STATISTICA, anno LXXX, n. 2, 2020

FURTHER RESULTS ON A CLASS OF DISTRIBUTIONS WHICH INCLUDES THE NORMAL ONES – LOOKING BACK

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1. Foreword

The initiative taken by the Editor of *Statistica*, Professor Simone Giannerini, of reprinting my paper published in 1986 is most welcome and gratifying. At the same time, it is sad that this initiative is connected with the celebration of the premature loss of Antonella Capitanio, a dear person and a solid scientist with whom I collaborated extensively. Starting from our first meeting in 1996 onwards, Antonella's contribution has been fundamental for the development of the literature connected to this paper.

This reprint offers the occasion for providing some retrospective considerations on the significance of the paper within the pertaining literature. The 1986 paper and its 1985 predecessor went essentially unnoticed in the literature of those years, apart from Henze (1986) and a few personal communications. Some interest appeared at least ten years later, initially in some isolated papers following the publication of Azzalini and Dalla Valle (1996), far more noticeably after the work of Azzalini and Capitanio (1999). At that point, researchers that wanted to access the 1986 paper did not always succeed, because of the limited international circulation of the journal in the final decades of the 20th century. So the paper remained far less visible than its 1985 companion. Hence, the present opportunity of reconsidering its role is most welcome, as already said.

In addition, this occasion allows me to amend some algebraic mistakes.

2. Annotations to the paper

The first sentence of the paper indicates that it represents "a natural continuation of a previous one of the author (1985)" and the second paragraph explains that the paper consists of two distinct parts. The first part, corresponding to Section 2, provides some general results on the nature and the properties of the 1985 proposal represented by the

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density functions given in equation (1) of the paper, namely

$$2\,G(\lambda y)\,f(y), \qquad (-\infty < y < \infty) \tag{1}$$

under symmetry conditions of the density f and the distribution function G. Although the general result (1) was presented in the 1985 paper, the focus there was almost entirely on the special case represented by the skew-normal distribution, which arises when f and G denote the standard normal density and distribution function, respectively. The second part of the paper, in Section 3, examines two specific instances of (1), to which we return later.

Examine first the main facts of Section 2. Proposition 1 states the following: if Y is a random variable with density f and Z has density (1), then $|Y|$ and $|Z|$ have the same distribution. This statement says that the modulation factor in (1), $G(\lambda y)$, is irrelevant when we consider the absolute value, $V = |Z|$, and clearly the same holds for any transformation of V. Hence, Proposition 1 represents an embryonic form of what several years later, after a few layers of generalization, has been called perturbation (or modulation) invariance property.

Propositions 2 and 3 provide stochastic representation of Z, in the two forms

$$Z = S_Y\, Y, \qquad Z = S_V\, V\,,$$

where S_Y and S_V are binary variables taking sign ± 1 with probability depending on Y and on V, respectively. These expressions constitute a step forward with respect to the 1985 paper, in two related ways. One was to provide a physical mechanism leading to density (1), which had been introduced as a merely mathematical construct. The other improvement was to provide a far more efficient method for sampling data from distribution (1), as compared to the acceptance-rejection technique presented in the 1985 paper. Again, these stochastic representations represent an embryonic form of results which have later been presented in a considerably more general formulation; see equation (13) of Azzalini and Capitanio (2003) and equation (8) of Wang *et al.* (2004).

The final part of Section 2 presents a more specific property. For the skew-normal distribution, an additional stochastic representation is possible in the additive form $Z = a\,|U_1| + b\,U_2$, where U_1 and U_2 are independent standard normal variables, and a, b are constants such that $a^2 + b^2 = 1$. This representation can be deduced by combining a result of Anděl *et al.* (1984) with Proposition 1 obtained earlier in the paper. The same result has been achieved in parallel independent work by Henze (1986), with two different arguments.

Although this additive representation represents a more specific results than the earlier ones in the paper, it turned out to be a result with much impact in the specialized literature later on, even more so in its subsequent extension to the multivariate case. One reason is its use as the key ingredient for building EM-type algorithms for maximum likelihood estimation. The other use arises in connection with Bayesian inference.

Section 3 takes a distinct direction, focusing on two constructions obtained by specific choices of f and G. In both cases, the baseline density f was the density (8), while the G ingredient was different in the two instances considered in Subsections 3.1 and 3.2. Density (8) is commonly called exponential power distribution or generalized error distribution; its earliest occurrence appears to be due to Subbotin (1923). The key reason for this choice of the baseline density is that it features a parameter ω which regulates the tail weight. Combining this aspect with the parameter λ in equation (1), the resulting distributions allow regulation of both asymmetry and kurtosis, in a slightly different way in the two cases examined.

The underlying plan was the construction of a parametric distribution capable to accommodating outlying observations occurring in real applications, with the possibility that outliers appear with unequal propensity in the two tails of distribution; see the paragraph preceding the one of equation (9). This logic would have provided a route towards robust methods, different from the the mainstream formulation, under strong development in those years.

This idea, however, was not followed upon in the paper, and not even in the immediate subsequent years. The main reason was dissatisfaction with the intersecting curves in the lower left corners of Figures 1 to 4, which point to difficulties with parameter estimation in the corresponding area of the parameter space. The plan re-emerged and was developed several years later, adopting a different baseline density, in the paper of Azzalini and Genton (2008).

FURTHER RESULTS ON A CLASS OF DISTRIBUTIONS WHICH INCLUDES THE NORMAL ONES

Adelchi Azzalini

1. INTRODUCTION

The present paper is the natural continuation of a previous one of the author (1985), which henceforth will be referred to as the *SN* paper, since in there a class of densities functions termed *SN* (a short-hand for skew-normal) has been studied. The chief motivation for considering the *SN* class of densities was the desirability of a parametric class of distributions which allows continuous variation from normality to non-normality.

The specific aim of this paper is two-fold: (i) to give some complementary results that help in understanding the nature of the *SN* distribution as well as some related distributions; (ii) to extend the *SN* class from one to two shape parameters, allowing for a wide range of the indexes of skewness and kurtosis; in the *SN* paper one such extension had briefly been considered, but it had a limited range of skewness and kurtosis. Additional comments and reference to related work will be made where appropriate.

2. ON THE REPRESENTATION OF *SN* AND RELATED DENSITIES

The starting point of the *SN* paper was the following result.

Lemma. Let $f(\cdot)$ be a density symmetric about 0 and $G(\cdot)$ an absolutely continuous distribution function such that $G'(\cdot)$ is a symmetric density. Then

$$2\,G(\lambda y)\,f(y) \qquad (-\infty < y < \infty) \tag{1}$$

is a density for all λ.

There are some simple but relevant properties of the density (1) which have not been remarked in the previous paper.

Proposition 1. If Y is a random variable (r.v.) with density f and Z has density (1), than $|Z|$ and $|Y|$ have the same density.

Proof. The density of $V = |Z|$ at v is

$$h(v) = 2f(v)\,G(\lambda v) + 2f(-v)\,G(-\lambda v)$$
$$= 2f(v)$$

which is equal to the density of $|Y|$. *QED*

In the *SN* paper, the emphasis was on the special case of (1), called $SN(\lambda)$ density,

$$\phi(z;\lambda) = 2\Phi(\lambda z)\,\phi(z) \tag{2}$$

where ϕ and Φ are the $N(0, 1)$ density and distribution function respectively. It has been shown in the *SN* paper that the square of a $SN(\lambda)$ r.v. is distributed as χ_1^2; this result is in fact a special case of Proposition 1 applied to (2).

Proposition 2. Under the condition of Lemma 1, let Y be a r.v. with density f and

$$Z = S_Y\,Y \tag{3}$$

where, conditionally on $Y = y$,

$$S_Y = \begin{cases} +1 & \text{with probability } G(\lambda y), \\ -1 & \text{with probability } 1 - G(\lambda y). \end{cases}$$

Then Z has density (1).

Proof. In fact the density of Z is

$$f(z)\,G(\lambda z) + f(-z)\,\{1 - G(-\lambda z)\} = 2f(z)\,G(\lambda z).$$

Proposition 3. Under the same conditions of the previous proposition, let $V = |Y|$ and S_V be defined similarly to S_Y. Then

$$Z = S_V\,V \tag{4}$$

has density (1).

Proof. If $z > 0$, the density of Z is

$$2f(z)\,P\{S_V = +1\} = 2f(z)\,G(\lambda z).$$

If $z < 0$, its density is

$$2f(-z)\,P\{S_V = -1\} = 2f(-z)\,\{1 - G(-\lambda z)\} = 2f(z)\,G(\lambda z). \quad QED$$

Representations (3) and (4) give a physical justification of density (1), which otherwise appeared artificial. These facts are also useful for computer random number sampling from (1); in fact it is sufficient to generate Y from f, and W from G, and put

$$Z = \begin{cases} Y & \text{if } W \leqslant \lambda Y \\ -Y & \text{if } W > \lambda Y \end{cases} \tag{5}$$

This generation method is twice more efficient than the acceptance-rejection technique proposed in the *SN* paper.

A further form of representation is possible for the special case of distribution (2) and it is implicit in a result of Andel et al. (1984), which can be rephrased as follows. Suppose that the stationary process $\{Z_t\}$ satisfies

$$Z_t = \delta \mid Z_{t-1} \mid + \epsilon_t \qquad \text{for } t = 0, \pm 1, \pm 2, \ldots \tag{6}$$

where $|\delta| < 1$ and $\{\epsilon_t\}$ is white noise $N(0,\ 1-\delta^2)$. Then the corresponding integral equation for the stationary density of Z_t is

$$g(z) = \int_{-\infty}^{\infty} \phi\left(\frac{z - \delta|t|}{\sqrt{1-\delta^2}}\right) g(t)\, dt \, \frac{1}{\sqrt{1-\delta^2}}$$

and its solution is given by (2) with

$$\lambda = \frac{\delta}{\sqrt{1-\delta^2}}, \; \delta = \frac{\lambda}{\sqrt{1+\lambda^2}}, \tag{7}$$

as it can be verified by direct substitution.

Since, by Proposition 1, the term $|Z_{t-1}|$ in (6) is distributed as the absolute value of $N(0, 1)$, then we have the corollary that, if U_1 and U_2 are independent $N(0, 1)$ variates, then

$$Z = \delta |U_1| + \sqrt{1-\delta^2}\ U_2$$

is $SN(\lambda)$.

3. FURTHER EXTENSION OF THE NORMAL DISTRIBUTION

One limitation of family (2) is that the parameter λ can only produce tails thinner than the normal ones, but not thicker, while the latter case would be more interesting in many applications. A class of densities which includes the normal ones and allows thick tails is

$$g(y; \omega) = C_\omega \exp\left\{-\frac{|y|^\omega}{\omega}\right\} \qquad (-\infty < y < \infty) \tag{8}$$

where ω is a positive parameter and

$$C_\omega = \{2\omega^{1/\omega - 1}\,\Gamma(1/\omega)\}^{-1}\;;$$

see Box (1953), Turner (1960), Vianelli (1963). The $g(y;\,2)$ density is the $N(0,\,1)$ density; $g(y;\,1)$ is the Laplace density; as $\omega \to \infty$, $g(y;\,\omega)$ converges to the uniform density in $(-1,\,1)$. Since low values of ω make the tails of (8) very heavy, it has been suggested to use (8) as a reference distribution in robustness studies.

Hill & Dixon (1982) have given evidence that, in real applications, the distribution of the data is often skew, while virtually all robust methods assume symmetry of the error distribution. Moreover, the distribution of real data is seldom so heavily tailed as the ones employed in theoretical robustness studies.

In order to introduce skewness in (8), two modifications of the form

$$2G(\lambda y)\,g(y;\,\omega) \tag{9}$$

will be considered, with G satisfying conditions of Lemma 1. Many choices of G are possible. The two choices considered are such that, at $\omega = 2$, the $SN(\lambda)$ density is obtained. Here and in the rest of the paper, we denote by Z a r.v. with density (9) and $V = |Z|$; then the following facts apply to all possible choices of G.

(i) Reversing the sign of λ in (9) gives the density of $-Z$; therefore there is no loss of generality in supposing $\lambda \geqslant 0$ in the following.

(ii) By Proposition 3,

$$E(Z^m) = E(V^m) = \omega^{m/\omega}\,\frac{\Gamma((m+1)/\omega)}{\Gamma(1/\omega)} \tag{10}$$

for m even. The second of these equalities holds also for m odd.

(iii) For m odd,

$$\begin{aligned} E(Z^m) &= E(S_V\,V^m) \\ &= E\{E(S_V \mid V)\,V^m\} \\ &= E\{(G(\lambda V) - G(-\lambda V))\,V^m\} \\ &= 2E\{(G(\lambda V) - 1/2)\,V^m\} \\ &= 2E\{G(\lambda V)\,V^m\} - E(V^m) \end{aligned} \tag{11}$$

3.1 *Distribution type I*

Consider the case that G in (9) is the integral of $g(y;\,\omega)$, namely

$$G(y) = \frac{1}{2}\{1 + \mathrm{sgn}(y)\,\gamma(|y|^\omega/\omega;\,1/\omega)/\Gamma(1/\omega)\} \tag{12}$$

where

$$\gamma(x; \nu) = \int_0^x t^{\nu - 1} e^{-t} dt$$

is the incomplete gamma function. Direct integration gives the m-th moment from the origin

$$\mu'_m = \frac{2C_\omega}{\Gamma(1/\omega)} \omega^{\frac{m+1}{\omega} - 1} \Gamma\left(\frac{m+2}{\omega}\right) \int_0^{\lambda^\omega} u^{1/\omega - 1} (1+u)^{-(m+2)/\omega} du$$

$$= 2C_\omega \omega^{\frac{m+1}{\omega} - 1} B\left(\frac{\lambda^\omega}{1+\lambda^\omega}; 1/\omega, (m+1)/\omega\right) \Gamma\left(\frac{m+1}{\omega}\right)$$

for m odd and positive λ; here $B(x; p, q)$ is the distribution function of a beta r.v. with parameters p, q. In general, the integral in the previous expression does not lend itself to explicit computation. In the special case that $(m+1)/\omega = n$ integer, then

$$\mu'_m = 2C_\omega \frac{\omega^{n-1} \Gamma(n)}{\Gamma(1/\omega)} \sum_{j=0}^{n-1} \frac{\Gamma(1/\omega + j)}{j! \lambda^{j\omega} (1 + \lambda^{-\omega})^{j+1/\omega}}$$

which simplifies further to

$$\mu'_m = \Gamma(m+1) \left\{1 - \frac{1}{(1+\lambda)^{m+1}}\right\}$$

if $\omega = 1$.

Using the first expression of μ'_m as well as (10), the indexes of skewness and kurtosis, γ_1 and γ_2, can be computed numerically. Figure 1 gives the plot of γ_2 versus γ_1 for ω ranging from 1 (top curve) to 2 (bottom curve) and λ ranging frm 0 to 10. It is seen that a wide range of (γ_1, γ_2) values is covered. Figure 2 gives the analogous plot for ω ranging from 1.9 to 2.5. In this case, we have the peculiar phenomenon that curves referring to $\omega > 2$ intersect: this implies that more than four moments are necessary to identify the member of the parametric class.

We want to show that (9) is log concave for the above choice of G and $\omega > 1$. The following preliminary result is probably well known, but the author did not find track of it in the literature.

Lemma 2. For any $x > 0$ and $0 < \alpha > 1$

$$\int_x^\infty e^{-t} t^{\alpha - 1} dt < e^{-x} x^{\alpha - 1}.$$

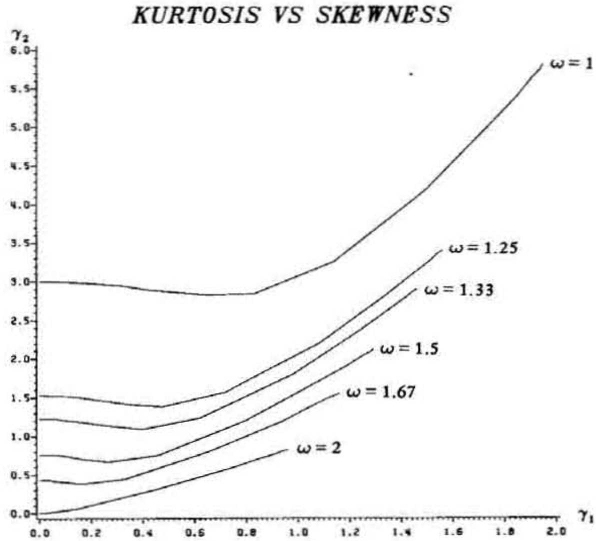
KURTOSIS VS SKEWNESS
γ2
6.0
5.5
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
γ1
ω = 1
ω = 1.25
ω = 1.33
ω = 1.5
ω = 1.67
ω = 2

Fig. 1

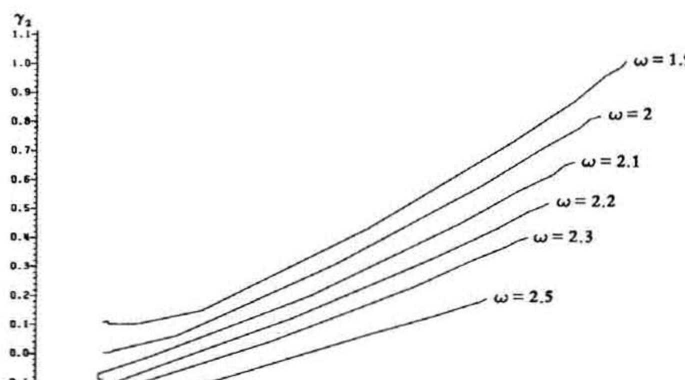
γ2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.1
0.1
0.3
0.5
0.7
0.9
γ1
ω = 1.9
ω = 2
ω = 2.1
ω = 2.2
ω = 2.3
ω = 2.5

Fig. 2

Proof. Integrating both sides of the inequality

$$e^{-t}\, t^{\alpha-1} < e^{-t}\,\{(1-\alpha)\, t^{\alpha-2} + t^{\alpha-1}\}$$

from x to ∞, we obtain the result. *QED*

Note that, setting $\alpha = 1/2$ and replacing x by $x^2/2$ in the lemma, we obtain the well known result

$$\Phi(-x) < \frac{\phi(x)}{x} \qquad \text{for } x > 0.$$

Returning to the problem of the log concavity of (9), it is easily seen that the second derivative of $g(y;\omega)$ is negative for $\omega > 1$. Consider then

$$\frac{d^2}{dy^2} \log G(\lambda y) = -\frac{\lambda^2\, g(t;\omega)}{G(t)} \left\{ \operatorname{sgn}(t)\, |t|^{\omega-1} + \frac{g(t;\omega)}{G(t)} \right\} \tag{13}$$

for $t = \lambda y$. This derivative is clearly negative for $t > 0$; for $t < 0$, put $x = -t > 0$ and consider

$$-x^{\omega-1} + \frac{g(x;\omega)}{1-G(x)}$$

with

$$1 - G(x) = C_\omega \int_x^\infty \exp(-u^\omega/\omega)\, du$$

$$= C_\omega\, \omega^{1/\omega-1} \int_{x^\omega/\omega}^\infty e^{-u}\, u^{1/\omega-1}\, du.$$

Using Lemma 2 with $\alpha = 1/\omega$ and x replaced by x^ω/ω, one obtains that (13) is negative and hence (9) is log concave.

For the purpose of random number generation, it is easy to see from the expression of G that, if X has gamma distribution with index $1/\omega$, then

$$W = \begin{cases} (\omega X)^{1/\omega} & \text{with probability } 1/2, \\ -(\omega X)^{1/\omega} & \text{with probability } 1/2 \end{cases}$$

has distribution (12). Then, using (5), one can sample from density (9).

3.2 *Distribution type II*

A second possible choice of G in (9) is

$$G(y) = \Phi\left\{ \operatorname{sgn}(y) \frac{|y|^\psi}{\sqrt{\psi}} \right\} \tag{14}$$

where $\psi = \omega/2$. This choice of G is the distribution function of $\mathrm{sgn}\,(U)\,|\sqrt{\psi}\,U|^{1/\psi}$ if U is $N(0, 1)$. The term $\sqrt{\psi}$ in the denominator of (14) is introduced to simplify some algebraic manipulations later on. Therefore the density considered now is

$$h(y) = 2C_{2\psi} \exp\left\{-\frac{|y|^{2\psi}}{2\psi}\right\} \Phi\left\{\mathrm{sgn}\,(\lambda y)\frac{|\lambda y|^{\psi}}{\sqrt{\psi}}\right\}. \tag{15}$$

It is not difficult to verify by direct computation that also this density is log concave for $\psi > 1/2$, i.e. $\omega > 1$.

To obtain in the odd moments of (15) we use (11). Then

$$E(V^m\, G(\lambda V)) = 2C_{2\psi}\, \psi^{s-1} \int_0^\infty t^{2s-1}\, e^{-t^2/2}\, \Phi(\lambda t)\, dt$$

$$= \frac{\omega^{m/\omega}}{\Gamma(1/\omega)} \int_0^\infty u^{2s-1}\, e^{-u}\, \Phi(\lambda\,\sqrt{2u}\,)\, du$$

$$= \frac{\omega^{m/\omega}}{\Gamma(1/\omega)} \left(\frac{1}{2}\Gamma(s) + \sum_{n=0}^{\infty} \frac{2^n\, \lambda^{2n+1}\, \Gamma(s+n+1/2)}{\sqrt{\pi}\,(2n+1)!!\,(1+\lambda^2)^{s+n+1/2}}\right)$$

having set $s = (m+1)/\omega$ and used the standard expansion

$$\Phi(x) = \frac{1}{2} + \phi(x) \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!!}\ .$$

Then, for m odd,

$$E(Z^m) = \frac{\omega^{s-1/\omega}\, 2\lambda}{\Gamma(1/\omega)\,\sqrt{\pi}\,(1+\lambda^2)^{s+1/2}} \sum_{n=0}^{\infty} \frac{(2\lambda^2)^n\, \Gamma(s+n+1/2)}{(2n+1)!!\,(1+\lambda^2)^n}\ .$$

This formula can be evaluated explicitly when s is integer, making use of the relation

$$\Gamma(n+1/2) = (2n-1)!!\, 2^{-n} \sqrt{\pi}\ .$$

For instance, if $\omega = 1$, one gets

$$E(Z) = \frac{1}{2}\,\delta\,(3-\delta^2)$$

where δ is as in (7).

Figure 3 and 4 give plots analogous to those of Figure 1 and 2 for the new choice of G. The values of ω in Figure 3 are 1, 4/3, 3/2, 2, 5/2, 8/3 and λ ranges from 0 to 10; in Figure 4 ω is 1.8, 1.9, 2, 2.1, 2.2, 2.4 and the same

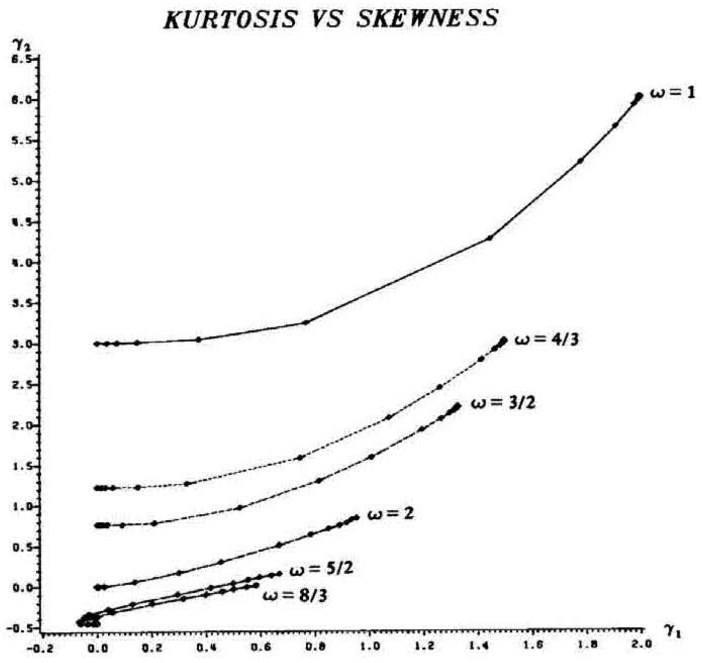
KURTOSIS VS SKEWNESS
γ_2
γ_1
ω = 1
ω = 4/3
ω = 3/2
ω = 2
ω = 5/2
ω = 8/3

Fig. 3

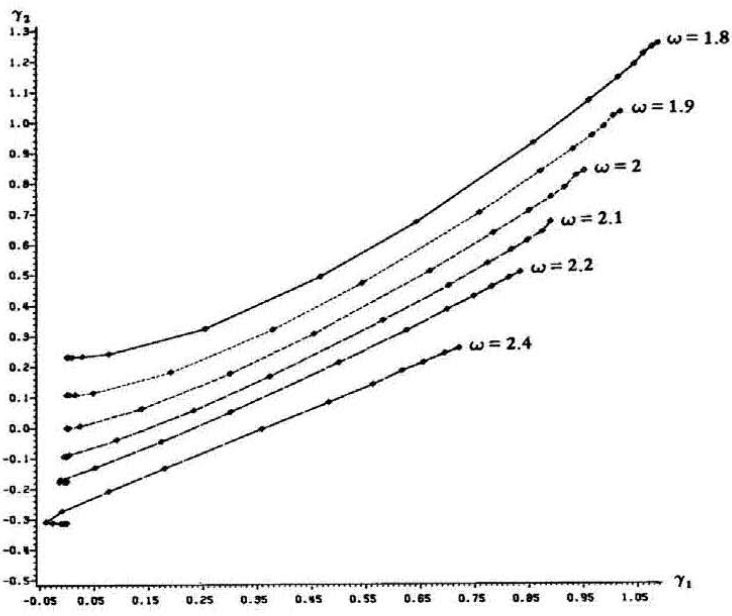
γ_2
γ_1
ω = 1.8
ω = 1.9
ω = 2
ω = 2.1
ω = 2.2
ω = 2.4

Fig. 4

values of λ. It is seen that the behaviour of these curves is similar to that of Figure 1 and 2.

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RIASSUNTO

Ulteriori risultati su una classe di distribuzioni che include quelle normali

Vengono presentati ulteriori risultati relativi ad una classe di funzioni di densità già trattate in un altro lavoro dell'autore (1985). Viene introdotto in particolare un ulteriore parametro di forma che consente un'ampia escursione agli indici di asimmetria e curtosi.

RÉSUMÉ

Résultats ultérieurs sur una classe de distributions qui comprend les distributions normales

On présente des résultats ultérieurs relatifs à une classe de fonctions de densité déjà traité dans un autre article de l'auteur (1985). On introduit en particulier un ultérieur paramètre de forme qui permet une large excurtion aux indices de asymétrie et de aplatissement.

AKNOWLEDGEMENT: I thank Dr. W. Ehm and Dr. W. Härdle for useful comments. This research was carried out during a visit of the author at the University of Heidelberg and was supported by the Deutsche Forschungsgemeinschaft, SFB 123, "Stochastiche Mathematische Modelle".

3. Corrigenda

There are various corrections to the original paper to be communicated. Of these, the first one is a rather obvious typo, while those in Subsection 3.2 are of more substantial nature.

1. In the statement of Lemma 2, change '$0 < \alpha > 1$' to '$0 < \alpha < 1$'.
2. The two lines following equation (15) must be removed.
3. Halfway on page 206, replace the line with 'Then, for m odd,' and the subsequent one with the following passage.

> Then, for any positive m and non-negative λ,
>
> $$E(Z^m) = \frac{2\,\omega^{m/\omega}\,\tilde{\lambda}}{\sqrt{\pi}\,\Gamma(1/\omega)\,(1+\tilde{\lambda}^2)^{s+1/2}} \sum_{n=0}^{\infty} \frac{\Gamma(s+n+1/2)}{(2n+1)!!} \left\{ \frac{2\tilde{\lambda}^2}{1+\tilde{\lambda}^2} \right\}^n$$
>
> where $\tilde{\lambda} = \lambda^{\omega/2}$ and $s = (m+1)/\omega$. If $\lambda < 0$, apply the above expression to $|\lambda|$ and then change the sign of the result.

4. An implication of the previous correction is that the subsequent expression on the same page, line 5 from bottom, must be changed to

$$E(Z) = \frac{\sqrt{\lambda}}{2\sqrt{1+\lambda}}\left(3 - \frac{\lambda}{1+\lambda}\right).$$

An implication of the above correction to $E(Z^m)$ is that, in principle, Figures 1 to 4 should be changed. However, when the corrected plots have been produced, there was no visible difference from the existing ones. It has then be decided not to insert new plots.

Acknowledgements

There are two distinct components in these acknowledgements. First, I must amend to my culpable omission of mentioning in the 1986 acknowledgements that, while I was working on the preliminary version of the paper at the University of Heidelberg (leading to the SFB 123 preprint Nr. 343, December 1985), I benefited from a very fruitful discussion with the late Professor Hermann Rost who draw my attention to the relevant paper of Anděl *et al.* (1984).

As for the present contribution, I like to thank Professor Balgobin Nandram (Worcester Polytechnic Institute, Massachusetts) for pointing out the incorrect statement of log-concavity of the 'type II' distribution. Once more, many thanks are due to the Editor of *Statistica*, Professor Simone Giannerini, for deciding to reprint the paper, and for the opportunity of complementing it with some remarks and a set of corrections.

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SUMMARY

The author's 1986 paper with the same title is reprinted here alongside some comments and corrections. The original abstract, here translated in English, was as follows: "Some further results are presented concerning a class of density functions already examined in another work of the author (1985). Specifically, an additional shape parameter is introduced which allows a wide range of the coefficients of asymmetry and kurtosis."

Keywords: Skew-symmetric distributions; Subbotin distribution; Symmetry-modulated distributions.