retests will indicate that all the batches in the pool are acceptable and that the retesting was not really needed. The criterion for retesting can be raised above $1/n$-th the limit of acceptability at the cost of a relatively small risk of accepting overly impure batches. The probability of failing to detect a defective batch when the retest criterion is raised in this manner will depend upon the form and parameters of the distribution of imperfection in single batches, as well as upon the number of batches in the pool. No simple general solution for this problem has been found.

FURTHER POINTS ON MATRIX CALCULATION AND SIMULTANEOUS EQUATIONS

By Harold Hotelling

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Since the publication of "Some new methods in matrix calculation" in the *Annals of Mathematical Statistics* (March, 1943, pp. 1–34), the following relevant points have come to the attention of the author.

A. T. Lonseth has improved substantially the limit of error for the efficient method of inverting a matrix described on p. 14. He writes:

> "Your use of the 'norm' of a matrix in the *Annals* paper especially interests me, as I was recently led to use it in solving the errors problem for infinite linear systems which are equivalent to Fredholm-type integral equations.
>
> "It is possible to replace the term $p^{\frac{1}{2}}$ in your inequality (7.5) by one, so that
>
> $$N(C_m - A^{-1}) \leqq N(C_0)k^{2^m}/(1 - k).$$
>
> To see this, one observes that from the developments on the bottom of p. 13 it follows that $(I - D)^{-1} = I + D^*$, where $N(D^*) < k/(1 - k)$. Then
>
> $$C_0(I - D)^{-1} = C_0 + C_0D^*$$
>
> so that
>
> $$N[C_0(I - D)^{-1}] \leqq N(C_0) + N(C_0)\, N(D^*) = N(C_0)\{1 + N(D^*)\},$$
>
> from which the result stated is seen to follow. I happen to have noticed this because the same thing has cropped up often in my recent work, and for the infinite case a bound $p^{\frac{1}{2}}$ is no bound at all.
>
> "Your paper has suggested improvements in my own proofs, for which I am grateful."

Dr. Lonseth's first formula above might well be written at the bottom of p. 14 of my article as a substitute for (7.5). It both simplifies and reduces the limit of error.

A method of solving normal equations by iteration, in which trial values of the unknown regression coefficients were applied to the values of the predictors and predictand in each of the N cases, and the results were used to improve the trial values, was orally suggested by John C. Flanagan in 1934. The plan involved the use of punched cards for the N substitutions in the trial regression equation at each stage. However, it seemed on further consideration and discussion that this would involve an unnecessarily large amount of work, since other methods require only as many substitutions at each stage as the number of unknowns, which is always less than N and usually very much less. I believe that Dr. Flanagan thereupon abandoned this plan and never published it.

Louis I. Guttman has proposed a similar method,[1] and has provided a proof of convergence in certain cases. In a final section he shows that the method can be modified by applying the same type of iterations to the normal, or product-sum, matrix instead of to the matrix of observations. This modification avoids the difficulty mentioned above. It is stated that one of these methods has been applied to a 64-variable problem.

The first method of section 10 of my paper for solving sets of linear equations is equivalent, in the case of normal equations, to the second method of Dr. Guttman. It is regrettable that reference to his study was omitted.

R. D. Gordon believes that the inequalities for principal components obtained at the end of the paper can be improved, but his entry into the army has prevented his fully working out his ideas. Paul A. Samuelson has some new and as yet unpublished ideas relating to calculation of principal components.

Merrill M. Flood, in "A computational procedure for the method of principal components," *Psychometrika*, Vol. 5 (1940), pp. 169–172, presents a method which appears to have considerable value, in that the number of vector multiplications is relatively small. However it requires solution of a system of $p - 1$ linear equations for each latent vector determined, and also of an additional such system. The relative value of this and other methods may depend on the relative costs of vector multiplication and of solving systems of linear equations. This in turn depends on the mechanical facilities available.

Paul Horst's paper, "A method for determining the coefficients of a characteristic equation" (*Annals of Mathematical Statistics*, Vol. 6 (1935), pp. 83–84) should have been referred to in connection with sections 11 and 12.

On p. 23 of "Some new methods in matrix calculation," in the sixth line from the bottom, *smaller* should be replaced by *greater*. On p. 32, the last expression in the third line should have r_t^2 in place of r_t. The last displayed formula on this page should read

$$w_{1t} + \cdots + w_{kt} \geq 1 - \frac{\nu_{2t} - \nu_{1t}^2}{(\eta_{k+1} - \nu_{1t})^2},$$

and the subscript $r + 1$ in the next line should be $k + 1$.

[1] "An iterative method for multiple correlation," *The Prediction of Personal Adjustment*, by Paul Horst and collaborators, Social Science Research Council, New York, 1941, pp. 313–318.