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FURTHER RESULTS ON THE GENERALIZED CUMULATIVE ENTROPY

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Recently, a new concept of entropy called generalized cumulative entropy of order n was introduced and studied in the literature. It is related to the lower record values of a sequence of independent and identically distributed random variables and with the concept of reversed relevation transform. In this paper, we provide some further results for the generalized cumulative entropy such as stochastic orders, bounds and characterization results. Moreover, some characterization results are derived for the dynamic generalized cumulative entropy. Finally, it is shown that the empirical generalized cumulative entropy of an exponential distribution converges to normal distribution.

Keywords: generalized cumulative entropy, lower record values, reversed relevation transform, stochastic orders, parallel system

Classification: 60E15, 62B10, 62N05

1. INTRODUCTION AND BACKGROUND

1.1. Basic notions

Let X be an absolutely continuous nonnegative random variable with probability density function (PDF) $f(x)$ and cumulative distribution function (CDF) $F(x) = \mathbb{P}(X \le x)$. It is well-known that the classical approach to the description of information related to X is based on Shannon information measure, defined by $H(f) = H(X) = -\mathbb{E}[\log f(X)]$, where $\mathbb{E}$ means expectation and 'log' stands for natural logarithm. Entropy, as a baseline concept in the field of information theory, was first introduced by Shannon [36]. It is also invoked to deal with information in the context of theoretical neurobiology, thermodynamics, and reliability theory. For some recent applications of Shannon's entropy to the ordering of coherent systems see Toomaj et al. [38] and references therein. In many realistic situations such as survival analysis and reliability engineering, one has information about the past lifetime, i. e. the time elapsed after failure till time t, given that the unit has already failed. The Shannon entropy applied to conditioned random variable is useful to measure uncertainty in such situations. Specifically, the inactivity time of a random lifetime X, also known as the reversed residual life or waiting time, is

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defined by

$$X_{[t]} = [t - X | X \leq t], \quad t > 0, \tag{1}$$

which has the PDF $f(x)/F(t)$, $0 \leq x \leq t$, where, as usual, $[X|B]$ denotes a random variable having the same distribution of X conditioned on B. Indeed, $X_{[t]}$ describes the length of the time interval occurring between the failure time X and an inspection time t, given that at time t, the system has been found failed. Di Crescenzo and Longobardi [13] considered the entropy for the past lifetime, called past entropy at time t of X, denoted by

$$\overline{H}(t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} \, \mathrm{d}x, \quad t > 0,$$

see also Muliere et al. [25]. In particular, $\lim_{t\to\infty} \overline{H}(t) = H(X)$ coincides with the classical Shannon entropy. Moreover, Di Crescenzo and Longobardi [14] proposed the cumulative entropy as an alternative measure of uncertainty for the inactivity time by replacing the probability density function with the cumulative distribution function in the Shannon entropy, that is

$$\mathcal{CE}(X) = -\int_0^\infty F(x) \log F(x) \, \mathrm{d}x = \int_0^\infty F(x) T(x) \, \mathrm{d}x, \tag{2}$$

where

$$T(x) = -\log F(x) = \int_x^\infty \tau(t) \, \mathrm{d}t, \quad x > 0, \tag{3}$$

denotes the cumulative reversed hazard function and $\tau(\cdot)$ stands for the reversed hazard rate function of X, defined by

$$\tau(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log F(t) = \frac{f(t)}{F(t)}, \quad t > 0. \tag{4}$$

To see the basic properties and information about the reversed hazard rate function, we refer the reader to e. g. Block et al. [8] and Chandra and Roy [11]. It follows that

$$\mathcal{CE}(X) = \mathbb{E}[\tilde{\mu}(X)], \tag{5}$$

where

$$\tilde{\mu}(t) = \mathbb{E}[X_{[t]}] = \mathbb{E}[t - X | X \leq t] = \frac{1}{F(t)} \int_0^t F(x) \, \mathrm{d}x, \quad t > 0, \tag{6}$$

is the mean inactivity time of X.

Several properties of cumulative entropy were discussed in Di Crescenzo and Longobardi [14]. They also considered the dynamic version of the cumulative entropy for the past lifetime and obtained various results such as characterization and stochastic ordering. Recently, Psarrakos and Navarro [30] introduced a new measure of uncertainty and called it generalized cumulative residual entropy (GCRE), defined by

$$\mathcal{E}_n(X) = \int_0^\infty \overline{F}(x) \frac{[-\log \overline{F}(x)]^n}{n!} \, \mathrm{d}x, \tag{7}$$

for $n = 0, 1, 2, \ldots$, where $\overline{F}(x) = 1 - F(x)$ is the survival function of X. A weighted (shift-dependent) version of $\mathcal{E}_n(X)$ has been studied recently by Kayal [21]. For $n = 1$, Eq. (7) coincides with the cumulative residual entropy, introduced by Rao et al. [33], whereas for $n = 0$ we have $\mathcal{E}_0(X) = \mathbb{E}(X)$. An analogue generalization was considered by Kayal [20] for past lifetime, called generalized cumulative entropy (GCE) of X and, due to (3), defined by

$$\mathcal{CE}_n(X) = \int_0^\infty F(x)\frac{[T(x)]^n}{n!}\,\mathrm{d}x = \int_0^\infty F(x)\frac{[-\log F(x)]^n}{n!}\,\mathrm{d}x, \tag{8}$$

for $n = 1, 2, \ldots$. Also, Kayal [20] considered the following quantity:

$$\begin{aligned} \mathcal{CE}_n(t) &= \mathcal{CE}_n(X;t) = \frac{1}{n!}\int_0^t \frac{F(x)}{F(t)}\left[-\log\frac{F(x)}{F(t)}\right]^n \mathrm{d}x, \\ &= \frac{1}{n!}\int_0^t \frac{F(x)}{F(t)}\,[T(x) - T(t)]^n\,\mathrm{d}x, \end{aligned} \tag{9}$$

for $n \geq 1$ and $t > 0$. We point out that $\mathcal{CE}_n(t)$ is the dynamic version of GCE for the inactivity time introduced in (1). Note that $\lim_{t\to\infty}\mathcal{CE}_n(t) = \mathcal{CE}_n(X)$, which coincides with the GCE of X. In Table 1, some examples of GCE and DGCRE concerning the uniform and Fréchet distributions are presented, where

$$\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t}\,\mathrm{d}x,$$

is the upper incomplete gamma function and $\Gamma(a, 0) = \Gamma(a)$ denotes the complete gamma function. It is not hard to verify that for all $k = 0, 1, \cdots, n$, for the case (c) of Table 1 we have

$$\int_0^t F(x)[T(x)]^k\,\mathrm{d}x = a^{\frac{1}{\gamma}}\Gamma\left(k - \frac{1}{\gamma}, \frac{a}{t^\gamma}\right), \qquad a > 0,\ \gamma > 1.$$

Hence, from Proposition 4.6 of Kayal [20], the results given for case (c) of Table 1 are obtained. Specifically, for $k = 0$ we have

$$\Gamma\left(-\frac{1}{\gamma}, \frac{a}{t^\gamma}\right) = te^{-\frac{a}{t^\gamma}} - a^{\frac{1}{\gamma}}\Gamma\left(1 - \frac{1}{\gamma}, \frac{a}{t^\gamma}\right).$$

Remark 1.1. Due to (8) and (9), $\mathcal{CE}_n(t)$ identifies with the generalized cumulative entropy of $[X|X \leq t]$, for $t > 0$.

1.2. Background on aging notions and stochastic orders

Aging notions and stochastic orders have many applications in various areas of sciences such as reliability and survival analysis, economics, insurance, actuarial and management sciences and coding theory; see Shaked and Shanthikumar [35] for a greater detail. In the following, we review some notions that are used in the sequel. Note that here and throughout this paper, the terms 'increasing' and 'decreasing' are used in a non-strict sense. Moreover, prime ($'$) denotes derivative.

$F(x)$	$\mathcal{CE}_n(X)$	$\mathcal{CE}_n(t)$
(a) $x,\ \ 0<x<1$	$\dfrac{1}{2^{n+1}}$	$\dfrac{t}{2^{n+1}}$
(b) $e^{-a/x},\ \ x>0$	$\dfrac{a}{n(n-1)}$	$\dfrac{ae^{\frac{a}{t}}}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(\dfrac{a}{t}\right)^{n-k}\Gamma\Big(k-1,\dfrac{a}{t}\Big)$
(c) $e^{-ax^{-\gamma}}$, $x>0$	$\dfrac{a^{1/\gamma}}{\gamma n!}\Gamma\Big(n-\dfrac{1}{\gamma}\Big)$	$\dfrac{a^{\frac{1}{\gamma}}e^{\frac{a}{t^\gamma}}}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(\dfrac{a}{t^\gamma}\right)^{n-k}\Gamma\Big(k-\dfrac{1}{\gamma},\dfrac{a}{t^\gamma}\Big)$

Tab. 1. The generalized cumulative entropy and the dynamic generalized cumulative entropy for some distribution functions, with $a>0$ and $\gamma>1$.

Definition 1.2. If X is an absolutely continuous random variable with support (l_X, u_X), CDF $F(x)$, PDF $f(x)$, survival function $\overline{F}(x)=1-F(x)$, hazard rate function $h_X(x)=\frac{\mathrm{d}}{\mathrm{d}x}\log\overline{F}(x)$ and reversed hazard rate function $\tau(x)=f(x)/F(x)$, then

- X has the increasing likelihood ratio (ILR) property if $f(x)$ is log-concave or, equivalently, if $f'(x)/f(x)$ is decreasing in $x\in(l_X,u_X)$;
- X has the decreasing likelihood ratio (DLR) property if $f(x)$ is log-convex or, equivalently, if $f'(x)/f(x)$ is increasing in $x\in(l_X,u_X)$;
- X has the decreasing reversed hazard rate (DRHR) property if $\tau(t)$ is decreasing in $t\in(l_X,u_X)$ or, equivalently, if $T(x)=-\log F(x)$ is convex;
- X is said to have decreasing failure rate (DFR) if $h_X(t)=\frac{\mathrm{d}}{\mathrm{d}t}\log\overline{F}(t)$ is decreasing in t.

Moreover, if Y is an absolutely continuous random variable with support (l_Y,u_Y), CDF $G(x)$, PDF $g(x)$, survival function $\overline{G}(x)=1-G(x)$, and hazard rate function $h_Y(x)=\frac{\mathrm{d}}{\mathrm{d}x}\log\overline{G}(x)$, then

- X is smaller than Y in the usual stochastic order (denoted by $X\le_{st}Y$) if $\overline{F}(t)\le\overline{G}(t),\ \forall\, t\in\mathbb{R}$, or equivalently $F(t)\ge G(t),\ \forall\, t\in\mathbb{R}$.
- X is smaller than Y in the likelihood ratio order (denoted by $X\le_{lr}Y$) if $f(x)g(y)\ge f(y)g(x)$ for all $x\le y$, with $x,y\in(l_X,u_X)\cup(l_Y,u_Y)$.
- X is smaller than Y in the up shifted likelihood ratio order (denoted by $X\le_{lr\uparrow}Y$) if $X-x\le_{lr}Y$ for all $x\ge0$ or equivalently for each $x\ge0$ we have $g(t)/f(t+x)$ is increasing in $t\in(l_X-x,u_X-x)\cup(l_Y,u_Y)$, where $a/0$ is taken to be equal to ∞ whenever $a>0$.
- X is smaller than Y in the hazard rate order (denoted by $X\le_{hr}Y$) if $h_X(x)\ge h_Y(x)$ for all $x\in\mathbb{R}$.

- X is smaller than Y in the dispersive order (denoted by $X \leq_d Y$) if $F^{-1}(v) - F^{-1}(u) \leq G^{-1}(v) - G^{-1}(u)$, $\forall\ 0 < u \leq v < 1$, where F^{-1} and G^{-1} are right continuous inverses of F and G, respectively, or equivalently

$$g(G^{-1}(v)) \leq f(F^{-1}(v)), \quad 0 < v < 1. \tag{10}$$

1.3. Motivation and plan of the paper

Several properties of the generalized cumulative entropy such as the effect of linear transformations, a two-dimensional version, a normalized version, bounds, stochastic ordering as well as the dynamic generalized cumulative entropy and various relationships with other functions are given by Kayal [20]. In this paper, we obtain some further results on the generalized cumulative entropy (8) and the dynamic generalized cumulative entropy (9). The new entropy measure is related to the mean time between lower record values of a sequence of independent and identically distributed (i.i.d.) random variables and the concept of reversed relevation transform. We also discuss some characterization results and stochastic ordering properties. Specifically, it is worth mentioning that the generalized cumulative entropy is equal to the difference between the means of consecutive recursive reversed relevation transforms, which in turn are of interest in the analysis of lower record values. For instance, their role in the analysis of a new measure of association based on the log-odds rate has been recently pinpointed by Asadi [2]. Hence, our new results can be applied within such fields. Moreover, as a further novelty, in the final part of the paper we introduce the empirical generalized cumulative entropy and show that

(i) it is an unbiased and consistent estimator for a random sample from an uniform distribution, and

(ii) it converges to the normal distribution for a random sample from an exponential distribution.

Therefore, the rest of this paper is organized as follows: In Section 2, the definitions, motivations and basic properties of n-fold reversed relevation transform are given. We also relate this notion and related results to some well-known concepts such as distortion functions, weighted random variables, and the proportional reversed hazard rates model. The new version of cumulative entropy, its stochastic ordering and aging classes properties are provided in Section 3. Characterization results and orderings for the generalized cumulative entropy of the maxima of random samples are also given. In Section 4, we study the monotonicity properties of the dynamic generalized cumulative entropy, and a characterization of a stochastic model related to family of distributions. Some examples and a central limit theorem for the empirical GCE are derived in Section 5. The summary given in Section 6 concludes the paper.

2. PROPERTIES OF N-FOLD REVERSED RELEVATION TRANSFORM

Consider a reliability (repairable) system, in which a failed unit can be repaired or replaced by a unit with the same age. Let X be a nonnegative random variables with support $[0, \infty)$, which denotes the lifetime of the first unit, with survival function $\overline{F}(t) =$

$1-F(t)$, and let Y be the lifetime of the second unit, with survival function $\overline{G}(t+x)/\overline{G}(x)$ given that $X = x$. Hence, the reliability of the relevation process is obtained by

$$\overline{G}\#\overline{F}(x) = \overline{G}(x) + \overline{F}(x) \int_0^x \frac{1}{\overline{F}(t)}\, \mathrm{d}G(t), \quad x > 0, \tag{11}$$

where the symbol $\#$ denotes the relevation transform of F and G introduced by Krakowski [23]; see also Baxter [7]. Equation (11) was discussed by Kapodistria and Psarrakos [19]; see also Burkschat and Navarro [9] and Psarrakos and Navarro [30] and the references therein. Recently, in analogy with (11), Di Crescenzo and Toomaj [16] introduced the *reversed relevation transform,* defined by

$$G\widetilde{\#}F(x) = G(x) + F(x) \int_x^\infty \frac{1}{F(t)}\, \mathrm{d}G(t), \quad x > 0, \tag{12}$$

where the symbol $\widetilde{\#}$ means reversed relevation transform of F and G. They provided some new connections of the cumulative entropy and the past lifetime by using the concept of reversed relevation transform. Let X and Y be independent random lifetimes with distribution function F and G, respectively. Denoting by $X[t] = [X|X \le t]$, the total time of X given that it is less than t, then the reversed relevation transform of F and G given in (12) can be viewed as the distribution function of $X[Y]$, i. e. the total time of X given that it is less than Y.

We remark that the analogy between the survival function (11) and the distribution function (12) holds also for the corresponding PDF's, say $g\#f(x)$ and $g\widetilde{\#}f(x)$, since for $x > 0$

$$g\#f(x) = f(x) \int_0^x \frac{1}{\overline{F}(t)}\, \mathrm{d}G(t) \qquad \text{and} \qquad g\widetilde{\#}f(x) = f(x) \int_x^\infty \frac{1}{F(t)}\, \mathrm{d}G(t), \tag{13}$$

where f and g denote the PDF's of X and Y, respectively. Let us now provide some ordering results involving such densities.

Proposition 2.1. Let X and Y be absolutely continuous nonnegative random variables with support $[0, \infty)$, and denote by $X\#Y$ and $X\widetilde{\#}Y$ the random variables having PDF's shown in (13), respectively. Then, one has

(i) $X\widetilde{\#}Y \le_{lr} X \le_{lr} X\#Y$;

(ii) $X\widetilde{\#}Y \le_{lr\uparrow} X \le_{lr\uparrow} X\#Y$ provided that X is ILR.

Proof. From (13) we have that the ratios $f(x)/g\widetilde{\#}f(x)$ and $g\#f(x)/f(x)$ are both increasing in $x > 0$. The proof of statement (i) thus follows recalling Definition 1.2. Similarly one can prove point (ii), noting that $f(x)/g\widetilde{\#}f(x+t)$ and $g\#f(x)/f(x)$ are both increasing in $x > 0$ for all $t > 0$, if and only if $f(x)$ is log-concave, i. e. X is ILR. □

Applying (12), we immediately have the following lemma concerning proper mixtures of distributions.

Lemma 2.2. Let us consider the cumulative distribution functions F, G and H. For $a \geq 0,\ b \geq 0$, and $a+b=1$, we have

$$(aF+bG)\widetilde{\#}H(x) = aF\widetilde{\#}H(x) + bG\widetilde{\#}H(x), \quad x>0.$$

It is worth pointing out that Lemma 2.2 still holds in the case of generalized mixtures. We recall that if $F, F_1, \ldots, F_k$ are distribution functions, then F is said a generalized mixture of $F_1, \ldots, F_k$ with weights $w_1, \ldots, w_k \in \mathbb{R}$ if $F(t) = w_1F_1(t) + \ldots + w_kF_k(t)$ for all $t \in \mathbb{R}$. For the basic properties of generalized mixtures, we refer the reader to e.g. Navarro and Rubio [28] and references therein.

Consider the reversed relevation transform (12) when $F = G$. In this case, recalling that $T(x) = -\log F(x)$, one has

$$F\widetilde{\#}F(x) = F(x) + F(x)\int_x^\infty \frac{1}{F(t)}\,\mathrm{d}F(t) = F(x)[1+T(x)], \quad x>0. \tag{14}$$

By iteration, the CDF of the n-fold recursive reversed relevation transform of F is defined as

$$F_n(x) = \begin{cases} F(x), & n=1 \\ F_{n-1}\widetilde{\#}F(x), & n\geq 2 \end{cases} \tag{15}$$

so that, recalling (14), we have $F_2(x) = F\widetilde{\#}F(x) = F(x)[1+T(x)]$.

Remark 2.3. We remark that in general $G\widetilde{\#}F(x) \neq F\widetilde{\#}G(x)$, i.e. the reversed relevation transform is not commutative. Indeed, in Theorem 1 of [16] it is specified that $G\widetilde{\#}F(x) = F\widetilde{\#}G(x)$ for all $x>0$ if, and only if, X and Y satisfy the proportional reversed hazard rates model. For instance, if $F(x) = 1-\frac{1}{x+1}$, $x \geq 0$, and $G(x) = 1-\frac{1}{(x+1)^2}$, $x \geq 0$, (so that $Y \leq_{st} X$), then for $x \geq 0$ one has

$$\begin{aligned} G\widetilde{\#}F(x) &= \frac{x(x+2(1+x)\log(1+1/x))}{(1+x)^2} \\ &\geq \frac{x[2(1+x)-(2+x)\log(x/(2+x))]}{2(1+x)^2} = F\widetilde{\#}G(x) \end{aligned}$$

(so that $Y\widetilde{\#}X \leq_{st} X\widetilde{\#}Y$). Moreover, due to (15) we have, for $x \geq 0$,

$$F\widetilde{\#}F_2(x) = F(x)[1+(1+T(x))\log(1+T(x))] \neq F(x)\Big[1+T(x)+\frac{1}{2}(-T(x))^2\Big] = F_2\widetilde{\#}F(x).$$

In Eq. (17) below we give the explicit expression of the n-fold recursive reversed relevation transform of F, defined in (15). Specifically, in order to provide an equivalent form for $F_n(x)$, hereafter we recall a useful notion. Let $q:[0,1]\to[0,1]$ be a continuous, non-decreasing and piecewise differentiable function, that satisfies $q(0)=0$ and $q(1)=1$. A function defined in this way is usually called distortion function. In applications such as insurance and reliability, distortion functions are commonly used for changing the probability measure. Indeed, for a given cumulative distribution function F, the transformation

$$F_q(x) = q[F(x)] = q \circ F(x)$$

defines a new distribution function associated to a certain random variable X_q, which is a distorted random variable induced by q; see, for instance, Sordo and Suarez-Llorens [37], and the references therein. For $n = 1, 2, \ldots$, let

$$q_n(x) = x \sum_{k=0}^{n-1} \frac{[-\log x]^k}{k!}, \qquad 0 \le x \le 1, \tag{16}$$

be a sequence of distortion functions. Hereafter we show that the CDF defined in (15) can be expressed in a straightforward manner in terms of the distortion function (16).

Remark 2.4. Given a nonnegative random variable X with CDF F, let X_n, $n \ge 1$, denote the n-fold recursive reversed relevation transform of F. Then, the cumulative distribution function of X_n, $n \ge 1$, is

$$F_n(x) = q_n[F(x)] = F(x) \sum_{k=0}^{n-1} \frac{[T(x)]^k}{k!}, \qquad x > 0, \tag{17}$$

where $T(\cdot)$ and $q_n(\cdot)$ are defined in (3) and (16), respectively. It should be noted that (17) is similar to relation (2) is Psarrakos and Navarro [30].

As mentioned earlier, $q_n(x)$ is a sequence of distortion functions and thus some ordering properties can be obtained from the results for distortion functions given in Navarro et al. [30] and Sordo and Suarez-Llorens [37]. Now, consider the definition of the lower record values and the related distribution (see, e. g., Chandler [10], or Arnold et al. [1]).

Definition 2.5. Suppose that $Y_1, Y_2, \ldots$ is a sequence of nonnegative i.i.d. random variables having the common CDF $F(x)$. We say that Y_k is a lower record value of this sequence if $Y_k < \min\{Y_1, Y_2, \ldots, Y_{k-1}\}$, with $k > 1$, and by definition Y_1 is a lower record value. Let $L(1) = 1$ and $L(n) = \min\{j : j > L(n-1), Y_j < Y_{L(n-1)}\}$ for $n > 1$, so that $L(n)$ denotes the index where the nth lower record value occurs. By defining $F_n(x)$ as the CDF of $Y_{L(n)}$, for $n \ge 1$ we have:

$$F_n(x) = F(x) \sum_{k=0}^{n-1} \frac{[-\log F(x)]^k}{k!}, \qquad x > 0. \tag{18}$$

One can see that (17) coincides with the distribution function of the nth lower record value given in (18). Hence, the study of n-fold recursive reversed relevation transform is equivalent to the study of lower record values. In the following proposition, with reference to the sequence defined in Remark 2.4, we prove that X_n converges to 0 (in probability) when n goes to infinity.

Proposition 2.6. Let $(X_n)_{n \ge 1}$ be a sequence of nonnegative random variables having CDF $F_n(x)$ given in (17), then $X_n \xrightarrow{p} 0$ as $n \to \infty$.

P r o o f. Recalling that $X_n \xrightarrow{d} 0$ if and only if $X_n \xrightarrow{p} 0$, when $n \to \infty$, it is sufficient to prove that

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases}$$

From (17), and recalling (3), we have

$$\begin{aligned}\lim_{n\to\infty} F_n(x) &= F(x)\sum_{k=0}^{\infty}\frac{[T(x)]^k}{k!}\\ &= F(x)e^{T(x)} = 1, \quad x > 0,\end{aligned}$$

and hence this completes the proof. □

If the CDF F is absolutely continuous, then the PDF of (17) is obtained as

$$f_n(x) = f(x)\frac{[T(x)]^{n-1}}{(n-1)!}, \qquad x > 0, \tag{19}$$

for all $n \geq 1$. Equation (19) shows that $f_n(x)$ is a sequence of weighted PDF's with a normalizing weight $w(x) = T^{n-1}(x)$, $x > 0$. We recall that, given an absolutely continuous random variable X having PDF f, and a nonnegative real function w, the associated weighted random variable X^w is

$$f^w(x) = \frac{w(x)f(x)}{\mathbb{E}[w(X)]}, \qquad x \in \mathbb{R},$$

provided that $0 < \mathbb{E}[w(X)] < \infty$. Some recent papers on weighted distributions are Bartoszewicz [6], Li et al. [24], Navarro et al. [29], among others.

Hereafter, we provide some ordering properties of the n-fold reversed relevation transform of F with CDF (17) and PDF (19). The results given hereafter, in this section, are similar to various results given in Section 3 of Di Crescenzo and Toomaj [16] and hence we omit their proofs.

Theorem 2.7. Let $X_1, X_2, \ldots$ be a sequence of nonnegative random variables defined as in Remark 2.4. Let $n \geq 1$.

(a) If X_n is ILR, then $X_n \geq_{lr\uparrow} X_{n+1}$.

(b) Let $n \geq 1$. If X is DLR, then X_n is DLR.

Remark 2.8. If X is ILR, then X_n is not necessarily ILR (see Example 4 of Di Crescenzo and Toomaj [16]).

Proposition 2.9. Let $\tau_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log F_n(x)$ denote the reversed hazard rate of X_n, with F_n given in (17) and let $q(x)$ be a nonnegative function of $x > 0$. If $q(x)\tau_1(x)$ is a decreasing function of $x > 0$, then $q(x)\tau_n(x)$ is also a decreasing function of $x > 0$, for all $n = 1, 2, \ldots$.

Corollary 2.10. If X is DRHR, then X_n is DRHR for all $n \geq 2$.

Let us now recall the following property.

Definition 2.11. Let X be an absolutely continuous random variable with support (l_X, u_X). We say that X has the decreasing reversed hazard rate in length-biased sense (LB-DRHR) if $x\,\tau(x)$ is decreasing in $x \in (l_X, u_X)$.

Equivalently, X is said to be LB-DRHR if and only if $X_v^* \leq_{st} X_w^*$ for all $0 < v < w < 1$, where $\mathbb{P}(X_v^* \leq x) = F^{-1}(vx)/F^{-1}(v)$, $0 < x < 1$, where F^{-1} is the right continuous inverse of the CDF of X (see Section 5.3 of Di Crescenzo et al. [15]). The following result also follows from Proposition 2.9.

Corollary 2.12. If X is LB-DRHR, then X_n is LB-DRHR for all $n \geq 2$.

Consider now the following stochastic order due to Rezaei et al. [34].

Definition 2.13. Let X and Y be nonnegative random variables with CDF's F and G and reversed hazard rates $\tau_X(x)$ and $\tau_Y(x)$, respectively. The random variable X is said to be smaller than Y in relative reversed hazard rate order (denoted by $X \leq_{RRH} Y$), if $\tau_Y(x)/\tau_X(x)$ is an increasing function of $x > 0$.

Let X and Y denote the lifetimes of two components. Given that the components have been found to be failed at the same time, then $X \leq_{\mathrm{R}RH} Y$ states that Y has been lived longer than X or equivalently X aged faster than Y.

Proposition 2.14. For the n-fold recursive reversed relevation transform of F, we have $X_n \leq_{\mathrm{R}RH} X$ for all $n \geq 1$.

Let X and Y be absolutely continuous nonnegative random variables with CDF's $F(x)$ and $G(x)$, and reversed hazard rate functions $\tau_X(x)$ and $\tau_Y(x)$, respectively. These variables satisfy the proportional reversed hazard rates model with proportionality constant $\theta > 0$, $\theta \neq 1$, if

$$G(x) = [F(x)]^{\theta}, \quad x > 0. \tag{20}$$

The parent distribution function can be expressed as $F(x) = e^{-T(x)}, \quad x > 0$, where $T(x)$ is defined in (3).

Theorem 2.15. Let $X_n, n \geq 1$, denote the n-fold recursive reversed relevation transform of F, and let Y be an absolutely continuous nonnegative random variable with CDF G. If X_1 and Y satisfy the proportional reversed hazard rates model given in (20), then $X_n \leq_{\mathrm{R}RH} Y$ for all $n \geq 1$.

P r o o f. The proof is similar to that of Theorem 6 of Di Crescenzo and Toomaj [16]. □

3. RESULTS ON THE GENERALIZED CUMULATIVE ENTROPY

In this section, we obtain some further results on the stochastic ordering properties of the generalized cumulative entropy (8).

First of all, recalling Remark 3.2 of Kayal [20], we note that the GCE is strictly related to the recursive reversed relevation transform. Indeed, from (8) and (17), or equivalently (18), we have

$$\mathcal{CE}_n(X) = \mathbb{E}(X_n) - \mathbb{E}(X_{n+1}), \qquad n = 1, 2, \ldots, \tag{21}$$

where $(X_n)_{n\geq 1}$ denotes the sequence of nonnegative random variables defined as in Remark 2.4.

Now we consider the dispersive and hazard rate orders. The proof of the first theorem is similar to the proof of Lemma 3 in Klein et al. [22] and hence we omit it.

Theorem 3.1. Let X and Y be absolutely continuous nonnegative random variables with CDFs F and G, respectively. If $X \leq_d Y$, then

$$\mathcal{CE}_n(X) \leq \mathcal{CE}_n(Y),$$

for all $n = 1, 2, \ldots$.

Theorem 3.2. Let X and Y be two independent nonnegative random variables. If X and Y have log-concave densities, then

$$\mathcal{CE}_n(X+Y) \geq \max\{\mathcal{CE}_n(X), \mathcal{CE}_n(Y)\},$$

for all $n = 1, 2, \ldots$.

Proof. Let X have a log-concave density. From Theorem 3.B.7 of Shaked and Shanthikumar [35], one can conclude that $X \leq_d X+Y$ for any random variable Y independent of X. Hence, Theorem 3.1 implies that $\mathcal{CE}_n(X+Y) \geq \mathcal{CE}_n(X)$. Similar result also holds when Y has a log-concave density i. e. $\mathcal{CE}_n(X+Y) \geq \mathcal{CE}_n(Y)$. Therefore, this completes the proof. □

Theorem 3.3. If $X \leq_{hr} Y$ and X or Y is DFR, then

$$\mathcal{CE}_n(X) \leq \mathcal{CE}_n(Y),$$

for all $n = 1, 2, \ldots$.

Proof. If $X \leq_{hr} Y$ and X or Y is DFR, then $X \leq_d Y$, due to Bagai and Kochar [3]. Therefore, from Theorem 3.1 the desired result follows. □

Remark 3.4. It is worth pointing out that for any nonnegative random variable X with decreasing reversed hazard rate (DRHR) property, it follows that $\mathcal{CE}_n(X)$ is decreasing in n, by Kayal [20]. Hence, the given upper bounds in Di Crescenzo and Longobardi [14] holds for $\mathcal{CE}_n(X)$ by noting that $\mathcal{CE}_n(X) \leq \mathcal{CE}_1(X)$.

In the sequel, we present some results related to various findings given in Psarrakos and Toomaj [31].

Proposition 3.5. For a nonnegative random variable X and $n = 1, 2, \ldots$, we have

$$\mathcal{CE}_n(X) \geq \frac{1}{n!}\left(\int_0^\infty F(x)\overline{F}(x)\,\mathrm{d}x\right)^n. \tag{22}$$

The proof is omitted, being similar to previous results. Indeed, the integral in the right-hand-side of (22) is already involved in other bounds for information measures see e. g., Proposition 4.3 of [14] and Proposition 1 of [32]; see also Remark 4.1 of [14] for its probabilistic interpretation. The next proposition gives a lower bound for the GCE, depending on the cumulative entropy.

Proposition 3.6. For a nonnegative random variable X and $n = 1, 2, \dots$, it holds that

$$\mathcal{CE}_n(X) \geq \frac{1}{n!}[\mathcal{CE}(X)]^n, \tag{23}$$

where $\mathcal{CE}(X)$ is defined in (2).

Proof. Since $F(x) \geq [F(x)]^n$, for all $x \in \mathbb{R}$ and for all $n = 1, 2, \dots$, from (8) we have

$$\mathcal{CE}_n(X) = \int_0^\infty F(x)\frac{[T(x)]^n}{n!}\,\mathrm{d}x \geq \frac{1}{n!}\int_0^\infty [F(x)\,T(x)]^n\,\mathrm{d}x \geq \frac{1}{n!}\left[\int_0^\infty F(x)T(x)\,\mathrm{d}x\right]^n,$$

where the last inequality is obtained from Jensen's inequality by noting that $g(x) = x^n,\ n \geq 1$, is a convex function. Then, the desired result follows by recalling (2). □

Making use of Proposition 4.2 of Di Crescenzo and Longobardi [14] and Proposition 3.6 another lower bound for the GCE is obtained hereafter in terms of the Shannon entropy.

Corollary 3.7. If X is an absolutely continuous nonnegative random variable, for all $n = 1, 2, \dots$, we have

$$\mathcal{CE}_n(X) \geq \frac{1}{n!}C^n e^{nH(X)},$$

where $C = \exp\left\{\int_0^1 \log(x|\log x|)\,\mathrm{d}x\right\} = e^{-1-\gamma} = 0.206549\dots$, γ being the Euler's constant.

Let us now consider the following problem: to express the generalized cumulative entropy of X in terms of quantities depending on X and Y, where Y is a random variable larger than X in the usual stochastic order. To this purpose, we introduce the following function:

$$R_n(x) = \frac{1}{n!}\int_x^\infty T^n(s)\,\mathrm{d}s, \quad x > 0, \tag{24}$$

where T is defined in (3).

Proposition 3.8. Let X and Y be nonnegative random variables with finite unequal means and such that $X \leq_{st} Y$. If the function defined in (24) is finite, and $\mathbb{E}[R_n(Y)]$ is finite, then for any $n = 1, 2, \dots$ we have

$$\mathcal{CE}_n(X) = \mathbb{E}[R_n(Y)] + \frac{1}{n!}\mathbb{E}\left[T^n(Z)\right]\left[\mathbb{E}(Y) - \mathbb{E}(X)\right], \tag{25}$$

where Z is an absolutely continuous nonnegative random variable having PDF

$$f_Z(x) = \frac{\mathbb{P}(Y > x) - \mathbb{P}(X > x)}{\mathbb{E}(Y) - \mathbb{E}(X)}, \qquad x > 0.$$

P r o o f . The proof follows from the probabilistic mean value theorem shown in Theorem 4.1 of Di Crescenzo [12], and making use of the identity $\mathcal{CE}_n(X) = \mathbb{E}[R_n(X)]$, which has been given in Lemma 3.1 of Kayal [20]. □

We note that Proposition 3.8 also provides a lower bound for the GCE of X. Indeed, since $T^n(\cdot) \geq 0$, due to (3), and since $\mathbb{E}(X) < \mathbb{E}(Y)$, from (25) we obtain

$$\mathcal{CE}_n(X) \geq \mathbb{E}[R_n(Y)].$$

From Eq. (21) we have $\mathbb{E}(X_n) = \mathbb{E}(X_{n+1}) + \mathcal{CE}_n(X)$, for $n \geq 1$. Let us now provide a generalization of such result.

Proposition 3.9. Let $(X_n)_{n\geq 1}$ denote the sequence of nonnegative random variables defined as in Remark 2.4. If $g(\cdot)$ is a measurable and differentiable function, with its derivative measurable and Riemann-integrable on the interval $[x, y]$ for all $y \geq x \geq 0$, such that $\mathbb{E}[g(X_n)]$ is finite for all $n \geq 1$, then we have

$$\mathbb{E}[g(X_n)] = \mathbb{E}[g(X_{n+1})] + \mathbb{E}[g'(Z_n)]\, \mathcal{CE}_n(X),$$

for any $n \geq 1$, where Z_n is an absolutely continuous nonnegative random variable having PDF

$$f_{Z_n}(x) = \frac{F(x)\,[T(x)]^n}{\mathcal{CE}_n(X)\, n!}, \qquad x > 0.$$

P r o o f . The proof follows from the probabilistic mean value theorem given in [12], by noting that the sequence $(X_n)_{n\geq 1}$ is decreasing in the usual stochastic order, and using Eqs. (17) and (21). □

Hereafter, we provide some characterization results of the GCE. First, we need the following well-known *Müntz-Szász* theorem; see for details Hwang and Lin [17] and Kamps [18].

Lemma 3.10. For any increasing sequence of positive integers $\{m_j, j \geq 1\}$, the sequence of polynomials $\{x^{m_j}\}$ is complete on $L(0, 1)$ if and only if

$$\sum_{j=1}^{+\infty} m_j^{-1} = +\infty, \quad 0 < m_1 < m_2 < \dots . \tag{26}$$

Hwang and Lin [17] extended the *Müntz-Szász* Theorem as follows:

Lemma 3.11. Let $f(x)$ be an absolutely continuous function on (a, b) with $f(a)f(b) \geq 0$, and let its derivative satisfy $f'(x) \neq 0$ a.e. on (a, b). Then, under the assumption (26), the sequence $\{f^{m_j}(x), j \geq 1\}$ is complete on $L(a, b)$ if and only if the function $f(x)$ is monotone on (a, b).

By using the proof techniques of Theorems 2.1 and 2.2 of Baratpour [4], and applying Lemmas 3.1 and 3.2, we obtain the following characterization results. Their proof is similar to the proof of Theorems 3.3 and 3.4 in Psarrakos and Toomaj [31] and hence we omit them. As usual, we denote by $X_{m:m}$, the maximum of a random sample having size m whose random variables are distributed as X, and similarly for $Y_{m:m}$. The random variable $X_{m:m}$ is known as the lifetime of the parallel system in engineering reliability and has many applications in engineering and reliability sciences; see e. g. Barlow and Proschan [5] for further details.

Theorem 3.12. Assume that $\mathcal{M} = \{m_j, j \geq 1\}$ is a strictly increasing sequence of positive integers such that (26) holds. Let X and Y be two nonnegative random variables with absolutely continuous CDF's F and G and PDF's f and g, respectively. Then F and G belong to the same family of distributions, but for a change of location and scale, if and only if

$$\frac{\mathcal{CE}_n(X_{m:m})}{\mathbb{E}(X_{m:m})} = \frac{\mathcal{CE}_n(Y_{m:m})}{\mathbb{E}(Y_{m:m})}, \tag{27}$$

for a fixed $n \geq 1$ and for all $m \in \mathcal{M}$.

Theorem 3.13. Let X and Y be two nonnegative random variables with PDF's f and g and absolutely continuous CDF's F and G, respectively. Then F and G belong to the same family of distributions, but for a change in location, if and only if

$$\mathcal{CE}_n(X_{m:m}) = \mathcal{CE}_n(Y_{m:m}), \tag{28}$$

for a fixed $n \geq 1$ and for all $m \in \mathcal{M}$, where $\mathcal{M}$ is defined as in Theorem 3.12.

The last theorem of this section concerns comparisons of generalized cumulative entropies of maxima.

Theorem 3.14. Let X and Y be two nonnegative random variables with PDF's f and g and absolutely continuous CDF's F and G, respectively, and let

$$A_1 = \big\{0 < v < 1 \big| f(F^{-1}(v)) \geq g(G^{-1}(v))\big\}, \;\; A_2 = \big\{0 < v < 1 \big| f(F^{-1}(v)) < g(G^{-1}(v))\big\}.$$

If $\mathcal{CE}_n(X) \leq \mathcal{CE}_n(Y)$ for all $n \geq 1$, and $A_1 = \emptyset$ or $A_2 = \emptyset$ or $\inf A_1 \geq \sup A_2$, then

$$\mathcal{CE}_n(X_{m:m}) \leq \mathcal{CE}_n(Y_{m:m})$$

for all $n = 1, 2, \ldots$ and for any positive integer m.

Proof. If $A_1 = \emptyset$ or $A_2 = \emptyset$, the proof is obvious. Hence, we suppose that $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. First note that since m is a positive integer and u^{m-1} is nondecreasing, then it holds that

$$\inf_{A_1} u^{m-1} \geq \sup_{A_2} u^{m-1} \quad \Leftrightarrow \quad \inf A_1 \geq \sup A_2. \tag{29}$$

Since, for n positive integer, $\mathcal{CE}_n(X) \leq \mathcal{CE}_n(Y)$ by assumption, we have

$$\mathcal{CE}_n(Y) - \mathcal{CE}_n(X) = \frac{1}{n!} \int_0^1 u D(u) \, \mathrm{d}u \geq 0, \tag{30}$$

where

$$D(u) := (-\log u)^n \left[\frac{1}{g(G^{-1}(u))} - \frac{1}{f(F^{-1}(u))}\right].$$

We note from (8) that

$$n!\,\mathcal{CE}_n(X_{m:m}) = m^n \int_0^1 \frac{u^m(-\log u)^n}{f(F^{-1}(u))}\,\mathrm{d}u. \tag{31}$$

Hence, by assumption, $D(u) \geq 0$ for $u \in A_1$ and $D(u) < 0$ for $u \in A_2$. Therefore, due to (31) we have

$$\begin{aligned}
&\mathcal{CE}_n(Y_{m:m}) - \mathcal{CE}_n(X_{m:m}) \\
&\quad = \frac{m^n}{n!}\int_0^1 u^m D(u)\,\mathrm{d}u \\
&\quad = \frac{m^n}{n!}\int_{A_1} u^m D(u)\,\mathrm{d}u + \frac{m^n}{n!}\int_{A_2} u^m D(u)\,\mathrm{d}u \\
&\quad \geq \frac{m^n}{n!}\left(\inf_{A_1} u^{m-1}\right)\int_{A_1} uD(u)\,\mathrm{d}u + \frac{m^n}{n!}\left(\sup_{A_2} u^{m-1}\right)\int_{A_2} uD(u)\,\mathrm{d}u \\
&\quad \geq \frac{m^n}{n!}\left(\sup_{A_2} u^{m-1}\right)\int_{A_1} uD(u)\,\mathrm{d}u + \frac{m^n}{n!}\left(\sup_{A_2} u^{m-1}\right)\int_{A_2} uD(u)\,\mathrm{d}u \\
&\quad = \frac{m^n}{n!}\left(\sup_{A_2} u^{m-1}\right)\int_0^1 uD(u)\,\mathrm{d}u \geq 0.
\end{aligned}$$

The second inequality in (32) holds by condition $\inf A_1 \geq \sup A_2$, due to (29), while the last inequality is obtained from (30). □

4. DYNAMIC GENERALIZED CUMULATIVE ENTROPY

In this section, we provide some further results on the dynamic version of the generalized cumulative entropy. As specified in (9), the dynamic version of GCE for the inactivity time (1) for $t > 0$ is given by

$$\mathcal{CE}_n(t) = \mathcal{CE}_n(X;t) = \frac{1}{n!}\int_0^t \frac{F(x)}{F(t)}\,[T(x) - T(t)]^n\,\mathrm{d}x = \frac{1}{n!}\int_0^t \frac{F(x)}{F(t)}\left[-\log\frac{F(x)}{F(t)}\right]^n\mathrm{d}x,$$

for all $n = 1, 2, \ldots$. In analogy to Proposition 3.6, we obtain

$$\mathcal{CE}_n(t) \geq \frac{1}{n!}\,[\mathcal{CE}_1(t)]^n, \quad t > 0.$$

Also, Corollary 3.7 can be stated as

$$\mathcal{CE}_n(t) \geq \frac{1}{n!}C^n e^{nH(t)}, \quad t > 0.$$

Hereafter, we obtain some characterization results based on the GCE as well as the dynamic GCE. To this purpose, we first need the following theorem due to Kayal [20].

Theorem 4.1. If X is an absolutely continuous nonnegative random variable, then

$$\mathcal{CE}'_n(t) = \tau(t)[\mathcal{CE}_{n-1}(t) - \mathcal{CE}_n(t)], \quad t > 0, \tag{32}$$

for all $n = 1, 2, \ldots$.

Note that in (32) we use

$$\mathcal{CE}_0(t) = \frac{1}{F(t)} \int_0^t F(x)\,\mathrm{d}x = \tilde{\mu}(t),$$

which is due to (6). As a consequence of the preceding theorem, we have the forthcoming characterization result which extends the result given in Theorem 6.2 (i) of Di Crescenzo and Longobardi [14].

Theorem 4.2. Let X be a random variable with support $[0, b]$, where b is finite. Then $\mathcal{CE}_n(X;t) = c\,\mathcal{CE}_{n-1}(X;t)$ holds for all $0 \le t \le b,\ 0 < c < 1$ and for a fixed $n = 1, 2, \ldots$, if and only if X has the distribution function

$$F(t) = \left(\frac{t}{b}\right)^{c/(1-c)}, \quad 0 \le t \le b,\ 0 < c < 1.$$

P r o o f. The necessity is trivial and hence it remains to prove the sufficiency part. We shall prove it by induction. For $n = 1$, it was proved by Di Crescenzo and Longobardi [14]. Now, assuming that the result is true for $n - 1$ ($n > 1$), we shall prove it for n. Let

$$\mathcal{CE}_n(t) = c\,\mathcal{CE}_{n-1}(t), \quad 0 \le t \le b,\ 0 < c < 1. \tag{33}$$

Thus, we obtain

$$\mathcal{CE}'_n(t) = c\,\mathcal{CE}'_{n-1}(t).$$

From (32), we have

$$c\,\mathcal{CE}'_{n-1}(t) = \tau(t)[\mathcal{CE}_{n-1}(t) - \mathcal{CE}_n(t)],$$

and then (33) implies

$$c\,\mathcal{CE}'_{n-1}(t) = (1-c)\tau(t)\mathcal{CE}_{n-1}(t).$$

Similarly, Eq. (32) for $n - 1$ yields

$$c\,\mathcal{CE}'_{n-1}(t) = c\tau(t)[\mathcal{CE}_{n-2}(t) - \mathcal{CE}_{n-1}(t)].$$

Therefore, we get

$$\mathcal{CE}_{n-1}(t) = c\,\mathcal{CE}_{n-2}(t),$$

and hence by induction hypothesis, we get the desired result. □

Now, we investigate the monotonicity of the function $\mathcal{CE}_n(t)$ with respect to $t > 0$, for any n. First, we need the following definition.

Definition 4.3. The cumulative distribution function F is said to have increasing dynamic generalized cumulative entropy of order n, shortly written as IDGCE_n, if for all $n = 0, 1, 2, \ldots$, $\mathcal{CE}_n(t)$ is an increasing function of $t > 0$.

Kayal [20] pointed out that the generalized dynamic cumulative entropy cannot be decreasing in t. Notice that IDGCE_0 is equivalent to say that F has an increasing mean inactivity time (IMIT), that is the mean inactivity time $\tilde{\mu}(t)$ is increasing with respect to $t > 0$. In the forthcoming theorem, we shall prove that the class of IDGCE_{n-1} is included in IDGCE_n for all $n = 1, 2, \ldots$. The proof of the results are similar the proof of Lemma 3.1 and Theorem 3.1 in Navarro and Psarrakos [27] and hence we omit their proof.

Lemma 4.4. Let X be an absolutely continuous nonnegative random variable such that $\mathcal{CE}_n(0) < \infty$, for a fixed $n = 1, 2, \ldots$. Then

$$\mathcal{CE}_n(t) = \frac{1}{F(t)} \int_0^t \mathcal{CE}_{n-1}(x) f(x)\, \mathrm{d}x, \quad t > 0.$$

Hence, under the assumptions of Lemma 4.1, the dynamic GCE can be expressed as

$$\mathcal{CE}_n(t) = \mathbb{E}[\mathcal{CE}_{n-1}(X) \,|\, X \le t], \qquad t > 0.$$

Theorem 4.5. If F is IDGCE_n, then F is IDGCE_{n+1}.

It is worth to point out that Theorem 4.5 generalizes Proposition 4.4 of Navarro et al. [26]. In this case, if X is IMIT, then

$$\mathrm{IDGCE}_1 \;\Rightarrow\; \ldots \;\Rightarrow\; \mathrm{IDGCE}_n. \tag{34}$$

Navarro et al. [26] gave an example where a cumulative distribution function F is either IMIT or IDGCE_1. Moreover, they also gave an example that a cumulative distribution function F is IDGCE_1 but not IMIT. In the following example, we show that F is not IMIT but it is IDGCE_n for $n = 1, 2, \ldots$.

Example 4.6. Let X have a cumulative distribution function

$$F(x) = \begin{cases} \dfrac{x}{2} & 0 \le x \le 1, \\ \dfrac{1}{2} & 1 \le x \le 2, \\ \dfrac{x-1}{2} & 2 \le x \le 3. \end{cases}$$

A straightforward calculation shows that the dynamic generalized cumulative entropy is

$$\mathcal{CE}_n(t) = \begin{cases} \dfrac{t}{2^{n+1}} & 0 \le t \le 1, \\ \dfrac{1}{2^{n+1}} & 1 \le t \le 2, \\ \dfrac{t-1}{2^{n+1}} + \dfrac{(\log(t-1))^n}{n!(t-1)} & 2 \le t \le 3, \end{cases}$$

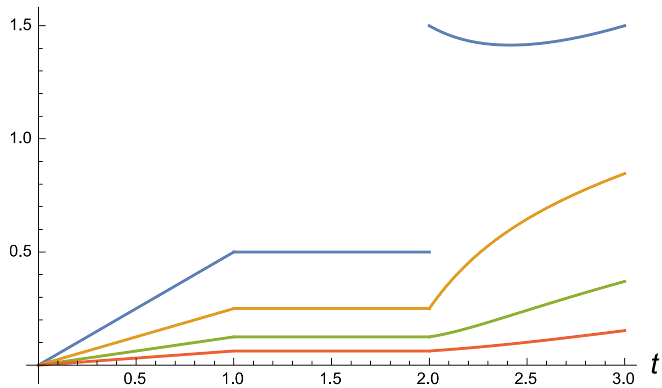

Fig. 1. The functions $\mathcal{CE}_n(t)$ of Example 4.6 for $n = 0, 1, 2, 3$ (from top to bottom).

for all $n = 0, 1, 2, \dots$. In Example 6.1 of Di Crescenzo and Longobardi [14], a special case for $\mathcal{CE}_0(t) = \tilde{\mu}(t)$ and $\mathcal{CE}_1(t)$ is given. The functions $\mathcal{CE}_n(t)$, are plotted in Figure 1 for $n = 0, 1, 2, 3$. It is seen that $\mathcal{CE}_0(t) > \mathcal{CE}_1(t) > \mathcal{CE}_2(t) > \mathcal{CE}_3(t)$, for all $t > 0$. Note that the functions $\mathcal{CE}_n(t)$ are increasing and continuous for all $n = 1, 2, \dots$, while $\mathcal{CE}_0(t)$ is not monotonic and is discontinuous at $t = 2$. As specified in (34), if F is IDGCE_1, then F is IDGCE_n for all $n = 2, 3, \dots$.

Now, we give an example where F is neither IMIT nor IDGCE_1, but it is included in the larger class IDGCE_n for all $n = 2, 3, \dots$.

Example 4.7. Let us suppose that a random variable X takes values in $[0, 1]$ with cumulative distribution function

$$F(x) = \begin{cases} \dfrac{1-p}{p} x & 0 \le x \le p, \\ \dfrac{p}{1-p} x + \dfrac{1-2p}{1-p} & p \le x \le 1, \end{cases}$$

where $0 < p < 1$. From (9), for $0 \le t \le p$, we have $\mathcal{CE}_n(t) = t/2^{n+1}$. However, it is not easy to compute the dynamic GCE when $p \le t \le 1$, and hence we proceed via numerical computations. The functions $\mathcal{CE}_n(t)$, $n = 0, 1, 2, 3, 4$ are displayed in Figure 2 for some values of t and for some choices of p. From Figure 2, it is seen that when $p = 0.7$, then $\mathcal{CE}_0(t)$ is not monotonic but $\mathcal{CE}_n(t)$ is increasing in t for all $n = 1, 2, \dots$, keeping in mind Eq. (34). Moreover, for $p = 0.8$, the functions $\mathcal{CE}_0(t)$ and $\mathcal{CE}_1(t)$ are not monotonic while $\mathcal{CE}_n(t)$ is increasing for all $n = 2, \dots$. In this case, we see that F is neither IMIT nor IDGCE_1, but it is included in the larger class IDGCE_n for all $n \in \{2, 3, \dots\}$. It is interesting to note that for $p = 0.9$, $\mathcal{CE}_n(t)$ is not monotonic for $n = 0, 1, 2$ however it is increasing for $n = 3, 4, \dots$ and finally F is not in the class IDGCE_n for $n = 0, 1, 2, 3$ when $p = 0.95$ while it is included in the larger class IDGCE_n for all $n \in \{4, 5, \dots\}$. According to the numerical findings, we expect that the dynamic

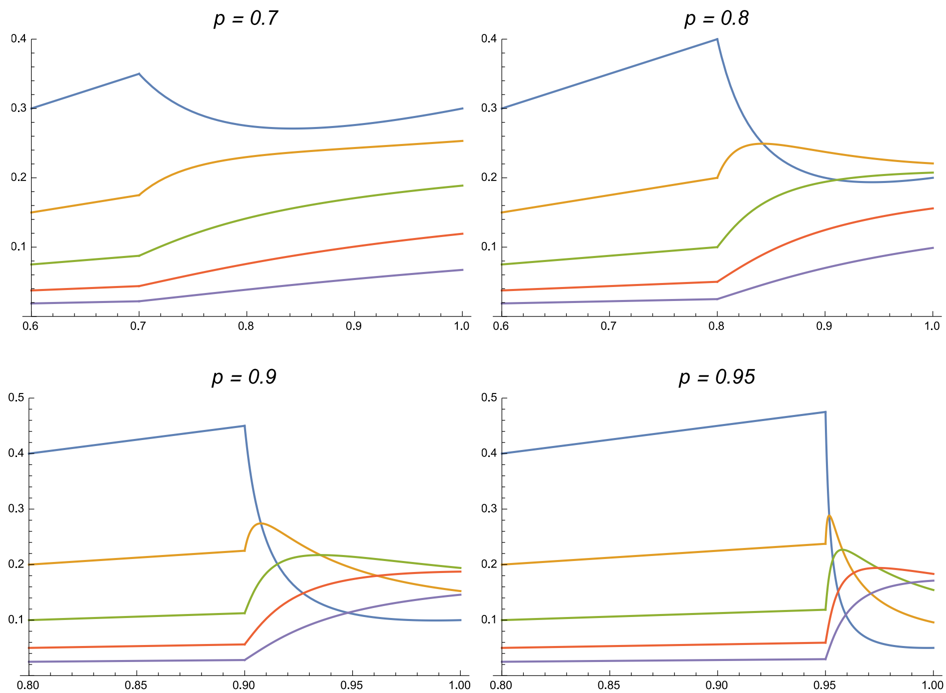

Fig. 2. Dynamic generalized cumulative entropy of Example 4.7 for $n = 0, 1, 2, 3, 4$ from top to bottom, near the origin for four choices of p.

generalized cumulative entropies $\mathcal{CE}_n(t)$, for some n, is not increasing for all t, when p is sufficiently large.

Theorem 4.8. Let X and Y be two nonnegative random variables with PDF's f and g and absolutely continuous CDF's F and G, respectively. Then F and G belong to the same family of distributions, but for a change in location and scale, if and only if

$$\mathcal{CE}_n(X_{m:m}; t) = \mathcal{CE}_n(Y_{m:m}; t),$$

for a fixed $n \geq 1$, for all $t \geq 0$, and for all $m \in \mathcal{M}$, where $\mathcal{M}$ is defined as in Theorem 3.12.

Proof. The necessity is trivial and hence it remains to prove the sufficiency part. If for a fixed $n \geq 1$ and for all $m \in \mathcal{M}$ we have $\mathcal{CE}_n(X_{m:m}; t) = \mathcal{CE}_n(Y_{m:m}; t)$, for all $t \geq 0$, then recalling Remark 1.1, by Theorem 3.13 we have that $[X|X \leq t]$ and $[Y|Y \leq t]$ have the same distribution but for a change in location parameter. Hence, we have $f_t(x) = g_t(x + d)$, $x > 0$, for all $t > 0$, where f_t and g_t are, respectively, the PDF's of $[X|X \leq t]$ and $[Y|Y \leq t]$. Thus, $f(x) = \frac{F(t)}{G(t)} g(x + d)$, $x > 0$, this meaning that F and G belong to the same family of distributions, but for a change in location and scale. $\square$

5. EMPIRICAL GENERALIZED CUMULATIVE ENTROPY

In this section, we provide some new results on the empirical GCE of a random variable defined by Kayal [20]. Let $X_1, X_2, \ldots, X_m$ be absolutely continuous nonnegative i.i.d. random variables, forming a random sample of size m taken from a population, which has CDF $F(x)$. We denote the empirical distribution of the sample by

$$\widehat{F}_m(x) = \sum_{i=1}^{m-1} \frac{i}{m} \mathbf{1}_{(X_{(i)}, X_{(i+1)}]}(x), \quad x \in \mathbb{R}, \tag{35}$$

where $X_{(1)} \le X_{(2)} \le \ldots \le X_{(m)}$ denote the associated order statistics of the sample, and $\mathbf{1}_A$ is the indicator function of A. According to (8) and (35), Kayal [20] defined the empirical generalized cumulative entropy for all fixed $n \ge 1$ as

$$\mathcal{CE}_n(\widehat{F}_m) = \frac{1}{n!} \sum_{i=1}^{m-1} U_{i+1} \frac{i}{m} \left[-\log \frac{i}{m} \right]^n, \tag{36}$$

where $U_{i+1} = X_{(i+1)} - X_{(i)}$, for $i = 1, 2, \ldots, m-1$, denote the sample spacings. The analogy between the empirical generalized cumulative entropy and the empirical cumulative entropy is straightforward. Indeed, for $n = 1$ the expression given in (36) identifies with Eq. (34) of [14].

Now, we focus our attention on the convergence of $\mathcal{CE}_n(\widehat{F}_m)$ as m tends to infinity, for all fixed $n \ge 1$. We recall that Glivenko–Cantelli theorem asserts that

$$\sup_{x \in \mathbb{R}} |\widehat{F}_m(x) - F(x)| \to 0 \quad \text{a.s. as } m \to \infty.$$

Example 5.1. Let $X_1, \ldots, X_m$ be a random sample taken from the uniform distribution in $[0, 1]$. For the uniform distribution, the sample spacing $U_{i+1} = X_{(i+1)} - X_{(i)}$ has the beta distribution with parameters 1 and m, i. e. $U_{i+1} \sim Beta(1, m)$ and hence $\mathbb{E}(U_{i+1}) = (m+1)^{-1}$. From (36), we obtain the mean and variance of the empirical generalized cumulative entropy:

$$\mathbb{E}[\mathcal{CE}_n(\widehat{F}_m)] = \frac{1}{n!(m+1)} \sum_{i=1}^{m-1} \frac{i}{m} \left[-\log \frac{i}{m} \right]^n, \tag{37}$$

$$Var[\mathcal{CE}_n(\widehat{F}_m)] = \frac{m}{n!(m+1)^2(m+2)} \sum_{i=1}^{m-1} \left(\frac{i}{m} \right)^2 \left[-\log \frac{i}{m} \right]^{2n}. \tag{38}$$

We notice that for $n = 1$, the mean (37) can be expressed in terms of the Riemann zeta function and the generalized Riemann zeta function (see Di Crescenzo and Longobardi [14] for further details). For all $n = 1, 2, \ldots$, from (37) and (38) it holds that

$$\lim_{m \to \infty} \mathbb{E}[\mathcal{CE}_n(\widehat{F}_m)] = \frac{1}{2^{n+1}},$$

and

$$\lim_{m \to \infty} Var[\mathcal{CE}_n(\widehat{F}_m)] = \frac{(2n)!}{n! 3^{2n+1}} \lim_{m \to \infty} \frac{m^2}{(m+1)^2(m+2)} = 0.$$

Hence, noting that $\mathcal{CE}_n(X_1) = \frac{1}{2^{n+1}}$, we immediately have that $\mathcal{CE}_n(\widehat{F}_m)$ is an unbiased and consistent estimator for the generalized cumulative entropy of a population uniformly distributed in $[0,1]$.

Example 5.2. Let $X_1, \dots, X_m$ be a random sample of exponentially distributed random variables with parameter λ. The sample spacing U_i, $1 \le i \le m$, are independent such that U_{i+1} has an exponential distribution with parameter $\lambda(m-i)$. Hence, from (36) we have

$$\mathbb{E}[\mathcal{CE}_n(\widehat{F}_m)] = \frac{1}{n!\lambda} \sum_{j=1}^{m-1} \frac{1}{m-j} \frac{j}{m} \left[-\log \frac{j}{m}\right]^n, \tag{39}$$

$$Var[\mathcal{CE}_n(\widehat{F}_m)] = \frac{1}{n!\lambda^2} \sum_{j=1}^{m-1} \frac{1}{m-j} \frac{j}{m} \left[-\log \frac{j}{m}\right]^{2n}. \tag{40}$$

We are now able to provide a central limit theorem for the empirical generalized cumulative entropy for random samples from the exponential distribution.

Theorem 5.3. If $X_1, X_2, \dots, X_n$ are i.i.d. random variables coming from the common exponential distribution, then for any fixed $n = 1, 2, \dots$,

$$Z_m = \frac{\mathcal{CE}_n(\widehat{F}_m) - \mathbb{E}[\mathcal{CE}_n(\widehat{F}_m)]}{\sqrt{Var[\mathcal{CE}_n(\widehat{F}_m)]}},$$

converges in distribution to the standard normal distribution as $m \to \infty$.

Proof. By (36) the empirical generalized cumulative entropy in this case can be expressed as the sum of independent exponential random variables W_i having mean

$$\mathbb{E}[W_i] = \frac{1}{m\lambda\left(\frac{1}{i/m} - 1\right)} \left[-\log \frac{i}{m}\right]^n.$$

Since $\mathbb{E}[|W_i - \mathbb{E}[W_i]|^3] = 2e^{-1}(6-e)[\mathbb{E}(W_i)]^3$ for any exponentially distributed random variable W_i (see Di Crescenzo and Longobardi [14]), we have

$$\sum_{i=1}^m Var(W_i) = \frac{1}{(m\lambda)^2} \sum_{i=1}^m \frac{1}{\left(\frac{1}{i/m} - 1\right)^2} \left[-\log \frac{i}{m}\right]^{2n} \approx \frac{c_{2,n}}{m\lambda^2},$$

$$\sum_{i=1}^m \mathbb{E}[|W_i - \mathbb{E}(W_i)|^3] = \frac{2(6-e)}{e(m\lambda)^3} \sum_{i=1}^m \frac{1}{\left(\frac{1}{i/m} - 1\right)^3} \left[-\log \frac{i}{m}\right]^{3n} \approx \frac{2(6-e)c_{3,n}}{em^2\lambda^3},$$

where, for $k = 2, 3$ and for a fixed $n = 1, 2, \dots$,

$$c_{k,n} = \int_0^1 \left(\frac{(-\log x)^n}{\frac{1}{x} - 1}\right)^k \mathrm{d}x < \infty.$$

Hence, Lyapunov's condition of the central limit theorem is satisfied, i. e.

$$\frac{\left(\sum_{i=1}^{m}\mathbb{E}[|W_i-\mathbb{E}(W_i)|^3]\right)^{1/3}}{\left(\sum_{i=1}^{m}Var(W_i)\right)^{1/2}} \approx \frac{[2(6-e)c_{3,n}]^{1/3}}{e^{1/3}c_{2,n}^{1/2}}m^{-1/6}\to 0 \qquad \text{as } m\to\infty.$$

The proof is thus completed. □

6. SUMMARY

In this paper, some further results of the GCE, introduced by Kayal [20], were discussed. First, we showed that the study of n-fold recursive reversed relevation transform is equivalent to the study of lower record values. Then, several ordering properties of n-fold recursive reversed relevation transform related to other concept of stochastic orders are presented. We also discussed some further results of the generalized cumulative entropy such as ordering properties, bounds, expressions and characterization results. Specifically, we investigated characterization based on the maximum of random variables. Similar results are obtained for the dynamic generalized cumulative entropy. The characterization problem from the cumulative entropy function was discussed in Di Crescenzo and Longobardi [13] and Navarro et al. [26]. Accordingly, in Theorem 4.2 we have analyzed the characterization problem for $\mathcal{CE}_n$, with $n = 1, 2, \dots$, which includes the cumulative entropy function as a particular case. Furthermore, we investigated some monotonicity properties of the GCE functions, and proved that the increasing dynamic GCE class of order n, denoted by IDGCE_n, is included in the IDGCE_{n+1} class. Finally, it was shown that for a random sample taken from the exponential distribution, the empirical generalized cumulative entropy converges to the normal distribution.

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