Further remarks on simple flows of fluids with pressure-dependent viscosities

Jaroslav Hron[a,2], Josef Málek[a,1], Vít Průša[a,2], K.R. Rajagopal[*,b,3]

[a]Mathematical Institute of Charles University, Sokolovská 83, Praha 8 - Karlín, Czech Republic
[b]Department of Mechanical Engineering, Texas A&M University, Texas, 77843, United States of America

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Abstract

Suslov and Tran [3] recently revisited the study carried out by Hron et al. [1] and they on the basis of their analysis claim that some of conclusions concerning one specific example, amongst the many considered by Hron et al. [1], are not justified. They claim that the class of simple flows of fluids with pressure dependent viscosity considered by Hron et al. [1] do not allow multiple solutions, and that the inflection velocity profiles reported in Hron et al. [1] cannot exist.

We have reexamined both papers, and we find that whether or not velocity profiles with inflection points exist depends on the class of functions to which the pressure belongs. If the pressure field is allowed to be discontinuous, which is in keeping with the class of functions to which pressure belongs to in the study of Hron et al. [1], such inflectional profiles are possible. However, if one requires the pressure field to be continuous then as Suslov and Tran [3] claim, such inflectional profiles are not possible. We provide a detailed explanation for this phenomenon that goes beyond the discussion presented in the paper by Suslov and Tran [3], and concerns subtle mathematical issues. Among other results we show that the solution with the inflectional profile is—interestingly—not a weak solution of the governing equations.

Concerning the non-uniqueness of the solution, we show that if we explicitly—instead of assuming that constants are fixed by an unknown procedure—specify a procedure for fixing all the integration constants in the solution, for example by fixing the pressure at two points or fixing the pressure gradient and the pressure at one point, we get a unique solution to the problem, provided all relevant quantities are continuous. On the other hand, if we relax the assumption on continuity, we can get multiple solutions.

Key words: pressure dependent viscosity, inflection velocity profiles, non-uniqueness
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1. Introduction

Recently Suslov and Tran [3] re-examined the several solutions established by Hron et al. [1] for the flow of fluids with pressure dependent viscosities, and claimed that one of them, namely the flow between parallel plates that showed the possibility of profiles with inflection, to be incorrect. In this study we re-study the problem and show that the conclusion of Suslov and Tran [3] is correct if one requires the pressure to be continuous, or if one requires that the constructed solution is a weak solution. However, on the other hand, if one allows for the pressure to be discontinuous, then such profiles with inflections are possible, and the conclusion drawn by Suslov and Tran [3] is incorrect. In this context, it ought to be borne in mind that the pressure field in the analysis of Hron et al. [1] belongs

*Corresponding author.
Email addresses: hron@karlin.mff.cuni.cz (Jaroslav Hron), malek@karlin.mff.cuni.cz (Josef Málek), prusv@karlin.mff.cuni.cz (Vít Průša), krajagopal@tamu.edu (K.R. Rajagopal)

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to the class of functions that could be discontinuous[4]. In fact one could find such mathematical solution even to the classical Navier–Stokes fluid if one allows the pressure field to be discontinuous—see Footnote 12 below for details. However for the Navier–Stokes fluids, the solution corresponding to the discontinuous pressure possesses a jump discontinuity in the second derivative of the velocity, while for the particular model of fluid with pressure dependent viscosity analyzed by Hron et al. [1], the velocity profile corresponding to the discontinuous pressure field has smooth derivatives of all order, and the same holds for the pressure at the line $x = 0$. Hron et al. [1] did not realize that the mathematically admissible solution they obtained (and that showed profiles with inflections) corresponded to the pressure being discontinuous.

The arguments given by Suslov and Tran [3] concerning existence of flows constructed by Hron et al. [1] are devoid of any discussion of the subtle mathematical and physical issues. However, they are to be commended for their intuition as they realized that the solutions with inflection points are possibly not physically realizable. In fact, their calculations unfortunately does not address the issue of the continuity or otherwise of the pressure. One of the aims of the present paper is to clarify the issue of existence of velocity profiles with inflection and its relation to continuity of the pressure. Moreover, we discuss whether the solutions with discontinuous pressure are weak solutions to the problem, and we find that the discontinuity in pressure contradicts the possibility that the solution is a weak solution. In our opinion this is a significant drawback of solutions with discontinuous pressure.

Similarly, whether one has uniqueness or non-uniqueness of solutions is tied intimately to whether the pressure field is required to be continuous or not. Thus, if one requires continuity of the pressure field, Suslov and Tran [3] are correct in their conclusion that there is no non-uniqueness in the velocity field (at least in the class of unidirectional flows). However, if the pressure field is allowed to be discontinuous, then Hron et al. [1] are correct in their claim that there are multiple solutions to the velocity field.

An interesting issue that is also at the heart of the different claims by Hron et al. [1] and Suslov and Tran [3] is the procedure that is used to fix a constant that appears in the explicit solution that is established. Requiring that the pressure field meet specific values at two distinct points, prescribing the pressure at one point and the pressure gradient, or the pressure at a specific point and the volumetric flux, lead to different solutions. We have to remember that the viscosity of the fluid depends on the pressure and thus specifying just the pressure gradient is insufficient to define the problem fully.

Finally, we observe that one could choose to require that the pressure field be continuous, allowing the velocity field to be discontinuous. This leads to interesting mathematical possibilities. Whether such solutions are physically meaningful and realizable is of course a legitimate question that can only be decided on the basis of observation.

The arrangement of the paper is as follows. In the next section we introduce the problem. This is followed by deriving the analytical expressions for the exact solution in §3. In the following section we develop analytical formulae for the velocity profiles corresponding to a discontinuous pressure field and we show that such solutions are not weak solutions. In §5 we discuss results concerning non-monotone velocity profiles where the pressure field is continuous and in §6 we delineate the range of values for a parameter that guarantees the existence of non-monotone velocity profiles. Section 7 consists of a discussion of the various methods of fixing an arbitrary constant that appears in the exact solution for the velocity–pressure pair, based on different physical requirements. In the final section, we end the paper with a brief discussion of the possibility of discontinuous velocity fields while pressure field is continuous.

[4]In the general mathematical analysis of initial and boundary value problems related to the equations of fluid mechanics the notion of a weak solution arises in a natural manner—see for example Málek and Rajagopal [2] and the references therein. In quite a large number of problems such a solution exists for a large class of data and for an arbitrarily large time interval. Within the framework of weak solutions, pressure is in general just an integrable function (there are cases connected with internal flows wherein the fluids adhere to the boundary where it is not known if such an integrable pressure exists). From mathematical standpoint, the integrability of pressure does not exclude the possibility of it being discontinuous, and from the mathematical point of view it is perfectly reasonable to consider discontinuous pressure fields.

From the physical point of view, pressure in an incompressible fluid is whatever it needs to be to enforce the constraint of incompressibility and thus once again a discontinuous pressure is not precluded. For instance, in problems in solid mechanics one looks for solutions with discontinuous deformation gradients though one usually expects deformation gradients to be continuous. Similarly, when one is concerned with discontinuities such as shocks in a compressible fluid, we do look for discontinuous density and velocity fields. Thus, it is legitimate to look for discontinuous pressure fields. However, within the purview of classical solutions one usually looks for continuous pressure and velocity and experiments suggest that such is indeed the case.

2. Problem description

In this section we briefly introduce the problem that was initially studied by Hron et al. [1] and then re-examined by Suslov and Tran [3]. The aim is to find a pair $\mathbf{U}$, Π, the velocity and the pressure, defined in the region between two infinite flat plates $\left\{[x, y] \in \mathbb{R}^2 |\, y \in [-1, 1]\right\}$ such that the pair solves[5]

$$\frac{\partial \mathbf{U}}{\partial t} + [\nabla \mathbf{U}]\,\mathbf{U} = -\nabla \Pi + \alpha \operatorname{div}\left(2\Pi\, |\mathbb{D}|^{p-2}\, \mathbb{D}\right), \tag{2.1a}$$

$$\operatorname{div} \mathbf{U} = 0, \tag{2.1b}$$

where $\mathbb{D}$ is the symmetric part of the velocity gradient, $\alpha > 0$ is a positive parameter (an analogue of the Reynolds number—if $p = 2$ we have $\alpha = \frac{1}{\mathrm{Re}}$), and the pair meets the following boundary conditions for the velocity

$$\mathbf{U}|_{y=-1} = 0, \tag{2.1c}$$

$$\mathbf{U}|_{y=1} = V\mathbf{e}_{\hat{x}}, \tag{2.1d}$$

and attains a given pressure at a fixed point

$$\Pi|_{[x,y]=[0,0]} = 1. \tag{2.1e}$$

A solution is sought in the form of an unidirectional flow $\mathbf{U} = U(y)\mathbf{e}_{\hat{x}}$ (see Figure 1) with velocity being an increasing[6] function of y.

3. Derivation of analytical formulae for the exact solution

Let us now focus on the particular case $p = \frac{3}{2}$ that was re-examined by Suslov and Tran [3], and let us first try to pin point the arguments that led Hron et al. [1] to claim existence of solutions with profiles that have an inflection for plane Couette flow. In this case the solution is—according to both papers—given by the following formula (we use notation used by Suslov and Tran [3])

$$U(y) = \frac{y+1}{\alpha^2} - \frac{4M_{1,2}}{\alpha^2 C_0} \frac{e^{C_0 y} - e^{-C_0}}{\left(M_{1,2} e^{-C_0} + 1\right)\left(M_{1,2} e^{C_0 y} + 1\right)}, \tag{3.1}$$

and on fixing boundary condition on the upper plate $U(y) = V$ we get the following algebraic equation to determine $M_{1,2}$ in terms of the remaining integration constant C_0

$$M_{1,2}^2 + 2\delta M_{1,2} + 1 = 0, \tag{3.2a}$$

$$\delta = \cosh C_0 + \frac{4}{\alpha^2 V - 2} \frac{\sinh C_0}{C_0}. \tag{3.2b}$$

We assume that $\alpha^2 V \neq 2$, for a discussion of the singular case see Suslov and Tran [3]. We can observe—and it will be important later—that from the quadratic equation (3.2a) we get

$$M_1 M_2 = 1, \tag{3.3a}$$

$$M_1 + M_2 = -2\delta, \tag{3.3b}$$

and moreover (3.2b) is an even function of C_0, i.e., $\delta(C_0) = \delta(-C_0)$. The pressure corresponding to velocity profile (3.1) is

$$\Pi(x, y) = \frac{M_{1,2} + e^{-C_0 y}}{M_{1,2} + 1} e^{C_0 \frac{x+y}{2}}. \tag{3.4}$$

[5]The viscosity takes form $\mu(\Pi) = \alpha\Pi$, this model could be however understood as an illustrative or representative model, since a realistic model for viscosity being a linear function of pressure should be $\mu(\Pi) = \beta(1 + \alpha\Pi)$.

[6]For solutions with non-monotone velocity profiles see §5.

Plots[7] given in Hron et al. [1] show inflection velocity profiles for $\alpha = 1$, $V = 1$, $C_0 = -1.9$ and these values correspond to $\alpha = 1$, $V = 1$ and $C_0 = -3.8$ in the notation used in Suslov and Tran [3] and in this paper. This particular value of C_0 falls in the interval[8] in which—according to Suslov and Tran [3]—solutions of form (3.1) are not possible, since the requirement on monotonicity for the velocity profile $U'(y) \geq 0$ can not be met.

Let us take formulae (3.1) and (3.4) and try to plug them into governing equations for the simple shear flow in a channel[9]

$$-\frac{\partial \Pi}{\partial x} + \alpha \frac{\partial}{\partial y}\left(\Pi \left(U'\right)^{p-1}\right) = 0, \tag{3.5a}$$

$$-\frac{\partial \Pi}{\partial y} + \alpha \frac{\partial}{\partial x}\left(\Pi \left(U'\right)^{p-1}\right) = 0, \tag{3.5b}$$

where $U' = \frac{\mathrm{d}U}{\mathrm{d}y}$, and observe the consequences. A simple calculation gives (we recall that $p = \frac{3}{2}$)

$$U'(y) = \frac{1}{\alpha^2}\left(\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1}\right)^2, \tag{3.6}$$

$$\left(U'(y)\right)^{p-1} = \frac{1}{\alpha}\left|\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1}\right|, \tag{3.7}$$

$$\frac{\partial \Pi}{\partial x} = \frac{C_0}{2}\Pi, \tag{3.8}$$

$$\frac{\partial \Pi}{\partial y} = \frac{C_0}{2}\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1}\Pi, \tag{3.9}$$

substituting these expressions into the governing equations (3.5) leads to the following pair of equations

$$-\frac{C_0}{2}\Pi + \frac{C_0}{2}\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1}\Pi\left|\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1}\right| + \Pi\frac{\partial}{\partial y}\left|\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1}\right| = 0, \tag{3.10a}$$

$$-\frac{C_0}{2}\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1}\Pi + \left|\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1}\right|\frac{C_0}{2}\Pi = 0. \tag{3.10b}$$

Clearly the second equation can be satisfied for all $y \in (-1, 1)$ if and only if

$$\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1} \geq 0. \tag{3.11}$$

If this condition holds then

$$\frac{\partial}{\partial y}\left|\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1}\right| = \frac{\partial}{\partial y}\left(\frac{M_{1,2}e^{C_0 y} - 1}{M_{1,2}e^{C_0 y} + 1}\right) = \frac{2M_{1,2}C_0 e^{C_0 y}}{\left(M_{1,2}e^{C_0 y} + 1\right)^2},$$

and substituting this identity into (3.10a) immediately shows that (3.10a) is also satisfied. Now[10] it is clear that the (non)existence of a solution depends on the sign of the term $M_{1,2}e^{C_0 y} - 1$ in (3.11). The solution to $M_{1,2}e^{C_0 y} - 1 = 0$, is

$$y_{\mathrm{crit}} = -\frac{1}{C_0}\ln M_{1,2}, \tag{3.12}$$

[7]Constant C_0 is not fixed by data (boundary conditions and value of pressure at the given point, conditions (2.1c), (2.1d) and (2.1e)), and this naturally means that the problem has multiple (infinitely many) solutions—in that different conditions used to fix C_0 possibly lead to different solutions, an issue that was not recognized or discussed by Suslov and Tran [3]. The authors rather tacitly—in the same way as Hron et al. [1] do—assume that C_0 is somehow fixed, for example by prescribing the value of the pressure at two different points or by fixing the pressure gradient. Later, after thorough discussion of fallacies of Suslov and Tran [3] and Hron et al. [1] papers, we will comment on this issue in some detail (see §7). Now, let us also assume that C_0 is fixed by an unknown procedure.

[8]See also §6 for a precise description of the parameter range that allows velocity profiles with inflection.

[9]Note that we have already used the assumption $U'(y) \geq 0$.

[10]For the parameter values we are interested in ($\alpha = 1$, $V = 1$, $C_0 = -3.8$) the constant δ is negative, and consequently $M_{1,2}$ are positive. Furthermore we can denote the roots of (3.2a) such that $M_1 \geq 1$ and $M_2 \leq 1$. For our particular parameter values we have $M_1 = 1.728107163$, $M_2 = 0.5786678174$ and $\delta = -1.15338749$.

and for the values of the parameter that we are interested in we get $y_{\text{crit}} \in (-1, 1)$, in other words the point where condition (3.11) fails lies in the channel. Finally we get[11]

$$\left|\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\right| = \begin{cases} \frac{M_1 e^{C_0 y}-1}{M_1 e^{C_0 y}+1}, & y \in \left[-1, -\frac{1}{C_0} \ln M_1\right], \\ -\frac{M_1 e^{C_0 y}-1}{M_1 e^{C_0 y}+1}, & y \in \left[-\frac{1}{C_0} \ln M_1, 1\right], \end{cases} \tag{3.13a}$$

$$\left|\frac{M_2 e^{C_0 y} - 1}{M_2 e^{C_0 y} + 1}\right| = \begin{cases} \frac{M_2 e^{C_0 y}-1}{M_2 e^{C_0 y}+1}, & y \in \left[-1, -\frac{1}{C_0} \ln M_2\right], \\ -\frac{M_2 e^{C_0 y}-1}{M_2 e^{C_0 y}+1}, & y \in \left[-\frac{1}{C_0} \ln M_2, 1\right]. \end{cases} \tag{3.13b}$$

where $-\frac{1}{C_0} \ln M_2 < 0 < -\frac{1}{C_0} \ln M_1$, and moreover in virtue of (3.3a), $-\frac{1}{C_0} \ln M_1 = -\left(-\frac{1}{C_0} \ln M_2\right)$. We also notice that $U'(y_{\text{crit}}) = 0$.

Consequently, the pair

$$U(y) = \frac{y+1}{\alpha^2} - \frac{4M_1}{\alpha^2 C_0} \frac{e^{C_0 y} - e^{-C_0}}{\left(M_1 e^{-C_0} + 1\right)\left(M_1 e^{C_0 y} + 1\right)}, \tag{3.14a}$$

$$\Pi(x, y) = \frac{M_1 + e^{-C_0 y}}{M_1 + 1} e^{C_0 \frac{x+y}{2}}, \tag{3.14b}$$

solves the governing equations (3.5) for $y \in \left(-1, -\frac{1}{C_0} \ln M_1\right)$, and the pair

$$U(y) = \frac{y+1}{\alpha^2} - \frac{4M_2}{\alpha^2 C_0} \frac{e^{C_0 y} - e^{-C_0}}{\left(M_2 e^{-C_0} + 1\right)\left(M_2 e^{C_0 y} + 1\right)}, \tag{3.15a}$$

$$\Pi(x, y) = \frac{M_2 + e^{-C_0 y}}{M_2 + 1} e^{C_0 \frac{x+y}{2}} \tag{3.15b}$$

solves governing equations (3.5) for $y \in \left(-1, -\frac{1}{C_0} \ln M_2\right)$, but neither (3.14) nor (3.15) solves governing equations in the whole domain, thus for all $y \in (-1, 1)$.

It therefore seems that velocity profiles with inflection do not indeed exist as concluded in Suslov and Tran [3] if we require the pressure to be continuous. The situation is however very interesting since we have an explicit formula for a solution of a system of partial differential equations and this formula is valid—as we will see in the next paragraphs—only if one of the terms is discontinuous! That is, if we relax the tacit assumption on the continuity of all the considered quantities, and we allow certain quantities (namely pressure) to be discontinuous, the formulae are valid. If we insist on having continuous pressure and velocity, we can however relax condition $U'(y) \geq 0$, and try to find a non-monotone velocity profile—this is done in Suslov and Tran [3] and reviewed in §5.

Let us now focus on one of the key equalities in the derivation of the analytical formula, namely on equation (20) in Suslov and Tran [3] that reads

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}y} \ln \left|\frac{1 + \alpha (U')^{p-1}}{1 - \alpha (U')^{p-1}}\right| = \frac{C_0}{2}, \tag{3.16}$$

and let us analyze the equation for our particular choice of parameters and solution (3.14). The argument of the logarithm is (we are using formulae (3.13))

$$\frac{1 + \alpha (U')^{p-1}}{1 - \alpha (U')^{p-1}} = \frac{1 + \left|\frac{M_1 e^{C_0 y}-1}{M_1 e^{C_0 y}+1}\right|}{1 - \left|\frac{M_1 e^{C_0 y}-1}{M_1 e^{C_0 y}+1}\right|} = \begin{cases} M_1 e^{C_0 y}, & y \in \left(-1, -\frac{1}{C_0} \ln M_1\right), \\ \frac{1}{M_1} e^{-C_0 y}, & y \in \left(-\frac{1}{C_0} \ln M_1, 1\right), \end{cases}$$

and for the derivative of the logarithm we get

$$\frac{\mathrm{d}}{\mathrm{d}y} \ln \left|\frac{1 + \alpha (U')^{p-1}}{1 - \alpha (U')^{p-1}}\right| = \begin{cases} C_0, & y \in \left(-1, -\frac{1}{C_0} \ln M_1\right), \\ \text{not exist}, & y = -\frac{1}{C_0} \ln M_1, \\ -C_0, & y \in \left(-\frac{1}{C_0} \ln M_1, 1\right). \end{cases}$$

[11] In all subsequent calculations we have to keep in mind that we assume $C_0 < 0$ and $M_{1,2} > 0$, since the sign of C_0 clearly the affects solution of the key inequality $M_{1,2} e^{C_0 y} - 1 \geq 0$. If $C_0 < 0$ then the inequality is satisfied for $y \in \left(-\infty, -\frac{1}{C_0} \ln M_{1,2}\right)$, on the other hand if $C_0 > 0$ then the inequality is satisfied for $y \in \left(-\frac{1}{C_0} \ln M_{1,2}, +\infty\right)$.

A similar calculation can be carried out for the other solution (3.15).

The simple observation sketched above shows that if the critical point y_{crit} lies in the channel, formula (3.16) may be wrong, or more precisely one should be cautious in assuming that $\ln\left|\frac{1+\alpha(U')^{p-1}}{1-\alpha(U')^{p-1}}\right|$ is differentiable in the whole domain. If such an assumption is not true, then the whole reasoning chain based on this equation (and this is precisely what is done in Suslov and Tran [3]) breaks down. Now we know that something interesting happens with regard to (3.16), and let us try to find a solution in the whole channel exploiting the non-differentiability of the logarithm in (3.16).

4. Analytical formulae for velocity profiles with with inflection corresponding to discontinuous pressure

Having identified the problem with regard to (3.16), we are almost ready to extend[12] the solution to the whole domain. First we can observe that formula (3.1) is invariant to a special change of constants. If we take $C_0 =_{\mathrm{def}} -C_0$ and $M_{1,2} =_{\mathrm{def}} \frac{1}{M_{1,2}}$ the formula will not change, indeed if

$$U(y; C_0, M_{1,2}) = \frac{y+1}{\alpha^2} - \frac{4M_{1,2}}{\alpha^2 C_0}\frac{e^{C_0 y} - e^{-C_0}}{\left(M_{1,2}e^{-C_0}+1\right)\left(M_{1,2}e^{C_0 y}+1\right)},$$

$$U\left(y; -C_0, \frac{1}{M_{1,2}}\right) = \frac{y+1}{\alpha^2} + \frac{4\frac{1}{M_{1,2}}}{\alpha^2 C_0}\frac{e^{-C_0 y} - e^{C_0}}{\left(\frac{1}{M_{1,2}}e^{C_0}+1\right)\left(\frac{1}{M_{1,2}}e^{-C_0 y}+1\right)},$$

then

$$U(y; C_0, M_{1,2}) = U\left(y; -C_0, \frac{1}{M_{1,2}}\right). \tag{4.1}$$

This identity holds for all $y \in [-1, 1]$ regardless of whether $M_{1,2}$ is related to C_0 by (3.2a). Formula (3.4) for the pressure is however invariant to the change of constants only for $x = 0$, indeed if

$$\Pi(x, y; C_0, M_{1,2}) = \frac{M_{1,2} + e^{-C_0 y}}{M_{1,2}+1}e^{C_0\frac{x+y}{2}},$$

$$\Pi\left(x, y; -C_0, \frac{1}{M_{1,2}}\right) = \frac{\frac{1}{M_{1,2}} + e^{C_0 y}}{\frac{1}{M_{1,2}}+1}e^{-C_0\frac{x+y}{2}},$$

[12] The extension procedure is however a little bit curious, and this is best seen if we consider the classical Navier–Stokes fluid undergoing combined Couette–Poiseuille flow (flow is driven by moving upper plane and by a pressure gradient). In this case we can also generate velocity profiles with inflection if we allow the pressure to be a discontinuous function. Indeed, if we want to solve problem with $V = 2$, $\Pi(0,0) = 3$ and if we prescribe pressure gradient in form

$$\frac{\partial \Pi}{\partial x} = \begin{cases} -2, & y \in (-1, 0)\,, \\ 2, & y \in [0, 1)\,, \end{cases}$$

then the corresponding velocity profile is given by

$$U(y) = \begin{cases} \left(1 - y^2\right), & y \in (-1, 0)\,, \\ 2 - \left(1 - y^2\right), & y \in [0, 1)\,, \end{cases}$$

and pressure field is

$$\Pi(x, y) = \begin{cases} -x + 3, & y \in (-1, 0)\,, \\ x + 3, & y \in [0, 1)\,. \end{cases}$$

The velocity profile is continuous, has continuous first derivatives and has an inflection point—in the sense that at this point the function changes form convex to concave—in the middle of the channel. The solution given above of course does not satisfy the Navier–Stokes equations pointwise, since $\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}$ and $\frac{\partial \Pi}{\partial y}$ have jump discontinuity at line $y = 0$.

This analogue with the classical Navier–Stokes fluid shows that the possibility of having discontinuous pressure is not a feature that is due to pressure-dependent viscosity. In the case of fluid with the pressure dependent viscosity we are studying, we still get a discontinuity in the pressure, nevertheless the velocity and its all derivatives are continuous.

then a simple manipulation yields

$$\Pi(x, y; C_0, M_{1,2}) = \Pi\left(x, y; -C_0, \frac{1}{M_{1,2}}\right) e^{C_0 x}, \tag{4.2}$$

and consequently for $x = 0$ we have $\Pi\left(0, y; C_0, M_{1,2}\right) = \Pi\left(0, y; -C_0, \frac{1}{M_{1,2}}\right)$, but for $x \neq 0$ we get $\Pi(x, y; C_0, M_{1,2}) \neq \Pi\left(x, y; -C_0, \frac{1}{M_{1,2}}\right)$. The observation above again holds regardless of whether $M_{1,2}$ is related to C_0 by (3.2a).

4.1. Formulae for negative C_0

Let us now take $C_0 < 0$ and the corresponding $M_1 > 1$ that solves equation (3.3) (we know that for our particular parameter values such a solution exists), and let us consider the following velocity profile

$$U(y) = \begin{cases} U(y; C_0, M_1), & y \in \left[-1, -\frac{1}{C_0} \ln M_1\right], \\ U(y; -C_0, \frac{1}{M_1}), & y \in \left[-\frac{1}{C_0} \ln M_1, 1\right]. \end{cases} \tag{4.3a}$$

From the analysis carried out above we already know that in fact $U(y) = U(y; C_0, M_1)$ in the whole domain, since the formula for velocity profile is invariant with respect to the change of parameters introduced above. Velocity profile (4.3a) is continuous, has a continuous first derivative and the first derivative is equal to zero at $y_{\mathrm{crit}} = -\frac{1}{C_0} \ln M_1$, furthermore the second derivative is also continuous and is equal to zero at y_{crit}. This can be clearly seen from formula derived in Hron et al. [1] and Suslov and Tran [3] (formula (30)).

Moreover we know that the velocity (4.3a) leads to non-differentiability in (3.16) at $y = -\frac{1}{C_0} \ln M_1$, but we can still demand at least

$$\frac{\mathrm{d}}{\mathrm{d}y} \ln \left| \frac{1 + \alpha (U')^{p-1}}{1 - \alpha (U')^{p-1}} \right| = \begin{cases} C_0, & y \in \left(-1, -\frac{1}{C_0} \ln M_1\right), \\ -C_0, & y \in \left(-\frac{1}{C_0} \ln M_1, 1\right), \end{cases}$$

and this equation is indeed satisfied by our velocity profile.

Now it is time to derive a corresponding formula for the pressure. On the part of the domain where we require the derivative above to be equal to C_0, we can repeat a step by step derivation of the formula for the pressure (see Hron et al. [1] or Suslov and Tran [3] for details), but we have to keep in mind that the formula for the pressure will hold only for $y \in \left[-1, -\frac{1}{C_0} \ln M_1\right)$, the pressure in this part of the domain will therefore be given by

$$\Pi(x, y; C_0, M_1) = \frac{M_1 + e^{-C_0 y}}{M_1 + 1} e^{C_0 \frac{x+y}{2}}, \; y \in \left[-1, -\frac{1}{C_0} \ln M_1\right). \tag{4.3b}$$

In the remaining part of the domain we require the derivative of the logarithm to be equal to $-C_0$. In this case we can again repeat step by step the procedure that gives the formula for the pressure, but we have to take $M_1 =_{\mathrm{def}} \frac{1}{M_1}$—this is necessary because we require the velocity to be continuous, see (4.3a). The formula for pressure will be

$$\Pi\left(x, y; -C_0, \frac{1}{M_1}\right) = \frac{\frac{1}{M_1} + e^{C_0 y}}{\frac{1}{M_1} + 1} e^{-C_0 \frac{x+y}{2}}, \; y \in \left(-\frac{1}{C_0} \ln M_1, 1\right], \tag{4.3c}$$

Summarizing formulae (4.3), we claim that pair

$$U(y) = \frac{y+1}{\alpha^2} - \frac{4 M_1}{\alpha^2 C_0} \frac{e^{C_0 y} - e^{-C_0}}{\left(M_1 e^{-C_0} + 1\right)\left(M_1 e^{C_0 y} + 1\right)}, \; y \in [-1, 1], \tag{4.4a}$$

$$\Pi(x, y) = \begin{cases} \Pi_L(x, y) = \frac{M_1 + e^{-C_0 y}}{M_1 + 1} e^{C_0 \frac{x+y}{2}}, & y \in \left[-1, -\frac{1}{C_0} \ln M_1\right), \\ \Pi_R(x, y) = \frac{\frac{1}{M_1} + e^{C_0 y}}{\frac{1}{M_1} + 1} e^{-C_0 \frac{x+y}{2}}, & y \in \left[-\frac{1}{C_0} \ln M_1, 1\right], \end{cases} \tag{4.4b}$$

solves pointwise the governing equations (3.5) in the whole domain except the set $\left\{[x, y] \in \mathbb{R}^2, y = -\frac{1}{C_0} \ln M_1\right\}$, the set where there is a jump discontinuity in the pressure (4.4b), and quantities appearing in the governing equations are not defined. Whether such a solution is physical or not can be disputed, but from mathematical point of view such a solution is acceptable. A plot of the solution is given in Figure 2.

If we want to make sure that pair (4.4) is indeed a solution to system (3.5), we can verify it by direct computation, we have

$$U'(y) = \frac{1}{\alpha^2}\left(\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\right)^2, \; y \in (-1, 1)$$

$$(U'(y))^{p-1} = \frac{1}{\alpha}\left|\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\right| = \begin{cases} \frac{1}{\alpha}\frac{M_1 e^{C_0 y}-1}{M_1 e^{C_0 y}+1}, & y \in \left(-1, -\frac{1}{C_0}\ln M_1\right), \\ -\frac{1}{\alpha}\frac{M_1 e^{C_0 y}-1}{M_1 e^{C_0 y}+1}, & y \in \left(-\frac{1}{C_0}\ln M_1, 1\right), \end{cases}$$

$$\frac{\partial \Pi}{\partial x} = \begin{cases} \frac{C_0}{2}\Pi_L, & y \in \left(-1, -\frac{1}{C_0}\ln M_1\right), \\ -\frac{C_0}{2}\Pi_R, & y \in \left(-\frac{1}{C_0}\ln M_1, 1\right), \end{cases}$$

$$\frac{\partial \Pi}{\partial y} = \begin{cases} \frac{C_0}{2}\frac{M_1 e^{C_0 y}-1}{M_1 e^{C_0 y}+1}\Pi_L, & y \in \left(-1, -\frac{1}{C_0}\ln M_1\right), \\ \frac{C_0}{2}\frac{M_1 e^{C_0 y}-1}{M_1 e^{C_0 y}+1}\Pi_R, & y \in \left(-\frac{1}{C_0}\ln M_1, 1\right). \end{cases}$$

substituting these expressions to governing equations (3.5)—see also (3.10)—gives for $y \in \left(-1, -\frac{1}{C_0}\ln M_1\right)$

$$-\frac{C_0}{2}\Pi_L + \frac{C_0}{2}\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\Pi_L \frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1} + \Pi_L \frac{\partial}{\partial y}\left(\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\right) = 0,$$

$$-\frac{C_0}{2}\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\Pi_L + \frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\frac{C_0}{2}\Pi_L = 0,$$

and for $y \in \left(-\frac{1}{C_0}\ln M_1, 1\right)$

$$-\left(-\frac{C_0}{2}\Pi_R\right) + \frac{C_0}{2}\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\Pi_R\left(-\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\right) + \Pi_R \frac{\partial}{\partial y}\left(-\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\right) = 0,$$

$$-\frac{C_0}{2}\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\Pi_R + \left(-\frac{M_1 e^{C_0 y} - 1}{M_1 e^{C_0 y} + 1}\right)\left(-\frac{C_0}{2}\Pi_R\right) = 0.$$

Introduction of a discontinuous pressure therefore balances the sign changes in (3.10) that occurs due to presence of $(U')^{p-1}$, and (4.4) is indeed a solution.

Now we can without doubts claim that for certain parameter values there exists a velocity profile U that has one inflection point, this means that the conclusion in Suslov and Tran [3] is not valid—provided we are willing to accept solutions with discontinuous pressure. On the other hand conclusions drawn in Hron et al. [1] should be faulted as the authors do not recognize that the pressure field that they are dealing with is discontinuous. It is clear from discussions in Hron et al. [1] that they are oblivious to the important role played by the discontinuous pressure gradient. Curiously enough, the plots of pressure in Hron et al. [1] are valid, since they are given only for $x = 0$ and in this case we have $\Pi_L(x, y) = \Pi_R(x, y)$ and pressure is on this line continuous.

Furthermore, we can see that for the construction above we do not need $M_1 > 1$, we can repeat the same procedure also for the other root of (3.2a) M_2 that is, in virtue of (3.3a), equal to $\frac{1}{M_1}$. The critical point (3.12) is now $y_{\text{crit}} = -\frac{1}{C_0}\ln M_2 = \frac{1}{C_0}\ln M_1$, therefore it has the same magnitude as the critical point for the solution found above, but lies in the opposite half plane ($-1 < -\frac{1}{C_0}\ln M_2 < 0 < -\frac{1}{C_0}\ln M_1 < 1$). Repeating step by step the solution procedure, we get (using the fact that $M_2 = \frac{1}{M_1}$)

$$U(y) = \frac{y+1}{\alpha^2} - \frac{4\frac{1}{M_1}}{\alpha^2 C_0}\frac{e^{C_0 y} - e^{-C_0}}{\left(\frac{1}{M_1}e^{-C_0} + 1\right)\left(\frac{1}{M_1}e^{C_0 y} + 1\right)}, \; y \in [-1, 1], \tag{4.5a}$$

$$\Pi(x, y) = \begin{cases} \Pi_L(x, y) = \frac{\frac{1}{M_1} + e^{-C_0 y}}{\frac{1}{M_1} + 1} e^{C_0 \frac{x+y}{2}}, & y \in \left[-1, \frac{1}{C_0}\ln M_1\right), \\ \Pi_R(x, y) = \frac{M_1 + e^{C_0 y}}{M_1 + 1} e^{-C_0 \frac{x+y}{2}}, & y \in \left[\frac{1}{C_0}\ln M_1, 1\right]. \end{cases} \tag{4.5b}$$

We claim that pair (4.5) is another solution to (3.5)—this claim can be again verified by direct computation—and that this solution is not equal to (4.4), therefore the claim of Suslov and Tran [3] that multiple solutions can not exist is wrong[13]. (We however again require the pressure to be a discontinuous function.) Plots of the alternative solution and the former solution are given in Figure 2, the functions are denoted as $U_2(x, y)$, $\Pi_2(x, y)$ and $U_1(x, y)$, $\Pi_1(x, y)$ respectively.

4.2. Formulae for positive C_0

If C_0 is positive, the outcomes of the analysis above are the same, but the formulae for the pressure are switched[14], and thus the analogue of (4.4) is

$$U(y) = \frac{y+1}{\alpha^2} - \frac{4M_1}{\alpha^2 C_0} \frac{e^{C_0 y} - e^{-C_0}}{(M_1 e^{-C_0} + 1)(M_1 e^{C_0 y} + 1)}, \; y \in [-1, 1], \tag{4.6a}$$

$$\Pi(x, y) = \begin{cases} \Pi_L(x, y) = \frac{\frac{1}{M_1} + e^{C_0 y}}{\frac{1}{M_1} + 1} e^{-C_0 \frac{x+y}{2}}, & y \in \left[-1, -\frac{1}{C_0} \ln M_1\right), \\ \Pi_R(x, y) = \frac{M_1 + e^{-C_0 y}}{M_1 + 1} e^{C_0 \frac{x+y}{2}}, & y \in \left[-\frac{1}{C_0} \ln M_1, 1\right], \end{cases} \tag{4.6b}$$

whereas now the critical point $y_{\text{crit}} = -\frac{1}{C_0} \ln M_1$ corresponding to M_1 is negative. If we take $M_2 = \frac{1}{M_1}$ instead of M_1, then we get analogue of (4.5), thus

$$U(y) = \frac{y+1}{\alpha^2} - \frac{4\frac{1}{M_1}}{\alpha^2 C_0} \frac{e^{C_0 y} - e^{-C_0}}{\left(\frac{1}{M_1} e^{-C_0} + 1\right)\left(\frac{1}{M_1} e^{C_0 y} + 1\right)}, \; y \in [-1, 1], \tag{4.7a}$$

$$\Pi(x, y) = \begin{cases} \Pi_L(x, y) = \frac{M_1 + e^{C_0 y}}{M_1 + 1} e^{-C_0 \frac{x+y}{2}}, & y \in \left[-1, \frac{1}{C_0} \ln M_1\right), \\ \Pi_R(x, y) = \frac{\frac{1}{M_1} + e^{-C_0 y}}{\frac{1}{M_1} + 1} e^{C_0 \frac{x+y}{2}}, & y \in \left[\frac{1}{C_0} \ln M_1, 1\right], \end{cases} \tag{4.7b}$$

whereas now the critical point $y_{\text{crit}} = \frac{1}{C_0} \ln M_1$ corresponding to M_2 is positive.

4.3. On the notion of solution

In the previous section we have constructed solutions (4.4) and (4.5) that do not satisfy the governing equations in the pointwise sense. One can ask whether these solutions—that in fact fulfill the governing equations pointwise up to a set of zero measure (the line $y = y_{\text{crit}}$)—are weak solutions to the problem. Let us recall that weak solution is a very natural concept of solution since it considers the governing equations in the sense of "averages" (see for example Málek and Rajagopal [2] for a notion of the weak solution to the governing equations for fluids with pressure dependent viscosity).

Concerning the problem investigated in this study, any weak solution (in the form of the unidirectional flow) to system (3.5) has to satisfy[15]

$$\forall \varphi \in \mathcal{D}(\mathcal{B}) \times \mathcal{D}(\mathcal{B}), \; \varphi = \varphi_1 \mathbf{e}_{\hat{x}} + \varphi_2 \mathbf{e}_{\hat{y}} : \quad (\Pi, \operatorname{div} \varphi)_{\mathcal{B}} = \left(\alpha \Pi \left(U'\right)^{p-1}, \frac{\partial \varphi_1}{\partial y} + \frac{\partial \varphi_2}{\partial x}\right)_{\mathcal{B}}, \tag{4.8}$$

where $\mathcal{B} \subset \overline{\mathcal{B}} \subset \left\{[x, y] \in \mathbb{R}^2 | \, y \in (-1, 1)\right\}$ is an open bounded set, $\mathcal{D}(\mathcal{B})$ denotes the space of infinitely differentiable functions with compact support in $\mathcal{B}$, and $(u, v)_{\mathcal{S}} =_{\text{def}} \int_{\mathcal{S}} uv \, \mathrm{d}\mathbf{x}$.

[13]But the "non-uniqueness" of this kind can be hardly interpreted as non-uniqueness since the issue of fixing C_0 has not been clarified, see §7 for further discussion on this.

[14]This is a consequence of sign change in C_0, because if C_0 changes sign, then the inequality $M_{1,2} e^{C_0 y} - 1 \geq 0$ is satisfied for $y > y_{\text{crit}} = -\frac{1}{C_0} \ln M_{1,2}$ instead of $y < y_{\text{crit}} = -\frac{1}{C_0} \ln M_{1,2}$ as in the case $C_0 < 0$, see also Footnote 11.

[15]The relation is a "weak" counterpart to the balance of linear momentum (3.5). It is formally obtained by multiplying (3.5a) by an arbitrary function φ_1 and (3.5b) by an arbitrary function φ_2, summing up both equations, integrating over a domain $\mathcal{B}$ and using integration by parts simultaneously with the assumption that functions $\varphi_{1,2}$ vanish on the boundary of the domain $\mathcal{B}$.

In particular, equation (4.8) must hold for any set $\mathcal{B}$ as drawn in Figure 1. Using this particular choice of $\mathcal{B}$ we will show in Lemma 4.1 that the jump discontinuity in the pressure

$$\Pi^+(x, y_{\mathrm{crit}}) \neq \Pi^-(x, y_{\mathrm{crit}}), \tag{4.9}$$

where $\Pi^\pm(x, y_{\mathrm{crit}}) =_{\mathrm{def}} \lim_{y \to y_{\mathrm{crit}}\pm} \Pi(x, y)$ are the limits of the pressure approaching the jump from "below" and "above", is not compatible with (4.8), although the pair Π and U is a solution to the governing equations (3.5) except the line $\Gamma = \left\{[x, y] \in \mathbb{R}^2 | \, y = y_{\mathrm{crit}}\right\}$ (thus up to a set of measure zero). This is an important observation, since it shows that "averaging" in the notion of a weak solution is in our case sensitive to discontinuities even if the discontinuities take place in a small set. The fact that (4.4) and other solutions with discontinuous pressure are not weak solutions is in our opinion a significant drawback of such solutions.

As we have already pointed out Hron et al. [1] were not aware of the fact that the solutions they constructed have a jump discontinuity in the pressure since they studied only the pressure in the cross-section $x = 0$, and the pressure in this cross-section is continuous and has continuous derivatives.

Lemma 4.1 (Solution with jump discontinuity in the pressure is not a weak solution)**.** *Let* $U(y)$*,* $\Pi(x, y)$ *be a weak solution to system* (3.5) *with appropriate boundary conditions—the solution in particular satisfies* (4.8)*—and let* $U'(y_{\mathrm{crit}}) = 0$*. Then*[16]

$$\forall x \in \mathbb{R} : \; \Pi^+(x, y_{\mathrm{crit}}) = \Pi^-(x, y_{\mathrm{crit}}). \tag{4.10}$$

Consequently, neither of the solutions (4.4)*,* (4.5)*,* (4.6) *and* (4.7) *is a weak solution.*

Proof. Let us integrate (4.8) by parts separately in $\mathcal{B}^+$ and $\mathcal{B}^-$ (see Figure 1), and use the fact that the governing equations hold in $\mathcal{B}^\pm$ pointwise and that $U'(y_{\mathrm{crit}}) = 0$. More explicitly

$$\begin{aligned} 0 = \left(\alpha \Pi \left(U'\right)^{p-1}, \frac{\partial \varphi_1}{\partial y} + \frac{\partial \varphi_2}{\partial x}\right)_{\mathcal{B}} - \left(\Pi, \operatorname{div} \varphi\right)_{\mathcal{B}} &= \left(-\frac{\partial}{\partial y}\left(\alpha \Pi \left(U'\right)^{p-1}\right) + \frac{\partial \Pi}{\partial x}, \varphi_1\right)_{\mathcal{B}^+} \\ + \left(-\frac{\partial}{\partial y}\left(\alpha \Pi \left(U'\right)^{p-1}\right) + \frac{\partial \Pi}{\partial x}, \varphi_1\right)_{\mathcal{B}^-} &+ \left(-\frac{\partial}{\partial x}\left(\alpha \Pi \left(U'\right)^{p-1}\right) + \frac{\partial \Pi}{\partial y}, \varphi_2\right)_{\mathcal{B}^+} \\ + \left(-\frac{\partial}{\partial x}\left(\alpha \Pi \left(U'\right)^{p-1}\right) + \frac{\partial \Pi}{\partial y}, \varphi_2\right)_{\mathcal{B}^-} &+ \left(\Pi^+ - \Pi^-, \varphi_2(\cdot, y_{\mathrm{crit}})\right)_\Gamma = \left(\Pi^+ - \Pi^-, \varphi_2(\cdot, y_{\mathrm{crit}})\right)_\Gamma \end{aligned}$$

hence $\forall \varphi_2 \in \mathcal{D}(\mathcal{B}) : \; \left(\Pi^+ - \Pi^-, \varphi_2(\cdot, y_{\mathrm{crit}})\right)_\Gamma = 0$ and (4.10) immediately follows. □

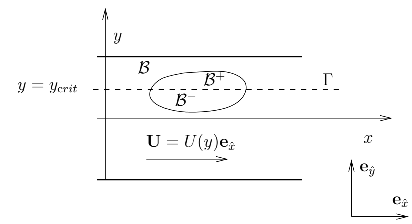

Figure 1: Problem geometry.

5. Analytical formulae for non-monotone velocity profiles with continuous pressure

5.1. Formulae for negative C_0

Suslov and Tran [3] also report another kind of solution, the so-called Poiseuille like solution. This solution arises form the assumption that the velocity profile is not monotone and that the derivative $U'(y)$ changes sign once

[16]By $\forall x \in \mathbb{R}$ we mean for all x up to a zero measure set in $\mathbb{R}$.

in $y \in (-1,1)$. Let us denote—following Suslov and Tran [3]—this point as Y, and let us assume that $U'(y) \geq 0$ in $y \in (-1, Y)$ and $U'(y) \leq 0$ in $y \in (Y, 1)$. In the part of the domain where we assume the derivative to be negative, the governing equations take form[17]

$$-\frac{\partial \Pi}{\partial x} - \alpha \frac{\partial}{\partial y}\left(\Pi\left(-U'\right)^{p-1}\right) = 0, \tag{5.1a}$$

$$-\frac{\partial \Pi}{\partial y} - \alpha \frac{\partial}{\partial x}\left(\Pi\left(-U'\right)^{p-1}\right) = 0, \tag{5.1b}$$

and the system can be solved once again analytically, and if we combine the solutions for $y \in (-1, Y)$ and $y \in (Y, 1)$, we get the following solution (we again assume that $C_0 < 0$)

$$U(y) = \begin{cases} \frac{y+1}{\alpha^2} + \frac{4}{\alpha^2 C_0} \frac{1-e^{C_0(y+1)}}{\left(1+e^{C_0(y-Y)}\right)\left(1+e^{C_0(Y+1)}\right)}, & y \in [-1, Y), \\ V + \frac{1-y}{\alpha^2} - \frac{4}{\alpha^2 C_0} \frac{1-e^{C_0(y-1)}}{\left(1+e^{C_0(y-Y)}\right)\left(1+e^{C_0(Y-1)}\right)}, & y \in [Y, 1], \end{cases} \tag{5.2a}$$

$$\Pi(x,y) = \frac{1+e^{C_0(Y-y)}}{1+e^{C_0 Y}} e^{C_0\left(\frac{x+y}{2}\right)}, \; y \in [-1, 1], \tag{5.2b}$$

where the unknown constant[18] Y is now determined using the continuity condition on the velocity, thus $\lim_{y \to Y+} U(y) = \lim_{y \to Y-} U(y)$. This leads to implicit equation[19]

$$\frac{4 \sinh(C_0 Y)}{\cosh(C_0 Y) + \cosh C_0} - 2 C_0 Y = -\alpha^2 V C_0. \tag{5.3}$$

The non-monotone velocity profile and corresponding pressure are plotted in Figure 2. Since (5.2) is a classical solution (the governing equations are satisfied pointwise), it is also a weak solution.

5.2. Formulae for positive C_0

One can also derive a similar solution even for $C_0 > 0$, but this result is not important for the following discussion. We refer the reader to Suslov and Tran [3] for details.

6. Parameter range for velocity profiles with inflection

In the previous sections we have shown that there exist one set of parameters that leads—provided we are willing to accept the discontinuity in the pressure—to velocity profiles with inflections and to multiple solutions to the governing equations. Now we can try to describe the whole parameter range where we can observe velocity profiles with inflection.

We have seen that for the existence of velocity profiles with inflections there must exist a critical point in the channel, we therefore require $y_{\text{crit}} \in (-1, 1)$. If this is true, then we immediately get—in virtue of (3.3a)—that there

[17] Note that if $U'(y) \leq 0$, then $|U'|^{p-2} U' = -|U'|^{p-1}$.

[18] In fact we set $M = e^{-C_0 Y}$ in the already derived formulae. If we go through the derivation of the solution for $U'(y) \geq 0$, then the only difference for the solution with $U'(y) \leq 0$ is that for $U'(y)$ we finally get

$$U'(y) = -\frac{1}{\alpha^2}\left(\frac{M e^{C_0 y} - 1}{M e^{C_0 y} + 1}\right)^2 = -\frac{1}{\alpha^2}\left(\frac{e^{C_0(y-Y)} - 1}{e^{C_0(y-Y)} + 1}\right)^2$$

instead of (3.6), thus the formula for the velocity is $U(y) = -\left(\frac{y-Y}{\alpha^2} + \frac{4}{\alpha^2 C_0} \frac{1}{e^{C_0(y-Y)}+1} + K\right)$, where the constant K must be fixed by the boundary condition (2.1d). The derivative of pressure using (5.2b) is obviously

$$\frac{\partial \Pi}{\partial y} = \frac{C_0}{2} \frac{1 - e^{C_0(Y-y)}}{1 + e^{C_0(Y-y)}} \Pi,$$

using this fact, one can immediately see that (5.2) is—for $y < Y$—indeed a solution to (2.1).

[19] The implicit equation has only one solution, since the implicit function is monotone in Y.

are in fact two such points, and that the situation is qualitatively the same (after proper labeling of the roots of (3.2a)) as in the previous sections. Therefore the whole problem of finding the parameter range that leads to velocity profiles with inflections reduces to problem whether for given α and V there exist C_0 such that $y_{\mathrm{crit}} = -\frac{1}{C_0} \ln M_1 \in (-1, 1)$.

Since M is a solution to (3.2a) we can plot y_{crit} as function of C_0, assuming the rest of parameters are being kept fixed. Let us again choose $\alpha = 1$, $V = 1$ and let us consider only the root $M = -\delta + \sqrt{\delta^2 - 1}$. Then we get the following plot of y_{crit} (see Figure 3). Point C_0^{inf} denotes value of C_0 below which we have a critical point y_{crit} in the channel, C_0^{ex} denotes value of C_0 where the formula for root of (3.2a) fails to provide real roots, thus the point where $\delta = -1$.

In our particular situation we have $C_0^{\mathrm{ex}} = -3.830016096$, $C_0^{\mathrm{inf}} = -1.915008048$, and in the whole range $\left(C_0^{\mathrm{ex}}, C_0^{\mathrm{inf}}\right)$ the situation is qualitatively the same as in §4, and all conclusions reached in this section are valid. We note that $C_0^{\mathrm{inf}} = -1.915008048$ is exactly the value of K that is used in Figure 4 in Suslov and Tran [3], and that is the value of C_0 for which the critical point lies on the channel wall—we can indeed see that one of the velocity profiles given in their Figure 4 has derivative equal to zero on the channel wall.

If one wants to get the formulae for C_0^{ex} and C_0^{inf}, it is possible to derive the following implicit equations (see also Suslov and Tran [3])

$$-2 \tanh C_0^{\mathrm{inf}} = \left(\alpha^2 V - 2\right) C_0^{\mathrm{inf}}, \tag{6.1a}$$

$$\cosh C_0^{\mathrm{ex}} + \frac{4}{\alpha^2 V - 2} \frac{\sinh C_0^{\mathrm{ex}}}{C_0^{\mathrm{ex}}} = -1. \tag{6.1b}$$

7. Realistic problem setting – fixing pressure at two points and other possibilities

As we have already noted—see the introductory part in §3—we have not fixed the constant C_0 using data (boundary conditions, value of the pressure at given point); we have assumed that C_0 is given *a priori*.

7.1. Fixing pressure at two points

If we—in addition to (2.1e)—try to fix the value of pressure at another point or fixing the pressure gradient, we can still have multiple solutions if we allow the pressure to be a discontinuous function. One can for example try to fix C_0 by the following additional requirement

$$\Pi|_{[x,y]=[1,0]} = e^{\frac{C}{2}}, \tag{7.1}$$

since we want the pressure to be positive, we can always rewrite the value of the pressure in the form given above.

7.1.1. Velocity profiles with inflection and non-monotone velocity profiles

Let us for simplicity assume that $C < 0$ and for the sake of illustration, let us pick $C = -3.8$, that is a value that falls into the parameter range for velocity profiles with inflection. If we want to get an inflection in the velocity profile, we have to choose an appropriate solution from options (4.4), (4.5), (4.6) and (4.7).

Let us now denote $C_0 =_{\mathrm{def}} C$. If we want $C_0 < 0$, then the available options are[20](4.4) and (4.5). But we can not use (4.5) since in this case the pressure at the centerline is given by $\Pi_R(x, y)$, and pressure gradient on the centerline is therefore positive, and consequently condition (7.1) can not be met. The only option is to use (4.4). Indeed, in this case the critical point $y_{\mathrm{crit}} = -\frac{1}{C_0} \ln M_1$ lies in the positive half line, and if we are interested in values of the pressure on the centerline, we always deal with $\Pi_L(x, y)$ in (4.4b), obviously the pressure fulfills (2.1e) and (7.1). We therefore have at least one solution

$$U(y) = \frac{y+1}{\alpha^2} - \frac{4M_1}{\alpha^2 C} \frac{e^{Cy} - e^{-C}}{(M_1 e^{-C} + 1)(M_1 e^{Cy} + 1)}, \; y \in [-1, 1], \tag{7.2a}$$

$$\Pi(x, y) = \begin{cases} \Pi_L(x, y) = \frac{M_1 + e^{-Cy}}{M_1 + 1} e^{C\frac{x+y}{2}}, & y \in \left[-1, -\frac{1}{C} \ln M_1\right), \\ \Pi_R(x, y) = \frac{\frac{1}{M_1} + e^{Cy}}{\frac{1}{M_1} + 1} e^{-C\frac{x+y}{2}}, & y \in \left[-\frac{1}{C} \ln M_1, 1\right]. \end{cases} \tag{7.2b}$$

[20]Suppose that we have already found solutions to (3.2a) and we have denoted them in such way that $M_1 > 1$ and $M_2 < 1$.

Let us try to generate another solution, let us now denote $C_0 =_{\text{def}} -C$, if we want $C_0 > 0$ then the available options are[21](4.6) and (4.7). Now we can not use alternative (4.6) since for this type of solution the pressure at the centerline is given by $\Pi_R(x, y)$ and pressure is therefore increasing along the centerline. Formula (4.7) however gives a solution, because in this case pressure at the centerline is given by $\Pi_L(x, y)$ and we can satisfy condition (7.1). The alternative solution is therefore

$$U(y) = \frac{y+1}{\alpha^2} + \frac{4\frac{1}{M_1}}{\alpha^2 C}\frac{e^{-Cy} - e^{C}}{\left(\frac{1}{M_1}e^{C} + 1\right)\left(\frac{1}{M_1}e^{-Cy} + 1\right)},\ y \in [-1, 1], \tag{7.3a}$$

$$\Pi(x, y) = \begin{cases} \Pi_L(x, y) = \frac{M_1 + e^{-Cy}}{M_1 + 1} e^{C\frac{x+y}{2}}, & y \in \left[-1, -\frac{1}{C}\ln M_1\right), \\ \Pi_R(x, y) = \frac{\frac{1}{M_1} + e^{Cy}}{\frac{1}{M_1} + 1} e^{-C\frac{x+y}{2}}, & y \in \left[-\frac{1}{C}\ln M_1, 1\right], \end{cases} \tag{7.3b}$$

but unfortunately, formulae (7.2) and (7.3) are the same, since the formulae for pressure are obviously identical, and we have (4.1) for the velocity.

Furthermore, we can also have a solution with a non-monotone velocity profile that is given by (5.2) with $C_0 = C$, thus

$$U(y) = \begin{cases} \frac{y+1}{\alpha^2} + \frac{4}{\alpha^2 C}\frac{1 - e^{C(y+1)}}{\left(1 + e^{C(y-Y)}\right)\left(1 + e^{C(Y+1)}\right)}, & y \in [-1, Y), \\ V + \frac{1-y}{\alpha^2} - \frac{4}{\alpha^2 C}\frac{1 - e^{C(y-1)}}{\left(1 + e^{C(y-Y)}\right)\left(1 + e^{C(Y-1)}\right)}, & y \in [Y, 1], \end{cases} \tag{7.4a}$$

$$\Pi(x, y) = \frac{1 + e^{C(Y-y)}}{1 + e^{CY}} e^{C\left(\frac{x+y}{2}\right)},\ y \in [-1, 1], \tag{7.4b}$$

where Y is a solution to implicit equation[22]

$$\frac{4\sinh(CY)}{\cosh(CY) + \cosh C} - 2CY = -\alpha^2 VC. \tag{7.5}$$

We therefore have two solutions, solution (7.4) with non-monotone velocity profile and continuous pressure and (7.2) with monotone velocity profile with inflection and discontinuous pressure.

7.1.2. Monotone velocity profiles without inflection point

If C is in the parameter range that does not lead to velocity profiles with inflection (let us again for definiteness fix $C = -1$, $C < 0$), then the situation is different from the situation in the previous paragraph, because now we do not have the possibility of dealing with a non-monotone velocity profile, since for this particular choice of C we—from (7.5)—would get $Y > 1$ (in our case $Y = 1.723618963$), but one of the assumptions that leads to a solution of type (7.4) was that $Y \in (-1, 1)$.

One should also note, that the value of C that gives $Y = 1$ is equal to C^0_{inf} given by (6.1a), this is obvious since for $Y = 1$, the implicit equation (7.5) coincides with (6.1a). Therefore whenever the critical point y_{crit} is within the channel, Y is also within the channel and *vice versa*—velocity profiles with inflection are always accompanied by non-monotone velocity profiles.

Now we can take $C_0 =_{\text{def}} C$ in (4.4) and (4.5), and a quick calculation gives $y_{\text{crit}} = -\frac{1}{C_0}\ln M_1 > 1$ in (4.4) and $y_{\text{crit}} = \frac{1}{C_0}\ln M_1 < -1$ in (4.5). In our particular case we have $M_1 = 6.152923926$ and consequently $y_{\text{crit}} = \pm 1.816927404$. Therefore, the pressure in (4.4) is, for all $y \in [-1, 1]$, given by $\Pi_L(x, y)$, whereas in (4.5) pressure in the whole domain is equal to $\Pi_R(x, y)$. More precisely, (4.4) is for a parameter range that does not lead to velocity profiles with inflection, and is given by

$$U(y) = \frac{y+1}{\alpha^2} - \frac{4M_1}{\alpha^2 C_0}\frac{e^{C_0 y} - e^{-C_0}}{(M_1 e^{-C_0} + 1)(M_1 e^{C_0 y} + 1)},\ y \in [-1, 1], \tag{7.6a}$$

$$\Pi(x, y) = \Pi_L(x, y) = \frac{M_1 + e^{-C_0 y}}{M_1 + 1} e^{C_0\frac{x+y}{2}},\ y \in [-1, 1], \tag{7.6b}$$

[21] Suppose that we have already found solutions to (3.2a) and we have denoted them in such way that $M_1 > 1$ and $M_2 < 1$.
[22] A brief inspection of the equation shows that the equation has an unique solution.

whereas (4.5) reduces to

$$U(y) = \frac{y+1}{\alpha^2} - \frac{4\frac{1}{M_1}}{\alpha^2 C_0} \frac{e^{C_0 y} - e^{-C_0}}{\left(\frac{1}{M_1} e^{-C_0} + 1\right)\left(\frac{1}{M_1} e^{C_0 y} + 1\right)}, \; y \in [-1,1]\,, \tag{7.7a}$$

$$\Pi(x,y) = \Pi_R(x,y) = \frac{M_1 + e^{C_0 y}}{M_1 + 1} e^{-C_0 \frac{x+y}{2}}, \; y \in [-1,1]\,. \tag{7.7b}$$

Obviously, if we want to satisfy condition (7.1), we have to choose a solution of the type (4.4), thus (7.6). Both "solutions" are compared in Figure 4. Now it is easy to understand why Hron et al. [1] claim that the problem has in this case multiple solutions. If we indeed solve the problem in such way that C_0 is almost treated as a material constant ("clearly related the pressure gradient along the x-direction") then both (7.6a) and (7.7) are solutions, nevertheless if we—instead of just assigning a value for C_0—require an explicit procedure for fixing its value (fixing pressure at two points, fixing pressure gradient and pressure at one point), then this procedure will automatically tell us which solution must be ignored, and which solution is the unique solution to the problem. If Hron et al. [1] had given plots of pressure on the channel centerline—Figure 4(f)—they would have been able to notice the importance of being more specific with regard to which procedure is to be used in fixing the remaining free parameter C_0.

7.2. *Fixing pressure gradient and pressure at one point*

Another possibility of how to make the problem determinate is to require the pressure to attain a given value at a certain point, (2.1e), and additionally to have a prescribed gradient at this point,

$$\left.\frac{\partial \Pi}{\partial x}\right|_{[x,y]=[0,0]} = \frac{C}{2}. \tag{7.8}$$

In this case the discussion is identical to the discussion in the previous section. Indeed if C allows velocity profile with inflection and a non-monotone velocity profile, then (7.2) and (7.4) are solutions to problem (2.1) with condition (2.1e) supplemented by (7.8). Similarly, if C does not allow velocity profiles with inflection, (7.6) is solution to problem (2.1) with condition (2.1e) supplemented by (7.8).

7.3. *Fixing volumetric flow rate and pressure at one point*

Another possibility is to fix the pressure at one point, (2.1e), and the volumetric flow rate, thus

$$\int_{-1}^{1} U(y)\, \mathrm{d}y = Q. \tag{7.9}$$

Although it is easy to get the formulae for volumetric flow rate, the equation for C_0 that arises form (7.9) is a complicated implicit equation. The numerical treatment of the problem is more suitable in this case, and numerical results have been given in Hron et al. [1].

8. Analytical formulae for discontinuous velocity profiles with continuous pressure

Now one can ask, whether we can for example relax the assumption on the continuity of velocity[23]. We can try to require pressure to take form (3.4), for definiteness we can fix pressure to have the same form as in §5 thus

$$\Pi(x,y) = \frac{1 + e^{C_0(Y-y)}}{1 + e^{C_0 Y}} e^{C_0\left(\frac{x+y}{2}\right)}, \; y \in [-1,1]\,. \tag{8.1a}$$

[23] This is again (see also Footnote 12) a procedure that we should probably avoid. We can in fact obtain discontinuous velocity profiles even for the classical Navier–Stokes fluid, indeed, if we consider Couette–Poiseuille channel flow of the classical Navier–Stokes fluid, then any parabolic velocity profile $U(y)$ solves the governing equations (if we fix the coefficient at y^2 to match the prescribed pressure gradient). Now we can divide the channel to two parts and we can use a different parabola in each part of the channel, the parabola in the bottom part of the channel will be fixed in such way that it fulfills the boundary condition on the bottom plane and the parabola in the upper part of the channel will be fixed such that it fulfills the boundary condition on the upper plane. But we still have enough degree of freedom to change for example slope of the parabolas, and if we do not require the velocity to be continuous we can chose slopes is such way, that the parabolas will not meet on the artificial boundary separating upper and bottom part of the channel.

From §5 we already know that velocity profile that corresponds to this pressure is (5.2a) thus

$$U(y) = \begin{cases} \frac{y+1}{\alpha^2} + \frac{4}{\alpha^2 C_0} \frac{1-e^{C_0(y+1)}}{\left(1+e^{C_0(y-Y)}\right)\left(1+e^{C_0(Y+1)}\right)}, & y \in [-1, Y)\,, \\ V + \frac{1-y}{\alpha^2} - \frac{4}{\alpha^2 C_0} \frac{1-e^{C_0(y-1)}}{\left(1+e^{C_0(y-Y)}\right)\left(1+e^{C_0(Y-1)}\right)}, & y \in [Y, 1]\,. \end{cases} \tag{8.1b}$$

When we were examining this non-monotone velocity profile, we required continuity of velocity at point Y, and this requirement lead to the implicit equation (5.3) that fixed the value of Y. But if we do not want the velocity to be continuous we can choose Y in (8.1) arbitrarily, and consequently we get a discontinuous velocity profile with jump discontinuity at point Y. A plot of such a solution is given in Figure 5.

References

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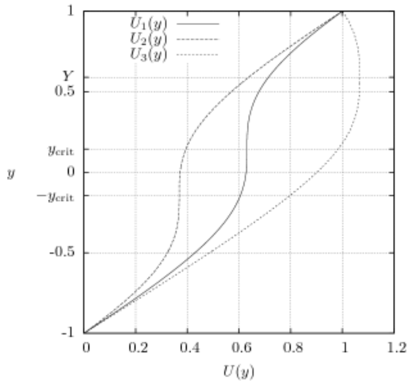
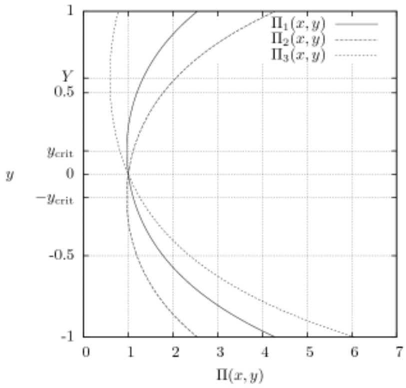
[2] Málek, J. and K. R. Rajagopal (2007). Mathematical properties of the solutions to the equations governing the flow of fluids with pressure and shear rate dependent viscosities. In S. Friedlander and D. Serre (Eds.), *Handbook of mathematical fluid dynamics*, Volume 4, pp. 407–444. Amsterdam: Elsevier.

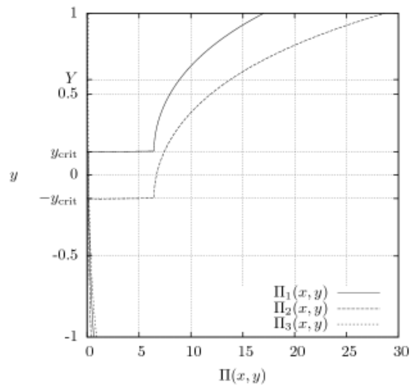
[3] Suslov, S. A. and T. D. Tran (2008, 154). Revisiting plane Couette-Poiseuille flows of a piezo-viscous fluid. *Journal of Non-Newtonian Fluid Mechanics 154*(2-3), 170–178.



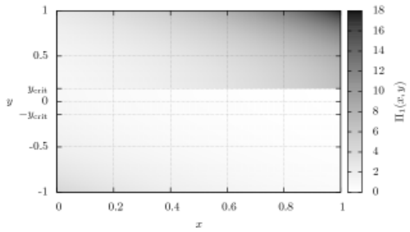
(a) Velocity profiles.



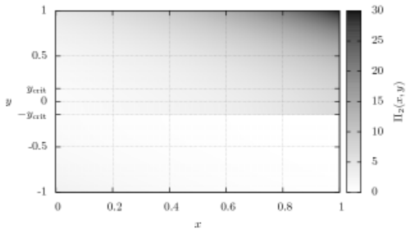
(b) Pressure at vertical section $x = 0$.



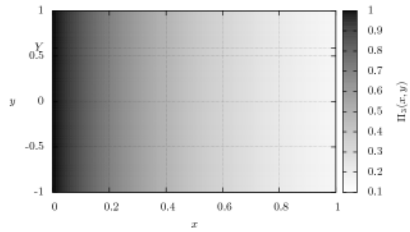
(c) Pressure at vertical section $x = 1$.



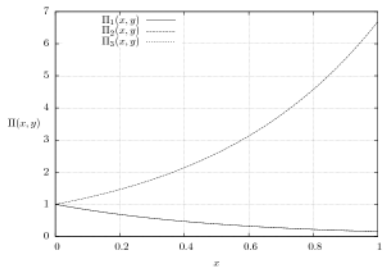
(d) Pressure in the whole domain, solution (4.4).



(e) Pressure in the whole domain, solution (4.5).



(f) Pressure in the whole domain, solution (5.2).



(g) Pressure at horizontal section $y = 0$. (Curve for $\Pi_1(x, y)$ coincides with the curve for $\Pi_3(x, y)$.)

Figure 2: Plots of the velocity profiles and corresponding pressure, solution (4.4) is denoted as $U_1(y)$, $\Pi_1(x, y)$, solution (4.5) as $U_2(y)$, $\Pi_2(x, y)$, and solution (5.2) as $U_3(y)$, $\Pi_3(x, y)$. ($\alpha = 1$, $V = 1$, $C_0 = -3.8$)

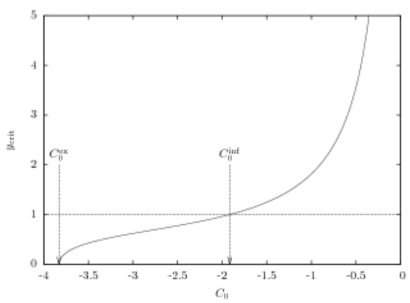
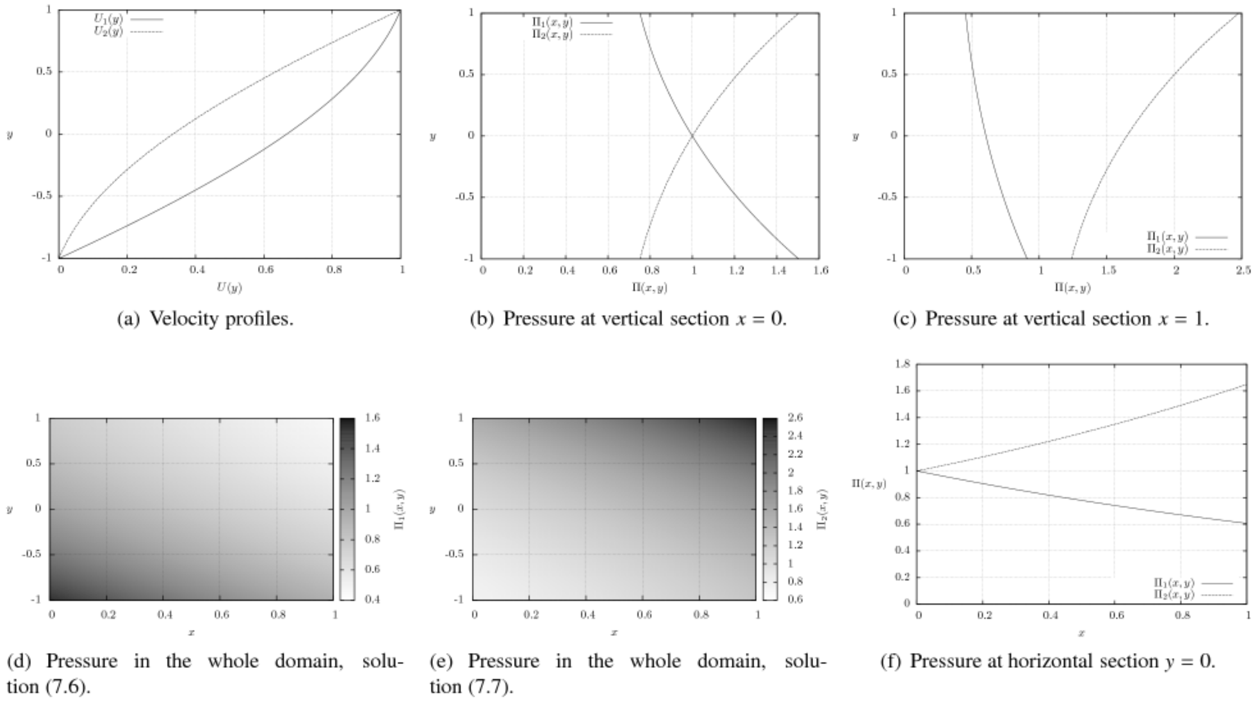
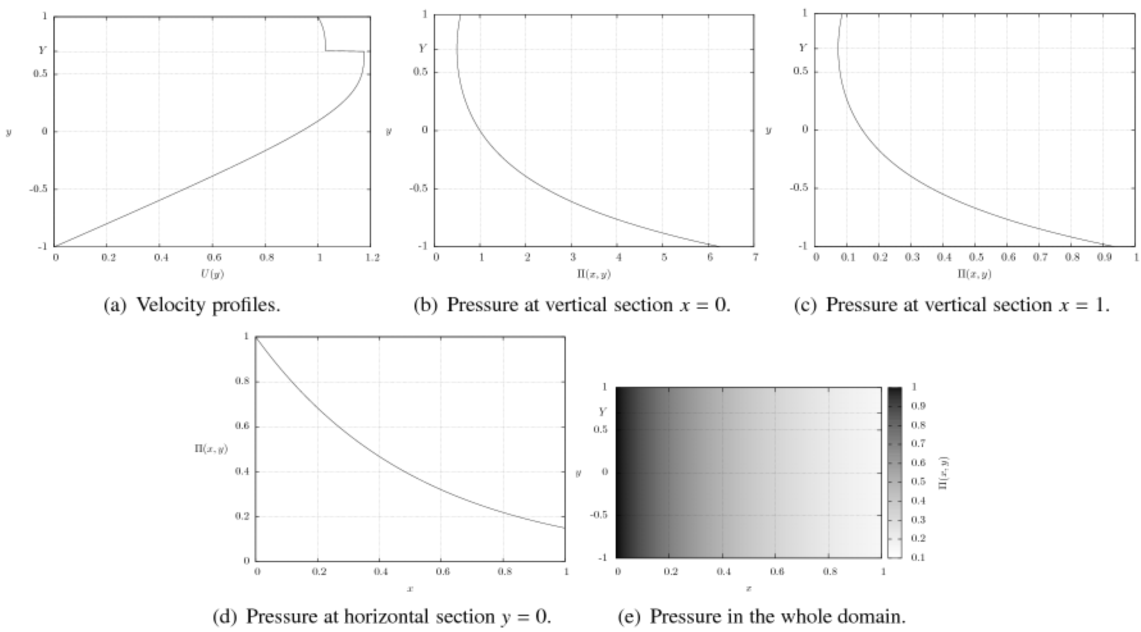


Figure 3: Plot of $y_{crit} = -\frac{1}{C_0} \ln\left(-\delta + \sqrt{\delta^2 - 1}\right)$, where δ is given by (3.2b). ($\alpha = 1$, $V = 1$)



(a) Velocity profiles.
(b) Pressure at vertical section $x = 0$.
(c) Pressure at vertical section $x = 1$.
(d) Pressure in the whole domain, solution (7.6).
(e) Pressure in the whole domain, solution (7.7).
(f) Pressure at horizontal section $y = 0$.

Figure 4: Plots of velocity profiles and corresponding pressure, comparison of (7.6) and (7.7). ($\alpha = 1$, $V = 1$, $C = -3.8$)



(a) Velocity profiles.
(b) Pressure at vertical section $x = 0$.
(c) Pressure at vertical section $x = 1$.
(d) Pressure at horizontal section $y = 0$.
(e) Pressure in the whole domain.

Figure 5: Plots of the velocity profile and corresponding pressure, solution (8.1). ($Y = 1$, $V = 1$, $\alpha = 1$, $C_0 = -3.8$)