FURTHER RESULTS ON E-COMPACT SPACES. I

BY

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Contents

There are two problems naturally connected with topological products: (a) given a space E find all spaces that are homeomorphic to subspaces of topological powers of E, and (b) given an E find all spaces that are homeomorphic to closed subspaces of powers of E. Problem (a) has been solved in [16]; to solve Problem (b) I have introduced in [17] the concept of E-compact spaces; the first systematic investigation of this concept has been given in [9]. The present paper is the first part of the summary of the author's further results in this direction. It contains a discussion concerning arbitrary spaces E; the second part will concern some particular cases of E. Some of the present results have been stated in [6], [19] and [20], some were announced in various issues of Notices Amer. Math. Soc. Various results included here have been obtained in cooperation with R. Blefko.

This paper is self-contained; all the results given in [16] and [9] are reproved, sometimes in a more general form and frequently with more efficient proofs.

In this paper, for the sake of logical simplicity, we define an E-completely regular (an E-compact) space as a space that is homeomorphic to a subspace (closed subspace, respectively) of some topological power $E^{\mathfrak{m}}$ of E. (Thus, our original definitions in [16],

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[17], and [9] now become necessary and sufficient conditions for E-complete regularity and E-compactness.) It should be pointed out that recently H. Herrlich [12] (1) has initiated a still more general approach to this type of problems. Herrlich considers a class $\mathfrak{E}$ of topological spaces and he calls a space X $\mathfrak{E}$-completely regular ($\mathfrak{E}$-compact) in case X is homeomorphic to a subspace (a closed subspace, respectively) of a product of spaces from the class $\mathfrak{E}$. We give only a brief comment on Herrlich's approach at the end of Ch. IV. Undoubtedly, this approach is very promising; it sometimes enables us to state some results and some problems in an essentially more general form.

The following terminology and notations are used.

The domain and the counter-domain of a function f is denoted by $D(f)$ and $C(f)$, respectively. $f: X \to Y$ ($f: X \xrightarrow[\text{onto}]{} Y$) stands for: f is a function with $D(f) = X$, $C(f) \subset Y$ ($C(f) = Y$, respectively). $f[A]$ and $f^{-1}[A]$ denote, respectively, the image and the counter-image of a set A under f. In general it is not assumed that $A \subset D(f)$ ($A \subset C(f)$); it is easy to see that $f[A] = f[A \cap D(f)]$, $f^{-1}[A] = f^{-1}[A \cap C(f)]$. The compositions of the functions f and g (i.e. the function h defined by $h(x) = f(g(x))$) is denoted by $f \circ g$. In general it is not assumed that $D(f) \supset C(g)$; consequently, $D(f \circ g)$ can be a proper subset of $D(g)$; in fact, we have $D(f \circ g) = g^{-1}[D(f)]$. id_X denotes the identity function on a set X.

By a regular, completely regular, normal space we mean a T_1-space which satisfies the corresponding separation axiom.

$X \subset_{\mathrm{top}} Y$ ($X \subset_{\mathrm{cl}} Y$) stands for: X is homeomorphic to a subspace (closed subspace, respectively) of Y.

A function f whose domain and range are topological spaces is said to be open (closed) provided that for every open (closed) subset A of the domain of f, $f[A]$ is open (closed) in the range of f. f is said to be quotient provided that for every subset of B of the range of f, B is open in the range of f iff $f^{-1}[B]$ is open in the domain of f.

$\mathcal{F}$ denotes the Alexandrov connected dyad; i.e., the two-point space $\{0, 1\}$ whose only proper non-empty open subset is the set $\{0\}$. $\mathcal{D}$ is the discrete dyad $\{0, 1\}$, $\mathcal{R}$ denotes the space of the reals, $\mathcal{N}$ denotes the space of non-negative integers (=the discrete space of cardinality $\aleph_0$). $\mathcal{P}$ and $\mathcal{Q}$ denote, respectively, the spaces of rational and irrational numbers. $\mathcal{L}_{\mathfrak{m}}$, where $\mathfrak{m}$ is a cardinal, denotes the space consisting of m points whose only closed proper subsets are the finite sets; $\mathcal{L}_{\mathfrak{m}}$ is a compact T_1-space.

Given two spaces X and E, $C(X, E)$ denotes the set of all continuous functions f with $f: X \to E$.

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(1) The manuscript of [12] was prepared already in 1965.

I. Extensions of topological spaces

An *extension* of a space X is a pair $(X, \varepsilon X)$, where εX is an arbitrary Hausdorff superspace of X such that X is dense in εX. An extension $(X, \varepsilon X)$ will usually be denoted by εX. An extension εX of X is called *proper* provided that $\varepsilon X \neq X$.

If $\varepsilon_1 X$ and $\varepsilon_2 X$ are two extensions of X, then by a *canonical map* of $\varepsilon_1 X$ into $\varepsilon_2 X$ we mean any continuous map $\varphi: \varepsilon_1 X \to \varepsilon_2 X$ such that $\varphi(p) = p$ for every $p \in X$. Clearly, a canonical map, if it exists, is unique.

We shall consider three relations between extensions $\varepsilon_1 X$ and $\varepsilon_2 X$ of X. We write

$\varepsilon_1 X =_{\text{ext}} \varepsilon_2 X$ provided that there exists a canonical map φ of $\varepsilon_1 X$ onto $\varepsilon_2 X$ and this map is a homeomorphism;

$\varepsilon_1 X \subset_{\text{ext}} \varepsilon_2 X$ provided that there exists a canonical map φ of $\varepsilon_1 X$ into $\varepsilon_2 X$ and this map is a homeomorphism;

$\varepsilon_1 X \leqslant_{\text{ext}} \varepsilon_2 X$ provided that there exists a canonical map of $\varepsilon_2 X$ onto $\varepsilon_1 X$.

It follows from the uniqueness of canonical maps that if a canonical map φ of $\varepsilon_1 X$ into $\varepsilon_2 X$ is not a homeomorphism, or φ is not onto $\varepsilon_2 X$, then $\varepsilon_1 X \neq_{\text{ext}} \varepsilon_2 X$.

The above relations have the following properties.

1.1. THEOREM. *For every three extensions* $\varepsilon_1 X$, $\varepsilon_2 X$, $\varepsilon_3 X$ *we have*

1. $\varepsilon_1 X =_{\text{ext}} \varepsilon_1 X$; *if* $\varepsilon_1 X =_{\text{ext}} \varepsilon_2 X$, *then* $\varepsilon_2 X =_{\text{ext}} \varepsilon_1 X$;
   if $\varepsilon_1 X =_{\text{ext}} \varepsilon_2 X$ *and* $\varepsilon_1 X =_{\text{ext}} \varepsilon_3 X$, *then* $\varepsilon_1 X =_{\text{ext}} \varepsilon_3 X$.
2. $\varepsilon_1 X \subset_{\text{ext}} \varepsilon_1 X$; *if* $\varepsilon_1 X \subset_{\text{ext}} \varepsilon_2 X$ *and* $\varepsilon_2 X \subset_{\text{ext}} \varepsilon_3 X$, *then* $\varepsilon_1 X \subset_{\text{ext}} \varepsilon_3 X$.
3. $\varepsilon_1 X \leqslant_{\text{ext}} \varepsilon_1 X$; *if* $\varepsilon_1 X \leqslant_{\text{ext}} \varepsilon_2 X$ *and* $\varepsilon_2 X \leqslant_{\text{ext}} \varepsilon_3 X$, *then* $\varepsilon_1 X \leqslant_{\text{ext}} \varepsilon_3 X$.
4. $\varepsilon_1 X =_{\text{ext}} \varepsilon_2 X$ *iff* $\varepsilon_1 X \subset_{\text{ext}} \varepsilon_2 X$ *and* $\varepsilon_2 X \subset_{\text{ext}} \varepsilon_1 X$ *iff* $\varepsilon_1 X \leqslant_{\text{ext}} \varepsilon_2 X$ *and* $\varepsilon_2 X \leqslant_{\text{ext}} \varepsilon_1 X$.
5. *If* $\varepsilon_1 X \subset_{\text{ext}} \varepsilon_2 X$ *and* $\varepsilon_1 X \leqslant_{\text{ext}} \varepsilon_2 X$, *then* $\varepsilon_1 X =_{\text{ext}} \varepsilon_2 X$;
   if $\varepsilon_1 X \subset_{\text{ext}} \varepsilon_2 X$ *and* $\varepsilon_2 X \leqslant_{\text{ext}} \varepsilon_1 X$, *then* $\varepsilon_1 X =_{\text{ext}} \varepsilon_2 X$.

Proof. The only non-trivial parts of this theorem (4. and the first statement in 5; the second statement in 5 follows from the uniqueness of canonical maps) follow from the following lemma.

1.2. LEMMA. *If f and g are continuous functions with $f: \varepsilon_1 X \to \varepsilon_2 X$, $g: \varepsilon_2 X \to \varepsilon_1 X$ and $f(p) = g(p) = p$ for every $p \in X$, then $f = g^{-1}$. Consequently, each of the functions f and g is onto and each of them is a homeomorphism.*

Proof of Lemma 1.2. Consider the compositions $f\circ g$ and $g\circ f$. We have $g\circ f\colon \varepsilon_1 X \to \varepsilon_1 X$, $f\circ g\colon \varepsilon_2 X \to \varepsilon_2 X$ and $f\circ g(p) = g\circ f(p) = p$ for every $p \in X$. Since X is dense in both $\varepsilon_1 X$ and $\varepsilon_2 X$, we have $g\circ f = \mathrm{id}_{\varepsilon_1 X}$ and $f\circ g = \mathrm{id}_{\varepsilon_2 X}$; consequently $f = g^{-1}$.

One could consider still another relation between extensions $\varepsilon_1 X$ and $\varepsilon_2 X$ asserting the existence of a canonical map of $\varepsilon_2 X$ into (but not necessarily onto) $\varepsilon_1 X$. However at the present moment, we did not find this relation very useful; furthermore, this relation can be expressed as the composition of the relations $\subset_{\mathrm{ext}}$ and $\leqslant_{\mathrm{ext}}$. Consequently, no special symbol will be introduced.

Note. The definition of an extension adopted in this paper is formally different from the one frequently used in the theory of compactifications. According to the latter an extension of X is a pair (h, Y), where h is a homeomorphism of X onto a dense subspace of Y. We have found, however, that our definition frequently makes various proofs formally simpler. If an extension of X (in our sense) is to be constructed via an embedding of X into a space Y, one can always appeal to the following.

1.3. THEOREM. (The formal theorem.) *If h is a homeomorphism of X into Y, then there exists a superspace X^* of X which is homeomorphic to Y by a homeomorphism h^* which is an extension of h.*

Clearly, the topological relations between $h[X]$ and Y are identical to those between X and X^* (for instance, $h[X]$ is dense, open, closed, etc., ..., in Y iff X is dense, closed, open, etc., ..., in X^*).

II. The embedding theorem

Let $\{E_\xi\colon \xi\in\Xi\}$ be a class of topological spaces; the topological (Tihonov) product of this class of spaces will be denoted by $\times\{E_\xi\colon \xi\in\Xi\}$. Elements of the product $\times\{E_\xi\colon \xi\in\Xi\}$ are functions e defined on Ξ and such that $e(\xi)\in E_\xi$ for every $\xi\in\Xi$. $e(\xi)$ is called the ξ-th coordinate of e and it is denoted by $\pi_\xi(e)$. The map π_ξ is called the *projection* of the product $\times\{E_\xi\colon \xi\in\Xi\}$ onto the ξ-th coordinate axis X_ξ. We shall also consider a more general type of projections: if Ξ_0 is a subset of Ξ, then we let $\pi_{\Xi_0}(e) = e\,|\,\Xi_0$ (= the restriction of e to Ξ_0) for every $e\in\times\{E_\xi\colon \xi\in\Xi\}$. π_{Ξ_0} is called the projection of $\times\{E_\xi\colon \xi\in\Xi\}$ onto the product $\times\{E_\xi\colon \xi\in\Xi_0\}$. (There is a formal difference between π_ξ and $\pi_{\{\xi\}}$.) Projections are continuous open maps. If all spaces E_ξ, $\xi\in\Xi$, are equal to a space E, then the product $\times\{E_\xi\colon \xi\in\Xi\}$ is denoted by $E^{\mathfrak{m}}$, where $\mathfrak{m} = \operatorname{card}\Xi$, and it is called the m-th (topological) power of E.

Sets of the form

(1) $\pi_{\xi_1}^{-1}[G_1] \cap \dots \cap \pi_{\xi_n}^{-1}[G_n]$,

where $\xi_1, \dots, \xi_n \in \Xi$ and G_i is an open subset of X_{ξ_i}, $i = 1, 2, \dots, n$, are called *elementary neighborhoods* in the product $\times\{E_\xi: \xi\in\Xi\}$; elementary neighborhoods form a base for the topology in this product. Every elementary neighborhood can be written in the form (1), where the indices $\xi_1, \xi_2, \ldots, \xi_n$ are all distinct.

Let X be a space and let $\mathfrak{F}=\{f_\xi: \xi\in\Xi\}$ be a collection of functions with

(2) $f_\xi: X\to E_\xi$ for every $\xi\in\Xi$;

let h be a map with

(3) $h: X\to\times\{E_\xi: \xi\in\Xi\}$

and consider the condition

(4) $\pi_\xi\circ h=f_\xi$ for every $\xi\in\Xi$.

Condition (4) can serve a dual role: if a map h with $h: X\to\times\{E_\xi: \xi\in\Xi\}$ is given, then (4) defines a class $\mathfrak{F}$ of functions $f_\xi: X\to E_\xi$. Conversely, if such a class $\mathfrak{F}$ of functions is given, then (4) defines a map $h: X\to\times\{E_\xi: \xi\in\Xi\}$ (i.e., there is one and only one map $h: X\to \times\{E_\xi: \xi\in\Xi\}$ satisfying (4)). This map h will be called the *parametric map* (of X into $\times\{E_\xi: \xi\in\Xi\}$) corresponding to the class $\mathfrak{F}$.

If $f_{\xi_1}, f_{\xi_2}, \ldots, f_{\xi_n}$ is a finite system of functions with $f_{\xi_i}: X\to E_{\xi_i}$, then by $\langle f_{\xi_1}, f_{\xi_2}, \ldots, f_{\xi_n}\rangle$ we shall denote a map whose value at a point $p\in X$, $\langle f_{\xi_1}, f_{\xi_2}, \ldots, f_{\xi_n}\rangle(p)$, is equal to the point $(f_{\xi_1}(p), f_{\xi_2}(p), \ldots, f_{\xi_n}(p))$ of the product $E_{\xi_1}\times E_{\xi_2}\times\ldots\times E_{\xi_n}$ (i.e., $\langle f_{\xi_1}, f_{\xi_2}, \ldots, f_{\xi_n}\rangle$ is the parametric map corresponding to the class $\{f_{\xi_1}, f_{\xi_2}, \ldots, f_{\xi_n}\}$).

Our main embedding theorem is as follows.

2.1. Theorem. (The Embedding Theorem.) *Let $\mathfrak{F}=\{f_\xi: \xi\in\Xi\}$ be a class of functions with $f_\xi: X\to E_\xi$, where X and E_ξ, $\xi\in\Xi$, are topological spaces. Let h be the parametric map corresponding to the class $\mathfrak{F}$ (i.e., h is a map of X into $\times\{E_\xi: \xi\in\Xi\}$ such that condition* (4) *is satisfied). We have*

a) *h is continuous if and only if each f_ξ is continuous;*

b) *h is one-to-one if and only if the class $\mathfrak{F}$ satisfies the following condition:*
   (i) *for every $p, q\in X$, $p\neq q$, there is an $f_\xi\in\mathfrak{F}$ with $f_\xi(p)\neq f_\xi(q)$;*

c) *h is a homeomorphism if and only if h is continuous and one-to-one and the class $\mathfrak{F}$ satisfies the following condition:*
   (ii) *for every closed subset A of X and for every $p\in X\setminus A$ there exists a finite system $f_{\xi_1}, \ldots, f_{\xi_n}$ of functions from $\mathfrak{F}$ such that $\langle f_{\xi_1}, \ldots, f_{\xi_n}\rangle(p)\notin \mathrm{cl}\langle f_{\xi_1}, \ldots, f_{\xi_n}\rangle[A]$, where* cl *stands for the closure in $E_{\xi_1}\times\ldots\times E_{\xi_n}$;*

d) *assume that the spaces E_ξ are all Hausdorff and assume that h is a homeomorphism. $h[X]$ is closed in $\times\{E_\xi: \xi \in \Xi\}$ if and only if the class $\mathfrak{F}$ satisfies the following condition:*

(iii) *there is no proper extension εX of X such that every function $f_\xi \in F$ admits a continuous extension $f_\xi^*: \varepsilon X \to E_\xi$.*

Furthermore, in condition (iii) *it suffices to consider only such extensions εX of X that $\varepsilon X \subset_{\text{top}} \times\{E_\xi: \xi \in \Xi\}$.*

Proof. Parts a) and b) of the theorem are well-known and are stated here only for completeness.

Part c) Assume that h is continuous and one-to-one and that the class $\mathfrak{F}$ satisfies condition (ii). Let A be a closed subset of X. For every finite system $\xi_1, \ldots, \xi_n$ of elements of Ξ we denote by $T_{\xi_1, \ldots, \xi_n}$ the set of all points e of the product $\times\{E_\xi: \xi \in \Xi\}$ such that $\pi_{\xi_i}(e) = f_{\xi_i}(p)$ for some $p \in A$ and for $i = 1, 2, \ldots, n$. It is now clear that condition (ii) is equivalent to the fact that $h[A]$ is the intersection of all sets of the form $h[X] \cap \overline{T}_{\xi_1, \ldots, \xi_n}$ where $\xi_1, \ldots, \xi_n$ ranges over all finite systems of elements of Ξ (and $\overline{T}_{\xi_1, \ldots, \xi_n}$ denotes the closure of $T_{\xi_1}, \ldots, T_{\xi_n}$ in $\times\{E_\xi: \xi \in \Xi\}$). Thus $h[A]$ is closed in $h[X]$, hence h is a homeomorphism.

Conversely, assume that h is a homeomorphism. Let A be a closed subset of X and let $p \in X \setminus A$. We have $h(p) \notin \overline{h[A]}$; consequently, there is an elementary neighborhood $U = \pi_{\xi_1}^{-1}[G_1] \cap \ldots \cap \pi_{\xi_n}^{-1}[G_n]$ (ξ_i are all distinct) with $h(p) \in U$ and $U \cap h[A] = \emptyset$. Set $\langle f_{\xi_1}, \ldots, f_{\xi_n} \rangle = \pi_{\Xi_0} \circ h$, where $\Xi_0 = \{\xi_1, \ldots, \xi_n\}$. It is clear that the system $f_{\xi_1}, \ldots, f_{\xi_n}$ satisfies the requirements of condition (ii).

Part d) Assume that $h[X]$ is closed in $\times\{E_\xi: \xi \in \Xi\}$. Let εX be a Hausdorff extension of X with the property that each f_ξ in $\mathfrak{F}$ admits a continuous extension $f_\xi^*: \varepsilon X \to E_\xi$. Let h^* be the parametric map of εX corresponding to the class $\mathfrak{F}^* = \{f_\xi^*: \xi \in \Xi\}$. Clearly, h^* is an extension of h. This implies (X is dense in εX) that $h^*[\varepsilon X] \subset \overline{h[X]} = h[X]$. In other words, h^* maps εX into $h[X]$. Consequently, setting $g = h^{-1} \circ h^*$, we see that g is a continuous function with $g: \varepsilon X \to X$ and $g(p) = p$ for every $p \in X$. Therefore $\varepsilon X = X$.(1) Consequently, no proper Hausdorff extension εX of X has the property expressed in (iii).

Conversely, assume that $h[X]$ is not closed in $\times\{E_\xi: \xi \in \Xi\}$. By the Formal Theorem (Theorem 1.3) there exists a superspace εX of X which is homeomorphic to the space $Y = \overline{h[X]}$ by a homeomorphism h^* with $h^* \supset h$. Clearly, εX is a proper extension of X with

(1) Here we use the following statement:

If P is a subset of a Hausdorff space S and g is a continuous function such that $g: S \to P$ and $g(p) = p$ for every $p \in P$, then P is closed in S.

The proof follows from the equality $P = \{p \in S: f(p) = g(p)\}$, where f is the identity function on S.

$\varepsilon X \subset_{\text{top}} \times\{E_\xi: \xi \in \Xi\}$ (so εX is Hausdorff). Furthermore, for every $\xi \in \Xi$, the formula $f_\xi^* = \pi_\xi \circ h^*$ defines a continuous extension of f_ξ with $f_\xi^*: \varepsilon X \to E_\xi$. Consequently, condition (iii) is not satisfied.

In connection with the Embedding Theorem we shall introduce the following definition.

2.2 *Definition.* An $\{E_\xi: \xi \in \Xi\}$*-distinguishing*, an $\{E_\xi: \xi \in \Xi\}$*-separating*, an $\{E_\xi: \xi \in \Xi\}$*-non-extendable* class for X is a class $\mathfrak{F} = \{f_\xi: \xi \in \Xi\}$ of continuous functions with $f_\xi: X \to E_\xi$ satisfying condition (i), (ii), (iii) of Theorem 2.1, respectively. If all the spaces E_ξ are equal to a fixed space E, then we shall use the terms: an *E-distinguishing*, an *E-separating*, an *E-non-extendable* class.

It is clear that if $\Xi = \Xi_1 \cup \Xi_2 \cup \Xi_3$ and $\mathfrak{F}_1 = \{f_\xi: \xi \in \Xi_1\}$, $\mathfrak{F}_2 = \{f_\xi: \xi \in \Xi_2\}$; $\mathfrak{F}_3 = \{f_\xi: \xi \in \Xi_3\}$ are, respectively, $\{E_\xi: \xi \in \Xi_1\}$-distinguishing, $\{E_\xi: \xi \in \Xi_2\}$-separating, $\{E_\xi: \xi \in \Xi_3\}$-non-extendable classes for X, then $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3 = \{f_\xi: \xi \in \Xi\}$ is an $\{E_\xi: \xi \in \Xi\}$-distinguishing, $\{E: \xi \in \Xi\}$-separating and $\{E_\xi: \xi \in \Xi\}$-non-extendable class for X. Ξ_i are not assumed to be disjoint.

2.3. *If X is a T_0-space, then an $\{E_\xi: \xi \in \Xi\}$-separating class is $\{E_\xi: \xi \in \Xi\}$-distinguishing.*

Proof. Given two distinct points of X there exists a closed set containing exactly one of them. Suffices to apply the definitions.

2.4. *If E_ξ, $\xi \in \Xi$, are Hausdorff and X is compact, then every $\{E_\xi: \xi \in \Xi\}$-distinguishing class for X is $\{E_\xi: \xi \in \Xi\}$-separating.*

Proof. Let $\mathfrak{F} = \{f_\xi: \xi \in \Xi\}$ be an $\{E_\xi: \xi \in \Xi\}$-distinguishing class for X, let $A \subset X$ be closed and $p_0 \in X \setminus A$. For every $q \in A$ we can find an $f_{\xi_q} \in F$ so that $f_{\xi_q}(p_0) \neq f_{\xi_q}(q)$. Select disjoint open set G_q and H_q of E_{ξ_q} so that $f_{\xi_q}(p_0) \in G_q$ and $f_{\xi_q}(q) \in H_q$. The class $\{f_{\xi_q}^{-1}[H_q]: q \in A\}$ is an open cover of A; let $f_{\xi_1}^{-1}[H_{q_1}], \ldots, f_{\xi_k}^{-1}[H_{q_k}]$ be a finite subcover. Let $\xi_1, \ldots, \xi_k$ be all the distinct indices out of $\xi_{q_1}, \ldots, \xi_{q_k}$; it is easy to see that the map $\langle f_{\xi_1}, \ldots, f_{\xi_k} \rangle$ satisfies the requirements of condition (ii) of Theorem 2.1.

It can easily be seen that the proof of part c) and Theorem 2.1 yield the following

2.5. *If $\mathfrak{F} = \{f_\xi: \xi \in \Xi\}$ is an $\{E_\xi: \xi \in \Xi\}$-separating class for X, then the corresponding parametric map h is continuous and closed* (*consequently, h is quotient*).

In the above statement we do not assume, in contrast to part c) of Theorem 2.1, that h is one-to-one (i.e., we do not assume that $\mathfrak{F}$ is an $\{E_\xi: \xi \in \Xi\}$-distinguishing class). However, the converse of 2.5 is false, i.e., if h is not one-to-one, then h being continuous and closed does not imply that the class $\mathfrak{F}$ is $\{E_\xi: \xi \in \Xi\}$-separating.

2.6. *Let E_ξ, $\xi \in \Xi$, be Hausdorff and let h be a homeomorphism of X into $\times\{E_\xi: \xi \in \Xi\}$. Let εX be a Hausdorff extension of X such that h admits an extension to a continuous map h^*: $\varepsilon X \to X\{E_\xi: \xi \in \Xi\}$. Then there exists an extension $\varepsilon_1 X$ such that*

a) $\varepsilon_1 X <_{\text{ext}} \varepsilon X$, $\varepsilon_1 X \subset_{\text{top}} \times\{E_\xi : \xi \in \Xi\}$, *and*

b) *h admits an extension to a homeomorphism $h_1 : \varepsilon_1 X \to \times\{E_\xi : \xi \in \Xi\}$.*

Proof. By the Formal Theorem, there is an extension $\varepsilon_1 X$ of X that is homeomorphic to $h^*[X]$ by a homeomorphism h_1 with $h \subset h_1$.

We shall conclude this section with the following criterion for non-extendability.

2.7. *Let cX be a compactification of X, and let εX_ξ be an extension of X_ξ; cX and εX_ξ are assumed to be Hausdorff. Let $\{g_\xi : \xi \in \Xi\}$ be a class of continuous functions with $g_\xi : cX \to \varepsilon X_\xi$ and $g_\xi[X] \subset E_\xi$. Let $f_\xi = g_\xi | X$. The class $\mathfrak{F} = \{f_\xi : \xi \in \Xi\}$ is $\{E_\xi : \xi \in \Xi\}$-non-extendable if and only if for every $p_0 \in cX \setminus X$ there exists a $\xi \in \Xi$ with $g_\xi(p_0) \notin E_\xi$.*

Proof. The necessity of the condition is obvious; we shall prove the sufficiency. Let εX be a proper extension of X, let $q_0 \in \varepsilon X \setminus X$. Let $\mathfrak{R}$ be the class of all open subsets of εX containing q_0. There exists a point $p_0 \in cX$ such that $p_0 \in \overline{G \cap X}^{cX}$ for every $G \in \mathfrak{R}$. Clearly, $p_0 \in cX \setminus X$. Take a $\xi \in \Xi$ such that $g_\xi(p_0) \notin E_\xi$. f_ξ does not admit a continuous extension $f^* : \varepsilon X \to E_\xi$.

Historical remarks on the Embedding Theorem. The method of producing maps into topological products via classes of functions into coordinate spaces is as old as parametric equations. Parametric equations of, say, a circle, $x = \cos 2\pi t$, $y = \sin 2\pi t$, $0 \leqslant t \leqslant 1$, give raise to a map h of the interval $0 \leqslant t \leqslant 1$ into the product of the ranges of the functions $x = \cos 2\pi t$, $y = \sin 2\pi t$; this map h assigns to every point t, $0 \leqslant t \leqslant 1$, the point $(\cos 2\pi t, \sin 2\pi t)$ of the plane. It is clear that "the parametric map corresponding to a class of functions" is constructed in exactly the same way. Urysohn [24], [25] to prove his famous metrization theorem, applied the technique of parametric maps to countable classes of functions with values in the unit interval $\mathfrak{J}$; subsequently, Tihonov [23] extended this procedure to classes of arbitrarily many $\mathfrak{J}$-valued functions. Both [25] and [23] contain sufficient conditions for a parametric map to be a homeomorphism. In [16] classes of functions with arbitrary ranges were considered; furthermore [16] contains a necessary and sufficient condition for a parametric map to be a homeomorphism. (In [16] only T_0-spaces were considered; consequently, conditions (i) of Theorem 2.1 was not mentioned; see 2.3 in the present paper.) Further comments on this topic can be found in [1], pp. 67–68, or [2], p. 42. Part d) of the Embedding Theorem was shown, in a somewhat less general form, in [9]; however, the proof given here follows that of [19]. The origin of part (d) can be traced to Kuratowski and Sierpiński [15] (1921) who, in fact, used one-element $\mathcal{R}$-non-extendable classes to prove that a difference of two closed subsets of a metric space X is homeomorphic to a closed subspace of $X \times \mathfrak{R}$. Subsequently, Kuratowski used countable $\mathcal{R}$-non-extendable classes to prove that a G_δ-subset of a metric space X is homeomorphic to a closed subspace of $X \times \mathfrak{R}^{\aleph_0}$. For further information see [14], pp. 240–241.

III. *E*-Completely regular spaces

3.1. *Definition.* Given two spaces X and E, we say that X is *E-completely regular* provided that $X \subset_{\mathrm{top}} E^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$. The class of all E-completely regular spaces will be denoted by $\mathfrak{C}(E)$. The smallest infinite cardinal $\mathfrak{m}$ for which $X \subset_{\mathrm{top}} E^{\mathfrak{m}}$ is called *the E-exponent of X* and it is denoted by $\exp_E X$ ($\exp_E X$ is defined only for an E-completely regular X). Classes $\mathfrak{C}(E)$ are called *classes of complete regularity* (i.e., a class $\mathfrak{C}$ of topological spaces is called a class of complete regularity provided that there exists a space E with $\mathfrak{C} = \mathfrak{C}(E)$). If $E \in \mathfrak{C}(E_1)$ and $E_1 \in \mathfrak{C}(E)$, then we say that E and E_1 has the same *degree of complete regularity.*

The following statements (3.2–3.7) are direct consequences of the definition.

3.2. $E \in \mathfrak{C}(E)$; *in fact,* $\exp_E E = \aleph_0$.

3.3. *If* $X \in \mathfrak{C}(E)$ *and* $X_0 \subset_{\mathrm{top}} X$, *then* $X_0 \in \mathfrak{C}(E)$; *in fact,* $\exp_E X_0 \leqslant \exp_E X$.

3.4. *If* $X_\xi \in \mathfrak{C}(E)$ *for every* $\xi \in \Xi$, *then* $\times\{E_\xi : \xi \in \Xi\} \in \mathfrak{C}(E)$; *in fact,* $\exp_E \times\{E_\xi : \xi \in \Xi\} \leqslant \Sigma\{\exp_E X_\xi : \xi \in \Xi\}$ ($\leqslant$ *cannot be replaced by* $=$ *and* $\Sigma\{\exp_E X_\xi : \xi \in \Xi\}$ *cannot be replaced by* $\sup\{\exp_E X_\xi : \xi \in \Xi\}$).

3.5. $\mathfrak{C}(E) \subset \mathfrak{C}(E_1)$ *if and only if* $E \in \mathfrak{C}(E_1)$.

3.6. $\mathfrak{C}(E) = \mathfrak{C}(E_1)$ *if and only if* $E \in \mathfrak{C}(E_1)$ *and* $E_1 \in \mathfrak{C}(E)$.

3.7. $\mathfrak{C}(E) = \mathfrak{C}(E^{\mathfrak{m}})$ *for every cardinal* $\mathfrak{m} > 0$.

From part c) of Theorem 2.1 we obtain the following characterization of E-completely regular spaces.

3.8. THEOREM. *A space X is E-completely regular if and only if the following two conditions are satisfied:*

(a) *for every* $p, q \in X$, $p \neq q$, *there is a continuous function* $f: X \to E$ *with* $f(p) \neq f(q)$;

(b) *for every closed subset A of X and for every* $p \in X \setminus A$ *there is a finite number n and a continuous function* $f: X \to E^n$ *such that* $f(p) \notin f[A]$.

Equivalently, X is E-completely regular if and only if C(X, E) is both an E-distinguishing and an E-separating class for X (or, if X admits a class which is both E-distinguishing and E-separating).

Remark. If X is a T_0-space, then, by 2.3, condition (a) can be ommitted. In this form Theorem 3.8 was stated in [16]. It was shown in [9] that in condition (b) it is not sufficient to consider only functions f with $f: X \to E$.

Another characterization of E-completely regular spaces (due to R. Blefko) is as follows.

3.9. *A T_0-space X is E-completely regular if and only if for every net* $\{x_n: n \in D\}$ *of points of X we have*

(1) $\qquad x_n \to x$ *if and only if* $f(x_n) \to f(x)$ *for every* $f \in C(X, E)$.

Proof. Assume (1) and let h be the parametric map corresponding to the class $C(X, E)$. If $x, y \in X$, $x \neq y$, then one of these points does not belong to the closure of the other; say $x \notin \overline{\{y\}}$. Then $x_n \not\to y$, where $x_n = x$ for $n = 1, 2, \ldots$. Consequently, by (1), $f(x) \neq f(y)$ for at least one $f \in C(X, E)$. It follows that h is one-to-one. It is now clear that (1) implies that h is a homeomorphism. Thus $X \subset_{\text{top}} E^{\mathfrak{m}}$, where $\mathfrak{m} = \text{card}\, C(X, E)$; consequently, X is E-completely regular. The converse follows immediately from Theorem 3.8.

In 3.9 the assumption of X being a T_0-space cannot be omitted. In fact, if X is an indiscrete space and E is a T_0-space, then condition (1) is always satisfied.

It is clear that $\mathfrak{C}(\mathfrak{J}) = \mathfrak{C}(\mathcal{R}) =$ the class of all completely regular spaces. In fact, we have

3.10. $\mathfrak{C}(E) =$ *the class of all completely regular space if and only if E is completely regular and* $\mathfrak{J} \subset_{\text{top}} E$.

Proof. Assume that $\mathfrak{C}(E) =$ the class of all completely regular spaces. Complete regularity of E is obvious. On the other hand, we have $\mathfrak{J} \subset_{\text{top}} E^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$. Let I_0 be a subspace of $E^{\mathfrak{m}}$ with $I_0 =_{\text{top}} \mathfrak{J}$; write $E^{\mathfrak{m}} = \times\{E_\xi: \xi \in \Xi_0\}$. For at least one $\xi_0 \in \Xi$, $\pi_{\xi_0}[I_0]$ contains more than one point. Now, $\pi_{\xi_0}[I_0]$ is a locally connected (metrizable) continuum; therefore $\pi_{\xi_0}[I_0]$ contains a homeomorph of I_0. Thus $\mathfrak{J} \subset_{\text{top}} E$.

The proof of the converse is obvious.

Similarly, we have

3.11. $\mathfrak{C}(\mathfrak{D}) =$ *the class of all 0-dimensional T_0-spaces. Furthermore,* $\mathfrak{C}(E) = \mathfrak{C}(\mathfrak{D})$ *if and only if E is a 0-dimensional T_0-space containing more than one point.*

3.12. $\mathfrak{C}(\mathfrak{F}) =$ *the class of all T_0-spaces. Furthermore,* $\mathfrak{C}(E) = \mathfrak{C}(\mathfrak{F})$ *if and only if E is T_0-space and E is not a T_1-space.*

Verification of 3.11 and 3.12 is straightforward (use Theorem 3.8). The first parts of 3.11 and 3.12 are classical results due, respectively, to N. B. Vedenisov and P. S. Alexandrov. For references see [1], p. 47, or [2], p. 42.

The first part of 3.12 implies, in particular, that for every T_0-space X the class $C(X, \mathfrak{F})$ is $\mathfrak{F}$-separating. In fact, a stronger statement holds:

3.13. *For every space X the class $C(X, \mathfrak{F})$ is $\mathfrak{F}$-separating.*

For the proof of 3.13 [16] can be consulted. 3.13 indicates the necessity of the assumption that X is a T_0-space in 2.3. In fact, the class $C(X, \mathfrak{I})$ (which by 3.13 is always $\mathfrak{I}$-separating) is $\mathfrak{I}$-distinguishing iff X is a T_0-space.

3.14. *Let* $\mathfrak{I}^*$ *be the space consisting of three points* 0, 1, 2 *in which* $\{0\}$ *is the only non-empty proper open subset.* $\mathfrak{C}(\mathfrak{I}^*)$ = *the class of all topological spaces.*

Proof. By 3.13 and the fact that $\mathfrak{I}$ is a subspace of $\mathfrak{I}^*$ we infer that (for every space X) the class $C(X, \mathfrak{I}^*)$ is $\mathfrak{I}^*$-separating. To prove that $C(X, \mathfrak{I}^*)$ is $\mathfrak{I}^*$-distinguishing one uses functions with values 1 and 2 (every such function is continuous).

In a similar way one can show that $\mathfrak{C}(E)$ = *the class of all topological spaces iff E contains a non-trivial* (*i.e., containing more than one point*) T_0-*subspace and a non-trivial indiscrete subspace.*

According to the above each of the following classes: the class of all topological spaces, the class of all T_0-spaces, the class of all 0-dimensional T_0-spaces, the class of all completely regular spaces, is a class of complete regularity. It was shown in [16] that the class of all T_1-spaces is not a class of complete regularity. This result has been strengthened by Bialynicki-Birula in 1958, who has shown (using Theorem 1 of [16]) that *there is no* T_1-*space* E *such that* $\mathfrak{C}(E)$ *contains all Hausdorff spaces* (1)). In [11], H. Herrlich has obtained a still stronger result showing that *there is no* T_1-*space* E *such that* $\mathfrak{C}(E)$ *contains all regular spaces.* These results include, of course, the result concerning the class of all T_1-spaces; they also imply that neither the class of all Hausdorff spaces nor the class of all regular spaces is a class of complete regularity. It follows from the above and from 3.12 that if $\mathfrak{C}(E)$ contains all regular spaces, then $\mathfrak{C}(E)$ contains all T_0-spaces. In other words, the class of all T_0-spaces it the smallest class of complete regularity containing all regular spaces.

3.15. *Definition.* A set $A \subset X$ is said to be *E-closed* (2) (*E-open*) in X provided that for some finite n there exists a closed (an open) subset T of E^n and a continuous function $f: X \to E^n$ such that $A = f^{-1}[T]$.

If $E = \mathfrak{I}$ or $E = \mathfrak{R}$, then the E-closed sets coincides with the so-called Vedenisov sets [26] or zero-sets of continuous real-valued functions.

3.16–3.20 follow directly from the definition.

3.16. $A \subset X$ *is E-closed if and only if* $X \setminus A$ *is E-open.*

3.17. *If* $f: X \to Y$ *is continuous and* A *is an E-closed* (*E-open*) *subset of* Y, *then* $f^{-1}[A]$ *is E-closed* (*E-open*) *in* X.

3.18. *A finite union and a finite intersection of E-closed* (*E-open*) *subsets of X is E-closed* (*E-open*) *in X.*

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(1) This result has never been published.

(2) This concept was introduced in [7].

3.19. *Let $\mathfrak{m}$ be a cardinal and assume that every closed subset of $E^{\mathfrak{m}}$ is E-closed. Then the intersection of $\mathfrak{m}$ E-closed subsets of X is E-closed in X.*

3.19 is a generalization of a well-known fact: the *intersection of countably many zero-sets is a zero-set*. Indeed, $\mathcal{J}^{\aleph_0}$ is a metric space.

3.20. *A T_0-space X is E-completely regular if and only if the class of all E-open subsets of X is a base (for open subsets). In particular, if X is E-completely regular, then every open (closed) subset of X is the union (the intersection) of E-open (E-closed) sets.*

3.21. *Definition.* A space X is said to be *E-Hausdorff* provided that the class $C(X, E)$ is E-distinguishing. X is said to be *E-normal* provided that for every two disjoint closed subsets A and B of X there exist two disjoint E-closed subsets A^1 and B^1 of X with $A \subset A^1$ and $B \subset B^1$. X is said to be *strongly E-normal* provided that for every two disjoint closed subsets A and B of X there exists a finite number n, a continuous function $f: X \to E^n$, and two disjoint closed subsets F_1 and F_2 of E^n such that $A \subset f^{-1}[F_1]$, $B \subset f^{-1}[F_2]$.

If $E = \mathcal{J}$ or $E = \mathcal{R}$, then E-normal as well as strongly E-normal T_1-spaces coincide with normal spaces (in the usual sense). However, the property of being $\mathcal{J}$-Hausdorff is stronger than the usual Hausdorff separation axiom; $\mathcal{J}$-Hausdorff spaces are sometimes called *functionally Hausdorff*.

3.22. *An E-completely regular space is E-Hausdorff; an E-normal T_1-space is E-completely regular; a strongly E-normal space is E-normal.*

An E-normal T_0-space need not to be E-completely regular; for instance, $\mathcal{F}$ is $\mathcal{J}$-normal but $\mathcal{F}$ is not completely regular. An E-normal space need not be strongly E-normal. Indeed, it is easy to see that if finite powers of E are normal, then a strongly E-normal space is normal. Consequently, a non-normal completely regular space is not strongly $\mathcal{J}^{\mathfrak{m}}$-normal for any cardinal $\mathfrak{m}$. On the other hand, every completely regular space is $\mathcal{J}^{\mathfrak{m}}$-normal for sufficiently large $\mathfrak{m}$.

3.23. *Let E be Hausdorff. If X is compact and E-Hausdorff, then X is strongly E-normal.*

Proof. By 2.4 we infer that X is E-completely regular. Let A and B be disjoint closed subsets of X. For every $p \in A$ there is a finite number n_p and a continuous function $f_p: X \to E^{n_p}$ with $f_p(p) \notin f_p[B]$. Since $f_p[B]$ is compact, we can find an open subset G_p of E^{n_p} with $f_p(p) \in G_p$ and $\bar{G}_p \cap f_p[B] = \emptyset$. The sets $f_p^{-1}[G_p]$, $p \in A$, form an open cover of A; let $f_{p_1}^{-1}[G_{p_1}], \ldots, f_{p_k}^{-1}[G_{p_k}]$ be a finite subcover of A. Set $f = \langle f_{p_1}, \ldots, f_{p_k} \rangle$, $F_1 = \bigcup_{i=1}^{k}(E^{n_{p_1}} \times \ldots \times \bar{G}_{p_i} \times \ldots \times E^{n_{p_k}})$, $F_2 = f[B]$. f is a continuous function of X into E^n, where $n = n_{p_1} + \ldots + n_{p_k}$, F_1 and F_2 are disjoint closed subsets of E^n and $A \subset f^{-1}[F_1]$, $B \subset f^{-1}[F_2]$. Thus X is strongly E-normal.

Consider now the following situation: Let φ be a continuous map with $\varphi: X \to_{\text{onto}} X^*$. φ induces a map $\overset{*}{\varphi}$ of $C(X^*, E)$ into $C(X, E)$ defined by

(2) $\overleftarrow{\varphi}(g) = g \circ p$ for every $g \in C(X^*, E)$.

The map $\overleftarrow{\varphi}$ is always one-to-one; in general, $\overleftarrow{\varphi}$ does not map $C(X^*, E)$ onto $C(X, E)$. $\overleftarrow{\varphi}$ is interesting from algebraic viewpoint; it is easy to see that $\overleftarrow{\varphi}$ is an isomorphism relative to all pointwisely defined operations in $C(X^*, E)$ and $C(X, E)$. More precisely, if $\otimes$ is an arbitrary operation (binary, for simplicity) in E; and if $\otimes_X$ and $\otimes_{X^*}$ are the corresponding pointwisely defined operations in $C(X, E)$ and $C(X^*, E)$, respectively (i.e., $(f_1 \otimes_X f_2)(p) = f_1(p) \otimes f_2(p)$ for every $p \in X$ and $(g_1 \otimes_{X^*} g_2)(q) = g_1(q) \otimes g_2(q)$ for every $q \in X^*$); then $\overleftarrow{\varphi}(g_1 \otimes_{X^*} g_2) = \overleftarrow{\varphi}(g_1) \otimes_X \overleftarrow{\varphi}(g_2)$ for every $g_1, g_2 \in C(X^*, E)$. Similarly, if ϱ is an arbitrary (binary, for simplicity) relation in E, then $g_1 \varrho_{X^*} g_2$ iff $\overleftarrow{\varphi}(g_1) \varrho_X \overleftarrow{\varphi}(g_2)$ for every $g_1, g_2 \in C(X^*, E)$; where $\varrho_X(\varrho_{X^*})$ is defined by $f_1 \varrho_X f_2 \equiv f_1(p) \varrho f_2(p)$ for every $p \in X$ $(g_1 \varrho_{X^*} g_2 \equiv g_1(q) \varrho g_2(q)$ for every $q \in C(X^*, E))$.

We shall now prove that one can always find an E-completely regular X^* for which $\overleftarrow{\varphi}$ maps $C(X^*, E)$ onto $C(X, E)$.

3.19. Theorem. (The identification theorem.) *For every space X there exists an E-completely regular space X^* and a continuous map $\varphi\colon X \to_{\text{onto}} X^*$ such that the map $\overleftarrow{\varphi}$ (defined by* (1)) *maps $C(X^*, E)$ onto $C(X, E)$.*

The pair (X^*, φ) with the above properties is called the E-transformation of X and X^* is called the E-modification of X.

The E-transformation of X is unique in the sense that if (X^*, φ_1) and (X^*, φ_2) are both E-transformations of X, then there exists a homeomorphism h of X^* onto X^* such that $\varphi_2 = \varphi_1 \circ h$.

Proof. Construction of an E-transformation. Let $\mathfrak{F} = C(X, E)$. It is easy to see that the pair (X^*, φ), where φ is the parametric map corresponding to the class $\mathfrak{F}$ and $X^* = \varphi[X]$, is the E-transformation of X.

Uniqueness of an E-transformation. Let (X_1^*, φ_1) and (X_2^*, φ_2) both be E-transformations of X. The map $\psi = (\overleftarrow{\varphi}_1)^{-1} \circ \overleftarrow{\varphi}_2$ is a one-to-one map of $C(X_2^*, E)$ onto $C(X_1^*, E)$. This map ψ induces, in a natural way, the map ψ_n of $C(X_2^*, E^n)$ onto $C(X_1^*, E^n)$; $n = 1, 2, \ldots$. ψ_n satisfies

(3) for every $f \in C(X_1^*, E^n)$ and every $g \in C(X_2^*, E^n)$ we have $f = \psi_n(g)$ iff $f(\varphi_1(p)) = g(\varphi_2(p))$ for every $p \in X$.

Using (3) and the E-complete regularity of X_1^* and X_2^* we obtain that

(4) for every $p_1, p_2 \in X$, $\varphi_1(p_1) = \varphi_1(p_2)$ iff $\varphi_2(p_1) = \varphi_2(p_2)$.

(4) enables us to define a one-to-one map h of X_1^* onto X_2^* such that $\varphi_2 = h \circ \varphi_1$. Using (3) again we prove that h is, in fact, a homeomorphism.

The above proof was obtained in cooperation with Blefko. Another proof of uniqueness (also due to Blefko), which is perhaps technically simpler and which, in particular, does not involve $C(X_1^*, E^n)$, can be based on 3.9.

According to Theorem 3.19, *every algebraic structure* $C(X, E)$ *of continuous functions* (in the sense of [21] and [22]) *is isomorphic to a structure* $C(X^*, E)$ *on an E-completely regular space* X^*.

It is easy to see that

3.20. *The following three conditions are equivalent*

(a) *X is E-Hausdorff;*

(b) *X admits a continuous one-to-one map onto an E-completely regular space;*

(c) *the map φ in the E-transformation (X^*, φ) of X is one-to-one.*

In connection with condition (b) note that if φ is a one-to-one continuous map of X onto an E-completely regular space X^*, then (X^*, φ) need not be an E-transformation of X.

It is easy to see that the E-transformation depends only upon the degree of complete regularity of E; i.e.,

3.21. *If* $\mathfrak{C}(E)=\mathfrak{C}(E_1)$ *and* (X^*, φ) *is the E-transformation of* X, *then* (X^*, φ) *is the* E_1-*transformation of* X.

Proof. X^* is E_1-completely regular. On the other hand, we can assume that E_1 is a subspace of $E^{\mathfrak{m}}$ (for some $\mathfrak{m}$). Let $f\in C(X, E_1)$. Then $f\in C(X, E^{\mathfrak{m}})$. Considering the coordinates of f, we infer that there is a $g\in C(X^*, E^{\mathfrak{m}})$ with $f=g\circ\varphi$. The last formula implies that the range of g is contained in E_1; thus, in fact, $g\in C(X^*, E_1)$. Thus (X^*, φ) is an E_1-transformation of X.

If $E=\mathfrak{I}$ or if $E=\mathfrak{R}$, then the E-transformation (the E-modification) of X will be called the *completely regular transformation (modification) of* X. The completely regular modification of X coincides with the space discussed by Čech in [8], p. 826, and denoted there by ϱX. The $\mathfrak{F}$-transformation ($\mathfrak{F}$-modification) of X will be called the T_0-transformation (T_0-modification) of X. T_0-modifications also were discussed in [8] (pp. 825–826, "... the theory of general topological spaces ... can be completely reduced to the theory of Kolmogoroff spaces"). To see this it suffices to show that

3.22. *Let φ be a continuous map with* $\varphi\colon X\to_{\text{onto}} X^*$. (X^*, φ) *is the* T_0-*transformation of* X *if and only if*

(a) $\varphi(p)=\varphi(q)$ *if and only if* $\overline{\{p\}}=\overline{\{q\}}$;

(b) *φ is a quotient map.*

Proof. Let (X^*, φ) be the $\mathfrak{F}$-transformation of X; we shall prove that (a) and (b) are satisfied. If $\overline{\{p\}}=\overline{\{q\}}$, then $p\in\overline{\{q\}}$ and $q\in\overline{\{p\}}$; consequently, by the continuity of

φ, $\varphi(p) \in \overline{\{\varphi(q)\}}$ and $\varphi(q) \in \overline{\{q(p)\}}$, hence $\overline{\{\varphi(p)\}} = \overline{\{\varphi(q)\}}$. But X^* is a T_0-space; consequently, the last equality implies $\varphi(p) = \varphi(q)$. On the other hand, if $\overline{\{p\}} \neq \overline{\{q\}}$, then one of these points does not belong to the closure of the other; say $p \notin \overline{\{q\}}$. The function f, defined by $f(s) = 1$ for $s \in \overline{\{q\}}$ and $f(s) = 0$ for $s \in X \setminus \overline{\{q\}}$, is a continuous function with $f\colon X \to \mathfrak{J}$. If follows that there exists a continuous function $g\colon X^* \to \mathfrak{J}$ with $f = g \circ \varphi$. Clearly, $g(\varphi(p)) = f(p) = 0$ and $g(\varphi(q)) = f(q) = 1$; consequently, $\varphi(p) \neq \varphi(q)$. Thus (a) is satisfied. On the other hand, by 3.13, the class $C(X, \mathfrak{J})$ is $\mathfrak{J}$-separating; consequently, by 2.7, φ is a closed map (note that φ is the parametric map corresponding to $C(X, \mathfrak{J})$); thus φ is quotient.

Conversely, if (X^*, φ) satisfies (a) and (b) then (X^*, φ) must be the $\mathfrak{J}$-transformation of X; indeed, conditions (a) and (b) uniquely determine the pair (X^*, φ) (in the sense of Theorem 3.19).

The concepts of an E-transformation and E-modification, as well as their uniqueness, deserve some comments. It can be easily seen that the E-transformation has the following maximality property:

3.23. *If (X^*, φ) is an E-transformation of X, then for every continuous map φ_1 of X onto an E-completely regular space X_1^* there exists a continuous map φ_2 of X^* onto X_1^* such that $\varphi_1 = \varphi_2 \circ \varphi$.*

In fact, the E-transformation is determined by the above maximality property:

3.24. *If φ is a continuous map of X onto an $X^* \in \mathfrak{C}(E)$ and the pair (X^*, φ) satisfies the conclusion of 3.23, then (X, φ) is the E-transformation of X.*

Now, the E-modification of X is, by definition, a space X^* such that there exists a continuous map $\varphi\colon X \to_{\text{onto}} X^*$ such that (X^*, φ) is an E-transformation of X. Consequently, from the uniqueness of E-transformations we obtain the uniqueness (up to homeomorphisms) of E-modifications. However, it is not true that if X^* is the E-modification of X and φ is an arbitrary continuous map of X onto X^*, then (X^*, φ) is an E-transformation of X. For instance, if $X = X^*$ is an E-completely regular space and φ is a continuous map with $\varphi\colon X \to_{\text{onto}} X^*$ and such that φ is not one-to-one, then X^* is the E-modification of X, but (X^*, φ) is not the E-transformation of X; indeed, $\overleftarrow{\varphi}$ does not map $C(X^*, E)$ onto $C(X, E)$.

From 3.23 we obtain a maximality property of the E-modifications:

3.25. *The E-modification X^* of X is a maximal E-completely regular continuous image of X; i.e. if X_1 is an arbitrary E-completely regular continuous image of X, then X_1 is a continuous image of X^*.*

In contrast to 3.24., the E-modification of X is not determined by its maximality property 3.25. Examples are trivial.

IV. E-compact spaces

Throughout this chapter all spaces will be assumed to be Hausdorff.

4.1. *Definition.* A space X is said to be *E-compact* provided that $X \subset_{\mathrm{cl}} E^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$. The smallest infinite cardinal $\mathfrak{m}$ for which $X \subset_{\mathrm{cl}} E^{\mathfrak{m}}$ is called *the large exponent of X relative to E* and it is denoted by $\mathrm{Exp}_E X$. ($\mathrm{Exp}_E X$ is defined only for an E-compact X). The class of all E-compact spaces is denoted by $\mathfrak{K}(E)$. Classes of the form $\mathfrak{K}(E)$ are called *classes of compactness.* If $E_1 \in \mathfrak{K}(E)$ and $E \in \mathfrak{K}(E_1)$, then we say that E and E_1 have *the same degree of compactness.*

4.2. $\mathfrak{K}(E) \subset \mathfrak{C}(E)$.

4.3. *$E \in \mathfrak{K}(E)$; in fact,* $\mathrm{Exp}_E E = \aleph_0$.

4.4. *If $X \in \mathfrak{K}(E)$ and $X_0 \subset_{\mathrm{cl}} X$, then $X_0 \in \mathfrak{K}(E)$; in fact* $\mathrm{Exp}_E X_0 \leqslant \mathrm{Exp}_E X$.

4.5. *If $X_\xi \in \mathfrak{K}(E)$ for every $\xi \in \Xi$, then $\times\{X_\xi : \xi \in \Xi\} \in \mathfrak{K}(E)$; in fact* $\mathrm{Exp}_E \times \{X_\xi : \xi \in \Xi\} \leqslant \Sigma\,\{\mathrm{Exp}_E X_\xi : \xi \in \Xi\}$ ($\leqslant$ cannot be replaced by $=$ and $\Sigma\,\{\mathrm{Exp}_E X_\xi : \xi \in \Xi\}$ cannot be replaced by $\sup\,\{\mathrm{Exp}_E X_\xi : \xi \in \Xi\}$).

4.6. *$\mathfrak{K}(E) \subset \mathfrak{K}(E_1)$ if and only if $E \in \mathfrak{K}(E_1)$.*

4.7. *$\mathfrak{K}(E) = \mathfrak{K}(E_1)$ if and only if $E \in \mathfrak{K}(E_1)$ and $E_1 \in \mathfrak{K}(E)$.*

4.8. *If X_ξ, $\xi \in \Xi$, are E-compact subspaces of a space X, then the intersection $\bigcap\{X_\xi : \xi \in \Xi\}$ is also E-compact; in fact,* $\mathrm{Exp}_E \bigcap\{X_\xi : \xi \in \Xi\} \leqslant \Sigma\{\mathrm{Exp}_E X_\xi : \xi \in \Xi\}$ ($\leqslant$ cannot be replaced by $=$ and $\Sigma\{\mathrm{Exp}_E X_\xi : \xi \in \Xi\}$ cannot be replaced by $\sup\{\mathrm{Exp}_E X_\xi : \xi \in \Xi\}$).

4.9. *Let X be an E-compact space and let f be a continuous function with $f: X \to Y$. If Y_0* is an *E-compact subspace of Y, then $f^{-1}[Y_0]$ is E-compact; in fact,* $\mathrm{Exp}_E f^{-1}[Y_0] \leqslant \mathrm{Exp}_E X + \mathrm{Exp}_E Y_0$.

Proof of 4.8 and 4.9. $\bigcap\{X_\xi : \xi \in \Xi\}$ is homeomorphic to the diagonal Δ of the product $\times\{E_\xi : \xi \in \Xi\}$ and Δ is closed in $\times\{E_\xi : \xi \in \Xi\}$. $f^{-1}[Y_0]$ is homeomorphic to the set $\{(x, y) : y = f(x),\ x \in X$, and $f(x) \in Y_0\}$ (the "graph" of f restricted to $f^{-1}[Y_0]$) and this set is closed in $X \times Y_0$. (These are classical facts; see [13], p. 144.)

From part d) of Theorem 2.1 we obtain

4.10. Theorem. *Let X be an E-completely regular space. X is E-compact if and only if for every proper extension εX of X there exists a continuous function $f: X \to E$ which cannot be extended to εX. In the above, it suffices to consider only E-completely regular extensions of X.*

Equivalently, an E-completely regular X is E-compact if and only if $C(X, E)$ is an E-non-extendable class for X (or if X admits an E-non-extendable class).

Clearly, $\mathfrak{K}(\mathfrak{I})$ = the class of all compact spaces and $\mathfrak{K}(\mathfrak{D})$ = the class of all 0-dimensional compact spaces. In fact, it can be easily shown (see the proof of 3.10) that

4.11. $\mathfrak{K}(E)=\mathfrak{K}(\mathfrak{J})$ *if and only if E is compact and* $\mathfrak{J}\subset_{\text{top}} E$.

4.12. $\mathfrak{K}(E)=\mathfrak{K}(\mathfrak{D})$ *if and only if E is a 0-dimensional compact space containing more than one point.*

There is a great variety of classes of compactness. Blefko has shown ([6], Ch. III) that

4.13. *Let ω_λ and ω_μ be initial ordinals. If $cf(\omega_\lambda)=cf(\omega_\mu)$, then $\mathfrak{K}(S(\omega_\lambda))=\mathfrak{K}(S(\omega_\mu))$, and if $cf(\omega_\lambda)\neq cf(\omega_\mu)$, then neither $\mathfrak{K}(S(\omega_\lambda))\subset\mathfrak{K}(S(\omega_\mu))$ nor $\mathfrak{K}(S(\omega_\mu))\subset\mathfrak{K}(S(\omega_\lambda))$.*

In the above, $S(\alpha)$ denotes the space of all ordinals $\xi<\alpha$ (with the order topology).

4.14. *Theorem on the existence and the uniqueness of $\beta_E X$.*

(a) *For every E-completely regular space X there exists an extension $\beta_E X$ such that*

(i) *$\beta_E X$ is E-compact;*

(ii) *every continuous function $f: X\to E$ admits a continuous extension $f^*: \beta_E X\to E$.*

(b) *$\beta_E X$ is uniquely determined by the above properties; i.e., if εX is an arbitrary extension of X that satisfies* (i) *and* (ii), *then* $\varepsilon X=_{\text{ext}}\beta_E X$.

(c) *$\beta_E X$ has also the following property: every continuous function $g: X\to Y$, where Y is an arbitrary E-compact space, admits a continuous extension $g^*: \beta_E X\to Y$.*

Proof. Part (a). The proof is a duplication of the famous Čech construction of βX [8]. Let $\mathfrak{F}=C(X, E)$, let h be the parametric map (see Ch. II) of X into $E^{\mathfrak{m}}=\times\{E_f: f\in\mathfrak{F}\}$, where $\mathfrak{m}=\text{card}\,\mathfrak{F}$ and $E_f=E$ for every $f\in\mathfrak{F}$, corresponding to the class $\mathfrak{F}$. There is a superspace $\beta_E X$ of X which is homeomorphic to $\overline{h[X]}$, the closure of $h[X]$ in $E^{\mathfrak{m}}$, by a homeomorphism h^* with $h\subset h^*$. It is easy to see that $\beta_E X$ satisfies (i) and (ii), in particular, if $f\in C(X, E)$, then $f^*=\pi_f\circ h^*$ is a continuous extension of f over $\beta_E X$.

Part (c). Suffices to embed Y into $E^{\mathfrak{m}}=\times\{E_\xi: \xi\in\Xi\}$ as a closed subspace and extend the functions $\pi_\xi\circ g$ and then take the parametric map corresponding to the class of these extensions.

Part (b). Assume that εX satisfies the assumptions of (b). Repeating the proof of (c), we see that εX has the property expressed in (c). This enables us, by extending the identity map of X onto itself, to define two continuous functions f and g with $f: \beta_E X\to\varepsilon X$, $g: \varepsilon X\to\beta_E X$, and $f(p)=g(p)=p$ for every $p\in X$. By Lemma 1.2, $\varepsilon X=_{\text{ext}}\beta_E X$.

4.15. COROLLARY. *Let $X\in\mathfrak{C}(E)$. X is E-compact if and only if $\beta_E X=X$.*

$\beta_{\mathfrak{J}} X$ coincides with the usual βX; βX is the largest, in the sense of $\leqslant_{\text{ext}}$ compactification of X. $\beta_{\mathfrak{D}} X$ is defined for every 0-dimensional space X; it was studied in [5]. $\beta_{\mathfrak{D}} X$ is the largest, in the sense of $\leqslant_{\text{ext}}$, 0-dimensional compactification of X. In general, if E is compact, then $\beta_E X$ is the largest E-compact (equivalently, E-completely regular and compact) extension of X. However, if E is not compact, then $\beta_E X$ need not be the largest

E-compact extension of X; in fact, an E-compact space can admit proper E-compact extensions. On the other hand, among extensions satisfying condition (ii) of Theorem 4.14, $\beta_E X$ can always be characterized as a largest or a maximal[1] extension in the sense of $\subset_{\mathrm{ext}}$.

4.16. *Let* $\mathfrak{E}(X)$ *be the class of all* E*-completely regular extensions of* X *satisfying condition* (ii) *of Theorem 4.14 and let* $\varepsilon X \in \mathfrak{E}(X)$. *The following conditions are equivalent*

(a) $\varepsilon X =_{\mathrm{ext}} \beta_E X$;

(b) εX *is the largest extension in* $\mathfrak{E}(X)$ (*in the sense of* $\subset_{\mathrm{ext}}$);

(c) εX *is a maximal extension in* $\mathfrak{E}(X)$ (*in the sense of* $\subset_{\mathrm{ext}}$).

Proof. (a) implies (b). $\beta_E(\varepsilon X)$ exists and it is an extension of X. It is easy to see that $\beta_E(\varepsilon X)$ satisfies 1. and 2. of Theorem 4.14. Consequently, by part (b) of this theorem $\beta_E(\varepsilon X) =_{\mathrm{ext}} \beta_E X$. But $\varepsilon X \subset \beta_E(\varepsilon X)$; consequently, $\varepsilon X \subset_{\mathrm{ext}} \beta_E X$. Thus $\beta_E X$ is the largest extension in $\mathfrak{E}(X)$.

Obviously, (b) implies (c).

(c) implies (a). Assume εX is maximal. If $\varepsilon X \neq_{\mathrm{ext}} \beta_E X$; then, by part b) of Theorem 4.14, εX is not E-compact. Consequently, $\beta_E(\varepsilon X)$ is a proper extension of εX and clearly, $\beta_E(\varepsilon X) \in \mathfrak{E}(X)$. Thus, εX is not maximal in $\mathfrak{E}(X)$.

In sequel, we shall compare the extensions $\beta_E X$ for different E's.

4.17. THEOREM. *Let* E_1 *and* E_2 *be two spaces with* $\mathfrak{C}(E_1) = \mathfrak{C}(E_2)$. $E_2 \in \mathfrak{K}(E_1)$ *if and only if* $\beta_{E_1} X \subset_{\mathrm{ext}} \beta_{E_2} X$ *for every* $X \in \mathfrak{C}(E_1)$.

In other words, the more compact E *the larger the extension* $\beta_E X$.

Proof. Assume $E_2 \in \mathfrak{K}(E_1)$. By part (c) of Theorem 4.14 we infer that $\beta_{E_1} X$ satisfies condition (ii) of this theorem relative to the space E_2. Consequently, by Theorem 4.16, $\beta_{E_1} X \subset_{\mathrm{ext}} \beta_{E_2} X$.

To prove the converse, note that, in particular, we have $E_2 \subset \beta_{E_1} E_2 \subset_{\mathrm{ext}} \beta_{E_2} E_2$. By Corollary 4.15, $\beta_{E_2} E_2 = E_2$, hence $\beta_{E_1} E_2 = E_2$; thus E_2 is E_1-compact.

As a particular case of the above theorem we obtain that $\beta_R X \subset_{\mathrm{ext}} \beta X$ for every completely regular space X and $\beta_N X \subset_{\mathrm{ext}} \beta_D X$ for every 0-dimensional space X.

To complete Theorem 4.17 we shall give an exact formula for $\beta_{E_1} X$ in terms of $\beta_{E_2} X$.

4.18. THEOREM. *Assume that* $\mathfrak{C}(E_1) = \mathfrak{C}(E_2)$ *and* $E_2 \in \mathfrak{K}(E_1)$. *Let* E_1^* *be an* E_2*-compact superspace of* E_1. *Then, for every* $X \in \mathfrak{C}(E_1)$,

---

(1) An element x of a partially ordered set P is said to be the *largest* element of P provided that $y \leqslant x$ for every $y \in P$. x is said to be a maximal element of P provided that there is no $y \in P$ with $x < y$. A largest element is a maximal one, but the converse is, in general, false.

$\beta_{E_1}X =_{\text{ext}} \{p \in \beta_{E_2}X \colon f(p) \in E_1$ *for every continuous function* f *with* $f\colon \beta_{E_2}X \to E_1^*$ *and* $f[X] \subset E_1\}$.

Proof. Denote the right-hand side of the above equality by εX. Let $\mathfrak{F}$ be the set of all continuous functions $f\colon \beta_{E_2}X \to E_1^*$ with $f[X] \subset E_1$. Clearly $\varepsilon X = \bigcap \{f^{-1}[E_1]\colon f \in \mathfrak{F}\}$. To complete the proof, it suffices to show that εX satisfies conditions (i) and (ii) of Theorem 4.14 (relative to E_1). By 4.9, $f^{-1}[E_1]$ is E_1-compact for every $f \in \mathfrak{F}$; consequently, by 4.8, εX is also E_1-compact. Let f_0 be an arbitrary continuous function with $f_0\colon X \to E_1$. By part (c) of Theorem 4.14 f_0 admits a continuous extension f with $\beta_{E_2}X \to E_1$. Clearly, $f[X] \subset E_1$, hence, by the very definition of εX, $f(p) \in E_1$ for every $p \in \varepsilon X$. Thus, $f^* = f \mid \varepsilon X$ is a continuous extension of f_0 with $f^*\colon \varepsilon X \to E_1$.

4.19. COROLLARY. *Under the assumption of Theorem 4.18, a space* $X \in \mathfrak{C}(E_1)$ *is* E_1-*compact if and only if for every* $p_0 \in \beta_{E_2}X \setminus X$ *there exists a continuous function* $f\colon \beta_{E_2}X \to E_1^*$ *such that* $f(p) \in E_1$ *for every* $p \in X$ *and* $f(p_0) \in E_1^* \setminus E_1$.

Proof. In view of Theorem 4.18, the condition of the above corollary is equivalent to the equality $\beta_{E_1}X = X$.

It is clear that $\mathfrak{K}(\mathcal{R}) = \mathfrak{K}((0, 1])$. Indeed, $(0, 1] \subset_{\text{cl}} \mathcal{R}$ and $\mathcal{R} \subset_{\text{cl}} (0, 1]^2$ ($\mathcal{R}$ is homeomorphic to the set $\{(x, y)\colon x, y \in (0, 1], x + y = 1\}$); thus $(0, 1] \in \mathfrak{K}(\mathcal{R})$ and $\mathcal{R} \in \mathfrak{K}((0, 1])$. Consequently, applying the above corollary with $E_1 = (0, 1]$, $E_2 = E_1^* = \mathcal{I}$, we obtain a known characterization of $\mathcal{R}$-compact spaces (see [18], p. 947, Proposition).

A completely regular space X *is* $\mathcal{R}$-*compact if and only if for every* $p_0 \in \beta X \setminus X$ *there exists a continuous function* $f\colon \beta X \to I$ *such that* $f(p) > 0$ *for every* $p \in X$ *and* $f(p_0) = 0$.

We shall now comment Herrlich's generalization of classes of complete regularity and classes of compactness [12]. Herrlich considers a class $\mathfrak{E}$ of topological spaces and he defines $X \in \mathfrak{C}(\mathfrak{E})$ ($X \in \mathfrak{K}(\mathfrak{E})$) iff X is homeomorphic to a subspace (closed subspace, respectively) of a product of spaces each of which is in $\mathfrak{E}$. Classes $\mathfrak{C}(\mathfrak{E})$ ($\mathfrak{K}(\mathfrak{E})$) will be called generalized classes of complete regularity (of compactness). Herrlich demonstrates that some of our considerations remain valid in this more general setting; this, in particular, concerns the E-transformation and the extension $\beta_E X$. Furthermore, Herrlich discusses these problems within the framework of the category theory.

The distinction between these two concepts can most conveniently be discussed within a system of set-theory which admits (proper) classes that are not sets. In what follows we shall adhere to this exact meaning of the terms "set" and "class". If a generalized class of complete regularity (of compactness) $\mathfrak{A}$ admits a set $\mathfrak{E}$ of spaces such that $\mathfrak{A} = \mathfrak{C}(\mathfrak{E})$ ($\mathfrak{A} = \mathfrak{K}(\mathfrak{E})$), then $\mathfrak{A}$ is a class of complete regularity (of compactness); in fact, it suffices to let E be the product of all spaces in $\mathfrak{E}$. Generalized classes of complete regularity and of compactness admit a simple characterization: every generalized class of complete regularity (of compactness) is closed under taking arbitarary products and arbitrary subspaces (arbitrary closed subspaces, respectively). The converse is also trivially true: for a class $\mathfrak{A}$ satisfying the above conditions we have $\mathfrak{A}=\mathfrak{C}(\mathfrak{A})$ ($\mathfrak{A}=\mathfrak{K}(\mathfrak{A})$, respectively). A natural problem is to find, for a given class $\mathfrak{A}$, a minimal class $\mathfrak{E}$ with $\mathfrak{A}=\mathfrak{C}(\mathfrak{E})$ ($\mathfrak{A}=\mathfrak{K}(\mathfrak{E})$). Such a minimal class $\mathfrak{E}$ can be characterized by the condition: $E\notin\mathfrak{C}(\mathfrak{E}\setminus\{E\})$ ($E\notin\mathfrak{K}(\mathfrak{E}\setminus\{E\})$) for every $E\in\mathfrak{E}$. The class $\mathfrak{A}$ of all T_1-spaces is trivially a generalized class of complete regularity (and $\mathfrak{A}$ is not a class of complete regularity); here we have $\mathfrak{A}=\mathfrak{C}(\mathfrak{E})$, where $\mathfrak{E}=\{\mathcal{L}_\mathfrak{m}\colon \mathfrak{m}$ is an arbitrary cardinal$\}$; this class $\mathfrak{E}$ is not minimal; in fact, no subclass $\mathfrak{E}_0$ of $\mathfrak{E}$ with $\mathfrak{C}(\mathfrak{E}_0)=\mathfrak{A}$ is minimal. I do not know if there is at all a minimal class $\mathfrak{E}'$ with $\mathfrak{C}(\mathfrak{E}')=$the class of all T_1-spaces.

We shall discuss another example (suggested by Blefko's result 4.13). Let $E\in\mathfrak{C}(\mathcal{D})$. It can easily be derived from Corollary 4.19 that if $S(\omega_\lambda)$, where ω_λ is a regular initial ordinal, is E-compact, then card $E\geqslant\aleph_\lambda$.(1) It follows that the generalized class of compactness $\mathfrak{K}(\mathfrak{E})$, where $\mathfrak{E}=\{S(\omega_\lambda)\colon \omega_\lambda$ is an initial ordinal$\}$, is not a class of compactness. However, it can easily be derived from Blefko's proof of 4.13 (see [6], Ch. 3) and from Theorem 2.1 that the class $\mathfrak{E}_0=\{S(\omega_\lambda)\colon \omega_\lambda$ is a regular initial ordinal$\}$ is a minimal class with $\mathfrak{K}(\mathfrak{E}_0)=\mathfrak{K}(\mathfrak{E})$ (this statement is stronger than 4.13). We thus see that Herrlich's approach leads us sometimes to a stronger formulation of some questions and results.

To conclude this chapter we shall state (without proofs) several results(2) which point out the difficulty involved in an attempt of a complete description of the totality of classes of compactness. Since $\mathfrak{K}(E)\subset\mathfrak{C}(E)$, it is natural to study classes of compactness contained in a given class of complete regularity. The smallest class of complete regularity (save for the trivial class consisting of the empty space and the one-point space) is the class of all 0-dimensional T_0-spaces; i.e., the class $\mathfrak{C}(\mathcal{D})$. The smallest class of compactness (save for the trivial class) contained in $\mathfrak{C}(\mathcal{D})$ is the class $\mathfrak{K}(\mathcal{D})$ of all compact 0-dimensional spaces. Another natural class is the class $\mathfrak{K}(\mathcal{N})$ of all $\mathcal{N}$-compact spaces (this class was first mentioned in [9]). Concerning this class the following can be proved:

4.20. *Let E be $\mathcal{N}$-compact. The following are equivalent:*

(a) $\mathfrak{K}(E)=\mathfrak{K}(\mathcal{N})$;

---

(1) It can easily be derived from 4.18 that *if $S(\omega_\lambda)$ is E-compact, then (a) there is a transfinite descending sequence*

$$F_0, F_1, \ldots, F_\xi, \ldots, \xi<\omega_\lambda,$$

of closed non-empty subsets of E with the total intersection empty.

(*a*) implies (in case ω_λ is regular) that (*b*) card $E\geqslant\aleph_\lambda$, (*c*) E does not have a base of cardinality $<\aleph_\lambda$, and (*d*) E is not $\aleph_\lambda$-compact.

(2) These results have been obtained in cooperation with Blefko and are stated in [6].

(b) $\mathcal{N} \subset_{\mathrm{cl}} E$;

(c) *E admits a continuous map onto* $\mathcal{N}$.

One easily derives from 4.20 that

4.21. *If E is* $\mathcal{N}$*-compact, but* $\mathcal{N}$ *is not E-compact* (*i.e., if E is more compact than* $\mathcal{N}$), *then E is* $\mathcal{D}$*-compact.*

4.21 asserts, in fact, that there is no class of compactness between $\mathfrak{K}(\mathcal{D})$ and $\mathfrak{K}(\mathcal{N})$; i.e., $\mathfrak{K}(\mathcal{N})$ is an immediate successor of $\mathfrak{K}(\mathcal{D})$ in the partially ordered (by set-theoretic inclusion) totality of 0-dimensional classes of compactness. Now, by Blefko's result, the classes $\mathfrak{K}(S(\omega_\lambda))$, where ω_λ is a regular initial ordinal, are mutually incomparable; each of these classes follows $\mathfrak{K}(\mathcal{D})$. Unfortunately, none of them, save for the class $\mathfrak{K}(S(\omega_0)) = \mathfrak{K}(\mathcal{N})$, is an immediate successor of $\mathfrak{K}(\mathcal{D})$. Thus, for instance, there are classes of compactness strictly between $\mathfrak{K}(\mathcal{D})$ and $\mathfrak{K}(S(\omega_1))$; however, not much is known about them. Here are some of the open questions. How many such classes are there? Are they (linearly) ordered (by $\subset$)? Does $\mathfrak{K}(\mathcal{D})$ have an immediate successor $\subset \mathfrak{K}(S(\omega_1))$? Does $\mathfrak{K}(S(\omega_1))$ have an immediate predecessor $\supset \mathfrak{K}(\mathcal{D})$?

V. Estimation of exponents

It is frequently useful to have estimations for $\exp_E X$ and $\operatorname{Exp}_E X$. Theorem 2.3 provides us with the following

5.1. *Let E be a* T_0*-space and let X be E-completely regular.* $\exp_E X \leqslant \mathfrak{m}$ *if and only if X has an E-separating class* $\mathfrak{F}$ *with* card $\mathfrak{F} \leqslant \mathfrak{m}$ ($\mathfrak{m}$ — *an infinite cardinal*).

5.2. *Let E be a Hausdorff space and let X be E-compact.* $\operatorname{Exp}_E X \leqslant \mathfrak{m}$ *if and only if X has a class* $\mathfrak{F}$ *such that* $\mathfrak{F}$ *is both E-separating and E-non-extendable and* card $\mathfrak{F} \leqslant \mathfrak{m}$.

Statement 5.1 enables us to relate $\exp_E X$ to the weight of X (weight X is the smallest infinite cardinal $\mathfrak{m}$ such that X has a base of cardinality $\leqslant \mathfrak{m}$).

5.3. *Let E be a* T_0*-space and let X be E-completely regular. We have*

(a) $\exp_E X \leqslant$ weight X;

(b) weight $X \leqslant \max \{\exp_E X,$ weight $E\}$.

Proof. (a) This proof duplicates the Tihonov procedure [23]. Let $\mathfrak{B}$ be a base in X with card $\mathfrak{B} \leqslant \mathfrak{m} =$ weight X. Let $\mathfrak{P}$ be the set of all pairs (U, V) such that $U, V \in \mathfrak{B}$ and there exists a finite n and a continuous function $g: X \to E^n$ such that

$$(1) \qquad \overline{g[U]} \cap \overline{g[X \setminus V]} = \varnothing$$

By the very definition, for every pair $(U, V) \in \mathfrak{P}$ there exists a finite set $\mathfrak{F}_{U,V}$ of continuous functions from X into E such that, if we let $\mathfrak{F}_{U,V} = \{f_1, f_2, ..., f_n\}$ and $g = \langle f_1, f_2, ..., f_n \rangle$, then g satisfies (1). It is easy to see that $\mathfrak{F}=\{\bigcup\{\mathfrak{F}_{U,V}: (U, V)\in\mathfrak{P}\}$ is an E-separating class for X. Clearly, card $\mathfrak{F}\leq\mathfrak{m}$; consequently, by 5.1, $\exp_E X\leq\mathfrak{m}$.

(b) is obvious.

Statement 5.2 is not so convenient. For one thing, it involves the estimation of both $\exp_E X$ and $\operatorname{Exp}_E X$. In many cases $\exp_E X$ is known and one would like to have a condition for $\operatorname{Exp}_E X$ which does not involve reconsideration of $\exp_E X$. On the other hand, the very magnitude of $\exp_E X$ and $\operatorname{Exp}_E X$ does not distinguish between the ordinary embedding and the closed one. For instance, we have $\exp_{\mathcal{N}}\mathcal{D}^{\aleph_0}=\operatorname{Exp}_{\mathcal{N}}\mathcal{D}^{\aleph_0}=\exp_{\mathcal{N}}Q=\operatorname{Exp}_{\mathcal{N}}Q=\aleph_0$ (in other words, $\exp_{\mathcal{N}}$ and $\operatorname{Exp}_{\mathcal{N}}$ do not distinguish between the Cantor set $\mathcal{D}^{\aleph_0}$ and the space of irrationals Q), but every embedding of $\mathcal{D}^{\aleph_0}$ into $\mathcal{N}^{\aleph_0}$ is a closed one; while there are embeddings of Q into $\mathcal{N}^{\aleph_0}$ which are very far from being closed. (1) However, Theorem 2.1 provides us with a concept which seems to satisfy the above requirements.

5.4 *Definition.* The *E-defect of X* (in symbols: $\operatorname{def}_E X$) is the smallest (finite or infinite) cardinal $\mathfrak{p}$ such that X has an E-non-extendable class $\mathfrak{F}$ with card $\mathfrak{F}=\mathfrak{p}$.

Thus, $\operatorname{def}_E X$ is defined only for an E-compact X. However, if X is not E-compact, then we will write $\operatorname{def}_E X=\infty$.

Clearly, we have

5.5 $\operatorname{Exp}_E X=\exp_E X+\operatorname{def}_E X$ *for every E-compact space X.*

We shall now give a complete product theoretic characterization of $\operatorname{def}_E X$.

5.6 *Definition.* A space E is said to be *admissible* provided that there exists a compact space E^* with $\mathfrak{C}(E^*)=\mathfrak{C}(E)$.

Non-admissible spaces exist. DeGroot [10] has proved the existence of a subspace E of the Euclidean plane such that E contains more than one point and every continuous function from E into E is either constant or it is the identity. It follows from the proof in [10] that E is dense in the plane (in fact, E intersects every Cantor set in the plane); consequently, E is not compact. Now it is easy to see that E is not admissible; in fact, the assumption that $\mathfrak{C}(E)=\mathfrak{C}(E^*)$ for some compact space E^* implies that, for some cardinal $\mathfrak{m}$, we have $E'\subset C\subset E^{\mathfrak{m}}$, where E' is homeomorphic to E and C is a compact subset of $E^{\mathfrak{m}}$. For at least one projection π_ξ of E, $\pi_\xi|E'$ is not constant; on the other hand, $\pi_\xi[C]$ is a proper subset of E (E is not compact). This yields a non-constant continuous map of E whose range is a proper subset of E.

5.7. THEOREM. *Let X be an E-compact space. The conditions below are related as follow:* (a) *implies* (b) *and* (c) *implies* (a). *If E is admissible, then* (b) *implies* (c), *hence, in this case,* (a), (b), *and* (c) *are equivalent.*

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(1) Recall that Q is homeomorphic to $\mathcal{N}^{\aleph_0}$.

(a) $\operatorname{def}_E X \leqslant \mathfrak{p}$;

(b) *for every homeomorphism* h *of* X *into* $E^{\mathfrak{m}}$ (*where* $\mathfrak{m}$ *is an arbitrary cardinal*) *there exists a homeomorphism* h' *of* X *into* $E^{\mathfrak{m}} \times E^{\mathfrak{p}}$ *such that* $h'[X]$ *is closed in* $E^{\mathfrak{m}} \times E^{\mathfrak{p}}$ *and* $h = \pi_1 \circ h'$, *where* π_1 *is the projection of* $E^{\mathfrak{m}} \times E^{\mathfrak{p}}$ *into* $E^{\mathfrak{m}}$;

(c) $X \subset_{\mathrm{cl}} C \times E^{\mathfrak{p}}$, *where* C *is a compact space with* $\mathfrak{C}(C) = \mathfrak{C}(E)$.

Remark 1. Condition (b) can be expressed in a form more resembling a classical theorem of Kuratowski ([13], p. 151, Théorème):

(b′) *for every embedding* X' *of* X *into* $E^{\mathfrak{m}}$ *there exists a continuous function* $f\colon X' \to E^{\mathfrak{p}}$ *such that the graph of* f *is closed in* $E^{\mathfrak{m}} \times E^{\mathfrak{p}}$.

These conditions can be expressed intuitively as follows: *every embedding of* X *into* $E^{\mathfrak{m}}$ *can be modified to a closed embedding by adding at most* $\mathfrak{p}$ *axes.*

The proof of the above theorem will be based on the following

5.8. Lemma. *Let* X *be a closed subspace of* $C \times E$, *where* C *is compact, and let* $\varepsilon X = X \cup \{p_0\}$ *be a one-point extension of* X ($p_0 \notin X$). *There exists a one-point extension* $\varepsilon_1 X = X \cup \{q_0\}$ *of* X ($q_0 \notin X$) *such that*

$$\varepsilon X \leqslant_{\mathrm{ext}} \varepsilon_2 X$$

and the function $g(p) = \pi(p)$ *for every* $p \in X$, *where* π *is the projection of* $C \times E$ *onto* C, *can be extended to a continuous function* $g^*\colon \varepsilon X \to C$.

Proof. Let $\mathfrak{G}$ be the class of all open subsets G of X such that $G \cup \{p_0\}$ is a neighborhood of p_0 in εX. There is a point $c_0 \in C$ such that $c_0 \in \overline{\pi[G]}$ for every $G \in \mathfrak{G}$. Add a new point q_0 to X taking as neighborhoods of q_0 in $\varepsilon_1 X = X \cup \{q_0\}$ all sets of the form $(\pi^{-1}[U] \cap G) \cup \{q_0\}$, where $G \in \mathfrak{G}$ and U is an arbitrary neighborhood of c_0 in C. $\varepsilon_1 X$ is the required extension. In fact, if $G \cup \{p_0\}$ is a neighborhood of p_0 in εX ($G \subset X$), then $G \cup \{q_0\}$ is a neighborhood of q_0 in $\varepsilon_1 X$. Thus $\varepsilon X \leqslant_{\mathrm{ext}} \varepsilon_1 X$. Furthermore, g can be continuously extended over $\varepsilon_1 X$ by setting $g^*(q_0) = c_0$.

Proof of Theorem 5.7. a) implies b). Let h be a homeomorphism of X into $E^{\mathfrak{m}} = \times\{E_\xi\colon \xi \in \Xi\}$. Let us set $f_\xi = \pi_\xi \circ h$ for every $\xi \in \Xi$ and $\mathfrak{F} = \{f_\xi\colon \xi \in \Xi\}$. Let $\mathfrak{F}_1 = \{f_\xi\colon \xi \in \Xi_1\}$, where $\operatorname{card} \Xi_1 \leqslant \mathfrak{p}$ and $\Xi \cap \Xi_1 = \emptyset$, be an E-non-extendable class for X. Suffices to define h' as the parametric map corresponding to the class $\mathfrak{F}_2 = \{f_\xi\colon \xi \in \Xi \cup \Xi_1\}$ and apply Theorem 2.1.

c) implies a). We shall start with the case $\mathfrak{p} = 1$. Assume that X is a closed subspace of $C \times E$; define $f(x, y) = y$ for every $(x, y) \in X$. We claim that $\{f\}$ is an E-non-extendable class for X. Indeed, let εX be a proper extension of X; we can assume that εX is a one-point extension. Suppose that f admits a continuous extension $f^*\colon \varepsilon X \to E$. Take the extension $\varepsilon_1 X$

described in Lemma 5.8. Since $\varepsilon X \leqslant_{\text{ext}} \varepsilon_1 X$, f admits a continuous extension $f_1^*: \varepsilon_1 X \to E$. On the other hand, (by Lemma 5.8) there is a continuous function $g^*: \varepsilon_1 X \to C$ such that $g^*(x, y) = x$ for every $(x, y) \in X$. Define $f(p) = (g^*(p), f_1^*(p))$ for every $p \in X$. h is a continuous map of $\varepsilon_1 X$ into $C \times E$; furthermore, $h(p) = p$ for every $p \in X$. Since X is closed in $C \times E$, the last condition implies $h: \varepsilon_1 X \to X$. This, however, contradicts the fact that $\varepsilon_1 X$ is a proper extension of X (see footnote on p. 166).

The general case can be reduced to the above as follows: from the above we obtain an existence of an $E^{\mathfrak{p}}$-non-extendable function f (strictly speaking, an existence of a one-element $E^{\mathfrak{p}}$-non-extendable class $\{f\}$), $f: X \to E^{\mathfrak{p}}$. Suffices to set $E^{\mathfrak{p}} = \times\{E_\xi: \xi \in \Xi\}$, $\operatorname{card} \Xi = \mathfrak{p}$, $E_\xi = E$ for every $\xi \in \Xi$, and $\mathfrak{F} = \{\pi_\xi \circ f: \xi \in \Xi\}$. $\mathfrak{F}$ is an E-non-extendable class for X.

b) implies c), provided that E is admissible. Let E_1 be a compact space with $\mathfrak{C}(E_1) = \mathfrak{C}(E)$. Let $X \subset_{\text{top}} E^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$. Since $\mathfrak{C}(E_1) = \mathfrak{C}(E)$, we have $E \subset_{\text{top}} E_1^{\mathfrak{m}_1} \subset_{\text{top}} E^{\mathfrak{m}_2}$ for some cardinals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Let C be a homeomorphic image of $E_1^{\mathfrak{m}_1}$ in $E^{\mathfrak{m}_2}$. Then $X \subset_{\text{top}} C$. Let X' be a homeomorphic image of X in C. We have $X' \subset E^{\mathfrak{m}_2}$; consequently, by b), there exists a closed subspace X'' of $E^{\mathfrak{m}_2} \times E^{\mathfrak{p}}$, such that X'' is homeomorphic to X and $\pi_1[X''] = X'$, where π_1 is the projection of $E^{\mathfrak{m}_2} \times E^{\mathfrak{p}}$ onto $E^{\mathfrak{m}_2}$. The equality $\pi_1[X''] = X'$ implies that $X'' \subset C \times E^{\mathfrak{p}}$. Thus $X \subset_{\text{cl}} C \times E^{\mathfrak{p}}$.

We shall conclude this section with a few simple remarks concerning $\mathcal{N}$- and $\mathcal{D}$-defects.

5.9. *If X is $\mathcal{N}$-completely regular (completely regular), then the following conditions are equivalent:*

(a) $\operatorname{def}_n X \leqslant 1$ $\quad (\operatorname{def}_R X \leqslant 1)$;

(b) $\operatorname{def}_n X < \aleph_0$ $\quad (\operatorname{def}_R X < \aleph_0)$;

(c) *X is locally compact and Lindelöf.*

Proof. We shall give the proof for $\mathcal{R}$-defect. Obviously, (a) implies (b). Assume (b). Since $\mathcal{R}$ is admissible; we infer from Theorem 5.7 that $X \subset_{\text{cl}} C \times \mathcal{R}^n$; where C is compact and n is finite; consequently, X is locally compact and Lindelöf. Assume (c). The point ∞ in the one-point compactification $\iota X = X \cup \{\infty\}$ of X satisfies the first axiom of countability; consequently, there is a continuous function $f: \iota X \to [0, 1]$ with $f(p) > 0$ for $p \in X$ and $f(\infty) = 0$. It is clear that the class $\{g\}$, where $g(p) = 1/f(p)$ for $p \in X$ is an $\mathcal{R}$-non-extendable function for X. (In the proof for $\mathcal{N}$-defect one has, using 0-dimensionality of ιX, to modify f so that its values on X are of the form $1/n$; see [9], proof of Lemma 2).

Clearly, $\operatorname{def}_n Q \leqslant \aleph_0$, where Q is the space of irrationals (in fact, $Q =_{\text{top}} \mathcal{N}^{\aleph_0}$); and, by Theorem 5.9, we infer that $\operatorname{def}_n Q = \aleph_0$. On the other hand, $\aleph_0 < \operatorname{def}_n \mathcal{D} \leqslant 2^{\aleph_0}$; where $\mathcal{D}$ is the space of rationals (in fact, $\mathrm{def}_{\mathcal{N}}\mathcal{D}\leq\aleph_0$ would imply that $\mathcal{D}\subset_{\mathrm{cl}}\mathcal{N}^{\aleph_0}$, but $\mathcal{D}$ does not admit a complete metric); I do not know if one can prove, without the continuum hypothesis, that $\mathrm{def}_{\mathcal{N}}\mathcal{D}=2^{\aleph_0}$.

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