FURTHER RESULTS RELATED TO A MINIMAX PROBLEM OF RICCERI

GIUSEPPE CORDARO

Received 23 December 2003

We deal with a theoric question raised in connection with the application of a three-critical points theorem, obtained by Ricceri, which has been already applied to obtain multiplicity results for boundary value problems in several recent papers. In the settings of the mentioned theorem, the typical assumption is that the following minimax inequality $\sup_{\lambda\in I}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda)) < \inf_{x\in X}\sup_{\lambda\in I}(\Phi(x)+\lambda\Psi(x)+h(\lambda))$ has to be satisfied by some continuous and concave function $h: I\to\mathbb{R}$. When $I=[0,+\infty[$, we have already proved, in a precedent paper, that the problem of finding such function h is equivalent to looking for a linear one. Here, we consider the question for any interval I and prove that the same conclusion holds. It is worth noticing that our main result implicitly gives the most general conditions under which the minimax inequality occurs for some linear function. We finally want to stress out that although we employ some ideas similar to the ones developed for the case where $I=[0,+\infty[$, a key technical lemma needs different methods to be proved, since the approach used for that particular case does not work for upper-bounded intervals.

1. Introduction

Here and throughout the sequel, E is a real separable and reflexive Banach space, X is a weakly closed unbounded subset of E, $I\subseteq\mathbb{R}$ an interval and Φ, Ψ are two (nonconstant) sequentially weakly lower semicontinuous functionals on X such that

$$\lim_{x\in X,\,\|x\|\to+\infty}(\Phi(x)+\lambda\Psi(x)) = +\infty \tag{1.1}$$

for all $\lambda\in I$.

In these settings, Ricceri showed that if there exists a continuous concave function $h: I\to\mathbb{R}$ such that

$$\sup_{\lambda\in I}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda)) < \inf_{x\in X}\sup_{\lambda\in I}(\Phi(x)+\lambda\Psi(x)+h(\lambda)), \tag{1.2}$$

then there is an open interval $J\subseteq I$ such that, for each $\lambda\in J$, the functional $\Phi+\lambda\Psi$ has

Journal of Inequalities and Applications 2005:5 (2005) 523–533
DOI: 10.1155/JIA.2005.523

a local nonabsolute minimum in the relative weak topology of X [11, 12, 13]. Under further assumptions, this fact leads to a three critical points theorem (see [13, Theorem 1] improving [12, Theorem 3.1]) which has been widely applied to get multiplicity results for nonlinear boundary value problems [1, 2, 3, 5, 6, 8, 9, 10, 11, 12, 13].

A natural way to get (1.2) is by a linear function. In view of applications of [12, Theorem 3.1], Ricceri gave useful conditions (see [12, Proposition 3.1]) under which (1.2), with $I = [0,+\infty[$, is satisfied by some linear function (see also [4]). In the same paper, Remark 5.2, Ricceri asked if (1.2) could be satisfied by a suitable continuous concave function also when this does not happen for linear ones. A complete and negative answer was given in [7, Theorem 1] but when the interval I is $[0,+\infty[$.

It is still an open and nontrivial problem if the same conclusion would hold for any interval $I \subseteq \mathbb{R}$. In this paper, an answer to this question is given.

Before our main result is stated, some notations are needed to be fixed. Let $\alpha \in \mathbb{R}$ and $\rho \in]-\sup_X \Psi, -\inf_X \Psi[$, we set

$$\begin{aligned} a(\rho,\alpha) &= \inf_{x\in\Psi^{-1}(]-\infty,-\rho])} \big(\Phi(x)+\alpha\big(\Psi(x)+\rho\big)\big), \\ b(\rho,\alpha) &= \inf_{x\in\Psi^{-1}(]-\rho,+\infty[)} \big(\Phi(x)+\alpha\big(\Psi(x)+\rho\big)\big). \end{aligned} \tag{1.3}$$

Moreover we put $a(\rho,-\infty) = \inf_{x\in\Psi^{-1}(-\rho)} \Phi(x)$ and $b(\rho,+\infty) = +\infty$. As usual, by definition, we put $\inf \emptyset = +\infty$.

Theorem 1.1. *Let $\alpha = \inf I$ and $\beta = \sup I$. Under the assumptions given above, the following assertions are equivalent:*

(i) *for each $\rho \in \mathbb{R}$, one has*

$$\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}} \inf_{x\in X} \big(\Phi(x)+\lambda\big(\Psi(x)+\rho\big)\big) = \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}} \big(\Phi(x)+\lambda\big(\Psi(x)+\rho\big)\big); \tag{1.4}$$

(ii) *for each $\rho \in]-\sup_X \Psi, -\inf_X \Psi[$, one has*

$$\begin{aligned} &\sup_{x\in\Psi^{-1}(]-\infty,-\rho[)} \frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)} \\ &\qquad \le \inf_{x\in\Psi^{-1}(]-\rho,+\infty[)} \frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)}; \end{aligned} \tag{1.5}$$

(iii) *for every concave function $h : I \to \mathbb{R}$ which is continuous in $I \setminus \{\alpha\}$, one has*

$$\sup_{\lambda\in I} \inf_{x\in X} \big(\Phi(x)+\lambda\Psi(x)+h(\lambda)\big) = \inf_{x\in X} \sup_{\lambda\in I} \big(\Phi(x)+\lambda\Psi(x)+h(\lambda)\big). \tag{1.6}$$

We want to stress out that the proof of Theorem 1.1 is not a straightforward consequence of the ideas developed in [7]. In fact, the proof of [7, Lemma 3] does not work when the interval I is upper-bounded. For this reason, different arguments are needed in order to prove Lemma 2.4 which is a key technical preliminary result.

2. Preliminary results

It is easily seen that $a(\rho,\cdot)$ is decreasingly monotone in $\mathbb{R}\cup\{-\infty\}$ and $b(\rho,\cdot)$ is increasingly monotone in $\mathbb{R}\cup\{+\infty\}$.

Theorem 2.1. *Let $\alpha,\beta\in\mathbb{R}\cup\{-\infty,+\infty\}$, with $\alpha<\beta$, and $\rho\in]-\sup_X\Psi,-\inf_X\Psi[$. The following assertions are equivalent:*

(i′) *one has*

$$\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}}\inf_{x\in X}\big(\Phi(x)+\lambda(\Psi(x)+\rho)\big)=\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}}\big(\Phi(x)+\lambda(\Psi(x)+\rho)\big);\tag{2.1}$$

(ii′) *one has*

$$\begin{aligned}&\sup_{x\in\Psi^{-1}(]-\infty,-\rho[)}\frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)}\\&\quad\le\inf_{x\in\Psi^{-1}(]-\rho,+\infty[)}\frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)}.\end{aligned}\tag{2.2}$$

Proof. First of all, we observe that

$$\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}}\big(\Phi(x)+\lambda(\Psi(x)+\rho)\big)=\min\{a(\rho,\alpha),b(\rho,\beta)\}.\tag{2.3}$$

(i′)⇒(ii′). Since

$$\lim_{|\lambda|\to+\infty}\inf_{x\in X}\big(\Phi(x)+\lambda(\Psi(x)+\rho)\big)=-\infty\tag{2.4}$$

and the upper semicontinuity of the function

$$\lambda\in[\alpha,\beta]\cap\mathbb{R}\longrightarrow\inf_{x\in X}\big(\Phi(x)+\lambda(\Psi(x)+\rho)\big),\tag{2.5}$$

there exists $\bar\lambda\in[\alpha,\beta]\cap\mathbb{R}$ such that

$$\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}}\inf_{x\in X}\big(\Phi(x)+\lambda(\Psi(x)+\rho)\big)=\inf_{x\in X}\big(\Phi(x)+\bar\lambda(\Psi(x)+\rho)\big).\tag{2.6}$$

Hence, by hypothesis (i′) and (2.3), it follows that

$$\inf_{x\in X}\big(\Phi(x)+\bar\lambda(\Psi(x)+\rho)\big)=\min\{a(\rho,\alpha),b(\rho,\beta)\}.\tag{2.7}$$

From (2.7), one has

$$\begin{aligned}&\sup_{x\in\Psi^{-1}(]-\infty,-\rho[)}\frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)}\\&\quad\le-\bar\lambda\le\inf_{x\in\Psi^{-1}(]-\rho,+\infty[)}\frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)}.\end{aligned}\tag{2.8}$$

(ii′)⇒(i′). Hypothesis (ii′) implies that $\min\{a(\rho,\alpha),b(\rho,\beta)\} \in \mathbb{R}$. Moreover we have that

$$\sup_{x\in\Psi^{-1}(]-\infty,-\rho[)} \frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)} \leq -\alpha, \tag{2.9}$$

$$\inf_{x\in\Psi^{-1}(]-\rho,+\infty[)} \frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)} \geq -\beta. \tag{2.10}$$

In fact, when $\alpha = -\infty$ (2.9) is true as strict inequality because of (ii′). So is (2.10) when $\beta = +\infty$. In the other cases, if (2.9) were not true then there would exist $\overline{x} \in \Psi^{-1}(]-\infty,-\rho[)$ such that

$$\Phi(\overline{x})+\alpha\big(\Psi(\overline{x})+\rho\big) < \min\{a(\rho,\alpha),b(\rho,\beta)\}, \tag{2.11}$$

that is absurd. Inequality (2.10) can be proved in analogous way.

By (2.9) and (2.10), which we have seen to be satisfied as strict inequalities when $\alpha = -\infty$ or $\beta = +\infty$, we can choose $\overline{\lambda} \in [\alpha,\beta] \cap \mathbb{R}$ such that

$$\begin{aligned} &\sup_{x\in\Psi^{-1}(]-\infty,-\rho[)} \frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)} \\ &\qquad \leq -\overline{\lambda} \leq \inf_{x\in\Psi^{-1}(]-\rho,+\infty[)} \frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)}. \end{aligned} \tag{2.12}$$

Hence one has

$$\inf_{x\in X}\big(\Phi(x)+\overline{\lambda}\big(\Psi(x)+\rho\big)\big) \geq \min\{a(\rho,\alpha),b(\rho,\beta)\}, \tag{2.13}$$

which, by (2.3), implies (i′). □

Corollary 2.2. *Let $\alpha,\beta \in \mathbb{R}\cup\{-\infty,+\infty\}$, with $\alpha<\beta$. The following assertions are equivalent:*

(a) *for every $\rho \in \mathbb{R}$, one has*

$$\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}}\ \inf_{x\in X}\big(\Phi(x)+\lambda\big(\Psi(x)+\rho\big)\big) = \inf_{x\in X}\ \sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}}\big(\Phi(x)+\lambda\big(\Psi(x)+\rho\big)\big); \tag{2.14}$$

(b) *for every $\rho \in]-\sup_X \Psi, -\inf_X \Psi[$, one has*

$$\begin{aligned} &\sup_{x\in\Psi^{-1}(]-\infty,-\rho[)} \frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)} \\ &\qquad \leq \inf_{x\in\Psi^{-1}(]-\rho,+\infty[)} \frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)}. \end{aligned} \tag{2.15}$$

Proof. (a)⇒(b). It directly follows from Theorem 2.1.

(b)⇒(a). It is enough to show that if $\rho \in \mathbb{R}\setminus]-\sup_X \Psi, -\inf_X \Psi[$, then

$$\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}} \inf_{x\in X} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho\right)\right) = \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho\right)\right). \tag{2.16}$$

Assume that $\rho \in \mathbb{R}$ is such that $\rho \le -\sup_X \Psi$. In this case, we have that

$$\inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho\right)\right) = \begin{cases} \inf_{x\in X} \left(\Phi(x)+\alpha\left(\Psi(x)+\rho\right)\right), & \alpha \in \mathbb{R}, \\ \inf_{x\in\Psi^{-1}(-\rho)} \Phi(x), & \alpha = -\infty. \end{cases} \tag{2.17}$$

Hence, it is clear that (2.16) holds when $\alpha \in \mathbb{R}$. When $\alpha = -\infty$ we proceed by contradiction. So we suppose that there exists $\gamma \in \mathbb{R}$ such that

$$\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}} \inf_{x\in X} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho\right) < \gamma < \inf_{\Psi^{1}(-\rho)} \Phi. \tag{2.18}$$

Then, there exist two sequences $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_-$, with $\lim_{n\to\infty}\lambda_n = -\infty$, and $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that, for every $n \in \mathbb{N}$, one has

$$\inf_{x\in X} \left(\Phi(x)+\lambda_n\left(\Psi(x)+\rho\right)\right) = \Phi(x_n)+\lambda_n\left(\Psi(x_n)+\rho\right)\right) < \gamma. \tag{2.19}$$

Being $\Psi(x_n) \le -\rho$, it results that

$$\Phi(x_n) < \gamma. \tag{2.20}$$

Taking into account the coerciveness of Φ, it follows that $\{x_n\}$ is bounded. By hypothesis E is a reflexive Banach space and X is weakly closed then there exist $x^* \in X$ and a subsequence $\{x_{n_k}\}$ weakly convergent to x^*. By (2.19) and $\lim_{n\to\infty}\lambda_n = -\infty$, it follows that $\Psi(x^*) = -\rho$. This is absurd if $\rho < -\sup_X \Psi$. If $\rho = -\sup_X \Psi$, we exploit the sequentially weakly lower semicontinuity and (2.20) to obtain the absurd $\Phi(x^*) \le \gamma < \inf_{\Psi^{-1}(-\rho)} \Phi$. So (2.16) holds.

By similar arguments, (2.16) can be proved when $\rho \ge -\inf_X \Psi$. □

Corollary 2.3. *Let $\alpha,\beta \in \mathbb{R}\cup\{-\infty,+\infty\}$, with $\alpha<\beta$, and $\rho\in\mathbb{R}$. Assume that*

$$\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}} \inf_{x\in X} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho\right)\right) = \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho\right)\right). \tag{2.21}$$

Then, for every $\gamma,\delta \in]\alpha,\beta[$ with $\gamma<\delta$, it results that

$$\sup_{\lambda\in[\gamma,\delta]\cap\mathbb{R}} \inf_{x\in X} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho\right)\right) = \inf_{x\in X} \sup_{\lambda\in[\gamma,\delta]\cap\mathbb{R}} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho\right)\right). \tag{2.22}$$

Proof. Our end is to apply Corollary 2.2. So, supposing $-\sup_X \Psi < \rho < -\inf_X \Psi$, we have to prove that

$$\begin{aligned}\sup_{x\in\Psi^{-1}(]-\infty,-\rho[)} & \frac{\Phi(x)-\min\{a(\rho,\gamma),b(\rho,\delta)\}}{\rho+\Psi(x)} \\ & \leq \inf_{x\in\Psi^{-1}(]-\rho,+\infty[)} \frac{\Phi(x)-\min\{a(\rho,\gamma),b(\rho,\delta)\}}{\rho+\Psi(x)}.\end{aligned} \tag{2.23}$$

By hypothesis and Corollary 2.2, it follows that

$$\begin{aligned}\sup_{x\in\Psi^{-1}(]-\infty,-\rho[)} & \frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)} \\ & \leq \inf_{x\in\Psi^{-1}(]-\rho,+\infty[)} \frac{\Phi(x)-\min\{a(\rho,\alpha),b(\rho,\beta)\}}{\rho+\Psi(x)}.\end{aligned} \tag{2.24}$$

Then (2.23) follows from $\min\{a(\rho,\gamma),b(\rho,\delta)\} \leq \min\{a(\rho,\alpha),b(\rho,\beta)\}$. □

Lemma 2.4. *Let $\alpha,\beta \in \mathbb{R}$, with $\alpha < \beta$, and suppose that*

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho\right)\right) = \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho\right)\right), \tag{2.25}$$

for every $\rho \in \mathbb{R}$. Consider a subdivision $\alpha = \alpha_1 < \alpha_2 < \cdots \alpha_n = \beta$ of the interval $[\alpha,\beta]$ with $n \geq 3$. Define the function $h : [\alpha,\beta] \to \mathbb{R}$,

$$h(\lambda) = \sum_{i=1}^{n-1} \chi_{[\alpha_i,\alpha_{i+1}]}(\lambda)\left(\rho_i\lambda + a_i\right) \quad \text{for each } \lambda \in [\alpha,\beta], \tag{2.26}$$

where $\{\rho_k\}_{1\leq k\leq n-1}$ is a nonincreasing finite sequence of real numbers and $a_{i+1} = a_i + (\rho_i - \rho_{i+1})\alpha_{i+1}$, for $1 \leq i \leq n-2$, with $a_1 \in \mathbb{R}$ arbitrarily chosen.

Then one has

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X} \left(\Phi(x)+\lambda\Psi(x)+h(\lambda)\right) = \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]} \left(\Phi(x)+\lambda\Psi(x)+h(\lambda)\right). \tag{2.27}$$

Proof. The proof is similar to that of [7, Lemma 3]. So, here we omit some passages that can be find in the cited article. The proof is divided into four steps. We prove only the first step and refer to [7] for the others.

By Corollary 2.3, we have

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X} \left(\Phi(x)+\lambda\Psi(x)+h(\lambda)\right) = \max_{1\leq i\leq n-1} \inf_{x\in X} \sup_{\lambda\in[\alpha_i,\alpha_{i+1}]} \left(\Phi(x)+\lambda\left(\Psi(x)+\rho_i\right)+a_i\right). \tag{2.28}$$

For convenience, denote

$$f_i(x,\lambda) = \Phi(x)+\lambda\left(\Psi(x)+\rho_i\right)+a_i, \tag{2.29}$$

for $1 \leq i \leq n-1$.

The first step. We prove the thesis when $\inf_X \Psi < -\rho_i < \sup_X \Psi$, for every $1 \le i \le n-1$.

Put, for $1 \le i \le n-2$,

$$\delta_i = \inf_{x \in \Psi^{-1}(]-\rho_i, -\rho_{i+1}])} f_i(x, \alpha_{i+1}). \tag{2.30}$$

In [7], it was proved that

$$\begin{aligned} &\inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) \\ &\quad = \min\left\{ \min_{1 \le i \le n-2} \delta_i, \inf_{x \in \Psi^{-1}(]-\infty, -\rho_1])} f_1(x, \alpha), \inf_{x \in \Psi^{-1}(]-\rho_{n-1}, +\infty[)} f_{n-1}(x, \beta) \right\}. \end{aligned} \tag{2.31}$$

By induction on $n \in \mathbb{N}$, we first prove the following inequalities:

$$\begin{aligned} &\inf_{x \in \Psi^{-1}(]-\infty, -\rho_{n-1}])} f_{n-1}(x, \alpha_1) \\ &\qquad \ge \min\left\{ \inf_{x \in \Psi^{-1}(]-\infty, -\rho_1])} f_1(x, \alpha_1), \min_{1 \le i \le n-2} \delta_i \right\} &\text{(2.32a)} \\ &\qquad \ge \inf_{x \in \Psi^{-1}(]-\infty, -\rho_{n-1}])} f_{n-1}(x, \alpha_{n-1}). &\text{(2.32b)} \end{aligned}$$

By definition of f_i and a_i, it is easily seen that

$$f_{i+1}(x, \alpha_{i+1}) = f_i(x, \alpha_{i+1}) \tag{2.33}$$

and, for $1 \le k \le n-1-i$,

$$f_i(x, \alpha_i) \le f_{i+k}(x, \alpha_i). \tag{2.34}$$

Let $n = 3$, one has

$$\begin{aligned} \inf_{x \in \Psi^{-1}(]-\infty, -\rho_2])} f_2(x, \alpha_2) &= \inf_{x \in \Psi^{-1}(]-\infty, -\rho_2])} f_1(x, \alpha_2) \\ &= \min\left\{ \inf_{x \in \Psi^{-1}(]-\infty, -\rho_1])} f_1(x, \alpha_2), \delta_1 \right\} \\ &\le \min\left\{ \inf_{x \in \Psi^{-1}(]-\infty, -\rho_1])} f_1(x, \alpha_1), \delta_1 \right\}. \end{aligned} \tag{2.35}$$

The last inequality follows from $a(\rho_1, \alpha_2) \le a(\rho_1, \alpha_1)$. Moreover,

$$\begin{aligned} &\min\left\{ \inf_{x \in \Psi^{-1}(]-\infty, -\rho_1])} f_1(x, \alpha_1), \delta_1 \right\} \\ &\quad \le \min\left\{ \inf_{x \in \Psi^{-1}(]-\infty, -\rho_1])} f_2(x, \alpha_1), \inf_{x \in \Psi^{-1}(]-\rho_1, -\rho_2])} f_2(x, \alpha_1) \right\} \\ &\quad = \inf_{x \in \Psi^{-1}(]-\infty, -\rho_2])} f_2(x, \alpha_1). \end{aligned} \tag{2.36}$$

Now suppose that (2.32a) and (2.32b) hold for $n = k-1$. Let $n = k$.
We have

$$
\begin{aligned}
\inf_{x\in\Psi^{-1}(]-\infty,-\rho_{k-1}])} f_{k-1}(x,\alpha_{k-1}) &= \inf_{x\in\Psi^{-1}(]-\infty,-\rho_{k-1}])} f_{k-2}(x,\alpha_{k-1}) \\
&= \min\Big\{\delta_{k-2}, \inf_{x\in\Psi^{-1}(]-\infty,-\rho_{k-2}])} f_{k-2}(x,\alpha_{k-1})\Big\} \\
&\leq \text{ being } a(\rho_{k-2},\alpha_{k-1}) \leq a(\rho_{k-2},\alpha_{k-2}),
\end{aligned}
\tag{2.37}
$$

$$
\begin{aligned}
&\min\Big\{\delta_{k-2}, \inf_{x\in\Psi^{-1}(]-\infty,-\rho_{k-2}])} f_{k-2}(x,\alpha_{k-2})\Big\} \\
&\qquad \leq \min\Big\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_1])} f_1(x,\alpha_1), \min_{1\leq i\leq k-2}\delta_i\Big\} \quad \text{(by hypothesis)}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\inf_{x\in\Psi^{-1}(]-\infty,-\rho_{k-1}])} f_{k-1}(x,\alpha_1) \\
&\qquad \geq (\text{by } (2.34)) \\
&\qquad \geq \min\Big\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_{k-2}])} f_{k-2}(x,\alpha_1), \inf_{x\in\Psi^{-1}(]-\rho_{k-2},-\rho_{k-1}])} f_{k-1}(x,\alpha_1)\Big\}.
\end{aligned}
\tag{2.38}
$$

Inequality (2.32a) follows from (2.38), the inductive hypothesis and the fact that

$$
\inf_{x\in\Psi^{-1}(]-\rho_{k-2},-\rho_{k-1}])} f_{k-1}(x,\alpha_1) \geq \delta_{k-2}. \tag{2.39}
$$

We still proceed by induction to prove (2.27).
Let $n = 3$. Then, owing to (2.28), one has

$$
\begin{aligned}
&\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda)) \\
&\quad = \max\Big\{\min\Big\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_1])} f_1(x,\alpha_1), \inf_{x\in\Psi^{-1}(]-\rho_1,+\infty[)} f_1(x,\alpha_2)\Big\}, \\
&\qquad\qquad \min\Big\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_2])} f_2(x,\alpha_2), \inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_2(x,\alpha_3)\Big\}\Big\} \\
&\quad = \max\Big\{\min\Big\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_1])} f_1(x,\alpha_1), \delta_1, \inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_2(x,\alpha_2)\Big\}, \\
&\qquad\qquad \min\Big\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_2])} f_2(x,\alpha_2), \inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_2(x,\alpha_3)\Big\}\Big\}.
\end{aligned}
\tag{2.40}
$$

By virtue of (2.32b) and having

$$
\inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_2(x,\alpha_2) \leq \inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_2(x,\alpha_3), \tag{2.41}
$$

it is enough to show that

$$\begin{aligned}&\max\left\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_2])} f_2(x,\alpha_2),\inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_1(x,\alpha_2)\right\}\\&\quad\geq \min\left\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_1])} f_1(x,\alpha_1),\inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_2(x,\alpha_3),\delta_1\right\}.\end{aligned}\tag{2.42}$$

We argue by contradiction. So suppose that (2.42) does not hold. Then one has

$$\inf_{x\in\Psi^{-1}(]-\infty,-\rho_2])} f_2(x,\alpha_2) < \min\left\{\delta_1,\inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_2(x,\alpha_3)\right\},\tag{2.43}$$

$$\inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_1(x,\alpha_2) < \min\left\{\delta_1,\inf_{x\in\Psi^{-1}(]-\infty,-\rho_1])} f_1(x,\alpha_1)\right\}.\tag{2.44}$$

If

$$\inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_1(x,\alpha_2) < \inf_{x\in\Psi^{-1}(]-\infty,-\rho_2])} f_2(x,\alpha_2),\tag{2.45}$$

then from (2.43) and (2.33) it results that $b(\rho_2,\alpha_2) < a(\rho_2,\alpha_2) < b(\rho_2,\alpha_3)$.
From $b(\rho_2,\alpha_2) < a(\rho_2,\alpha_2)$ it follows that

$$\inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} \frac{\Phi(x)-a(\rho_2,\alpha_2)}{\rho_2+\Psi(x)} < -\alpha_2.\tag{2.46}$$

Consequently, from $a(\rho_2,\alpha_2) < b(\rho_2,\alpha_3)$ and condition (ii′) of Theorem 2.1 it follows that

$$\sup_{x\in\Psi^{-1}(]-\infty,-\rho_2[)} \frac{\Phi(x)-a(\rho_2,\alpha_2)}{\rho_2+\Psi(x)} < -\alpha_2.\tag{2.47}$$

This is absurd. In fact it implies that $a(\rho_2,\alpha_2) = \inf_{\Psi^{-1}(-\rho_2)}\Phi$ which contradicts (2.43) because $\delta_1 \leq \inf_{\Psi^{-1}(-\rho_2)}\Phi + a_2$.
If

$$\inf_{x\in\Psi^{-1}(]-\infty,-\rho_2])} f_2(x,\alpha_2) \leq \inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} f_1(x,\alpha_2),\tag{2.48}$$

then, by (2.44), (2.32a), and (2.33), it follows that $a(\rho_2,\alpha_2) \leq b(\rho_2,\alpha_2) < a(\rho_2,\alpha_1)$. Furthermore, since (2.42) does not hold, it also results that $b(\rho_2,\alpha_2) < b(\rho_2,\alpha_3)$. Consequently $b(\rho_2,\alpha_2) < \min\{a(\rho_2,\alpha_1),b(\rho_2,\alpha_3)\}$. This implies that

$$\inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} \frac{\Phi(x)-\min\{a(\rho_2,\alpha_1),b(\rho_2,\alpha_3)\}}{\rho_2+\Psi(x)} < -\alpha_2.\tag{2.49}$$

Hence, by (ii′) of Theorem 2.1, it results that

$$a(\rho_2,\alpha_2) \geq \min\{a(\rho_2,\alpha_1),b(\rho_2,\alpha_3)\}.\tag{2.50}$$

Then, being $a(\rho_2,\alpha_2) < a(\rho_2,\alpha_1)$, it follows that $b(\rho_2,\alpha_3) < a(\rho_2,\alpha_1)$.

Now, fix $\epsilon > 0$ such that

$$a(\rho_2,\alpha_2) < b(\rho_2,\alpha_2) + \epsilon < b(\rho_2,\alpha_3). \tag{2.51}$$

Then, one has

$$\sup_{x\in\Psi^{-1}(]-\infty,-\rho_2[)} \frac{\Phi(x) - (b(\rho_2,\alpha_2)+\epsilon)}{\rho_2 + \Psi(x)} > -\alpha_2. \tag{2.52}$$

We exploit condition (ii′) of Theorem 2.1 to obtain

$$\inf_{x\in\Psi^{-1}(]-\rho_2,+\infty[)} \frac{\Phi(x) - b(\rho_2,\alpha_3)}{\rho_2 + \Psi(x)} > -\alpha_2. \tag{2.53}$$

From the previous inequality it follows the absurd $b(\rho_2,\alpha_2) \geq b(\rho_2,\alpha_3) > b(\rho_2,\alpha_2) + \epsilon$. This completes the proof for $n = 3$.

Suppose that (2.27) hold for $n = k-1$, then one has

$$\begin{aligned}
&\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X} \left(\Phi(x) + \lambda\Psi(x) + h(\lambda)\right) \\
&\quad= \max\Bigg\{\min\Bigg\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_1])} f_1(x,\alpha_1), \min_{1\leq i\leq k-2} \delta_i, \inf_{x\in\Psi^{-1}(]-\rho_{k-1},+\infty[)} f_{k-2}(x,\alpha_{k-1})\Bigg\}, \\
&\qquad\quad \min\Bigg\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_{k-1}])} f_{k-1}(x,\alpha_{k-1}), \inf_{x\in\Psi^{-1}(]-\rho_{k-1},+\infty[)} f_{k-1}(x,\alpha_k)\Bigg\}\Bigg\}.
\end{aligned} \tag{2.54}$$

So the conclusion follows since it results that

$$\begin{aligned}
&\max\Bigg\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_{k-1}])} f_{k-1}(x,\alpha_{k-1}), \inf_{x\in\Psi^{-1}(]-\rho_{k-1},+\infty[)} f_{k-2}(x,\alpha_{k-1})\Bigg\} \\
&\quad\geq \min\Bigg\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_1])} f_1(x,\alpha_1), \min_{1\leq i\leq k-2} \delta_i, \inf_{x\in\Psi^{-1}(]-\rho_{k-1},+\infty[)} f_{k-1}(x,\alpha_k)\Bigg\}.
\end{aligned} \tag{2.55}$$

Except for obvious changes, (2.55) can be proved by same arguments used for (2.42). □

Lemma 2.5. *Let $\alpha,\beta \in \mathbb{R}$, with $\alpha < \beta$, and $g : [\alpha,\beta] \to \mathbb{R}$ be a concave function such that $\max\{|g'_d(\alpha)|, |g'_s(\beta)|\} \neq +\infty$. There exists a nonincreasing sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ pointwise convergent to g on $[\alpha,\beta]$ such that, for every $n \in \mathbb{N}$, g_n is formally defined as the function h in Theorem 2.1.*

For the proof, refer to the proof of [7, Lemma 4].

3. Proof of Theorem 1.1

Proof. (i)⇔(ii). It follows from Corollary 2.3.

(i)⇒(iii). Except for obvious changes, the proof is analogous to that of its counterpart in [7, Theorem 1].

(iii)⇒(i). It immediately follows from the fact that

$$\sup_{\lambda\in I}\inf_{x\in X}\left(\Phi(x)+\lambda\Psi(x)+h(\lambda)\right)=\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}}\left(\Phi(x)+\lambda\Psi(x)+h(\lambda)\right),$$
$$\inf_{x\in X}\sup_{\lambda\in I}\left(\Phi(x)+\lambda\Psi(x)+h(\lambda)\right)=\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]\cap\mathbb{R}}\left(\Phi(x)+\lambda\Psi(x)+h(\lambda)\right),\tag{3.1}$$

when h is linear. □

References

[1] G. Barletta and R. Livrea, *Existence of three periodic solutions for a non autonomous second order system*, to appear in Le Matematiche.

[2] G. Bonanno, *Existence of three solutions for a two point boundary value problem*, Appl. Math. Lett. **13** (2000), no. 5, 53–57.

[3] ———, *Multiple solutions for a Neumann boundary value problem*, J. Nonlinear Convex Anal. **4** (2003), no. 2, 287–290.

[4] ———, *Some remarks on a three critical points theorem*, Nonlinear Anal. **54** (2003), no. 4, 651–665.

[5] G. Bonanno and R. Livrea, *Multiplicity theorems for the Dirichlet problem involving the p-Laplacian*, Nonlinear Anal. **54** (2003), no. 1, 1–7.

[6] P. Candito, *Existence of three solutions for a nonautonomous two point boundary value problem*, J. Math. Anal. Appl. **252** (2000), no. 2, 532–537.

[7] G. Cordaro, *On a minimax problem of Ricceri*, J. Inequal. Appl. **6** (2001), no. 3, 261–285.

[8] ———, *Three periodic solutions to an eigenvalue problem for a class of second-order Hamiltonian systems*, Abstr. Appl. Anal. **2003** (2003), no. 18, 1037–1045.

[9] R. Livrea, *Existence of three solutions for a quasilinear two point boundary value problem*, Arch. Math. (Basel) **79** (2002), no. 4, 288–298.

[10] S. A. Marano and D. Motreanu, *On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems*, Nonlinear Anal. Ser. A: Theory Methods **48** (2002), no. 1, 37–52.

[11] B. Ricceri, *A new method for the study of nonlinear eigenvalue problems*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 3, 251–256.

[12] ———, *Existence of three solutions for a class of elliptic eigenvalue problems*, Math. Comput. Modelling **32** (2000), no. 11-13, 1485–1494.

[13] ———, *On a three critical points theorem*, Arch. Math. (Basel) **75** (2000), no. 3, 220–226.

Giuseppe Cordaro: Department of Mathematics, University of Messina, 98166 Sant'Agata, Messina, Italy

E-mail address: cordaro@dipmat.unime.it