forall t \in X$. Hence, $f^{-1}(U)$ is *ps-ro* open fuzzy set on X, for every *ps-ro* open fuzzy set U on Y. Therefore, f is *ps-ro* fuzzy continuous function. But, f is not fuzzy continuous as $f^{-1}(E)$ is not fuzzy open on X. Clearly, every *ps-ro* open fuzzy set is fuzzy open but not conversely, as for an example here A is fuzzy open but not *ps-ro* open fuzzy on X. This implies that a fuzzy continuous function need not be *ps-ro* fuzzy continuous. Hence, *ps-ro* fuzzy continuous and fuzzy continuous functions are independent of each other.

The following couple of results give characterizations of *ps-ro* fuzzy continuous functions.

Theorem 3.2. Let (X, τ) and (Y, σ) be two fts. For a function $f : X \to Y$, the following are equivalent:
(a) f is *ps-ro* fuzzy continuous.
(b) Inverse image of each *ps-ro* open fuzzy set on Y under f is *ps-ro* open on X.
(c) For each fuzzy point x_α on X and each *ps-ro* open *nbd*. V of $f(x_\alpha)$, there exists a *ps-ro* open fuzzy set U on X, such that $x_\alpha \leqslant U$ and $f(U) \leqslant V$.
(d) For each *ps-ro* closed fuzzy set F on Y, $f^{-1}(F)$ is *ps-ro* closed on X.
(e) For each fuzzy point x_α on X, the inverse image under f of every *ps-ro nbd*. of $f(x_\alpha)$ on Y is a *ps-ro nbd*. of x_α on X.
(f) For all fuzzy set A on X, $f(ps\text{-}cl(A)) \leqslant ps\text{-}cl(f(A))$.
(g) For all fuzzy set B on Y, $ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(ps\text{-}cl(B))$.
(h) For all fuzzy set B on Y, $f^{-1}(ps\text{-}int(B)) \leqslant ps\text{-}int(f^{-1}(B))$.

Proof. $(a) \Rightarrow (b)$ Let f be *ps-ro* fuzzy continuous and μ be any *ps-ro* open fuzzy set on Y. Then $\mu = \vee \mu_i$, where μ_i is pseudo regular open fuzzy set on Y, for each i. Now, $f^{-1}(\mu) = f^{-1}(\vee_i \mu_i) = \vee_i f^{-1}(\mu_i)$. f being *ps-ro* fuzzy continuous, $f^{-1}(\mu_i)$ is *ps-ro* open fuzzy set and consequently,

$f^{-1}(\mu)$ is *ps-ro* open fuzzy set on X.
$(b) \Rightarrow (a)$ Let the inverse image of each *ps-ro* open fuzzy set on Y under f be *ps-ro* open fuzzy set on X. Let U be a pseudo regular open fuzzy set on Y. Every pseudo regular open fuzzy set being *ps-ro* open fuzzy set, the result follows.
$(b) \Rightarrow (c)$ Let V be any *ps-ro* open *nbd*. of $f(x_\alpha)$ on Y. Then there is a *ps-ro* open fuzzy set V_1 on Y such that $f(x_\alpha) \leqslant V_1 \leqslant V$. By hypothesis, $f^{-1}(V_1)$ is *ps-ro* open fuzzy set on X. Again, $x_\alpha \leqslant f^{-1}(V_1) \leqslant f^{-1}(V)$. So, $f^{-1}(V)$ is a *ps-ro nbd*. of x_α, such that $f(f^{-1}(V)) \leqslant V$, as desired.
$(c) \Rightarrow (b)$ Let V be any *ps-ro* open fuzzy set on Y and $x_\alpha \leqslant f^{-1}(V)$. Then $f(x_\alpha) \leqslant V$ and so by given condition, there exists *ps-ro* open fuzzy set U on X such that $x_\alpha \leqslant U$ and $f(U) \leqslant V$. Hence, $x_\alpha \leqslant U \leqslant f^{-1}(V)$. i.e., $f^{-1}(V)$ is a *ps-ro nbd*. of each of the fuzzy points contained in it. Thus $f^{-1}(V)$ is *ps-ro* open fuzzy set on X.
$(b) \Leftrightarrow (d)$ Obvious.
$(b) \Rightarrow (e)$ Suppose, W is a *ps-ro* open *nbd*. of $f(x_\alpha)$. Then there exists a *ps-ro* open fuzzy set U on Y such that $f(x_\alpha) \leqslant U \leqslant W$. Then $x_\alpha \leqslant f^{-1}(U) \leqslant f^{-1}(W)$. By hypothesis, $f^{-1}(U)$ is *ps-ro* open fuzzy set on X and hence the result is obtained.
$(e) \Rightarrow (b)$ Let V be any *ps-ro* open fuzzy set on Y. If $x_\alpha \leqslant f^{-1}(V)$ then $f(x_\alpha) \leqslant V$ and so $f^{-1}(V)$ is a *ps-ro nbd*. of x_α.
$(d) \Rightarrow (f)$ *ps-cl*$(f(A))$ being a *ps-ro* closed fuzzy set on Y, $f^{-1}($*ps-cl*$(f(A)))$ is *ps-ro* closed fuzzy set on X. Again, $f(A) \leqslant$ *ps-cl*$(f(A))$. So, $A \leqslant f^{-1}($*ps-cl*$(f(A)))$. As *ps-cl*(A) is the smallest *ps-ro* closed fuzzy set on X containing A, *ps-cl*$(A) \leqslant f^{-1}($*ps-cl*$(f(A)))$. Hence, $f($*ps-cl*$(A)) \leqslant ff^{-1}($*ps-cl*$(f(A))) \leqslant$ *ps-cl*$(f(A))$.
$(f) \Rightarrow (d)$ For any *ps-ro* closed fuzzy set B on Y, $f($*ps-cl*$(f^{-1}(B))) \leqslant$ *ps-cl*$(f(f^{-1}(B))) \leqslant$ *ps-cl*$(B) = B$. Hence, *ps-cl*$(f^{-1}(B)) \leqslant f^{-1}(B) \leqslant$ *ps-cl*$(f^{-1}(B))$. Thus, $f^{-1}(B)$ is *ps-ro* closed fuzzy set on X.
$(f) \Rightarrow (g)$ For any fuzzy set B on Y, $f($*ps-cl*$(f^{-1}(B))) \leqslant$ *ps-cl*$(f(f^{-1}(B))) \leqslant$ *ps-cl*(B). Hence, *ps-cl*$(f^{-1}(B)) \leqslant f^{-1}($*ps-cl*$(B))$.
$(g) \Rightarrow (f)$ Let $B = f(A)$ for some fuzzy set A on X. Then *ps-cl*$(f^{-1}(B)) \leqslant f^{-1}($*ps-cl*$(B)) \Rightarrow$ *ps-cl*$(A) \leqslant$ *ps-cl*$(f^{-1}(B)) \leqslant f^{-1}($*ps-cl*$(f(A)))$. So, $f($*ps-cl*$(A)) \leqslant$ *ps-cl*$(f(A))$.
$(b) \Rightarrow (h)$ For any fuzzy set B on Y, $f^{-1}($*ps-int*$(B))$ is *ps-ro* open fuzzy set on X. Also, $f^{-1}($*ps-int*$(B)) \leqslant f^{-1}(B)$. So, $f^{-1}($*ps-int*$(B)) \leqslant$ *ps-int*$(f^{-1}(B))$.
$(h) \Rightarrow (b)$ Let B be any *ps-ro* open fuzzy set on Y. So, *ps-int*$(B) = B$. Now, $f^{-1}($*ps-int*$(B)) \leqslant$ *ps-int*$(f^{-1}(B)) \Rightarrow f^{-1}(B) \leqslant$ *ps-int*$(f^{-1}(B)) \leqslant f^{-1}(B)$. Hence, $f^{-1}(B)$ is *ps-ro* open fuzzy set on X.

Theorem 3.3. Let (X, τ) and (Y, σ) be two fts. A function $f : X \to Y$ is f is *ps-ro* fuzzy continuous iff for every fuzzy point x_α on X and every *ps-ro* open fuzzy set V on Y with $f(x_\alpha)qV$ there exists a *ps-ro* open fuzzy set U on X with $x_\alpha qU$ and $f(U) \leqslant V$.
Proof. Let f be *ps-ro* fuzzy continuous and x_α a fuzzy point on X, V a *ps-ro* open fuzzy set V on Y with $f(x_\alpha)qV$. So, $V(f(x)) + \alpha > 1 \Rightarrow f^{-1}(V)(x) + \alpha > 1$. So, $x_\alpha q(f^{-1}(V))$. Now, $ff^{-1}(V) \leqslant V$ is always true. Choosing $U = f^{-1}(V)$ we have, $f(U) \leqslant V$ with $x_\alpha qU$.
Conversely, let the condition hold. Let V be any *ps-ro* open fuzzy set on Y. To prove $f^{-1}(V)$ is *ps-ro* open fuzzy set on X, we shall prove $1 - f^{-1}(V)$ is *ps-ro* closed fuzzy set on X. Let x_α be any fuzzy point on X such that $x_\alpha > 1_X - f^{-1}(V)$. So, $(1 - f^{-1}(V))(x) < \alpha \Rightarrow V(f(x)) + \alpha > 1$. So,

$f(x_\alpha)qV$. By given condition, there exists a *ps-ro* open fuzzy set on U such that $x_\alpha qU$ and $f(U) \leqslant V$. Now, $U(t) + (1 - f^{-1}(V))(t) \leqslant V(f(t)) + 1 - V(f(t)) = 1, \forall\, t$. Hence, $U \not{q}(1 - f^{-1}(V))$. Consequently, x_α is not a fuzzy *ps*-cluster point of $1 - f^{-1}(V)$. This proves $1 - f^{-1}(V)$ is a *ps-ro* closed fuzzy set on X

Theorem 3.4. Let X, Y, Z be *fts*. For any functions $f_1 : Z \to X$ and $f_2 : Z \to Y$, a function $f : Z \to X \times Y$ is defined as $f(x) = (f_1(x), f_2(x))$ for $x \in Z$, where $X \times Y$ is endowded with the product fuzzy topology. If f is *ps-ro* fuzzy continuous then f_1 and f_2 are both *ps-ro* fuzzy continuous.

Proof. Let U_1 be a *ps-ro q-nbd*. of $f_1(x_\alpha)$ on X, for any fuzzy point x_α on Z. Then $U_1 \times 1_Y$ is a *ps-ro q-nbd*. of $f(x_\alpha) = (f_1(x_\alpha), f_2(x_\alpha))$ on $X \times Y$. By *ps-ro* continuity of f, there exists *ps-ro q-nbd*. V of x_α on Z such that $f(V) \leqslant U_1 \times 1_Y$. Then $f(V)(t) \leqslant (U_1 \times 1_Y)(t) = U_1(t) \wedge 1_Y(t) = U_1(t), \forall\, t \in Z$. So, $f_1(V) \leqslant U_1$. Hence, f_1 is *ps-ro* fuzzy continuous. Similarly, it can be shown that f_2 is also *ps-ro* fuzzy continuous.

Theorem 3.5. Let $f : X \to Y$ be a function from a *fts* X to another *fts* Y and $g : X \to X \times Y$ be the graph of the function f. Then f is *ps-ro* fuzzy continuous if g is so.

Proof. Let g be *ps-ro* fuzzy continuous and B be *ps-ro* open fuzzy set on Y. By Lemma 2.4 of (Azad, 1981), $f^{-1}(B) = 1_X \wedge f^{-1}(B) = g^{-1}(1_X \times B)$. Now, as $1_X \times B$ is *ps-ro* open fuzzy set on $X \times Y$, $f^{-1}(B)$ becomes *ps-ro* open fuzzy set on X. Hence, f is *ps-ro* fuzzy continuous.

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