FURTHER RESULTS ON THE REVERSE ORDER LAW FOR GENERALIZED INVERSES

Dragan S. Djordjević

Abstract. The reverse order rule $(AB)^\dagger = B^\dagger A^\dagger$ for the Moore-Penrose inverse is established in several equivalent forms. Results related to other generalized inverses are also proved.

1. Introduction

Throughout this paper H, K, L denote arbitrary Hilbert spaces. We use $\mathcal{L}(H, K)$ to denote the set of all linear bounded operators from H to K. Also, $\mathcal{L}(H) = \mathcal{L}(H, H)$. For $A \in \mathcal{L}(H, K)$ we use $\mathcal{R}(A)$ to denote the range, and $\mathcal{N}(A)$ to denote the null-space of A. The Moore-Penrose inverse of A is denoted by $A^\dagger$. It is well-known that the Moore-Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed. We assume that the reader is familiar with the properties of the Moore-Penrose inverse (see, for example, [BIG], [C], [He], [K], [N], [NV]). We also assume that the following classes of operators are well-known: $A\{1\}$, $A\{1,3\}$, $A\{1,4\}$, $A\{1,2,3\}$, $A\{1,2,4\}$.

Some equivalent conditions of the reverse order rule

$$(AB)^\dagger = B^\dagger A^\dagger \tag{1}$$

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are well-known (see all references). We shall prove some new conditions, which are equivalent to (1). Also, conditions

$$B\{1,3\} \cdot A\{1,3\} \subset (BA)\{1,3\}$$
$$B\{1,4\} \cdot A\{1,3\} \subset (BA)\{1,4\}$$
$$B^\dagger A^\dagger \in (AB)\{1,2,3\}$$
$$B^\dagger A^\dagger \in (AB)\{1,2,4\}$$
$$B^\dagger A^\dagger \in (AB)\{1,3\}$$
$$B^\dagger A^\dagger \in (AB)\{1,4\}$$

will be investigated. By now, some of these conditions are investigated for complex matrices.

The aim of this paper is to prove some equivalence results for linear bounded Hilbert space operators, and thus obtain well-known results connected to the reverse order rule (1).

2. Results

We begin with the following auxiliary result, which can be found in [BIG] for complex matrices. For the completeness, we give its proof.

Lemma 2.1. *Let $A \in \mathcal{L}(H,K)$ have a closed range and $B \in \mathcal{L}(K,H)$. Then the following statements are equivalent:*

(1) *$ABA = A$ and $(AB)^* = AB$;*

(2) *there exists some $X \in \mathcal{L}(K,H)$, such that $B = A^\dagger + (I - A^\dagger A)X$.*

Proof. (2) $\Longrightarrow$ (1): Obvious.

(1) $\Longrightarrow$ (2): Since $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$, where A_1 is invertible, it follows that $A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. An elementary calculation shows that $B = \begin{bmatrix} A_1^{-1} & 0 \\ U & V \end{bmatrix}$, where U, V are arbitrary linear and bounded. Now, take $X = \begin{bmatrix} X_1 & X_2 \\ U & V \end{bmatrix}$, for arbitrary X_1, X_2 linear and bounded. $\square$

Now, we prove the main result of the paper.

Theorem 2.2. *Let $A \in \mathcal{L}(H,K)$ and $B \in \mathcal{L}(K,L)$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(1) $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$;
(2) $B\{1,3\} \cdot A\{1,3\} \subset (AB)\{1,3\}$;
(3) $B^\dagger A^\dagger \in (AB)\{1,3\}$;
(4) $B^\dagger A^\dagger \in (AB)\{1,2,3\}$.

Proof. The operator B has the following matrix form with respect to the orthogonal sum of subspaces: $B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}$, where B_1 is invertible. From the proof of Lemma 2.1 it follows that any $B^{(1,3)} \in B\{1,3\}$ has the form $\begin{bmatrix} B_1^{-1} & 0 \\ U & V \end{bmatrix}$. The operator A has the following form: $A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$. Now, $A^* = \begin{bmatrix} A_1^* & 0 \\ A_2^* & 0 \end{bmatrix}$ and $AA^* = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, where $D = A_1A_1^* + A_2A_2^*$ is positive and invertible in $\mathcal{L}(\mathcal{R}(A))$. We obtain $A^\dagger = A^*(AA^*)^\# = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}$. Let $A^{(1,3)} \in A\{1,3\}$. By Lemma 2.1 it follows that there exists some $X \in \mathcal{L}(L,K)$, such that $A^{(1,3)} = A^\dagger + (I - A^\dagger A)X$. Let X have the form $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}$. We get the following

$$A^{(1,3)} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

and

$$ABB^{(1,3)}A^{(1,3)} = \begin{bmatrix} A_1Z_{11} & A_1Z_{12} \\ 0 & 0 \end{bmatrix},$$

where

$$Z_{11} = A_1^*D^{-1} + (I - A_1^*D^{-1}A_1)X_{11} - A_1^*D^{-1}A_2X_{21},$$
$$Z_{12} = (I - A_1^*D^{-1}A_1)X_{12} - A_1^*D^{-1}A_2X_{22},$$
$$Z_{21} = A_2^*D^{-1} - A_2^*D^{-1}A_1X_{11} + (I - A_2^*D^{-1}A_2)X_{21},$$
$$Z_{22} = -A_2^*D^{-1}A_1X_{12} + (I - A_2^*D^{-1}A_2)X_{22}.$$

Notice also that $A^*AB = \begin{bmatrix} A_1^*A_1B_1 & 0 \\ A_2^*A_1B_1 & 0 \end{bmatrix}$.

(1) $\Longrightarrow$ (2): The inclusion $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ is equivalent to $BB^\dagger A^*AB = A^*AB$. Now, $BB^\dagger A^*AB = \begin{bmatrix} A_1^*A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $BB^\dagger A^*AB = A^*AB$ is

equivalent to $A_2^*A_1B_1 = 0$. Since B_1 is invertible, we obtain $A_2^*A_1 = 0$, or, equivalently, $A_1^*A_2 = 0$. It follows that $\mathcal{R}(A_2) \subset \mathcal{N}(A_1^*)$. We have the following orthogonal decomposition: $\mathcal{R}(A) = \overline{\mathcal{R}(A_1)} \oplus \mathcal{N}(A_1^*)$. Now,

$$\begin{aligned}\mathcal{R}(A) &= \left\{ \left[\begin{smallmatrix} A_1x+A_2y \\ 0 \end{smallmatrix}\right] : x \in \mathcal{R}(B), y \in \mathcal{N}(B^*) \right\} = \mathcal{R}(A_1) + \mathcal{R}(A_2) \\ &= \mathcal{R}(A_1) \oplus \mathcal{R}(A_2),\end{aligned}$$

knowing that $\mathcal{R}(A_2) \subset \mathcal{N}(A_1^*)$. Since $\mathcal{R}(A)$ is closed, we get that both $\mathcal{R}(A_1)$ and $\mathcal{R}(A_2)$ are closed. Consider the following decompositions of A_1 and A_2: $A_1 = \left[\begin{smallmatrix} A_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{R}(A_1^*) \\ \mathcal{N}(A_1) \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{R}(A_1) \\ \mathcal{N}(A_1^*) \end{smallmatrix}\right]$, where A_{11} is invertible, and $A_2 = \left[\begin{smallmatrix} 0 & 0 \\ A_{22} & 0 \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{R}(A_2^*) \\ \mathcal{N}(A_2) \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{R}(A_1) \\ \mathcal{N}(A_1^*) \end{smallmatrix}\right]$. We have the following: $0 < D = A_1A_1^* + A_2A_2^* = \left[\begin{smallmatrix} A_{11}A_{11}^* & 0 \\ 0 & A_{22}A_{22}^* \end{smallmatrix}\right]$, implying that both $A_{11}A_{11}^*$ and $A_{22}A_{22}^*$ are invertible. Hence, $D^{-1} = \left[\begin{smallmatrix} (A_{11}A_{11}^*)^{-1} & 0 \\ 0 & (A_{22}A_{22}^*)^{-1} \end{smallmatrix}\right]$. Notice that $A_1^*D^{-1}A_1 = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $A_1(I - A_1^*D^{-1}A_2) = 0$ and $A_1^*D^{-1}A_2 = 0$. Now, it follows that

$$A_1[(I - A_1^*D^{-1}A_1)X_{12} - A_1^*D^{-1}A_2X_{22}] = 0$$

and

$$A_1[A_1^*D^{-1} + (I - A_1^*D^{-1}A_1)X_{11} - A_1^*D^{-1}A_2X_{21}] = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

is selfadjoint. An elementary computation shows that $ABB^{(1,3)}A^{(1,3)}AB = AB$.

(2) $\Longrightarrow$ (3): Obvious.

(3) $\Longrightarrow$ (1): From the proof of the implication (1) $\Longrightarrow$ (2), it follows that the condition $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ is equivalent to $A_2^*A_1 = 0$. Now, $ABB^\dagger A^\dagger = \left[\begin{smallmatrix} A_1A_1^*D^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is selfadjoint, implying that $[A_1A_1^*, D^{-1}] = 0 = [A_1A_1^*, D]$ (here $[U,V] = UV - VU$). Also, $\left[\begin{smallmatrix} A_1B_1 & 0 \\ 0 & 0 \end{smallmatrix}\right] = AB = ABB^\dagger A^\dagger AB = \left[\begin{smallmatrix} A_1A_1^*D^{-1}B_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, implying that $A_1B_1 = A_1A_1^*D^{-1}A_1B_1 = D^{-1}A_1A_1^*A_1B_1$. Hence, we get $DA_1B_1 = A_1A_1^*A_1B_1$ and consequently $A_2A_2^*A_1B_1 = 0$. Since

B_1 is invertible, we obtain $A_2A_2^*A_1 = 0$ and $\mathcal{R}(A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$. It follows that $A_2^*A_1 = 0$.

(4) $\Longrightarrow$ (3): Obvious.

(1) $\Longrightarrow$ (4): If $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$, we have to prove that $B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$. Notice that $AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B^\dagger A^\dagger = \begin{bmatrix} B_1^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. Using previously proved facts: D commutes with $A_1A_1^*$ (the implication (3) $\Longrightarrow$ (1)) and matrix forms of A_1 and D (the implication (1) $\Longrightarrow$ (2)), we compute as follows:

$$\begin{aligned} B_1^{-1}A_1^*D^{-1}A_1B_1B_1^{-1}A_1^*D^{-1} &= B_1^{-1}A_1^*A_1A_1^*D^{-2} \\ &= B_1^{-1}A_1^* \begin{bmatrix} A_{11}A_{11}^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (A_{11}A_{11}^*)^{-2} & 0 \\ 0 & (A_{22}A_{22}^*)^{-2} \end{bmatrix} \\ &= B_1^{-1}A_1^* \begin{bmatrix} (A_{11}A_{11}^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = B_1^{-1}A_1^*D^{-1}. \end{aligned}$$

Now, it obviously follows that $B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$ is satisfied. $\square$

In the same manner we can prove the following result:

Theorem 2.3. *Let $A \in \mathcal{L}(H,K)$ and $B \in \mathcal{L}(K,L)$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(1) $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$*;*
(2) $B\{1,4\} \cdot A\{1,4\} \subset (AB)\{1,4\}$*;*
(3) $B^\dagger A^\dagger \in (AB)\{1,4\}$*;*
(4) $B^\dagger A^\dagger \in (AB)\{1,2,4\}$.

For complex matrices see the following literature: the equivalence (1) $\Longleftrightarrow$ (4) in both Theorem 2.2 and Theorem 2.3 is proved in [T2]; conditions (2) in both Theorem 2.2 and Theorem 2.3 are investigated in [WG].

Now, as a corollary, we obtain the following result.

Corollary 2.4. *Let $A \in \mathcal{L}(H,K)$ and $B \in \mathcal{L}(K,L)$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(1) $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$*;*

(2) $B\{1,3\} \cdot A\{1,3\} \subset AB\{1,3\}$ *and* $B\{1,4\} \cdot A\{1,4\} \subset AB\{1,4\}$*;*
(3) $B^\dagger A^\dagger \in AB\{1,3,4\}$*;*
(4) $B^\dagger A^\dagger = (AB)^\dagger$.

It is important to mention that the equivalence (1) $\Longleftrightarrow$ (4) is a classical result, proved for complex matrices in [G], and for bounded operators on Hilbert spaces in [B1], [B2] and [I].

Remark 2.5. The equivalence (3) $\Longleftrightarrow$ (4) in Theorem 2.2, Theorem 2.3 and Corollary 2.4, suggests that the "$\{2\}$ - property" is implied by the rest. For matrices, this follows from a rank argument. If X is a $\{1\}$-inverse of A, then X is also a $\{2\}$-inverse if and only if $\operatorname{rank} X = \operatorname{rank} A$. Since we can not talk about "rank" here, we resolve this situation using the special partition of operators.

Results which are related to the reverse order rule for generalized inverses follow. Multiple matrix products are considered in [Hw] and [T1]. General condition to the reverse order rule for inner inverses are given in [W2] and for outer inverses in [D]. The reverse order rule for the weighted Moore-Penrose inverse is investigated in [SW].

Finally, we find that results of this paper are closely connected with the results of H. J. Werner [W1]. Although in [W1] the finite dimensional technique is used, the results which will be presented here, are valid in arbitrary Hilbert spaces also.

In [W1] the geometric approach is involved, taking the range and the null space of the generalized inverses. Among other things, the following result is proved in [W1, Theorem 5.5] (interpreted in an infinite dimensional settings).

Theorem 2.6. *Let* $B \in \mathcal{L}(H,K)$ *and* $A \in \mathcal{L}(K,L)$*, such that* A*,* B *and* $C = AB$ *have closed ranges. Let* T *be a closed subspace of* H *such that* $T \stackrel{\bullet}{+} \mathcal{N}(B) = H$ *(the sum is not necessarily orthogonal) and* $\mathcal{R}(C^*) \subset T$*.*

Then the following statements are equivalent:

(1) *There exist some operators* A^- *and* B^- *satisfying:* $AA^-A = A$, $A^-A = P_{\mathcal{R}(A^*),\mathcal{N}(A)}$, $BB^-B = B$, $B^-B = P_{T,\mathcal{N}(B)}$, $BB^- = P_{\mathcal{R}(B),\mathcal{N}(B^*)}$, *such that the following is satisfied:* $D = B^-A^-$, $CDC = C$ *and* $DC = P_{\mathcal{R}(C^*),\mathcal{N}(C)}$.

(2) $\mathcal{R}(BB^*A) \subset \mathcal{R}(A^*)$;

(3) *For each operators* A^- *and* B^- *satisfying:* $AA^-A = A$, $A^-A = P_{\mathcal{R}(A^*),\mathcal{N}(A)}$, $BB^-B = B$, $B^-B = P_{T,\mathcal{N}(B)}$, $BB^- = P_{\mathcal{R}(B),\mathcal{N}(B^*)}$, *the following holds:* $D = B^-A^-$, $CDC = C$ *and* $DC = P_{\mathcal{R}(C^*),\mathcal{N}(C)}$.

We see that for $C = AB$ the condition $\mathcal{R}(C^*) \subset \mathcal{R}(B^*)$ holds. Hence, for $T = \mathcal{R}(B^*)$ we get the result closely related to our Theorem 2.3. Now, the corollary is stated according to our notations.

Corollary 2.7. *Let* $B \in \mathcal{L}(H,K)$ *and* $A \in \mathcal{L}(K,L)$, *such that* A, B *and* $C = AB$ *have closed ranges. Then the following statements are equivalent:*

(1) *There exist some* $A^- \in A\{1,4\}$ *and some* $B^- \in B\{1,3,4\}$ *such that* $B^-A^- \in C\{1,4\}$.

(2) $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$;

(3) $A\{1,4\} \cdot B\{1,3,4\} \subset C\{1,4\}$.

We see that Corollary 2.7 contains a weaker result than our Theorem 2.3.

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References

[BIG] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd Edition, Springer, New York, 2003.

[B1] R. H. Bouldin, *The pseudo-inverse of a product*, SIAM J. Appl. Math.**25** (1973), 489–495.

[B2] R. H. Bouldin, *Generalized inverses and factorizations*, Recent applications of generalized inverses, Pitman Ser. Res. Notes in Math. No. 66 (1982), 233–248.

[C] S. R. Caradus, *Generalized Inverses and Operator Theory*, Queen's Papers in Pure and Applied Mathematics, 50, Queen's University, Kingston, 1978.

[D] D. S. Djordjević, *Unified approach to the reverse order rule for generalized inverses*, Acta Sci. Math. (Szeged) **l67** (2001), 761–776.

[G] T. N. E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev. **8** (1966), 518–521.

[He] R. E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, New York, Marcel Dekker, 1988.

[Hw] R. E. Hartwig, *The reverse order law revisited*, Linear Algebra Appl. **76** (1986), 241–246.

[I] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. **34** (1982), 43–52.

[K] J. J. Koliha, *The Drazin and Moore-Penrose inverse in C^*-algebras*, Math. Proc. R. Ir. Acad. **99A** (1999), 17-27.

[N] M. Z. Nashed, *Inner, outer and generalized inverses in Banach and Hilbert spaces*, Numer. Funct. Anal. Optim. **9** (1987), 261–325.

[NV] M. Z. Nashed and G. F. Votruba, *A unified operator theory of generalized inverses*, Generalized Inverses and Applications, (M. Z. Nashed, ed.), Academic Press, New York, 1976, 1–109.

[SW] W. Sun and Y. Wei, *Inverse order rule for weighted generalized inverse*, SIAM J. Matrix Anal. Appl. **19** (1998), 772–775.

[T1] Y. Tian, *Reverse order laws for the generalized inverses of multiple matrix products. Generalized inverses*, Linear Algebra Appl. **211** (1994), 85–100.

[T2] Y. Tian, *Using rank formulas to characterize equalities for Moore-Penrose inverses of matrix products*, Appl. Math. Comput. **147** (2004), 581–600.

[WG] M. Wei and W. Guo, *Reverse order laws for least squares g-inverses and minimum norm g-inverses of products of two matrices*, Linear Algebra Appl. **342** (2002), 117–132.

[W1] H. J. Werner, *G-inverses of matrix products*, Data Analysis and Statistical Inference, Ed. S. Schach and G. Trenkler, Verlag Josef Eul, 1992, pp. 531–546.

[W2] H. J. Werner, *When is B^-A^- a generalized inverse of AB?*, Linear Algebra Appl. **210** (1994), 255–263.

Department of Mathematics and Informatics
Faculty of Sciences and Mathematics
University of Niš
P. O. Box 224
18000 Niš
Serbia

E-mail: `dragan@pmf.ni.ac.yu` `ganedj@EUnet.yu`