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FURTHER RESULTS FOR SEMIREGENERATIVE PHENOMENA

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Abstract

A theory of semiregenerative phenomena was developed by the author (Prabhu [11]). The set of points at which such a phenomenon occurs is called a semiregenerative set. There is a correspondence between a semiregenerative set and the range of a Markov subordinator with a unit drift (or a Markov renewal process in the discrete time case). Prabhu, Tang and Zhu [14] showed that the properties of semiregenerative sets associated with Markov random walks completely characterize the fluctuation behaviour of these processes in the nondegenerate case and also established a Wiener-Hopf factorization based on these sets. These results are surveyed in this paper.

Key words: Borel measures, fluctuation behaviour, linked regenerative phenomena, Markov-additive process, Markov random walk, Markov renewal process, quasi-Markov chain, recurrent and regenerative phenomena, semiregenerative phenomena, semiregenerative sets, subordinator, Wiener-Hopf factorization.

1 Introduction

In the literature semiregenerative processes are defined as those having an imbedded Markov renewal process (MRP). Adopting a different approach the author developed a theory of semiregenerative phenomena and gave some examples (Prabhu [11]). The connecting idea is that of the semiregenerative set whose elements are (roughly speaking) the points at which the phenomenon occurs. In discrete time this set coincides with the range of an MRP. If the observed range of a process contains a semiregenerative set as a subset, then the process is semiregenerative. It is important, however, to study the phenomenon itself independently of the process.

The main results of the theory of semiregenerative phenomena are direct extensions of Feller's [4] theory of recurrent events (phenomena) in discrete time and Kingman's [9] theory of regenerative phenomena in continuous time. It is known that a recurrent set is the range of a renewal process and a regenerative set is the range of a subordinator with a unit drift. A random walk on the real line gives rise to two recurrent sets consisting of the ascending and descending ladder points. Gut and Prabhu [5] showed that the Wiener-Hopf factorization is essentially a property of Borel measures based on these sets and is thus free of transform techniques that have been traditionally used in the literature. These authors also derived the fluctuation theory of random walks as a consequence of the properties of these sets being terminating or nonterminating.

It turns out that the approach used by Gut and Prabhu [5] can also be used for Markov random walks (MRWs). The Wiener-Hopf factorization result here goes back to Presman [16], while several other authors have also investigated the same. The techniques used by these authors are heavily analytical, including those from operator theory (see, however, Asmussen [1]). Prabhu, Tang and Zhu [14] established this factorization as a property of positive Borel measures based on semirecurrent sets associated with the process. They also gave a complete characterization of the fluctuation behavior of MRW's by using the properties of these sets.

In this paper we first give a brief sketch of the theory of semiregenerative phenomena (section 2). We next deal with MRWs, which are discrete time versions of Markov-additive processes (MAPs). Section 3 contains the basic definitions and a description of the fluctuation behaviour of a nondegenerate MRW. The semirecurrent phenomena associated with an MRW are treated in section 4, where the results of Prabhu, Tang and Zhu [14] are surveyed. In the final section 5 we consider continuous time phenomena and explore the connection between a semiregenerative set and the range of a Markov subordinator with a unit drift (this is an MAP whose additive component has nondecreasing sample functions).

The references cited in this paper contain references to other approaches to semiregenerative sets, Wiener-Hopf factorization of MRWs and theory of Markov renewal and Markov-additive processes (however, for a recent survey of these processes see Prabhu [12]). For queueing applications see Asmussen [2], Prabhu and Tang [13], and Prabhu and Zhu [15]. Additional references on Wiener-Hopf factorization in continuous time are Barlow, Rogers and Williams [3], Kaspi [6] and Kennedy and Williams [7].

2 Semiregenerative phenomena

Let the set T be either $[0,\infty)$ or $\{0,1,2,\ldots\}$, $\mathcal{E}$ a countable set and $(\Omega,\mathcal{F},P)$ a probability space.

Definition 1 *A semiregenerative phenomenon* $Z=\{Z_{t\ell},(t,\ell)\in T\times\mathcal{E}\}$ *on a probability space* $(\Omega,\mathcal{F},P)$ *is a stochastic process taking values 0 or 1 and such that for* $(t_r,\ell_r)\in T\times\mathcal{E}$ $(r\geq 1)$, *with* $0=t_0\leq t_1\leq\ldots\leq t_r$, $j\in\mathcal{E}$ *we have*

$$P\{Z_{t_1\ell_1}=Z_{t_2\ell_2}=\ldots=Z_{t_r\ell_r}=1|Z_{0j}=1\}$$

$$=\prod_{i=1}^{r}P\{Z_{t_i-t_{i-1},\ell_i}=1|Z_{0,\ell_{i-1}}=1\}\quad(\ell_0=j). \tag{1}$$

For each $\ell\in\mathcal{E}$, denote $Z_\ell=\{Z_{t\ell},t\in T\}$. Since

$$P\{Z_{t_1\ell}=Z_{t_2\ell}=\ldots=Z_{t_r\ell}=1|Z_{0j}=1\}$$

$$=P\{Z_{t_1\ell}=1|Z_{0j}=1\}\prod_{i=2}^{r}P\{Z_{t_i-t_{i-1},\ell}=1|Z_{0\ell}=1\}$$

Z_ℓ is a (possibly delayed) regenerative phenomenon in the sense of Kingman [9] in the continuous time case $T=[0,\infty)$, and a recurrent event (phenomenon) in the sense of Feller [4] in the discrete time case $T=\{0,1,2,\ldots\}$.

The family $Z' = \{Z_\ell, \ell \in \mathcal{E}\}$ is a family of linked regenerative phenomena, for which a theory was developed by Kingman [8] in the case of finite $\mathcal{E}$; later he reformulated the results in terms of quasi-Markov chains (Kingman [9]). This concept is explained below.

Example 1 Let $J = \{J_t,\ t \in T\}$ be a time-homogeneous Markov chain on the state space $\mathcal{E}$ and C a fixed subset of $\mathcal{E}$. Denote

$$Z_{t\ell} = 1_{\{J_t=\ell\}} \text{ for } (t,\ell) \in T \times C. \tag{2}$$

The random variables $Z_{t\ell}$ satisfy the relation (1) and thus $Z = \{Z_{t\ell}, (t,\ell) \in T \times C\}$ is a semiregenerative phenomenon. In the case of a finite subset C define

$$K_t = J_t \text{ if } J_t \in C, \text{ and } = 0 \text{ if } J_t \notin C. \tag{3}$$

Then $\{K_t,\ t \in T\}$ is defined to be a quasi-Markov chain on the state space $C \cup \{0\}$. □

We specify the initial distribution $\{a_j, j \in \mathcal{E}\}$, where

$$P\{Z_{0j} = 1\} = a_j \tag{4}$$

with $a_j \geq 0, \sum a_j = 1$. The relation (1) determines all finite dimensional distributions of Z. It can also be proved that Z is strongly regenerative (that is, (1) holds for stopping times). We shall write P_j and E_j for the probability and the expectation conditional on the event $\{Z_{0j} = 1\}$.

Associated with a semiregenerative phenomenon Z is the set

$$\zeta = \{(t,\ell) \in T \times \mathcal{E} : Z_{t\ell} = 1\} \tag{5}$$

which is called a semiregenerative set. The main result of Prabhu [11] is the correspondence between the set ζ and the range of an MRP (in the discrete time case) and of a Markov subordinator with a unit drift (in the continuous time case).

In this section we consider semirecurrent (that is, discrete time regenerative) phenomena. Accordingly, let $\mathcal{N}_+ = \{0, 1, 2, \ldots\}$, $\mathcal{L} = \mathcal{N}_+ \times \mathcal{E}$ and ζ the semi-recurrent set defined by (5). Let

$$u_{jk}(n) = P\{Z_{nk} = 1 | Z_{0j} = 1\} \tag{6}$$

where $u_{jk}(0) = \delta_{jk}$.

Definition 2 *Let $T_0 = 0$ and for $r \geq 1$*

$$T_r = \min\{n > T_{r-1} : (n, \ell) \in \zeta \ \text{ for some } \ \ell\}. \tag{7}$$

We shall call T_r the semirecurrence times of the phenomenon Z.

Let $J_r = \ell$ when $Z_{T_r,\ell} = 1$. This ℓ is unique because of definition (1). We have then the following.

Theorem 1 *The process $\{(T_r, J_r),\ r \geq 0\}$ is an MRP on the state space $\mathcal{L}$. Denote by*

$$q_{jk}^{(r)}(n) = P_j\{T_r = n, J_r = k\} \tag{8}$$

the transition probabilities of this MRP, where we shall write q_{jk} for $q_{jk}^{(1)}(n)$. Then the probabilities $\{u_{jk}(n), n \in \mathcal{N}_+, j, k \in \mathcal{E}\}$ defined by (6) form the unique solution of the Markov renewal equation

$$x_{jk}(n) = q_{jk}(n) + \sum_{m=1}^{n-1} \sum_{\ell \in \mathcal{E}} q_{j\ell}(m) x_{\ell k}(n-m) \tag{9}$$

with $0 \leq x_{jk}(n) \leq 1$. This solution is given by

$$u_{jk}(n) = \sum_{r=1}^{n} q_{jk}^{(r)}(n). \qquad \square \tag{10}$$

Theorem 1 states that a semirecurrent phenomenon gives rise to an MRP with lifetimes $T_r - T_{r-1}$ $(r \geq 1)$ concentrated on the set $\{1, 2, \ldots\}$. The following theorem states that this is the only way that a semirecurrent phenomenon can occur.

Theorem 2 *(i) Let $\{(T_r, J_r),\ r \geq 0\}$ be an MRP on the state space $\mathcal{L}$, with lifetime distribution concentrated on $\{1, 2, \ldots\}$. Denote by*

$$\mathcal{R} = \{(n, \ell) \in \mathcal{L} : (T_r, J_r) = (n, \ell) \ \text{ for some } \ r \geq 0\} \tag{11}$$

the range of this process and $Z'_{n\ell} = 1_{(n,\ell) \in \mathcal{R}}$. Then $Z' = \{Z'_{n\ell}, (n, \ell) \in \mathcal{L}\}$ is a semirecurrent phenomenon.

(ii) Conversely, any semirecurrent phenomenon Z is equivalent to a phenomenon Z' generated in the above manner in the sense that Z and Z' have the same $\{u_{jk}(n)\}$ sequence. $\square$

Example 1 (continuation). For a quasi-Markov chain in discrete time the semirecurrence times are the hitting times of the set C. If $K_0 = j \in C$, the process spends one unit of time in j and $T_1 - 1$ units outside of C before returning to C. If $C = \mathcal{E}$, then $T_r = r$ almost surely ($a.s.$) for all $r \geq 0$. □

Example 2 Let $\{(S_n, J_n),\ n \in \mathcal{N}_+\}$ be an MRW on the state space $\mathbb{R}_+ \times \mathcal{E}$. A detailed definition of these processes will be given in section 3. Let

$$M_n = \max(0, S_1, S_2, ..., S_n), \quad m_n = \min(0, S_1, S_2, ..., S_n). \tag{12}$$

Considering M_n first, denote

$$Z_{n\ell} = 1_{\{M_n - S_n = 0,\ J_n = \ell\}}, \quad (n, \ell) \in \mathcal{L}. \tag{13}$$

We have $P\{Z_{0j} = 1\} = P\{J_0 = j\}$. For $0 = n_0 \leq n_1 \leq \ldots \leq n_r \quad (r \geq 1)$ it can be easily verified that the random variables $Z_{n\ell}$ satisfy the property

$$\begin{aligned} P\{Z_{n_1\ell_1} = Z_{n_2\ell_2} = \ldots = Z_{n_r\ell_r} = 1 | Z_{0j} = 1\} \\ = u_{j\ell_1}(n_1) u_{\ell_1\ell_2}(n_2 - n_1) \ldots u_{\ell_r - \ell_{r-1}}(n_r - n_{r-1}) \end{aligned} \tag{14}$$

where

$$u_{jk}(n) = P\{S_m \leq S_n \quad (0 \leq m \leq n), \quad J_n = k \mid J_0 = j\}. \tag{15}$$

Thus $Z = \{Z_{n\ell}\}$ is a semirecurrent phenomenon. We obtain a second such phenomenon by considering a minimum functional m_n. These two phenomena (or the associated recurrent sets) determine the fluctuation behaviour of the MRW, as will be seen in section 4. □

3 Markov random walks

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and denote $\mathbb{R} = (-\infty, \infty)$ and $\mathcal{E}$ = a countable set.

Definition 3 *A Markov random walk* $(S, J) = \{(S_n, J_n),\ n \geq 0\}$ *is a Markov process on the state space* $\mathbb{R} \times \mathcal{E}$ *whose transition distribution measure has the property*

$$\begin{aligned} P\{(S_{m+n}, J_{m+n}) \in A \times \{k\} | (S_m, J_m) = (x, j)\} \\ = P\{(S_{m+n} - S_m, J_{m+n}) \in (A - x) \times \{k\} | J_m = j\} \end{aligned} \tag{16}$$

for $j, k \in \mathcal{E}$ and a Borel subset A of $\mathbb{R}$. We shall only consider the time-homogeneous case in which the second probability in (16) does not depend on m; we denote it as $Q_{jk}^{(n)}\{A - x\}$, so that

$$Q_{jk}^{(n)}\{A\} = P\{(S_n, J_n) \in A \times \{k\} | J_0 = j\}. \tag{17}$$

To complete the definition we need to define the initial measure $\{a_j\}$, where

$$\begin{aligned} P\{(S_0, J_0) \in A \times \{j\}\} &= a_j \text{ if } 0 \in A \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{18}$$

The marginal process $J = \{J_n, n \geq 0\}$ is a Markov chain on $\mathcal{E}$ with transition probabilities $P_{jk}^{(n)} = Q_{jk}^{(n)}\{\mathbb{R}\}$. We shall assume that J is irreducible.

If the measure Q is concentrated on $[0, \infty) \times \mathcal{E}$, then the process reduces to an MRP.

Suppose $J_0 = j$. The process obtained by looking at $\{S_n\}$ at the epochs of successive visits of the Markov chain J is a random walk imbedded in the MRW. The MRW may be viewed as a family of such random walks indexed by $\mathcal{E}$.

We need the time-reversed version of the given MRW. This is defined as the MRW $(\hat{S}, \hat{J}) = \{(\hat{S}_n, \hat{J}_n), \ n \geq 0\}$ whose transition distribution measure $\hat{Q}$ is given by

$$\hat{Q}_{jk}\{A\} = \frac{\pi_k}{\pi_j} Q_{kj}\{A\} \tag{19}$$

where $\{\pi_j\}$ is the stationary measure of J (and also of $\hat{J}$). From (19) the n-step transition distribution measure of the time-reversed MRW is obtained as

$$\hat{Q}_{jk}^{(n)}\{A\} = \frac{\pi_k}{\pi_j} Q_{kj}^{(n)}\{A\} \quad (n \geq 0). \tag{20}$$

If all the random walks imbedded in the MRW are degenerate (that is, the distribution of the increments is concentrated at the origin), then we shall say that the MRW itself is degenerate. The fluctuation behaviour of degenerate MRWs has no analogue in the theory of random walks. Thus it cannot be claimed that a degenerate MRW is uniformly bounded (that is, $\liminf S_n$ and $\limsup S_n$ are both finite), the exception being the case of finite $\mathcal{E}$.

For nondegenerate MRWs with J persistent the fluctuation behaviour is identical with that of random walks. Thus they are of the following mutually exclusive types:

(i) $M_n \to \infty,\ m_n \to m > -\infty,\ S_n \to +\infty$ a.s.

(ii) $M_n \to M < \infty,\ m_n \to -\infty,\ S_n \to -\infty$ a.s.

(iii) $M_n \to \infty,\ m_n \to -\infty, -\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = \infty.$

If the mean dirft

$$\mu = \sum_{k\in\xi} \pi_k E_k(S_1)$$

exists and is finite, then (i), (ii) and (iii) hold according as $\mu > 0, < 0$ or $= 0$ (Prabhu, Tang and Zhu [14], Theorems 7 and 13).

4 Semirecurrent phenomena in Markov random walks

Let M_n and m_n be the maximum and minimum functionals of $\{S_n\}$ as in (12) and define the random sets

$$\zeta_+ = \{(n,j) : M_n - S_n = 0,\ J_n = j\} \tag{21}$$

$$\zeta_- = \{(n,j) : S_n - m_n = 0,\ J_n = j\}. \tag{22}$$

The results of Example 2 can be expressed as follows.

Theorem 3 *The sets ζ_+ and ζ_- are semirecurrent sets.* □

The elements of ζ_+ and ζ_- are respectively $\{(T_k, J_k),\ k \geq 0\}$ and $\{(\overline{T}_k, J_{\overline{T}_k}),\ k \geq 0\}$, where $T_0 = \overline{T}_0 = 0$ a.s. and for $k \geq 1$

$$T_k = \min\{n > T_{k-1} : S_n \geq S_{T_{k-1}}\} \tag{23}$$

$$\overline{T}_k = \min\{n > \overline{T}_{k-1} : S_n \leq S_{\overline{T}_{k-1}}\}. \tag{24}$$

Thus $\{T_k\}$ and $\{\overline{T}_k\}$ are respectively the ascending and descending ladder epochs of $\{S_n\}$. We call ζ_+ and ζ_- the ladder sets of the MRW. Since every semirecurrent set is the range of a (unique) MRP by Theorem 2, it follows that the ladder processes $\{T_k, J_{T_k}\}$ and $\{\overline{T}_k, J_{\overline{T}_k}\}$ are both MRPs.

The fluctuation behaviour of the MRW is related to the properties of its ladder sets being terminating and nonterminating sets, as shown by the following.

Theorem 4 *For a nondegenerate MRW (S, J) with J irreducible and persistent it is impossible for both ζ_+ and ζ_- to be terminating. Furthermore we have the following:*

(i) ζ_+ is nonterminating and ζ_- is terminating $\Leftrightarrow M_n \to \infty$, $m_n \to m > -\infty \Leftrightarrow \lim S_n = \infty$;

(ii) ζ_+ is terminating and ζ_- nonterminating $\Leftrightarrow M_n \to M < \infty$, $m_n \to -\infty \Leftrightarrow \lim S_n = -\infty$;

(iii) ζ_+ and ζ_- are both nonterminating $\Leftrightarrow M_n \to \infty$, $m_n \to -\infty \Leftrightarrow \limsup S_n = \infty$, $\liminf S_n = -\infty$. □

The Wiener-Hopf factorization of the MRW turns out to be a property of the associated ladder sets. First we need the ladder sets of the time-reversed MRW, which we shall denote by $\hat{\zeta}_+$ and $\hat{\zeta}_-$. Thus (with obvious notation)

$$\hat{\zeta}_- = \{(n, j) : \hat{S}_n - \hat{m}_n = 0,\ \hat{J}_n = j\}. \tag{25}$$

Also, denote by ρ_n the first epoch among $(0, 1, ..., n)$ at which the maximum M_n is attained in the given MRW. We redefine the ladder set ζ_+ and write

$$\zeta_+ = \{(n, j) : M_n - S_n = 0,\ J_n = j,\ \rho_n = n\}. \tag{26}$$

With ζ_+ so defined as $\hat{\zeta}_-$ as in (25) we associate the following measures:

$$\mu_{ij}^+\{I\} = \sum_{n=0}^{\infty} E_i[s^n; (n, j) \in \zeta_+,\ S_n \in I] \tag{27}$$

$$\hat{\mu}_{ij}^-\{I\} = \sum_{n=0}^{\infty} E_i[s^n; (n, j) \in \hat{\zeta}_-,\ \hat{S}_n \in I] \tag{28}$$

for fixed $s \in (0, 1)$ and every finite interval I. Also

$$\nu_{ij}^-\{I\} = \frac{\pi_j}{\pi_i} \hat{\mu}_{ji}^-\{I\}. \tag{29}$$

Finally, let

$$\mu_{ij}\{I\} = \sum_{n=0}^{\infty} s^n Q_{ij}^{(n)}\{I\}. \tag{30}$$

These measures are all finite. We now introduce the matrix-valued measures in which the elements are indexed by $\mathcal{E} \times \mathcal{E}$:

$$\mu_s^+\{I\} = (\mu_{ij}^+\{I\}),\ \nu_s^-\{I\} = (\nu_{ij}^-\{I\}),\ \mu_s\{I\} = (\mu_{ij}\{I\}). \tag{31}$$

Then we have the following.

Theorem 5 *We have*

$$\mu_s = \mu_s^+ * \nu_s^-. \quad \square \tag{32}$$

The factorization obtained by Presman [16] in terms of transforms is a direct consequence of the above theorem.

Taking $I = \mathbb{R}$ in (32) we obtain the Wiener-Hopf factorization for the marginal chain J as follows.

Corollary 1 *Let $P = (P_{jk})$ be the transition probability matrix of the marginal chain J. For $s \in (0,1)$ we then have*

$$I - sP = U(s)V(s) \tag{33}$$

where

$$U(s) = \mu_s^+\{[0,\infty)\},\ V(s) = \nu_s^-\{(-\infty, 0]\}. \tag{34}$$

5 Continuous time phenomena

We now turn our attention to continuous time semi-regenerative phenomena. For such a phenomenon $Z = \{Z_{t\ell}\}$ let

$$P_{jk}(t) = P\{Z_{tk} = 1 | Z_{0j} = 1\} \quad (t \geq 0) \tag{35}$$

where $P_{jk}(0) = \delta_{jk}$. The phenomenon is standard if

$$P_{jk}(t) \to \delta_{jk} \text{ as } t \to 0+. \tag{36}$$

For a standard phenomenon it is known that the limit

$$\lim_{t\to 0+} \frac{1-P_{jj}(t)}{t} \quad (j \in \mathcal{E}) \tag{37}$$

exists but may be possibly infinite. We shall only consider stable semiregenerative phenomena, where the limit (37) is finite for every $j \in \mathcal{E}$.

The main result here is that the associated semi-regenerative set (5) is the range of a Markov subordinator with a unit drift. This is a Markov additive process (MAP) $(X, J) = \{X(\tau), J(\tau), \tau \geq 0\}$ on the state space $\mathbb{R}_+ \times \mathcal{E}$, whose additive component X has non-decreasing sample functions and is thus the continuous time analogue of an MRP. Although there exists a fairly extensive literature on MAPs (see the references cited by Prabhu [11] and [12]), for our purpose we need the following constructive definition of a Markov subordinator given by Prabhu [11] along the lines suggested by Neveu [10]. See also Prabhu and Zhu [15].

Let $J = \{J(\tau), \tau \geq 0\}$ be a time-homogeneous Markov chain on the state space $\mathcal{E}$, all of whose states are stable. Let $T_0 = 0$, T_n $(n \geq 1)$ the epochs of successive jumps in J and denote $J_n = J(T_n)$ $(n \geq 0)$. We define a sequence of continuous time processes $\{X_n^{(1)}, n \geq 1\}$ and a sequence of random variables $\{X_n^{(2)}, n \geq 1\}$ as follows:

(i) On $\{T_n \leq \tau < T_{n+1}, J_n = j\}$, $X_{n+1}^{(1)}(\tau)$ is a subordinator with a unit drift and Lévy measure μ_{jj}.

(ii) Given $X_m^{(1)}, X_m^{(2)}, J_m$ $(1 \leq m \leq n)$, J_0, the increment $X_{n+1}^{(1)}(\tau) - X_{n+1}^{(1)}(T_n)$ and the random variables $(X_{n+1}^{(2)}, T_{n+1}, J_{n+1})$ depend only on J_n.

(iii) Given J_n, $X_{n+1}^{(1)}(\tau) - X_{n+1}^{(1)}(T_n)$ and $(X_{n+1}^{(2)}, T_{n+1}, J_{n+1})$ are conditionally independent with

$$P\{X_{n+1}^{(1)}(\tau) - X_{n+1}^{(1)}(T_n) \in A | J_n = j\} = H_j\{\tau - T_n; A\} \tag{38}$$

and

$$P\{X_{n+1}^{(2)} \in A, T_{n+1} \in ds, J_{n+1} = k | J_n = j\} = e^{-q_{jj}s} q_{jk} M_{jk}\{A\} ds \tag{39}$$

for any Borel subset A of $\mathbb{R}_+$. Here the M_{jk} are concentrated on $[0, \infty)$, while H_j is concentrated on $[0, \infty]$; q_{jk} $(j \neq k)$ are the transition rates of J and $q_{jj} = \sum_{k \neq j} q_{jk}$ $(0 < q_{jj} < \infty)$.

According to this construction, on intervals where $J(\tau) = j$, the process X evolves as an ordinary subordinator with Lévy measure μ_{jj} depending on j. In addition, when J jumps from j to k, X receives a jump with distribution M_{jk}; these are the Markov-modulated jumps.

Now let

$$S_0 = 0, S_n = \sum_{m=1}^{n} [X_m^{(1)}(T_m) - X_m^{(1)}(T_{m-1}) + X_m^{(2)}] \quad (n \geq 1) \tag{40}$$

and

$$L' = \sup_{n \geq 0} T_n, \;\; L = \sup_{n \geq 0} S_n. \tag{41}$$

Then the Markov subordinator (X, J) that we need is defined as follows:

$$\begin{aligned} \{X(\tau), J(\tau)\} &= \{S_n + X_{n+1}^{(1)}(\tau) - X_{n+1}^{(1)}(T_n), J_n\} \;\; \text{for} \;\; T_n \leq \tau < T_{n+1} \\ &= (L, \Delta) \;\; \text{for} \;\; t \geq L' \end{aligned} \tag{42}$$

where Δ is a point of compactification of the set $\mathcal{E}$.

The transition distribution measure of (X, J) can be expressed in terms of the Markov renewal measure associated with the $MRP\{(S_n, T_n, J_n), \; n \geq 0\}$. We are here interested in the range of this process, which is,

$$\mathcal{R} = \{(t, \ell) \in I\!\!R_+ \times \mathcal{E} : (X(\tau), J(\tau)) = (t, \ell) \text{for some } \; \tau \geq 0\}. \tag{43}$$

Let

$$Z_{t\ell} = 1_{(t,\ell) \in \mathcal{R}}, P_{jk}(t) = P\{Z_{tk} = 1 | Z_{0j} = 1\}. \tag{44}$$

These probabilities $P_{jk}(t)$ will be expressed in terms of the Markov renewal measure associated with the MRP $\{(S_n, J_n), n \geq 0\}$, namely,

$$U_{jk}\{A\} = \sum_{n=0}^{\infty} P_j\{S_n \in A, J_n = k\}. \tag{45}$$

Our construction makes it clear that

$$\mathcal{R} = \bigcup_{n=0}^{\infty} [\mathcal{R}_{X_{n+1}^{(1)}} \times \{J_n\}] \tag{46}$$

where $\mathcal{R}_{X_{n+1}^{(1)}}$ is the range of the subordinator $X_{n+1}^{(1)}$. We know that each $\mathcal{R}_{X_{n+1}^{(1)}}$ is a regenerative set (Kingman [9], section 4.2) with its p-function given by

$$p_j(t) = P\{t \in \mathcal{R}_{X_{n+1}^{(1)}} | J_n = j\} \tag{47}$$

We have then the following.

Theorem 6 *The family $Z = \{Z_{t\ell}, (t, \ell) \in \mathbb{R}_+ \times \mathcal{E}\}$ is a standard semiregenerative phenomenon for which*

$$P_{jk}(t) \geq \int_0^t U_{jk}\{ds\} p_k(t-s). \tag{48}$$

The equality in (48) holds iff $L = \infty$. □

Theorem 6 states that the range of the Markov subordinator constructed above is a semiregenerative phenomenon. We ask whether semiregenerative phenomena can arise only in this manner. From our approach it turns out that for finite $\mathcal{E}$, the answer to this question is in the affirmative. This is also true in special cases with $\mathcal{E}$ countable (Prabhu [11]).

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