FURTHER RESULTS ON THE CRITICAL GALTON-WATSON PROCESS WITH IMMIGRATION

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1. Introduction

Consider a Galton-Watson process in which each individual reproduces independently of all others and has probability a_j $(j = 0, 1, \cdots)$ of giving rise to j progeny in the following generation, and in which there is an independent immigration component where b_j $(j = 0, 1, \cdots)$ is the probability that j individuals enter the population at each generation. Defining X_n $(n = 0, 1, \cdots)$ to be the population size at the n-th generation, it is known that $\{X_n\}$ defines a Markov chain on the non-negative integers.

When $|x| < 1$, let $A(x) = \sum_{j=0}^{\infty} a_j x^j$, $B(x) = \sum_{j=0}^{\infty} b_j x^j$ and $P_i^{(n)}(x) = \sum_{j=0}^{\infty} p_{ij}^{(n)} x^j$ where $\{p_{ij}^{(n)}\}$ $(i, j, n = 0, 1, \cdots)$ are the n-step transition probabilities of the Markov chain $\{X_n\}$. We shall assume that $0 < a_0, b_0 < 1$. Denote the means of the offspring and immigration distributions by $\alpha = A'(1-)$ and $\beta = B'(1-)$ respectively. We always assume that $\beta < \infty$ and, unless otherwise stated, $\alpha = 1$. In this case the variance of the offspring distribution is given by $2\gamma = A''(1-)$ and we assume that $0 < \gamma < \infty$. Finally, let $\sigma = \beta/\gamma$.

Pakes [6] has shown that if $\sum_{j=1}^{\infty} a_j j^2 \log j$, $B''(1-) < \infty$ then $n^{\sigma} p_{00}^{(n)} \to \mu_0$, $(n \to \infty)$ where $0 < \mu_0 < \infty$. For the case where $\{X_n\}$ is irreducible and aperiodic, this result shows it to be null-recurrent when $\sigma \leqq 1$ and transient otherwise. In section 2 we shall show that $n^{\sigma} P_i^{(n)}(x)$ converges to a function $U(x)$ which is regular in the open unit disc and which generates the invariant measure, $\{\mu_j\}$, of $\{X_n\}$. Seneta [9] has demonstrated the existence and uniqueness (up to a constant multiple) of an invariant measure under very weak hypotheses. A discussion of the asymptotic behaviour of $\{p_{ij}^{(n)}\}$ and $\{\mu_j\}$ is given in section 2. Some results on the asymptotic behaviour of the Green's function $G_{ij} = \sum_{n=0}^{\infty} p_{ij}^{(n)}$, which exists under the conditions of theorem 1 below and if $\sigma > 1$, are given in section 3.

It was shown in Pakes [6] that X_n/n converges weakly to a gamma distributed random variable. A related problem for $Y_n = \sum_{m=0}^{n} X_m$, the total number of individuals which have existed in the population up to the n-th generation, is considered in section 4. More specifically, we show that Y_n/n^2 converges weakly to a random variable Y, where $E(e^{-\theta Y}) = [\operatorname{sech}((\gamma\theta)^{\frac{1}{2}})]^{\sigma}$. Finally, we briefly examine the rate of convergence of generating functions of the total population in a non-super-critical Galton-Watson process without immigration.

2. The asymptotic form of the transition probabilities

In the proof of theorem 1, we shall need the following result.

LEMMA 1. *Let* $K(x) = \sum_{j=0}^{\infty} k_j x^j$ *be a probability generating function. If* $0 < d_n < 1$ *and* $1-d_n \sim a/n$, $0 < a < \infty$, *then*

$$\sum_{n=1}^{\infty} (1-K(d_n))/n < \infty \text{ iff} \sum_{j=1}^{\infty} k_j \log j < \infty.$$

PROOF. Since there exists finite positive constants c_1 and c_2 such that $c_1/n < 1-d_n < c_2/n$ if $n > N$, it suffices to show that

$$S = \sum_{n=N}^{\infty} (1-K(1-c/n))/n < \infty \text{ iff} \sum_{j=1}^{\infty} k_j \log j < \infty,$$

where c is a finite positive constant and N is so large that $c/(N-1) < 1$. Fubini's theorem yields

$$S = \sum_{j=0}^{\infty} k_j \sum_{n=N}^{\infty} (1-(1-c/n)^j)/n.$$

Let $S_j = \sum_{n=N}^{\infty} (1-(1-c/n)^j)/n$. For fixed j the terms of this series are monotone decreasing and so

$$0 \leqq S_j - I_j \leqq [1-(1-c/(N-1))^j]/(N-1) \leqq 1.$$

where

$$I_j = \int_N^{\infty} [1-(1-c/x)\,]/x\, dx = \int_{1-c/N}^{1} (1-y^j)/(1-y)\, dy$$
$$= \log j + L_j - M_j,$$

and where $L_j = \sum_{k=1}^{j} 1/k - \log j$ satisfies $0 < L_j < 1$ and

$$0 \leqq M_j = \sum_{k=1}^{j} (1-c/N)^k/k < (N/c)-1.$$

The lemma now follows on observing that

$$0 \leqq \sum_{j=0}^{\infty} k_j I_j \leqq S \leqq \sum_{j=0}^{\infty} k_j I_j + 1.$$

From the definition of the Markov chain $\{X_n\}$ it is easily seen that

$$P_i^{(n)}(x) = (A_n(x))^i \prod_{m=0}^{n-1} B(A_m(x)) \tag{1}$$

where $A_0(x) = x$ and $A_{n+1}(x) = A(A_n(x))$.

THEOREM 1. *If, in addition to the conditions of section* 1, $\sum_{j=1}^{\infty} a_j j^2 \log j$, $\sum_{j=1}^{\infty} b_j j \log j < \infty$, *then the sequence of functions* $\{n^\sigma P_i^{(n)}(x)\}$ *converges to* $U(x)$ $(|x| < 1)$ *where* $U(x)$ *satisfies the functional equation*

$$B(x)U(A(x)) = U(x). \tag{2}$$

The convergence is uniform over compact subsets of the open unit disc. Denoting the power series representation of $U(x)$ *by* $\sum_{j=0}^{\infty} \mu_j x^j$, *the n-step transition probabilities are given by*

$$p_{ij}^{(n)} = n^{-\sigma}(\mu_j + r_{ij}(n)) \qquad (i, j = 0, 1, \cdots) \tag{3}$$

where $r_{ij}(n) = o(1)$ $(n \to \infty)$.

REMARK. Since completing this work, the author has found that Karlin and McGregor [4] have obtained the first part of this theorem but on assuming that $B(x)$ is regular at $x = 1$ and $A'''(1-) < \infty$. When $x = 0$, the above theorem slightly strengthens theorem 1 of Pakes [6].

PROOF. It is well known that under our hypotheses $A_n(x) \uparrow 1 (n \to \infty)$; see Harris [3]. Thus we need only consider

$$D_n(x) \equiv n^\sigma P_0^{(n)}(x) = B(x) \prod_{m=1}^{n-1} (1+1/m)^\sigma B(A_m(x)). \tag{4}$$

For the present consider a fixed $x \in [0, 1]$. The existence of a finite positive limit of this sequence of functions is equivalent to the convergence of the series $\sum_{m=1}^{\infty} (d_m(x) - 1)$, where

$$d_m(x) = (1+1/m)^\sigma B(A_m(x)).$$

Using $(1+1/m)^\sigma = 1 + \sigma/m + r_m$ where $r_m = 0(1/m^2)$ we have

$$\begin{aligned} d_m(x) - 1 &= \sigma/m - (1 - B(A_m(x)) - \sigma(1 - B(A_m(x)))/m + r_m B(A_m(x)) \\ &= \sigma/m - \beta(1 - A_m(x)) + (1 - A_m(x))(\beta - B'(\eta_m)) - \sigma(1 - B(A_m(x)/m \\ &\quad + r_m B(A_m(x)) \end{aligned} \tag{5}$$

where $A_m(x) < \eta_m < 1$, and we have used the mean value theorem to obtain the second equality. Theorem 1 of Kesten *et al.* [5] shows that $1 - A_n(x) \sim n/\gamma$ and so the third and fourth terms of equation (5) are $0(1/m^2)$. Clearly, the second term of (5) is non-negative and is dominated by $(1 - A_m(x))(\beta - B'(A_m(x))$. Application of lemma 1 with $K(x) = B'(x)/\beta$ shows that

$$\sum_{m=1}^{\infty} (1 - A_m(x))(\beta - B'(\eta_m)) < \infty \text{ if} \sum_{j=1}^{\infty} b_j \, j \log j < \infty.$$

Writing

$$1 - A_m(x) = [m\gamma - h_m(x) + 1/(1-x)]^{-1}$$

we obtain

$$\sigma/m-\beta(1-A_m(x)) = \sigma\cdot\frac{1/(1-x)-h_m(x)}{m(m-h_m(x)+1/(1-x))} = -\sigma h_m(x)/\gamma m^2+0(1/m^2)$$

where we have used $h_m(x) = o(m)$; see Kesten et al. [5]. Indeed it is shown in this reference that $h_m(x) = \sum_{k=0}^{m-1}\delta(A_k(x))$ where $\delta(x)$ satisfies the (corrected) inequality

$$-\gamma^2(1-x)/(1-A(0)) \leqq \delta(x) \leqq \varepsilon(x) \tag{6}$$

where $0 \leqq \varepsilon(x) = \gamma-(A(x)-x)/(1-x)^2 \leqq \gamma$, and $\varepsilon(x)$ is monotone non-increasing on $0 \leqq x < 1$ and $\varepsilon(x)\downarrow 0(x\uparrow 1)$. Thus we have

$$-\frac{\gamma^2}{1-A(0)}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{k=0}^{m-1}(1-A_k(0)) \leqq \sum_{m=1}^{\infty}\frac{h_m(x)}{m^2} \leqq \sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{t=0}^{m-1}\varepsilon(A_k(0)). \tag{7}$$

For sufficiently large M there exist positive constants a, b such that $a/m < 1-A_m(0) < b/m$ $(m \geqq M)$ and so we see that the terms of the series on the left of equation (7) are $0[(\log m)/m^2]$ for large m.

Observing that $\varepsilon(x) \leqq \gamma-A''(x)$, use of lemma 1 with $K(x) = A''(x)/2\gamma$ shows that the series on the right of (7) will converge if $\sum_{j=0}^{\infty}a_j j^2\log j < \infty$. Thus $D_n(x) \to U(x)$, say, $(n\to\infty)$ and $0 < U(x) < \infty$ when $0 \leqq x < 1$.

If $0 < R < 1$, equation (4) and the result of the last paragraph implies $|n^\sigma P_i^{(n)}(x)| \leqq D_n(R) < M(R) < \infty$ $(|x| \leqq R)$ where $M(R)$ is a constant depending on R. Vitali's theorem (Titchmarsh [10]) shows that $n^\sigma P_i^{(n)}(x)$ converges uniformly over compact subsets of the disc $|x| < R$, and thus uniformly over compact subsets of the open unit disc, to a function $\sum_{j=0}^{\infty}\mu_j x^j$ which coincides with $U(x)$ for $0 \leqq x < 1$ and thus defines $U(x)$ for all $|x| < 1$. It is clear that $\mu_j \geqq 0$. Equation (3) follows from the uniform convergence.

From equation (4) we have

$$(1+1/n)^\sigma B(x)D_n(A(x)) = D_{n+1}(x)$$

and this implies equation (2), thus completing the proof.

Equation (3) clearly demonstrates the absence of the geometric ergodicity property; compare with the situation when $\alpha \neq 1$ in Pakes [7].

Theorem 1 allows us to write

$$P_i^{(n)}(x) = n^{-\sigma}[U(x)+r_i^{(n)}(x)] \quad (0 \leqq x < 1;\quad n = 1, 2, \cdots) \tag{8}$$

where $r_i^{(n)}(x) = o(1)$ $(n\to\infty)$. We can obtain some information on the asymptotic form of $r_i^{(n)}(x)$. Equation 1, theorem 1 and the fact that $1-A_n(x) \sim n/\gamma$ (Kesten et al. [5]) shows that $n^{\sigma+1}(P_0^{(n)}(x)-P_i^{(n)}(x)) = iU(x)(1+\zeta_i^{(n)}(x))/\gamma$ where $\zeta_i^{(n)}(x) = o(1)$ $(n\to\infty)$. This yields

$$P_i^{(n)}(x) = n^{-\sigma}(U(x)+r_0^{(n)}(x))-iU(x)n^{-\sigma-1}(1+\zeta_i^{(n)}(x))/\gamma.$$

We shall now show that, in general, $n^{-\sigma}r_0^{(n)}(x)$ tends to zero much less rapidly than does $n^{-\sigma-1}$. This is evident from the following:

COROLLARY. *If we assume that* $B''(1-)$, $A^{iv}(1-) < \infty$ *then*

$$r_0^{(n)}(x) = -(\Omega U(x) \log n)\, n + 0(1/n)$$

where

$$\Omega = \beta(\gamma^2 - A'''(1-)/6)\gamma^3.$$

PROOF. From theorem 1 we have

$$r_0^{(n)}(x) = D_n(x)(1 - \prod_{m=n}^{\infty}(1+1/m)^{\sigma} B(A_m(x))).$$

Now

$$V_n(x) = 1 - \prod_{m=n}^{\infty}(1+1\ m)^{\sigma} B(A_m(x) = \sum_{m=n}^{\infty}(1-d_m(x)) + W_n(x) \tag{9}$$

where $W_n(x)$ is an error term which will be examined subsequently. The proof of theorem 1 and the finiteness of $B''(1-)$ show that

$$1 - d_m(x) = \sigma h_m(x)/\gamma m^2 + 0(1/m^2).$$

Since $A^{iv}(1-) < \infty$ it follows from lemma 10.1 and case c of the proof of theorem 11.1 in Harris [3] that

$$1 - d_m(x) = -\Omega[\log(1+m\gamma(1-x))]/m^2 + 0(1/m^2) \qquad (0 \leqq x < 1). \tag{10}$$

If m is sufficiently large, then $[\log(1+am)]/m^2$ $(0 < a < \infty)$ becomes monotone non-increasing and so

$$\int_n^{\infty}[\log(1+ay)]/y^2\, dy \leqq \sum_{m=n}^{\infty}[\log(1+am)]/m^2 \leqq \int_{n-1}^{\infty}[\log(1+ay)]/y^2\, dy$$

for n sufficiently large. This shows, after some manipulation, that

$$\sum_{m=n}^{\infty}(1-dm(x)) = -(\log n)/n + 0(1/n).$$

Equation (10) shows that for $\Omega \neq 0$ and m sufficiently large, $1-d_m(x)$ is either positive or negative, the sign being determined by that of Ω. Use of the appropriate one of the inequalities

$$\sum_{m=n}^{\infty} x_m - (\sum_{m=n}^{\infty} x_m)^2/2 \leqq 1 - \prod_{m=n}^{\infty}(1-x_m) \leqq \sum_{m=n}^{\infty} x_m + \sum_{m=n}^{\infty}[x_m^2/(1-x_m)] \quad (0 \leqq x_m < 1)$$

$$\sum_{m=n}^{\infty} x_m - (\sum_{m=n}^{\infty} x_m)^2/2(1+\sum_{m=n}^{\infty} x_m) \leqq 1 - \prod_{m=n}^{\infty}(1-x_m) \leqq \sum_{m=n}^{\infty} x_m + (\sum_{m=n}^{\infty} x_m^2)/2 \quad (-1 < x_m \leqq 0)$$

where $0 < \prod_{m=n}^{\infty}(1-x_m) < \infty$, shows that $W_n(x) = 0(1/n)$.

If $\Omega = 0$, then applying the last two inequalities to the bounds of $1-d_m(x)$ implied by (10) proves the corollary directly. This completes the proof.

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Observe that this corollary gives the rate at which $n^\sigma P_i^{(n)}(x)$ approaches $U(x)$. We now briefly examine the asymptotic behaviour of the invariant measure.

THEOREM 2. *If the conditions of theorem* 1 *are satisfied, then*

$$\sum_{i=0}^{j} \mu_j \sim j^\sigma/\Gamma(\sigma+1) \qquad (j \to \infty)$$

where $\Gamma(\cdot)$ *is the Gamma function.*

PROOF. From theorem 1 we have

$$U(x) = B(x)\prod_{m=1}^{\infty}(1+1/m)^\sigma B(A_m(x)) \quad (0 \leqq x < 1). \tag{11}$$

Following Kesten et al. [5] we consider

$$U(A_n(0)) = B(A_n(0))\prod_{m=1}^{\infty}(1+1/m)^\sigma B(A_{m+n}(0))$$
$$= B(A_n(0))\prod_{m=1}^{n}(1+1/m)^\sigma \prod_{m=n+1}^{\infty}(1+1/m)^\sigma B(A_m(0)).$$

The convergence of the infinite product (11) implies that the second product in the last expression $\to 1$ $(n \to \infty)$ thus giving

$$U(A_n(0) \sim (n+1)^\sigma \qquad (n \to \infty).$$

Since $U(x)$ is monotone on $0 \leqq x \leqq 1$, we have

$$U(x) \sim (1-x)^{-\sigma} \qquad (x \uparrow 1)$$

and the theorem now follows from Karamata's theorem (Feller [2]).

3. Asymptotic properties of the Green's functions

We define the Green's functions by

$$G_{ij} = \sum_{n=0}^{\infty} p_{ij}^{(n)} \qquad (i, j = 0.\ 1, \cdots) \tag{12}$$

where, by equation (3), the series converges if $\sigma > 1$. In this section we shall obtain some information on the behaviour of G_{ij} when i is fixed and $j \to \infty$ and when j is fixed and $i \to \infty$. We turn now to the first case, the approach being similar to that used in the proof of theorem 2.

THEOREM 3. *If* $\sigma > 1$ *and the conditions of theorem* 1 *are fulfilled then* $\sum_{k=0}^{j} G_{ik} \sim j/(\sigma-1)$ $(j \to \infty)$.

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PROOF. Equation (12) yields

$$G_i(x) = \sum_{j=0}^{\infty} G_{ij} x^j = \sum_{n=0}^{\infty} P_i^{(n)}(x) \quad (i = 0, 1, \cdots; \; 0 \leqq x < 1). \tag{13}$$

which implies, on using (1),

$$G_i(A_m(0)) = \sum_{n=0}^{\infty} p_{i0}^{(n+m)}/p_{00}^{(m)}.$$

Since $n^\sigma p_{i0}^{(n)} \to \mu_0 > 0$ $(n \to \infty; i = 0, 1, \cdots)$, then for each $\varepsilon > 0$ there exists $M(\varepsilon)$ such that

$$(1-\varepsilon)(m/(m+n))^\sigma < p_{i0}^{(n+m)}/p_{00}^{(m)} < (1+\varepsilon)(m/(m+n))^\sigma \tag{14}$$
$$(m > M(\varepsilon); \; n = 0, 1, \cdots).$$

Observing that the terms of the series $\sum_{n=0}^{\infty}(m+n)^{-\sigma}$ are monotone decreasing, comparison with $\int_0^\infty (x+m)^{-\sigma} dx$ eventually yields

$$1-\varepsilon \leqq (\sigma-1)G_i(A_m(0))/m \leqq (1+\varepsilon)(m/(m-1))^\sigma < (1+\varepsilon)^\sigma$$

when $m > \max(M(\varepsilon), 1+1/\varepsilon)$. Thus $G_i(A_m(0)) \sim m/(\sigma-1)$ and since $G_i(x)$ is monotone on $0 \leqq x < 1$,

$$G_i(x) \sim 1/(\sigma-1)(1-x) \qquad (x \uparrow 1)$$

and the theorem now follows on applying Karamata's theorem.

In considering the behaviour of G_{ij} for large i, we need the following result generalising lemma 1 of [5] (which was stated without proof).

LEMMA 2. *If* $0 < \alpha_n < 1$, $\alpha_n \sim c/n$ *and* $\beta_n \sim bn^{-\sigma}$ *where* $0 < b, c < \infty$ *and* $\sigma > 1$, *then*

$$\lim_{i\to\infty} \sum_{n=1}^{\infty} (1-\alpha_n)^i i^{\sigma-1} \beta_n = bc^{1-\sigma}\Gamma(\sigma-1).$$

PROOF. For each $\varepsilon > 0$ we have

$$(b-\varepsilon)n^{-\sigma} \leqq \beta_n \leqq (b+\varepsilon)n^{-\sigma} \qquad (n \geqq N = N(\varepsilon)) \tag{15}$$

so that

$$(b-\varepsilon)\{(1/i)\sum_{n=N}^{\infty}(i/n)^\sigma e^{-ic/n}\} \leqq \sum_{n=N}^{\infty} i^{\sigma-1}\beta_n e^{-ic/n}$$
$$\leqq (b+\varepsilon)\{(1/i)\sum_{n=N}^{\infty}(i/n)^\sigma e^{-ic/n}\}.$$

The expression in curly brackets on the left and right hand side of this inequality tends to $\int_0^\infty x^{-\sigma}e^{-c/x}dx = c^{1-\sigma}\Gamma(\sigma-1)$ and since ε is arbitrary we have

$$\lim_{i\to\infty} \sum_{n=1}^{\infty} i^{\sigma-1}\beta_n e^{-ic/n} = bc^{1-\sigma}\Gamma(\sigma-1) \tag{16}$$

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Thus it suffices to show that

$$\lim_{i\to\infty}\sum_{n=1}^{\infty}(e^{-ic/n}-(1-\alpha_n)^i)i^{\sigma-1}\beta_n = 0.$$

For each $\varepsilon > 0$ there exists $M = M(\varepsilon)$ such that both (15) and

(17) $0 < 1-(1+\varepsilon)c/n \leqq 1-\alpha_n \leqq 1-(1-\varepsilon)c/n$

hold for $n \geqq M$. The last inequality gives

$$\begin{aligned} e^{-ic/n}-(1-\alpha_n)^i &\leqq [e^{-c/n}-(1-(1+\varepsilon)c/n)]\sum_{k=0}^{i-1}e^{-kc/n}(1-\alpha_n)^{i-1-k} \\ &\leqq (\varepsilon c\ n+c^2/2n^2)\sum_{k=0}^{i-1}e^{-kc/n}(1-(1-\varepsilon)c/n)^{i-1-k} \\ &\leqq Ji(\varepsilon c/n+c^2/2n^2)e^{-ic/n} \end{aligned}$$

where J is a positive constant. Thus we obtain

$$\begin{aligned} &\lim_{i\to\infty}\sup\sum_{n=1}^{\infty}(e^{-ic/n}-(1-\alpha_n)^i)i^{\sigma-1}\beta_n \\ &\qquad\leqq J(b+\varepsilon)\lim_{i\to\infty}\sum_{n=M}^{\infty}(\varepsilon c/n-c^2/2n^2)e^{-ic/n}(i/n)^{\sigma} \\ &\qquad= \varepsilon c^{1-\sigma}\Gamma(\sigma)J(b+\varepsilon). \end{aligned}$$

But since ε is arbitrary the limit superior is non-positive. Similarly, use of the left hand sides of (15) and (17) shows that the limit inferior is non-negative, thus completing the proof.

THEOREM 4. *Let the conditions of theorem* 1 *be satisfied and let* $\sigma > 1$. *Then for each* $0 \leqq x < 1$,

(18) $$G_i(x) \sim (\gamma/i)^{\sigma-1}\Gamma(\sigma-1)U(x) \qquad (i\to\infty),$$

and for each $j = 0, 1, \cdots$,

(19) $$G_{ij} \sim (\gamma/i)^{\sigma-1}\Gamma(\sigma-1)\mu_j \qquad (i\to\infty).$$

PROOF. Recalling that $P_0^{(n)}(x) \sim n^{-\sigma}U(x)$ and that $A_n(x) = 1-\alpha_n$ where $\alpha_n \sim 1/(n\gamma)$ we see, using equation (1), that lemma 2 can be applied to equation (13) thus yielding (18).

Setting $x = 0$ in (18) gives (19) when $j = 0$ and this yields $G_i(x)/G_{i0} \to U(x)/\mu_0$ $(i\to\infty)$. Recalling that $\mu_0 > 0$,

$$G_{ij}/G_{i0} \leqq x^{-j}G_i(x)/G_{i0} < M < \infty.$$

This enables us to apply the dominated convergence theorem to the series $G_i(y)/G_{i0}$ with $0 \leqq y < x$ and obtain $G_{ij}/G_{i0} \to \mu_j/\mu_0$ and this implies (19).

4. A limit theorem for the total number of individuals

Let $Y_n = \sum_{m=0}^{n} X_m$ be the total number of individuals that have existed up to and including the n-th generation in our Galton-Watson process with immigration. Assuming $X_0 = i$, it is not difficult to show that

$$Q_i^{(n)}(x) = E(x^{Y_n}) = (g_n(x))^i \prod_{m=0}^{n-1} B(g_m(x)) \tag{20}$$

where $g_n(x)$ is the generating function of the total number of individuals up to and including the n-th generation that are descended from a single anscestor (and including this individual) in the zero-th generation of an ordinary Galton-Watson process. It is well known that

$$g_{n+1}(x) = xA(g_n(x)),\ g_0(x) = x \qquad (n = 0, 1, \cdots) \tag{21}$$

and that if $\alpha \leqq 1$,

$$g_n(x) \downarrow g(x) = xA(g(x)) \qquad (n \to \infty) \tag{22}$$

where $g(x)$ generates an honest probability distribution; see [3] p. 32. (Monotone convergence is not proven in [3], however see equation (32) below).

To see that (20) is true, consider a sequence $\{\mathscr{B}_n\}_0^\infty$, of independent Galton-Watson processes having an offspring distribution generated by $A(x)$ and initial distributions generated by x^i when $n = 0$, and $B(x)$ when $n = 1, 2, \cdots$. Defining $y_{n,m}$ to be the total number of individuals that have existed up to and including the m-th generation of $\mathscr{B}_n$, it is clear that

$$Y_n = \sum_{m=0}^{n} y_{m,n-m} \tag{23}$$

and this is equivalent to (20).

Logarithmic differentiation of (20) and use of (21) yields

$$E(Y_n) = i(n+1) + \beta n(n+1)/2,$$

which suggests considering the convergence of Y_n/n^2 in some sense.

THEOREM 5. *Under the assumptions made in section* 1, *the random variable* Y_n/n^2 *converges weakly to a random variable* Y *whose distribution is defined by the Laplace-Stieltjes transform*

$$E(e^{-\theta Y}) = [\operatorname{sech}(\gamma\theta)^{\frac{1}{2}}]^{\sigma}. \tag{24}$$

REMARKS. When $\sigma = 1$, the distribution defined by (24) arises in the context of first passage time distributions for the Wiener-Lévy process, see for example Bharucha-Reid [1] p. 152. It is clear from (23) that Y is the weak limit of a system of infinitesimal random variables and, as such, it has an infinitely divisible distribution. The Kolmogorov canonical representation of the characteristic function of Y is given by

$$i\beta t/2+\sigma\int_0^\infty (e^{iut}-1-iut)\theta_2(0, e^{iu\pi/\gamma})/(2u)\,du$$

where $\theta_2(\cdot,\cdot)$ is the second theta function; the example on p. 534 of [2] is relevant to the derivation of this result.

We now obtain two results which are needed for the proof of theorem 5.

LEMMA 3. *Let* $u \equiv u(x) = xA'(g(x))$. *Then,*

$$x-g(x) \sim [(1-x)/\gamma]^{\frac{1}{2}}$$

and

$$1-u(x) \sim 2[\gamma(1-x)]^{\frac{1}{2}}, \quad (x\uparrow 1).$$

PROOF. Using a four term Taylor expansion of the right hand side of (22) we have

$$g(x) = \frac{\gamma x(1-g(x))^2}{1-x}+\frac{(1-g(x))^3}{1-x}\,\frac{A'''(\eta)}{6}(g(x)<\eta<1).$$

But $g(x)\uparrow 1(x\uparrow 1)$ and so the first part of the lemma follows. The second part follows from

$$1-A'(g(x)) = 2\gamma(1-g(x))-A''(\zeta)(1-g(x)^2/2 \qquad (g(x)<\zeta<1).$$

LEMMA 4. *Let* $\theta_n = e^{-\theta/n^2}$ $(\theta>0)$ *and let* $\phi(x)\to 1$ *as* $x\uparrow 1$. *Then*

$$\lim_{n\to\infty}(1-u(\theta_n))\sum_{m=1}^{n-1}\frac{\phi(\theta_n)(u(\theta_n))^m}{1+\phi(\theta_n)(u(\theta_n))^m} = -\log\left(\frac{1+e^{-2(\gamma\theta)^{1/2}}}{2}\right). \tag{25}$$

PROOF. By hypothesis we have

$$\phi(\theta_n) = 1+t_n, \quad t_n = o(1) \qquad (n\to\infty) \tag{26}$$

and using the second part of lemma 3 it is easily seen that

$$1-u(\theta_n) = 2(\gamma\theta)^{\frac{1}{2}}/n+r_n/n, \quad r_n = o(1) \qquad (n\to\infty). \tag{27}$$

Writing $\delta = 2(\gamma\theta)^{\frac{1}{2}}$ and $\eta_{m,n} = (1-\delta/n)^m$, it is not difficult to show that

$$\left|\frac{\phi(\theta_n)(u(\theta_n))^m}{1+\phi(\theta_n)(u(\theta_n))^m}-\frac{\exp(-\delta m/n)}{1+\exp(-\delta m/n)}\right| \leqq |\eta_{m,n}-(u(\theta_n))^m| + |t_n|+|\exp(-\delta m/n)-\eta_{m,n}|. \tag{28}$$

Observing that (27) is equivalent to $u(\theta_n) = \eta_{1,n}-r_n/n$, we have

$$|\eta_{m,n}-(u(\theta_n))^m| \leqq m|r_n|/n.$$

Noting that $e^{-\delta/n} = \eta_{1,n}+v_n$ where $|v_n| \leqq \delta/2n^2$, yields

$$|\exp(-\delta m/n)-\eta_{m,n}| \leqq \delta m/2n^2.$$

Summing (28) from $m = 0$ to $m = n-1$, and using the last two estimates proves the equality of the left hand side of (25) (if it exists) and

$$\lim_{n\to\infty} \frac{\delta}{n} \sum_{m=1}^{n-1} \frac{\exp(-\delta m/n)}{1+\exp(-\delta m/n)}.$$

However this limit does exist and is clearly equal to

$$\delta \int_0^1 e^{-\delta x}/(1+e^{-\delta x})\,dx$$

thus completing the proof.

PROOF OF THEOREM 5. Recalling that $\theta_n = e^{-\theta/n^2}$ we have

$$E(\exp(-\theta Y_n/n^2)) = Q_i^{(n)}(\theta_n)$$

and $g_n(\theta_n) \to 1$ $(n \to \infty)$, and so it suffices to consider the case $i = 0$. We have

$$\log Q_0^{(n)}(\theta_n) = -\sum_{m=0}^{n-1}(1-B(g_m(\theta_n))) - \sum_{m=0}^{n-1} R_{m,n}(\theta) \tag{29}$$

where

$$\begin{aligned} 0 \leqq R_{m,n}(\theta) &\leqq [1-B(g_m(\theta_n))]^2/B(g_m(\theta_n)) \\ &\leqq [1-B(g(\theta_n))][1-B(g_m(\theta_n))]/B(g(\theta_n)). \end{aligned}$$

If the first sum in (29) is bounded as $n \to \infty$, then the second sum, denoted by $R_1^{(n)}(\theta)$, tends to zero.

Using the expansion $B(x) = 1-\beta(1-x)+(1-x)r(x)$, where $r(x)$ is monotone non-increasing on $[0, 1]$ and $r(x) \downarrow 0$ as $x \uparrow 1$, we obtain

$$\log Q_0^{(n)}(\theta_n) = -\beta \sum_{m=0}^{n-1}(1-g_m(\theta_n)) + R_1^{(n)}(\theta) + R_2^{(n)}(\theta) \tag{30}$$

where

$$\begin{aligned} 0 \leqq R_2^{(n)}(\theta) &= \sum_{m=0}^{n-1}(1-g_m(\theta_n))r(g_m(\theta_n)) \\ &\leqq r(g(\theta_n)) \sum_{m=0}^{n-1}(1-g_m(\theta_n)) \end{aligned}$$

and the last quantity is $o(1)$ if the sum in (30) remains bounded as $n \to \infty$. Putting the sum in (30) into the form

$$-\sum_{m=0}^{n-1}(1-g_m(\theta_n)) = -\sum_{m=0}^{n-1}(1-g(\theta_n)) + \sum_{m=0}^{n-1}[g_m(\theta_n)-g(\theta_n)] \tag{31}$$

and using the first half of lemma 3 we see that the limit of the first expression on the right hand side of (31) equals $-(\theta/\gamma)^{\frac{1}{2}}$. Thus we need only find the limit of the second sum on the right hand side of (31). We shall do this by obtaining upper and lower bounds for $g_n(x)-g(x)$ which will enable us to invoke lemma 4. Our approach is based on that of Seneta [8].

Equation (21) and the mean value theorem yields, for $0 \leqq x < 1$

(32) $$0 < g_{n+1}(x) - g(x) = x(g_n(x) - g(x))A'(\eta_n)$$
$$(n = 0, 1, \cdots; g(x) < \eta_n < g_n(x)).$$

The left hand inequality is clearly true for $n = 0$, and so for all n, by induction. The same argument shows that $g_{n+1}(x) \leqq g_n(x)(0 \leqq x \leqq 1)$ and thus if we let h be a fixed non-negative integer and write $b_n = 1/(g_n(x) - g(x))$, then the following inequality obtains for $0 \leqq h \leqq n$,

(33) $$u = xA'(g(x) \leqq b_n/b_{n+1} \leqq xA'(g_h(x)).$$

Moreover,

$$g_{n+1}(x) = x[A(g(x)) + (g_n(x) - g(x))A'(g(x)) + (g_n(x) - g(x))^2 A''(\delta_n)]$$

where $g(x) < \delta_n < g_n(x)$, and this yields

$$b_{n+1} = \frac{b_n}{u} - \frac{xA''(\delta_n)}{2u} \cdot \frac{b_{n+1}}{b_n}.$$

Noting that $A''(g(x)) < A''(\delta_n) \leqq A''(g_n(x)) \leqq A''(g_h(x))$, and combining the last equation with (33) finally yields

$$\frac{b_n}{u} - \frac{xA''(g_h(x))}{2u^2} \leqq b_{n+1} \leqq \frac{b_n}{u} - \frac{A''(g(x))}{2uA'(g_h(x))}.$$

Proceeding us in [8], we iterate this inequality and invert the result to obtain finally,

(34) $$\frac{2A'(g_h(x))(1-u)}{A''(g(x))} \cdot \frac{\phi u^n}{1+\phi u^n} \leqq g_{h+n}(x) - g(x)$$
$$\leqq \frac{2u(1-u)}{xA''(g_h(x))} \cdot \frac{\tau u^n}{1+\tau u^n}$$

where

$$\phi = \phi(x) = \frac{A''(g(x))}{2A''(g_h(x))b_h(1-u) - A''(g(x))},$$
$$\tau = \tau(x) = \frac{xA''(g(x))}{2u(1-u)b_h - xA''(g_h(x))}.$$

We now take $h = 0$. Using lemma 3, it is clear that $\phi(x)$, $\tau(x) \to 1(x \uparrow 1)$, and so lemma 4 is applicable and we obtain

$$\lim_{n\to\infty} \sum_{m=0}^{n-1} (g_m(\theta_n)) - g(\theta_n)) = -\frac{1}{\gamma} \log\left(\frac{1+\exp[-2(\gamma\theta)^{\frac{1}{2}}]}{2}\right).$$

Thus our boundedness requirements are satisfied and the error terms in equation (30) are $o(1)$. The theorem now follows on applying the continuity theorem for Laplace-Stieltjes transforms; see Feller [2].

5. The rate of convergence of $g_n(x)$

We shall briefly look at the rate at which $g_n(x) \to g(x)$ in a non-supercritical Galton-Watson process, that is, $\alpha \leqq 1$. Iteration of equation (32) gives

$$g_n(x)-g(x) = (x-g(x))x^n \prod_{m=0}^{n-1} A'(\eta_m) \leqq (x-g(x))(\alpha x)^n$$

which yields

$$u^{-n}(g_n(x)-g(x)) = (x-g(x)) \prod_{m=0}^{n-1} [1-x(A'(g(x))-A'(\eta_m))/u]. \tag{35}$$

The left hand side of this equation will have a limit iff the series $\sum_{m=0}^{\infty}(A'(\eta_m) -A'(g(x)))$ is convergent. We have

$$\begin{aligned} 0 \leqq A'(\eta_m)-A'(g(x)) &\leqq A'(g_m(x))-A'(g(x)) \\ &\leqq (g_m(x)-g(x))A''(g_m(x)) \\ &\leqq (x-g(x))(\alpha x)^m A''(g_m(x)). \end{aligned}$$

Thus, even if $A''(1-) = \infty$, the series converges for $0 \leqq x \leqq 1$ and uniformly so on $0 \leqq x \leqq a < 1$. If $A''(1-) < \infty$ then the convergence is uniform on $[0, 1]$. The same applies to the limit of the product in equation (35), and so we have

$$\lim_{n\to\infty} (xA'(g(x)))^{-n}(g_n(x)-g(x)) = G(x)$$

where $G(x)$ is continuous and non-negative for $0 \leqq x < 1$. Equation (34) yields bounds for $G(x)$ namely

$$\frac{2\phi(x)A'(g_h(x))(1-u)}{u^h A''(g(x))} \leqq G(x) \leqq \frac{2\tau(x)u(1-u)}{xu^h A''(g_h(x))}.$$

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Added in proof: The density function in the statement of Theorem 3 (page 480) is incorrect, substitute γ for β.

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