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Fusing Uncertain, Imprecise and Paradoxist Information (DSmT)

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FUSING UNCERTAIN, IMPRECISE AND PARADOXIST INFORMATION (DSMT)

The scientific area of sensor information fusion is one of the most significant, fundamental, and actively researched areas of information fusion. Generally, it has the advantage of: (1) using redundant information, (2) using the complementarities of the available information, (3) getting more reliable information, and (4) improving the decision-making process. Its main purpose is to gather, manipulate, and interpret the information accurately and efficiently at various levels of abstraction, which occur in the multi-sensor integration problem, due to the fact that the data are input at one level of abstraction (sensor-specific level) and displayed at another – generalized level of abstraction (nature/class/type). The overall aim is to ensure that the sensors remain in balance for as much of the time as possible, so that they are capable of detecting all possible changes in the problem under consideration. Without fusion, it is difficult to guarantee that our perception of a problem, in time, would remain accurate, adequate, and consistent.

In general, sensor data is inaccurate, incomplete and uncertain, or even paradoxical or/and conflicting. Imperfections pervade real-world scenarios and have to be incorporated into every information system that attempts to provide a complete and accurate model of the real world. The nature of uncertainty depends on the mathematical theory, within which problem situations are modeled. Any mathematical theory is capable of capturing only certain particular type/aspects of it. The classical mathematical approaches for characterizing situations under uncertainty are Probability theory, Dempster-Shafer evidential reasoning theory, Fuzzy set/logic theory. However, none of the models for imperfect data available to date can fit to all forms of imprecise information. These models are complementary, not concurrent.

The real challenge is in recognizing the nature of imprecision, uncertainty, conflicts and paradoxes encountered in the particular problem. This volume of *Information & Security: An International Journal* is devoted to some new advances and applications of the Theory of Plausible and Paradoxical Reasoning (DSmT), developed by Dr. Jean Dezert (ONERA, France) and Prof. Florentin Smarandache (University of New Mexico, USA). It proposes a new flexible general approach to managing both uncertainty and conflicts/paradoxes for a wide class of static or dynamic fusion problems, where the information to combine is modeled as a finite set of belief functions provided by independent sources of information. Its important contribution consists in overcoming the practical limitations of the Dempster-Shafer theory, related to the acceptance of the law of the third excluded middle. DSmT works for any model, which fits adequately with the true nature of the fusion problem under consideration and that way can be interpreted as a general and direct extension of Probability theory and Dempster-Shafer theory.

The first part of this I&S volume is oriented to some theoretical advances of the Theory of Plausible and Paradoxical Reasoning: an introduction of the important fusion of both quantitative and qualitative beliefs; a generalization of the classic combination rules to DSm hyper-power sets; and a new class of fusion rules based on T-Conorm and T-Norm fuzzy operators. The second part of this journal volume presents some interesting applications of DSm theory, including: ordered DSmT and its application to the definition of continuous DSm models; robot map building from sonar sensors and DSmT; and human expert fusion for image classification. This dual presentation makes this volume important and interesting for both theoretical and applied scientists.

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INTRODUCTION TO THE FUSION OF QUANTITATIVE AND QUALITATIVE BELIEFS

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Abstract: The efficient management and combination of uncertain and conflicting sources of information are of great importance for the development of reliable information fusion systems. Advanced fusion systems have to deal with both the quantitative and qualitative aspects of beliefs expressed by the different sources of information (sensors, expert systems, human reports, etc). This article introduces the theory of plausible and paradoxical reasoning, known as DSmT (Dezert-Smarandache Theory) in the literature, developed originally for dealing with imprecise, uncertain, and potentially highly conflicting sources of information providing quantitative beliefs on a given set of possible solutions of a given problem. The authors propose here also new ideas on a possible extension of DSmT to the combination of uncertain and conflicting qualitative information in order to deal directly with beliefs expressed with linguistic labels instead of numerical values so as to get closer to the nature of the information expressed in natural language and available directly from human experts.

Keywords: Dezert-Smarandache Theory, DSmT, Information Fusion, Quantitative Belief, Qualitative Belief, Conflict Management.

Introduction

The development of the Dezert-Smarandache Theory (DSmT)[1] arises from the necessity to overcome the inherent limitations of Dempster-Shafer's Theory (DST),[2] which are closely related to the acceptance of Shafer's model (i.e., working with an *homogeneous* frame of discernment Θ defined as a finite set of *exhaustive* and *exclusive* hypotheses θ_i, $i = 1, \dots, n$), the third excluded middle principle, and Dempster's rule for the combination of independent sources of evidence. The limitations of DST are well reported in the literature[3] and several alternative rules to Dempster's rule of combination have been proposed,[4] including several developments announced recently.[5]

DSmT provides a new mathematical framework for the fusion of quantitative or qualitative beliefs, which appears less restrictive and more general than the basis and constraints of DST.

In general, DSmT is based on the refutation of the principle of the third excluded middle and the Shafer's model, since for a wide class of fusion problems the hypotheses one has to deal with can have different intrinsic nature and also appear only vague and imprecise in such a way that precise refinement is just impossible to be obtained in reality so that the exclusive elements θ_i cannot be properly identified and defined. Many problems involving fuzzy/vague continuous and relative concepts described in natural language with different semantic contents and having no absolute interpretation belong to this category. Although DSmT was initially developed for the fusion of quantitative beliefs (i.e. numbers/masses in $[0, 1]$ satisfying a given set of constraints), we will show later how it can be extended quite directly to the fusion of qualitative beliefs (i.e., when precise numbers are replaced by imprecise linguistic labels).

DSmT begins with the notion of *free DSm model* and considers Θ as a frame of exhaustive elements only that can potentially overlap and have different intrinsic nature and that can also change with time when new information and evidence is received about the model. DSmT provides flexibility on the structure of the model under consideration. When the free DSm model holds, the conjunctive consensus is used. If the free model does not describe the reality adequately since it is known that some subsets of Θ contain elements truly exclusive but also possibly truly non-existing at all at a given time (in dynamic fusion[6]), new fusion rules have to be used to take into account these integrity constraints. The constraints can be explicitly introduced into the free DSm model in order to fit adequately with our current knowledge of the reality; we actually construct a *hybrid DSm model*, on which the combination will be efficiently performed. In fact, Shafer's model corresponds to a very specific hybrid (and homogeneous) DSm model, including all possible exclusivity constraints. DSmT has been developed to work with any model and to combine imprecise, uncertain, and potentially highly conflicting sources for static and dynamic information fusion. DSmT refutes the idea that the sources provide their (quantitative or qualitative) beliefs with the same absolute interpretation of elements of Θ; what is considered good by someone can be considered bad by somebody else. This paper is a revised and extended version of other publications of the authors.[7]

After a short presentation of the notion of hyper-power set and DSm models in this section, the next section will present the main combination rules for the fusion of quantitative precise or imprecise beliefs, i.e., the Classic DSm (DSmC), the Hybrid DSm (DSmH), and the Proportional Conflict Redistribution (PCR) rules of combination. The quantitative fusion rules will be extended then to their qualitative counterparts. Such an extension makes it possible to deal directly with beliefs expressed with linguistic labels extracted from natural language.

Notion of Hyper-Power Set

Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a finite set (called frame) of n exhaustive elements.[8] The free Dedekind's lattice denoted *hyper-power set* D^Θ[9] is defined as:

1. $\emptyset, \theta_1, \ldots, \theta_n \in D^\Theta$.

2. If $A, B \in D^\Theta$, then $A \cap B$ and $A \cup B$ belong to D^Θ.

3. No other elements belong to D^Θ, except those obtained by using rules 1 or 2.

If $|\Theta| = n$, then $|D^\Theta| \leq 2^{2^n}$. How to generate D^Θ has been described by the authors in another publication.[10] Since for any given finite set Θ, $|D^\Theta| \geq |2^\Theta|$, we call D^Θ the *hyper-power set* of Θ. $|D^\Theta|$ for $n \geq 1$ follows the sequence of Dedekind's numbers: 1, 2, 5, 19, 167, ... An analytical expression of Dedekind's numbers has been obtained by Tombak and colleagues.[11]

Example: If $\Theta = \{\theta_1, \theta_2, \theta_3\}$, then its hyper-power set D^Θ includes the following nineteen elements: $\emptyset$, $\theta_1 \cap \theta_2 \cap \theta_3$, $\theta_1 \cap \theta_2$, $\theta_1 \cap \theta_3$, $\theta_2 \cap \theta_3$, $(\theta_1 \cup \theta_2) \cap \theta_3$, $(\theta_1 \cup \theta_3) \cap \theta_2$, $(\theta_2 \cup \theta_3) \cap \theta_1$, $(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$, θ_1, θ_2, θ_3, $(\theta_1 \cap \theta_2) \cup \theta_3$, $(\theta_1 \cap \theta_3) \cup \theta_2$, $(\theta_2 \cap \theta_3) \cup \theta_1$, $\theta_1 \cup \theta_2$, $\theta_1 \cup \theta_3$, $\theta_2 \cup \theta_3$, and $\theta_1 \cup \theta_2 \cup \theta_3$.

Free and Hybrid DSm Models

$\Theta = \{\theta_1, \ldots, \theta_n\}$ denotes the finite set of hypotheses, characterizing the fusion problem. D^Θ constitutes the *free DSm model* $\mathcal{M}^f(\Theta)$ and allows to work with fuzzy concepts, which depict a continuous and relative intrinsic nature. Such kinds of concepts cannot be precisely refined with an absolute interpretation because of the unapproachable universal truth. When all θ_i are truly exclusive discrete elements, D^Θ reduces to the classical power set 2^Θ. This is what we call the Shafer's model, denoted $\mathcal{M}^0(\Theta)$. Between the free DSm model and the Shafer's model, there exists a wide class of fusion problems represented in terms of DSm hybrid models, where Θ involves both fuzzy continuous concepts and discrete hypotheses. In such a class, some exclusivity constraints and possibly some non-existential constraints (especially in dynamic fusion) have to be taken into account. Each hybrid fusion problem is then characterized by a proper hybrid DSm model $\mathcal{M}(\Theta)$ with $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$ and $\mathcal{M}(\Theta) \neq \mathcal{M}^0(\Theta)$. The main differences between DST and DSmT are: (1) the model one works with, and (2) the choice of the combination rule. We use here the generic notation G for denoting either D^Θ (when DSmT is considered) or 2^Θ (when DST is considered). We denote with G^* the set G, from which the empty set is excluded ($G^* = G \setminus \{\emptyset\}$).

- *A 3D Example of free DSm model*: When $\Theta = \{\theta_1, \theta_2, \theta_3\}$, the free-model $\mathcal{M}^f(\Theta)$ corresponds to the Venn diagram presented in Figure 1 where all elements can overlap partially but with vague boundaries in such a way that no exact/precise refinement is possible.

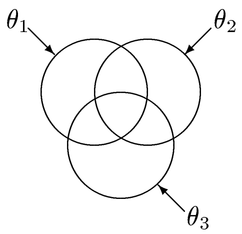

Figure 1: Venn Diagram for the Free DSm Model $\mathcal{M}^f(\Theta)$.

- *A 3D Example of a hybrid DSm model*: Let us consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and only the exclusivity constraint of θ_3 with respect to θ_1 and θ_2; then one gets the Venn diagram presented in Figure 2 for this specific hybrid DSm model $\mathcal{M}(\Theta)$ defined by Θ and the chosen (integrity) constraint.

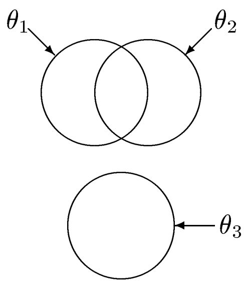

Figure 2: Venn Diagram for a Hybrid DSm Model $\mathcal{M}(\Theta)$.

- *A 3D Example of Shafer's model*: Let us consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$. The Shafer's model, denoted $\mathcal{M}^0(\Theta)$, assumes all elements of Θ being truly exhaustive and exclusive. Its corresponding Venn diagram is illustrated in Figure 3.

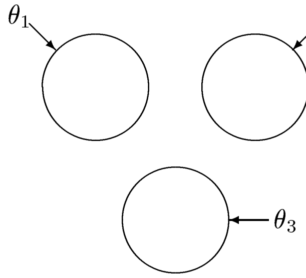

Figure 3: Venn Diagram for the Shafer's Model $\mathcal{M}^0(\Theta)$.

Fusion of Quantitative Beliefs

Quantitative Belief Functions

In the DSmT framework, a (precise) quantitative basic belief assignment[12] (bba) associated with a given source of information (body of evidence) about a frame Θ is defined as a precise mapping $m(.)$ from G into $[0,1]$, i.e., $m(.) : G \to [0,1]$ satisfying:

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{A \in G} m(A) = 1. \tag{1}$$

From $m(.)$, we define the (quantitative) credibility and plausibility functions as:

$$\text{Bel}(A) \triangleq \sum_{\substack{B \subseteq A \\ B \in G}} m(B) \qquad \text{and} \qquad \text{Pl}(A) \triangleq \sum_{\substack{B \cap A \neq \emptyset \\ B \in G}} m(B) \tag{2}$$

These definitions remain compatible with the definitions of Bel(.) and Pl(.) given in DST when $\mathcal{M}^0(\Theta)$ holds[13] since in that case $G = D^\Theta$ reduces to the classical power-set 2^Θ.

Combinations of Precise Quantitative Beliefs

The three main DSm fusion rules proposed in the DSmT framework for the combination of precise quantitative beliefs will be presented in this section. The simplest rule is the Classic DSm rule (DSmC), which corresponds to the consensus operator on hyper-power set when the free DSm model holds. The second and more sophisticated one is the DSm hybrid rule (DSmH),[14] which allows to work on any static or dynamic hybrid model and also to work on the Shafer's model whenever this model holds. DSmH is a direct extension of Dubois & Prade's rule[15] for dealing with dynamic/temporal fusion (i.e., when the frame and its model/constraints change with time). Then, the authors present the proportional conflict redistribution rule #5 (PCR5), which proposes a more subtle transfer of the conflicting masses than (DSmH).[16] The DSmH and PCR rules are mathematically well defined and work both with any models and independently of the value of the degree of conflict. In practice, when the reliabilities of the sources are known, we can easily take them into account in all DSm-based fusion rules by discounting them by the proper discounting factor and using the classical discounting approach of beliefs.[17] Here, the authors will not present the details of well-known discounting techniques since, in authors' opinion, the issue of combination remains more important. The authors emphasize here that this preprocessing/discounting step, although very important from practical point of view, does not have to however appear as a substitute or as an artificial *engineering trick* to circumvent the inherent deficiencies of a chosen combination rule. Even if the DSm-based rules work for any degree of conflict between sources, the authors do not claim that they should be applied blindly in practice when conflict becomes very large, without trying first to analyze the origins of the partial conflicts, estimate and take into account (when it is possible) the reliability of each source prior to their combination. But once all these necessary preliminary steps (deep analysis of the problems, refinement of the model, and reliability assessment of each source) have been done, one has always to choose what s/he considers the most relevant combination rule for application. The DSm-based rules provide possible new solutions and valuable alternatives for the combination of uncertain, imprecise, and conflicting information. Comparisons between the different main quantitative rules of combination with several examples can be found in other works performed by the authors and their colleagues.[18]

Classic DSm Fusion Rule (DSmC)

When the free DSm model $\mathcal{M}^f(\Theta)$ holds, the conjunctive consensus, called DSm Classic (DSmC) rule, is performed on D^Θ. DSmC of two independent[19] sources associated with gbba $m_1(.)$ and $m_2(.)$ is thus given for $\forall C \in D^\Theta$ by [20]:

$$m_{DSmC}(C) = \sum_{\substack{A,B \in D^\Theta \\ A \cap B = C}} m_1(A) m_2(B). \tag{3}$$

Due to the fact that D^Θ is closed under $\cup$ and $\cap$ operators, the DSmC guarantees that $m(.)$ is a proper gbba. DSmC is commutative and associative and can be used for the fusion of sources involving fuzzy concepts whenever $\mathcal{M}^f(\Theta)$ holds. It can be easily extended to the fusion of $k > 2$ independent sources.[21]

Example for DSmC

Let us consider a generalization of Zadeh's example[22] and take $\Theta = \{\theta_1, \theta_2, \theta_3\}$; let $0 < \epsilon_1, \epsilon_2 < 1$ be two positive numbers and assume that two experts provide the quantitative and precise bba $m_1(\theta_1) = 1 - \epsilon_1$, $m_1(\theta_2) = 0$, $m_1(\theta_3) = \epsilon_1$, $m_2(\theta_1) = 0$, $m_2(\theta_2) = 1 - \epsilon_2$, and $m_2(\theta_3) = \epsilon_2$.

If one adopts the free-DSm model for Θ (i.e., we accept the non-exclusivity of hypotheses), using the DSmC rule one gets zero for all masses of D^Θ except the following ones:

$$m_{DSmC}(\theta_3) = \epsilon_1\epsilon_2$$

$$m_{DSmC}(\theta_1 \cap \theta_2) = (1 - \epsilon_1)(1 - \epsilon_2)$$

$$m_{DSmC}(\theta_1 \cap \theta_3) = (1 - \epsilon_1)\epsilon_2$$

$$m_{DSmC}(\theta_2 \cap \theta_3) = (1 - \epsilon_2)\epsilon_1$$

Hybrid DSm Fusion Rule (DSmH)

When $\mathcal{M}^f(\Theta)$ does not hold (some integrity constraints exist), one deals with a proper DSm hybrid model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. The DSm Hybrid rule (DSmH) for $k \geq 2$ independent sources is thus defined for all $A \in D^\Theta$ as [23]:

$$m_{DSmH}(A) \triangleq \phi(A) \cdot \Big[S_1(A) + S_2(A) + S_3(A)\Big], \tag{4}$$

where $\phi(A)$ is the *characteristic non-emptiness function* of a set A, i.e. $\phi(A) = 1$ if $A \notin \boldsymbol{\emptyset}$ and $\phi(A) = 0$ otherwise, where $\boldsymbol{\emptyset} \triangleq \{\emptyset_{\mathcal{M}}, \emptyset\}$. $\emptyset_{\mathcal{M}}$ is the set of all elements of D^Θ which have been forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical/universal empty set. $S_1(A) \equiv m_{\mathcal{M}^f(\theta)}(A)$, $S_2(A)$, and $S_3(A)$ are defined by:

$$S_1(A) \triangleq \sum_{\substack{X_1, X_2, \ldots, X_k \in D^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_k = A}} \prod_{i=1}^{k} m_i(X_i) \tag{5}$$

$$S_2(A) \triangleq \sum_{\substack{X_1, X_2, \ldots, X_k \in \boldsymbol{\emptyset} \\ [\mathcal{U} = A] \vee [(\mathcal{U} \in \boldsymbol{\emptyset}) \wedge (A = I_t)]}} \prod_{i=1}^{k} m_i(X_i) \tag{6}$$

$$S_3(A) \triangleq \sum_{\substack{X_1, X_2, \ldots, X_k \in D^\Theta \\ X_1 \cup X_2 \cup \ldots \cup X_k = A \\ X_1 \cap X_2 \cap \ldots \cap X_k \in \boldsymbol{\emptyset}}} \prod_{i=1}^{k} m_i(X_i), \tag{7}$$

where all sets involved in the formulas are in canonical form and $\mathcal{U} \triangleq u(X_1) \cup \ldots \cup u(X_k)$, where $u(X)$ is the union of all θ_i that compose X, and $I_t \triangleq \theta_1 \cup \ldots \cup \theta_n$ is the total ignorance. $S_1(A)$ is nothing but the DSmC rule for k independent sources based on $\mathcal{M}^f(\Theta)$; $S_2(A)$ is the mass of all relatively and absolutely empty sets that is transferred to the total or relative ignorances associated with non existential constraints (if any, like in some dynamic problems); $S_3(A)$ transfers the sum of relatively empty sets directly onto the canonical disjunctive form of non-empty sets. DSmH generalizes DSmC and makes it possible to work with the Shafer's model. It is definitely not equivalent to Dempster's rule since these rules are different. DSmH works for any model (free DSm, Shafer's or hybrid models) when dealing with *precise* bba. The reader may refer to a recent report on DSmT including MatLab[24] codes.[25]

Example for (DSmH)

Let us consider the previous example with $\Theta = \{\theta_1, \theta_2, \theta_3\}$; let $0 < \epsilon_1, \epsilon_2 < 1$ be two positive numbers and let two experts provide the quantitative and precise bba $m_1(\theta_1) = 1 - \epsilon_1$, $m_1(\theta_2) = 0$, $m_1(\theta_3) = \epsilon_1$, $m_2(\theta_1) = 0$, $m_2(\theta_2) = 1 - \epsilon_2$, and $m_2(\theta_3) = \epsilon_2$. Now, assume that the Shafer's model holds, i.e., we assume that θ_1, θ_2, and θ_3 are truly exclusive.

- Based on the DSmH fusion rule (4), on gets:

$$m_{DSmH}(\theta_3) = \epsilon_1 \epsilon_2$$

$$m_{DSmH}(\theta_1 \cup \theta_2) = (1 - \epsilon_1)(1 - \epsilon_2)$$

$$m_{DSmH}(\theta_1 \cup \theta_3) = (1 - \epsilon_1)\epsilon_2$$

$$m_{DSmH}(\theta_2 \cup \theta_3) = (1 - \epsilon_2)\epsilon_1$$

  All other masses are zero. This result makes sense since it depends truly on the values of ϵ_1 and ϵ_2, contrariwise to Dempster's rule according to the next item.

- Using Dempster-Shafer's (DS) rule of combination,[26] one gets:

$$m_{DS}(\theta_3) = \frac{(\epsilon_1 \epsilon_2)}{(1 - \epsilon_1) \cdot 0 + 0 \cdot (1 - \epsilon_2) + \epsilon_1 \epsilon_2} = 1$$

  which is absurd (or at least counter-intuitive). Note that whatever positive values for ϵ_1 and ϵ_2 are chosen, the Dempster's rule gives *always the same result* (one), which is not normal. The only acceptable and correct result obtained by the Dempster's rule is obtained only in the trivial case when $\epsilon_1 = \epsilon_2 = 1$, i.e., when the two sources agree in θ_3 with certainty, which is obvious.

When $\epsilon_1 = \epsilon_2 = 1/2$, one obtains:

$$m_1(\theta_1) = 1/2 \quad m_1(\theta_2) = 0 \quad m_1(\theta_3) = 1/2$$
$$m_2(\theta_1) = 0 \quad m_2(\theta_2) = 1/2 \quad m_2(\theta_3) = 1/2$$

Dempster's rule still yields $m_{DS}(\theta_3) = 1$, while DSmH based on the same Shafer's model yields now $m_{DSmH}(\theta_3) = 1/4$, $m_{DSmH}(\theta_1 \cup \theta_2) = 1/4$, $m_{DSmH}(\theta_1 \cup \theta_3) = 1/4$, $m_{DSmH}(\theta_2 \cup \theta_3) = 1/4$, which is more acceptable in authors' opinion. A detailed discussion on this example (and on others) with answers to a recent criticism by Haenni[27] can be found in another publication by the authors.[28]

Proportional Conflict Redistribution Rule no. 5 (PCR5)

Instead of applying a direct transfer of partial conflicts to partial uncertainties as is the case with DSmH, the idea behind the Proportional Conflict Redistribution (PCR) rule[29] is to transfer (total or partial) conflicting masses to non-empty sets involved in the conflicts proportionally with respect to the masses assigned to them by the sources as follows:

1. Calculation of the conjunctive rule of the belief masses of sources;
2. Calculation of the total or partial conflicting masses;
3. Redistribution of the (total or partial) conflicting masses to the non-empty sets involved in the conflicts proportionally to their masses assigned by the sources.

Actually, the way the conflicting mass is redistributed leads to several versions of PCR rules. These PCR fusion rules work for any degree of conflict, any DSm model (Shafer's model, free DSm model, or any hybrid DSm model) and both in DST and DSmT framework for static or dynamic fusion situations. The authors present here the most comprehensive proportional conflict redistribution rule (rule #5) denoted PCR5.[30] PCR5 is what the authors consider the most efficient PCR fusion rule for the combination of two sources. A more intuitive version of PCR5 for $s >= 3$ sources, denoted PCR6, has been recently proposed by Martin and Osswald.[31] PCR6 coincides with PCR5 in the two-source case, but differs from PCR5 when combining more than two sources.

The PCR5 rule redistributes the partial conflicting mass to the elements involved in the partial conflict, considering the conjunctive normal form of the partial conflict. PCR5 is according to the authors the most interesting redistribution of conflicting mass to non-empty sets, following the logic of the conjunctive rule. PCR5 performs a better redistribution of the conflicting mass than Dempster's rule since it goes backwards on the tracks of the conjunctive rule and redistributes the conflicting mass only to the sets involved in the conflict and proportionally to their masses put in the conflict. The PCR5 rule is quasi-associative and preserves the neutral impact of the vacuous belief assignment because in any partial conflict, as well in the total conflict (which is a sum of all partial conflicts), the conjunctive normal form of each partial conflict does not include Θ, since Θ is a neutral element for intersection (conflict); therefore Θ gets no mass after the redistribution of the conflicting mass. The authors have proved the continuity of the PCR5 result with continuous variations of bba to combine.[32] The general PCR5 formula for $s \geq 2$ sources, $m_{PCR5}(\emptyset) = 0$ and $\forall X \in G \setminus \{\emptyset\}$ is given by [33]:

$$m_{PCR5}(X) = m_{12\dots s}(X) + \sum_{\substack{2\leq t\leq s\\ 1\leq r_1,\dots,r_t\leq s\\ 1\leq r_1<r_2<\dots<r_{t-1}<(r_t=s)}} \sum_{\substack{X_{j_2},\dots,X_{j_t}\in G\setminus\{X\}\\ \{j_2,\dots,j_t\}\in\mathcal{P}^{t-1}(\{1,\dots,n\})\\ X\cap X_{j_2}\cap\dots\cap X_{j_s}=\emptyset\\ \{i_1,\dots,i_s\}\in\mathcal{P}^s(\{1,\dots,s\})}}$$

$$\frac{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(X)^2)\cdot[\prod_{l=2}^{t}(\prod_{k_l=r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]}{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(X)) + [\sum_{l=2}^{t}(\prod_{k_l=r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]} \tag{8}$$

where G corresponds to the classical power-set 2^Θ if the Shafer's model is used or G corresponds to a constrained hyper-power set D^Θ if any other hybrid DSm model is used instead; i, j, k, r, s, and t in (8) are integers.

$$m_{12\dots s}(X) \equiv m_\cap(X) = \sum_{\substack{X_1,\dots,X_s\in G\\ X_1\cap\dots\cap X_s=X}} \prod_{i=1}^{s} m_i(X_i)$$

corresponds to the conjunctive consensus on X between s sources and where all denominators are different from zero. If a denominator is zero, that fraction is discarded; the set of all subsets of k elements from $\{1, 2, \dots, n\}$ (permutations of k out of n elements) is denoted $\mathcal{P}^k(\{1, 2, \dots, n\})$; the order of the elements is not important.

When $s = 2$ (fusion of only two sources), the previous PCR5 formula reduces to the following simple fusion formula:

$$m_{PCR5}(\emptyset) = 0 \quad \text{and} \quad \forall X \in G \setminus \{\emptyset\}$$

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y\in G\setminus\{X\}\\ X\cap Y=\emptyset}} \left[\frac{m_1(X)^2 m_2(Y)}{m_1(X)+m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X)+m_1(Y)}\right] \tag{9}$$

For $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ with the Shafer's model and $s = 2$ Bayesian equi-reliable sources, i.e., when the quantitative bba $m_1(.)$ and $m_2(.)$ reduce to subjective probability measures $P_1(.)$ and $P_2(.)$, it can be shown[34] that after elementary algebraic derivations, the previous PCR5 formula reduces to the following simple formula, with $P_{12}^{PCR5}(\emptyset) = 0$ and $\forall \theta_i \in \Theta$:

$$\begin{aligned} P_{12}^{PCR5}(\theta_i) &= P_1(\theta_i) \sum_{j=1}^{n} \frac{P_1(\theta_i)P_2(\theta_j)}{P_1(\theta_i) + P_2(\theta_j)} + P_2(\theta_i) \sum_{j=1}^{n} \frac{P_2(\theta_i)P_1(\theta_j)}{P_2(\theta_i) + P_1(\theta_j)} \\ &= \sum_{s=1,2} P_s(\theta_i) \left[\sum_{j=1}^{n} \frac{P_s(\theta_i)P_{s' \neq s}(\theta_j)}{P_s(\theta_i) + P_{s' \neq s}(\theta_j)} \right] \end{aligned} \tag{10}$$

Moreover, it could be verified that $P_{12}^{PCR5}(.)$ defines a subjective-combined probability measure, satisfying all axioms of the classical Probability Theory.

Examples for PCR5

- *Example 1*: Let us take $\Theta = \{A, B\}$ of exclusive elements (Shafer's model) and the following bba:

	A	B	$A \cup B$
$m_1(.)$	0.6	0	0.4
$m_2(.)$	0	0.3	0.7
$m_\cap(.)$	0.42	0.12	0.28

The conflicting mass is $k_{12} = m_\cap(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18$. Therefore, A and B are the only focal[35] elements involved in the conflict. Hence, according to the PCR5 hypothesis only A and B deserve a part of the conflicting mass and $A \cup B$ does not. With PCR5, one redistributes the conflicting mass $k_{12} = 0.18$ to A and B proportionally to the masses $m_1(A)$ and $m_2(B)$ assigned to A and B, respectively. Let x be the conflicting mass to be redistributed to A and y the conflicting mass redistributed to B, then

$$\frac{x}{0.6} = \frac{y}{0.3} = \frac{x+y}{0.6+0.3} = \frac{0.18}{0.9} = 0.2$$

hence $x = 0.6 \cdot 0.2 = 0.12$, $y = 0.3 \cdot 0.2 = 0.06$. Thus, the final result using the PCR5 rule is:

$$\begin{cases} m_{PCR5}(A) = 0.42 + 0.12 = 0.54 \\ m_{PCR5}(B) = 0.12 + 0.06 = 0.18 \\ m_{PCR5}(A \cup B) = 0.28 \end{cases}$$

For comparison, here are the results obtained using Dempster's (DS), DSmH, and PCR5 rules:

	A	B	$A \cup B$
m_{DS}	0.512	0.146	0.342
m_{DSmH}	0.420	0.120	0.460
m_{PCR5}	0.540	0.180	0.280

- *Example 2*: Let us modify Example 1 and consider:

	A	B	$A \cup B$
$m_1(.)$	0.6	0	0.4
$m_2(.)$	0.2	0.3	0.5
$m_\cap(.)$	0.50	0.12	0.20

The conflicting mass $k_{12} = m_\cap(A \cap B)$ as well as the distribution coefficients for PCR5 remain the same as in the previous example, but one obtains now:

	A	B	$A \cup B$
m_{DS}	0.609	0.146	0.231
m_{DSmH}	0.500	0.120	0.380
m_{PCR5}	0.620	0.180	0.200

- *Example 3*: Let us modify Example 2 and consider:

	A	B	$A \cup B$
$m_1(.)$	0.6	0.3	0.1
$m_2(.)$	0.2	0.3	0.5
$m_\cap(.)$	0.44	0.27	0.05

The conflicting mass $k_{12} = 0.24 = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.24$ is now different from the previous examples, which means that $m_2(A) = 0.2$ and $m_1(B) = 0.3$ did make an impact on the conflict. Therefore, A and B are the only focal elements involved in the conflict and thus only A and B deserve a part of the conflicting mass. PCR5 redistributes the partial conflicting mass 0.18 to A and B proportionally to the masses $m_1(A)$ and $m_2(B)$ and also the partial conflicting mass 0.06 to A and B proportionally to the masses $m_2(A)$ and $m_1(B)$. After all derivations (the details can be found elsewhere[36]), one finally gets:

	A	B	$A \cup B$
m_{DS}	0.579	0.355	0.066
m_{DSmH}	0.440	0.270	0.290
m_{PCR5}	0.584	0.366	0.050

It could be clearly seen that $m_{DS}(A \cup B)$ receives some part of the conflicting mass although $A \cup B$ does not deserve any part of the conflicting mass (according to the hypothesis of PCR5) due to the fact that $A \cup B$ is not involved in the conflict (only A and B are involved in the conflicting mass). According to the authors, Dempster's rule appears less correct than the PCR5 and Inagaki's rules.[37] It could be demonstrated [38] that the Inagaki's fusion rule[39] (with an optimal choice of tuning parameters) can become in some cases very close to PCR5, however, in authors' opinion, the PCR5 result is more precise (at least less ad-hoc than the Inagaki's one).

- *Example 4*: Zadeh's example[40]

  Let us consider $\Theta = \{M, C, T\}$ as the frame of three potential origins of possible diseases of a patient (M standing for *meningitis*, C for *concussion*, and T for *tumor*), the Shafer's model and the following two belief assignments provided independently by two doctors after examination of the patient:

$$m_1(M) = 0.9 \qquad m_1(C) = 0 \qquad m_1(T) = 0.1$$
$$m_2(M) = 0 \qquad m_2(C) = 0.9 \qquad m_2(T) = 0.1$$

  The total conflicting mass is high since it is given by:

$$m_1(M)m_2(C) + m_1(M)m_2(T) + m_2(C)m_1(T) = 0.99$$

  - With Dempster's rule and Shafer's model (DS), one gets the following counter-intuitive result (the reader may find justifications[41] and criticism[42] in various other publications): $m_{DS}(T) = 1$
  - With Yager's rule[43] and Shafer's model: $m_Y(M \cup C \cup T) = 0.99$ and $m_Y(T) = 0.01$
  - With DSmH and Shafer's model:

$$m_{DSmH}(M \cup C) = 0.81 \qquad m_{DSmH}(T) = 0.01$$
$$m_{DSmH}(M \cup T) = m_{DSmH}(C \cup T) = 0.09$$

  - The Dubois & Prade's rule (DP)[44] based on Shafer's model gives for Zadeh's example the same result as DSmH, because DP and DSmH coincide in all static fusion problems.[45]
  - With PCR5 and the Shafer's model:

$$m_{PCR5}(M) = m_{PCR5}(C) = 0.486 \qquad m_{PCR5}(T) = 0.028$$

It could be seen that when the total conflict between sources becomes high, DSmT is able (in authors' opinion) to control either through DSmH or PCR5 rules more adequately than Dempster's rule the combination of information, even when working with Shafer's model – which is only a specific hybrid model. The DSmH rule is in agreement with the DP rule in static fusion, but DSmH and DP rules differ in general (for non-degenerate cases) in dynamic fusion, while the PCR5 rule seems more precise owing to the proper proportional conflict redistribution of partial conflicts only to elements involved in the partial conflicts. Besides this particular example, the authors have shown elsewhere[46] the existence of several infinite classes of counter-examples to Dempster's rule, which can be solved by DSmT.

Combination of Imprecise Quantitative Beliefs

When the sources are unable to provide precise quantitative basic belief assignments (bba) $m(.)$, they can in some cases at least express their quantitative belief assignment on a frame Θ in an imprecise manner as *admissible imprecise* quantitative basic belief assignments $m^I(.)$, whose values are real subunitary intervals of $[0, 1]$, or even more general as real subunitary sets (i.e. sets, not necessarily intervals). In the general case, these sets can be unions of (closed, open, or half-open/half-closed) intervals and/or scalars all in $[0, 1]$.

Definition of Imprecise Quantitative Basic Beliefs Assignment

An imprecise quantitative bba $m^I(.)$ is mathematically defined as $m^I(.) : D^\Theta \to \mathcal{P}([0, 1]) \setminus \{\emptyset\}$, where $\mathcal{P}([0, 1])$ is the set of all subsets of the interval $[0, 1]$. $m^I(.)$ over D^Θ is said to be *admissible* if and only if there exists for every $X \in D^\Theta$ at least one real number $m(X) \in m^I(X)$, such that $\sum_{X \in D^\Theta} m(X) = 1$. $m^I(.)$ is a normal extension of $m(.)$ from scalar values to set values. For example, if a source $m(.)$ is not sure about the scalar value $m(A) = 0.3$, it may be considered an imprecise source, which gives a set value $m^I(A) = [0.2, 0.4]$, for example.

Operators on Sets

The following simple commutative operators on sets (addition $\boxplus$ and multiplication $\boxdot$) are required[47] for fusion of imprecise bba:

- Addition:

$$\mathcal{X}_1 \boxplus \mathcal{X}_2 \triangleq \{x \mid x = x_1 + x_2, x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\} \tag{11}$$

- Multiplication:

$$\mathcal{X}_1 \boxdot \mathcal{X}_2 \triangleq \{x \mid x = x_1 \cdot x_2, x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\}. \quad (12)$$

These operators are generalized for the summation and products of $n \geq 2$ sets as follows:

$$\boxed{\sum_{k=1,\ldots,n}} \mathcal{X}_k = \{x \mid x = \sum_{k=1,\ldots,n} x_k, \;\; x_1 \in \mathcal{X}_1, \ldots, x_n \in \mathcal{X}_n\} \quad (13)$$

$$\boxed{\prod_{k=1,\ldots,n}} \mathcal{X}_k = \{x \mid x = \prod_{k=1,\ldots,n} x_k, \;\; x_1 \in \mathcal{X}_1, \ldots, x_n \in \mathcal{X}_n\}. \quad (14)$$

Considering these operators, one can easily generalize the DSmC and DSmH fusion rules from scalars to sets[48] to obtain their imprecise counterparts. In order to extend PCR5 to its imprecise counterpart, i.e., the imp-PCR5 fusion rule, for dealing with imprecise quantitative belief assignments, the division operator on sets needs to be introduced as follows:

- Division (for the case when $0 \notin \mathcal{X}_2$, $\inf(\mathcal{X}_2) \neq 0$, and $\sup(\mathcal{X}_2) \neq 0$):

$$\mathcal{X}_1 \boxslash \mathcal{X}_2 \triangleq \{x \mid x = x_1/x_2, \text{where } x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\} \quad (15)$$

The operations with sets are associative and commutative similarly to the operations with numbers. Thus, for $a, b, c, d \geq 0$ and $e, f > 0$, if one computes $((a,b) \boxdot (c,d)) \boxslash (e,f)$, one gets:

$$((a,b) \boxdot (c,d)) \boxslash (e,f) = (ac, bd) \boxslash (e,f) = (ac/f, bd/e)$$

and, the same result will be obtained if we compute $(a,b) \boxdot ((c,d) \boxslash (e,f))$ because

$$(a,b) \boxdot ((c,d) \boxslash (e,f)) = (a,b) \boxdot (c/f, d/e) = (ac/f, bd/e).$$

In the following examples, the authors will preferably compute the division operations at the end, considering the fact that they often do not give exact values but approximations; and, early approximations in calculations will grow in inaccuracy.

Imprecise Classic DSm Fusion Rule (imp-DSmC)

The Imprecise Classic DSm fusion rule (imp-DSmC), which extends the Classic DSm fusion rule (DSmC) for combining imprecise (admissible) quantitative basic belief assignments is given for $k \geq 2$ sources by $m^I_{DSmC}(\emptyset) = 0$ and $\forall A \neq \emptyset \in D^\Theta$,

$$m^I_{DSmC}(A) = \boxed{\sum_{\substack{X_1, X_2, \ldots, X_k \in D^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_k = A}}} \boxed{\prod_{i=1,\ldots,k}} m^I_i(X_i). \quad (16)$$

Imprecise Hybrid DSm Fusion Rule (imp-DSmH)

Similarly, one can generalize DSmH from scalars to sets for the combination of $k \geq 2$ sources by $m^I_{DSmH}(\emptyset) = 0$ and $\forall A \neq \emptyset \in D^\Theta$,

$$m^I_{DSmH}(A) \triangleq \phi(A) \boxdot \left[S^I_1(A) \boxplus S^I_2(A) \boxplus S^I_3(A)\right] \tag{17}$$

with

$$S^I_1(A) \triangleq \boxed{\sum}_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_k = A}} \boxed{\prod}_{i=1,\ldots,k} m^I_i(X_i) \tag{18}$$

$$S^I_2(A) \triangleq \boxed{\sum}_{\substack{X_1,X_2,\ldots,X_k \in \boldsymbol{\emptyset} \\ [\mathcal{U}=A] \vee [(\mathcal{U} \in \boldsymbol{\emptyset}) \wedge (A = I_t)]}} \boxed{\prod}_{i=1,\ldots,k} m^I_i(X_i) \tag{19}$$

$$S^I_3(A) \triangleq \boxed{\sum}_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_k = A \\ (X_1 \cap X_2 \cap \ldots \cap X_k) \in \boldsymbol{\emptyset}}} \boxed{\prod}_{i=1,\ldots,k} m^I_i(X_i) \tag{20}$$

The imp-DSmC and imp-DSmH fusion rules are just natural extensions of DSmC and DSmH from scalar-valued to set-valued sources of information. It has been proved that (16) and (17) provide an admissible imprecise belief assignment (the reader may refer to the Theorem of Admissibility and its proof in Chapter 6, p.138, of the book "Advances and Applications of DSmT for Information Fusion" edited by Smarandache and Dezert[49]). In other words, DSm combination of two admissible imprecise bba produces also an admissible imprecise bba. As their precise counterparts, the imprecise DSm combination rules are *quasi-associative*, i.e. one stores in the computer's memory the conjunctive rule's result and, when new evidence comes in, it is combined with the result from the application of the conjunctive rule. In this way, associativity is preserved.

Imprecise PCR5 Fusion Rule (imp-PCR5)

The imp-PCR5 formula is a direct extension of the PCR5 formula using addition, multiplication, and division operators on sets. For the combination of $s \geq 2$ sources, it is given by $m^I_{PCR5}(\emptyset) = 0$ and $\forall X \in G \setminus \{\emptyset\}$:

$$m^I_{PCR5}(X) = [\boxed{\sum}_{\substack{X_1,X_2,\ldots,X_s \in G\\(X_1\cap X_2 \cap\ldots\cap X_s)=X}} \boxed{\prod}_{i=1,\ldots,s} m^I_i(X_i)]$$
$$\boxplus [\boxed{\sum}_{\substack{2\leq t\leq s\\1\leq r_1,\ldots,r_t\leq s\\1\leq r_1<r_2<\ldots<r_{t-1}<(r_t=s)}} \boxed{\sum}_{\substack{X_{j_2},\ldots,X_{j_t}\in G\setminus\{X\}\\ \{j_2,\ldots,j_t\}\in\mathcal{P}^{t-1}(\{1,\ldots,n\})\\ X\cap X_{j_2}\cap\ldots\cap X_{j_s}=\emptyset\\ \{i_1,\ldots,i_s\}\in\mathcal{P}^s(\{1,\ldots,s\})}} [Num^I(X) \boxslash Den^I(X)]] \quad (21)$$

where $Num^I(X)$ and $Den^I(X)$ are defined by

$$Num^I(X) \triangleq [\boxed{\prod}_{k_1=1,\ldots,r_1} m^I_{i_{k_1}}(X)^2] \boxdot [\boxed{\prod}_{l=2,\ldots,t} (\boxed{\prod}_{k_l=r_{l-1}+1,\ldots,r_l} m^I_{i_{k_l}}(X_{j_l})] \quad (22)$$

$$Den^I(X) \triangleq [\boxed{\prod}_{k_1=1,\ldots,r_1} m^I_{i_{k_1}}(X)] \boxplus [\boxed{\sum}_{l=2,\ldots,t} (\boxed{\prod}_{k_l=r_{l-1}+1,\ldots,r_l} m^I_{i_{k_l}}(X_{j_l})] \quad (23)$$

where all denominator-sets $Den^I(X)$ involved in (21) are different from zero. If a denominator-set $Den^I(X)$ is such that $\inf(Den^I(X)) = 0$, then the fraction is discarded. When $s = 2$ (i.e., fusion of only two sources), the previous imp-PCR5 formula reduces to the following simple fusion formula: $m^I_{PCR5}(\emptyset) = 0$ and $\forall X \in G \setminus \{\emptyset\}$

$$m^I_{PCR5}(X) = m^I_{12}(X) + \boxed{\sum}_{\substack{Y\in G\setminus\{X\}\\c(X\cap Y)=\emptyset}} [(m^I_1(X)^2 m^I_2(Y)) \boxslash (m^I_1(X) + m^I_2(Y))] \boxplus [(m^I_2(X)^2 m^I_1(Y)) \boxslash (m^I_2(X) + m^I_1(Y))] \quad (24)$$

with

$$m^I_{12}(X) \triangleq \boxed{\sum}_{\substack{X_1,X_2\in G\\X_1\cap X_2=X}} m^I_1(X_1) \boxdot m^I_2(X_2)$$

Table 1: Inputs for Fusion with Imprecise bba.

$A \in D^\Theta$	$m_1^I(A)$	$m_2^I(A)$
θ_1	$[0.1, 0.2] \cup \{0.3\}$	$[0.4, 0.5]$
θ_2	$(0.4, 0.6) \cup [0.7, 0.8]$	$[0, 0.4] \cup \{0.5, 0.6\}$

Example for imp-DSmC

Let us consider $\Theta = \{\theta_1, \theta_2\}$, two independent sources with the imprecise admissible bba given in Table 1.

Using imp-DSmC, i.e. the DSm classic rule for sets, one obtains[50]:

$$\begin{aligned} m^I_{DSmC}(\theta_1) &= ([0.1, 0.2] \cup \{0.3\}) \boxdot [0.4, 0.5] \\ &= ([0.1, 0.2] \boxdot [0.4, 0.5]) \cup (\{0.3\} \boxdot [0.4, 0.5]) \\ &= [0.04, 0.10] \cup [0.12, 0.15] \end{aligned}$$

$$\begin{aligned} m^I_{DSmC}(\theta_2) &= ((0.4, 0.6) \cup [0.7, 0.8]) \boxdot ([0, 0.4] \cup \{0.5, 0.6\}) \\ &= [0, 0.40] \cup [0.42, 0.48] \end{aligned}$$

$$\begin{aligned} m^I_{DSmC}(\theta_1 \cap \theta_2) &= [([0.1, 0.2] \cup \{0.3\}) \boxdot ([0, 0.4] \cup \{0.5, 0.6\})] \\ &\quad \boxplus [[0.4, 0.5] \boxdot ((0.4, 0.6) \cup [0.7, 0.8])] \\ &= (0.16, 0.58] \end{aligned}$$

Hence, finally, the fusion admissible result is given in Table 2.

Table 2: Fusion Result with imp-DSmC.

$A \in D^\Theta$	$m^I_{DSmC}(A) = [m_1^I \oplus m_2^I](A)$
θ_1	$[0.04, 0.10] \cup [0.12, 0.15]$
θ_2	$[0, 0.40] \cup [0.42, 0.48]$
$\theta_1 \cap \theta_2$	$(0.16, 0.58]$
$\theta_1 \cup \theta_2$	0

Example for imp-DSmH

If it happens[51] that $\theta_1 \cap \theta_2 \overset{\mathcal{M}}{\equiv} \emptyset$ (this is the hybrid model $\mathcal{M}$ one has to deal with), then the imprecise hybrid DSm rule (imp-DSmH) for sets (17) is used, and, therefore, the imprecise belief mass $m^I_{DSmC}(\theta_1 \cap \theta_2) = (0.16, 0.58]$ is then directly transferred to $\theta_1 \cup \theta_2$ and the other imprecise masses are not changed. Finally, the result obtained with the imp-DSmH rule is given in Table 3.

It can be easily checked that for source 1, there exist precise masses $(m_1(\theta_1) = 0.3) \in ([0.1, 0.2] \cup \{0.3\})$ and $(m_1(\theta_2) = 0.7) \in ((0.4, 0.6) \cup [0.7, 0.8])$, such that $0.3 + 0.7 = 1$; and for source 2, there exist precise masses $(m_1(\theta_1) = 0.4) \in ([0.4, 0.5])$ and $(m_2(\theta_2) = 0.6) \in ([0, 0.4] \cup \{0.5, 0.6\})$, such that $0.4 + 0.6 = 1$. Therefore, both sources associated with $m_1^I(.)$ and $m_2^I(.)$ are admissible imprecise sources of information. Also, it can be easily checked that DSmC yields the paradoxical basic belief assignment $m_{DSmC}(\theta_1) = [m_1 \oplus m_2](\theta_1) = 0.12$, $m_{DSmC}(\theta_2) = [m_1 \oplus m_2](\theta_2) = 0.42$, and $m_{DSmC}(\theta_1 \cap \theta_2) = [m_1 \oplus m_2](\theta_1 \cap \theta_2) = 0.46$. Table 2 demonstrates that the admissibility is satisfied since there exists at least a bba (here $m_{DSmC}(.)$) with $(m_{DSmC}(\theta_1) = 0.12) \in m^I_{DSmC}(\theta_1)$, $(m_{DSmC}(\theta_2) = 0.42) \in m^I_{DSmC}(\theta_2)$ and $(m_{DSmC}(\theta_1 \cap \theta_2) = 0.46) \in m^I_{DSmC}(\theta_1 \cap \theta_2$, such that $0.12 + 0.42 + 0.46 = 1$.

Table 3: Fusion Result with imp-DSmH for $\mathcal{M}(\Theta)$.

$A \in D^\Theta$	$m^I_{DSmH}(A) = [m_1^I \oplus m_2^I](A)$
θ_1	$[0.04, 0.10] \cup [0.12, 0.15]$
θ_2	$[0, 0.40] \cup [0.42, 0.48]$
$\theta_1 \cap \theta_2 \overset{\mathcal{M}}{\equiv} \emptyset$	0
$\theta_1 \cup \theta_2$	$(0.16, 0.58]$

Similarly, if it occurs that $\theta_1 \cap \theta_2 = \emptyset$, then one uses DSmH and $m_{DSmH}(\theta_1 \cap \theta_2) = 0$, and $m_{DSmH}(\theta_1 \cup \theta_2) = 0.46$ is obtained in result; the other masses remain unchanged. The admissibility still holds because one can pick at least one number in each subset $m^I_{DSmH}(.)$, such that the sum of these numbers is 1. This approach can be also used in a similar manner to obtain imprecise pignistic probabilities from $m^I_{DSmH}(.)$ for decision-making under quantitative uncertain, paradoxical, and imprecise sources of information as well.[53]

Table 4: Inputs for Fusion with Imprecise bba.

$m_1^I(\theta_1) = [0.1, 0.2] \cup \{0.3\}$	$m_1^I(\theta_2) = (0.4, 0.6) \cup [0.7, 0.8]$
$m_2^I(\theta_1) = [0.4, 0.5]$	$m_2^I(\theta_2) = [0, 0.4] \cup \{0.5, 0.6\}$

Examples for imp-PCR5

Example no. 1

Let us consider $\Theta = \{\theta_1, \theta_2\}$, Shafer's model and two independent sources with the same imprecise admissible bba as those given in Table 1 (see Table 4).

Working with sets, one gets for the conjunctive consensus:

$$m^I_{12}(\theta_1) = [0.04, 0.10] \cup [0.12, 0.15] \qquad m^I_{12}(\theta_2) = [0, 0.40] \cup [0.42, 0.48]$$

while the conflicting imprecise mass is given by:

$$k^I_{12} \equiv m^I_{12}(\theta_1 \cap \theta_2) = [m^I_1(\theta_1) \boxdot m^I_2(\theta_2)] \boxplus [m^I_1(\theta_2) \boxdot m^I_2(\theta_1)] = (0.16, 0.58].$$

Using the PCR5 rule for Proportional Conflict redistribution,

- One redistributes the partial imprecise conflicting mass $m^I_1(\theta_1) \boxdot m^I_2(\theta_2)$ to θ_1 and θ_2 proportionally to $m^I_1(\theta_1)$ and $m^I_2(\theta_2)$. Using the fraction bar symbol instead of $\boxslash$ for convenience to denote the division operator on sets, one has:

$$\begin{aligned}
\frac{x^I_1}{[0.1, 0.2] \cup \{0.3\}} &= \frac{y^I_1}{[0, 0.4] \cup \{0.5, 0.6\}} \\
&= \frac{([0.1, 0.2] \cup \{0.3\}) \boxdot ([0, 0.4] \cup \{0.5, 0.6\})}{([0.1, 0.2] \cup \{0.3\}) \boxplus ([0, 0.4] \cup \{0.5, 0.6\})} \\
&= \big[[0, 0.08] \cup [0.05, 0.10] \cup [0.06, 0.12] \\
&\qquad\qquad \cup [0, 0.12] \cup \{0.15, 0.18\}\big] \\
&\quad \boxslash \big[[0.1, 0.6] \cup [0.6, 0.7] \cup [0.7, 0.8] \\
&\qquad\qquad \cup [0.3, 0.7] \cup \{0.8, 0.9\}\big] \\
&= \frac{[0, 0.12] \cup \{0.15, 0.18\}}{[0.1, 0.8] \cup \{0.9\}}
\end{aligned}$$

whence

$$\begin{aligned}
x^I_1 &= \left[\frac{[0, 0.12] \cup \{0.15, 0.18\}}{[0.1, 0.8] \cup \{0.9\}}\right] \boxdot ([0.1, 0.2] \cup \{0.3\}) \\
&= \frac{[0, 0.024] \cup [0.015, 0.030] \cup [0.018, 0.036] \cup [0, 0.036] \cup \{0.045, 0.048\}}{[0.1, 0.8] \cup \{0.9\}} \\
&= \frac{[0, 0.036] \cup \{0.045, 0.048\}}{[0.1, 0.8] \cup \{0.9\}} \\
&= \left[\frac{0}{0.8}, \frac{0.036}{0.1}\right] \cup \left[\frac{0}{0.9}, \frac{0.036}{0.9}\right] \cup \left[\frac{0.045}{0.8}, \frac{0.045}{0.1}\right] \cup \left[\frac{0.048}{0.8}, \frac{0.048}{0.1}\right] \\
&= [0, 0.36] \cup [0, 0.04] \cup [0.05625, 0.45000] \cup [0.06, 0.48] = [0, 0.48]
\end{aligned}$$

$$
\begin{aligned}
y_1^I &= \left[\frac{[0,0.12]\cup\{0.15,0.18\}}{[0.1,0.8]\cup\{0.9\}}\right] \boxdot (0,0.4]\cup\{0.5,0.6\}) \\
&= \big[[0,0.048]\cup[0,0.060]\cup[0,0.072]\cup[0,0.6]\cup[0,0.072] \\
&\quad \cup\{0,075,0.090,0.090,0.108\}\big] \boxslash [0.1,0.8]\cup\{0.9\} \\
&= \frac{[0,0.072]\cup\{0,075,0.090,0.108\}}{[0.1,0.8]\cup\{0.9\}} \\
&= \left[\frac{0}{0.8},\frac{0.072}{0.1}\right]\cup\left[\frac{0}{0.9},\frac{0.072}{0.9}\right]\cup\left[\frac{0.075}{0.8},\frac{0.075}{0.1}\right] \\
&\quad \cup\left[\frac{0.090}{0.8},\frac{0.090}{0.1}\right]\cup\left[\frac{0.108}{0.8},\frac{0.108}{0.1}\right]\cup\left\{\frac{0.075}{0.9},\frac{0.090}{0.9},\frac{0.108}{0.9}\right\} \\
&= [0,0.72]\cup[0,0.08]\cup[0.09375,0.75]\cup[0.1125,0.9]\cup[0.135,1.08] \\
&\quad \cup\{0.083333,0.1,0.12\} \\
&= [0,1.08]\approx[0,1]
\end{aligned}
$$

- One redistributes the partial imprecise conflicting mass $m_1^I(\theta_2)\boxdot m_2^I(\theta_1)$ to θ_1 and θ_2 proportionally to $m_1^I(\theta_2)$ and $m_2^I(\theta_1)$. Now, the following proportionalization is obtained:

$$
\begin{aligned}
\frac{x_2^I}{[0.4,0.5]} &= \frac{y_2^I}{(0.4,0.6)\cup[0.7,0.8]} \\
&= \frac{([0.4,0.5]\boxdot((0.4,0.6)\cup[0.7,0.8])}{([0.4,0.5]\boxplus((0.4,0.6)\cup[0.7,0.8])} \\
&= \frac{(0.16,0.30)\cup[0.28,0.40]}{(0.8,1.1)\cup[1.1,1.3]} = \frac{(0.16,0.40]}{(0.8,1.3]}
\end{aligned}
$$

whence

$$
\begin{aligned}
x_2^I &= \frac{(0.16,0.40]}{(0.8,1.3]}\boxdot[0.4,0.5] \\
&= \frac{(0.064,0.200]}{(0.8,1.3]} \\
&= \left(\frac{0.064}{1.3},\frac{0.200}{0.8}\right) = (0.049231,0.250000)
\end{aligned}
$$

$$\begin{aligned}
y_2^I &= \frac{(0.16, 0.40]}{(0.8, 1.3]} \boxdot (0.4, 0.6) \cup [0.7, 0.8] \\
&= \frac{(0.064, 0.240) \cup (0.112, 0.320]}{(0.8, 1.3]} \\
&= \frac{(0.064, 0.320]}{(0.8, 1.3]} = \left(\frac{0.064}{1.3}, \frac{0.320}{0.8}\right) \\
&= (0.049231, 0.400000).
\end{aligned}$$

Hence, one finally gets with the imprecise PCR5:

$$\begin{aligned}
m_{PCR5}^I(\theta_1) &= m_{12}^I(\theta_1) \boxplus x_1^I \boxplus x_2^I \\
&= ([0.04, 0.10] \cup [0.12, 0.15]) \boxplus [0, 0.48] \boxplus (0.049231, 0.250000) \\
&= ([0.04, 0.10] \cup [0.12, 0.15]) \boxplus (0.049231, 0.73) \\
&= (0.089231, 0.83) \cup (0.169231, 0.88) \\
&= (0.089231, 0.88)
\end{aligned}$$

$$\begin{aligned}
m_{PCR5}^I(\theta_2) &= m_{12}^I(\theta_2) \boxplus y_1^I \boxplus y_2^I \\
&= ([0, 0.40] \cup [0.42, 0.48]) \boxplus [0, 1] \boxplus (0.049231, 0.400000) \approx [0, 1]
\end{aligned}$$

$$m_{PCR5}^I(\theta_1 \cap \theta_2) = 0.$$

Example no. 2

Let us consider a simpler example with $\Theta = \{\theta_1, \theta_2\}$, Shafer's model, and two independent sources with the following imprecise admissible bba:

$m_1^I(\theta_1) = (0.2, 0.3)$	$m_1^I(\theta_2) = [0.6, 0.8]$
$m_2^I(\theta_1) = [0.4, 0.7)$	$m_2^I(\theta_2) = (0.5, 0.6]$

Working with sets, one gets for the conjunctive consensus:

$$m_{12}^I(\theta_1) = (0.08, 0.21) \qquad m_{12}^I(\theta_2) = (0.30, 0.48)$$

The total (imprecise) conflict between the two imprecise quantitative sources is given by:

$$\begin{aligned}
k_{12}^I \equiv m_{12}^I(\theta_1 \cap \theta_2) &= [m_1^I(\theta_1) \boxdot m_2^I(\theta_2)] \boxplus [m_1^I(\theta_2) \boxdot m_2^I(\theta_1)] \\
&= ((0.2, 0.3) \boxdot (0.5, 0.6]) \boxplus ([0.4, 0.7] \boxdot [0.6, 0.8]) \\
&= (0.10, 0.18) \boxplus [0.24, 0.56) = (0.34, 0.74).
\end{aligned}$$

Using the PCR5 rule for Proportional Conflict redistribution of the partial (imprecise) conflict $m_1^I(\theta_1) \boxdot m_2^I(\theta_2)$, one has:

$$\frac{x_1^I}{(0.2, 0.3)} = \frac{y_1^I}{(0.5, 0.6]} = \frac{(0.2, 0.3) \boxdot (0.5, 0.6]}{(0.2, 0.3) \boxplus (0.5, 0.6]} = \frac{(0.10, 0.18)}{(0.7, 0.9)}$$

whence

$$\begin{aligned} x_1^I &= \frac{(0.10, 0.18)}{(0.7, 0.9)} \boxdot (0.2, 0.3) \\ &= \frac{(0.02, 0.054)}{(0.7, 0.9)} \\ &= (\frac{0.02}{0.9}, \frac{0.054}{0.7}) \\ &= (0.022222, 0.077143) \end{aligned}$$

$$\begin{aligned} y_1^I &= \frac{(0.10, 0.18)}{(0.7, 0.9)} \boxdot (0.5, 0.6] \\ &= \frac{(0.050, 0.108)}{(0.7, 0.9)} \\ &= \left(\frac{0.050}{0.9}, \frac{0.108}{0.7}\right) \\ &= (0.055556, 0.154286). \end{aligned}$$

Using the PCR5 rule for Proportional Conflict redistribution of the partial (imprecise) conflict $m_1^I(\theta_2) \boxdot m_2^I(\theta_1)$, one has:

$$\frac{x_2^I}{[0.4, 0.7)} = \frac{y_2^I}{[0.6, 0.8]} = \frac{[0.4, 0.7) \boxdot [0.6, 0.8]}{[0.4, 0.7) \boxplus [0.6, 0.8]} = \frac{[0.24, 0.56)}{[1, 1.5)}$$

whence

$$x_2^I = \frac{[0.24, 0.56)}{[1, 1.5)} \boxdot [0.4, 0.7) = \frac{[0.096, 0.392)}{[1, 1.5)} = \left(\frac{0.096}{1.5}, \frac{0.392}{1}\right) = (0.064, 0.392)$$

$$y_2^I = \frac{[0.24, 0.56)}{[1, 1.5)} \boxdot [0.6, 0.8] = \frac{[0.144, 0.448)}{[1, 1.5)} = \left(\frac{0.144}{1.5}, \frac{0.448}{1}\right) = (0.096, 0.448).$$

Hence, one finally gets with the imprecise PCR5:

$$\begin{aligned} m^I_{PCR5}(\theta_1) &= m^I_{12}(\theta_1) \boxplus x^I_1 \boxplus x^I_2 \\ &= (0.08, 0.21) \boxplus (0.022222, 0.077143) \boxplus (0.064, 0.392) \\ &= (0.166222, 0.679143) \end{aligned}$$

$$\begin{aligned} m^I_{PCR5}(\theta_2) &= m^I_{12}(\theta_2) \boxplus y^I_1 \boxplus y^I_2 \\ &= (0.30, 0.48) \boxplus (0.055556, 0.154286) \boxplus (0.096, 0.448) \\ &= (0.451556, 1.08229) \approx (0.451556, 1] \end{aligned}$$

$$m^I_{PCR5}(\theta_1 \cap \theta_2) = 0.$$

Fusion of Qualitative Beliefs

Different qualitative methods for reasoning under uncertainty have been developed mainly in the Artificial Intelligence field in the past few decades. These methods have attracted the attention of more and more people in the Information Fusion community, especially those working on development of modern multi-source[54] systems for defense applications. George Polya was the first mathematician who in 1954 attempted a formal characterization of qualitative human reasoning,[55] followed by the work of Lotfi Zadeh.[56] The interest in qualitative reasoning methods is based on the need to deal with decision-making situations where the precise numerical methods are not appropriate (when the information/inputs are not directly expressed in numbers). Several formalisms for qualitative reasoning have been proposed as extensions of the frames of probability, possibility, and/or evidence theories.[57] The limitations of the numerical techniques are discussed thoroughly by Parsons and Hunter.[58] The authors' intention in this article is not to browse and write a survey of all techniques dealing with qualitative information, but only to briefly mention the main attempts proposed to solve the combination problem. A good presentation of these techniques can be found in Parsons' milestone book.[59] Among all available techniques, one has to however give credit to Wellman's works,[60] who proposed a general characterization of "qualitative probability" to relax precision in representation and reasoning within the probabilistic framework. His "Qualitative" Probabilistic Networks (QPN) based on a Qualitative Probability Language (QPL) defined by a set of numerical underlying probability distributions belongs actually to the family of imprecise probability[61] and probability bounds analysis (PBA) methods[62] and cannot be considered truly as a qualitative approach since it deals with quantitative (imprecise) probability distributions. Based on Dempster-Shafer Theory, Wong and Lingras[63] proposed a method for generating a (numerical) basic belief function from preference relations between each pair of propositions specified qualitatively. Their method, however, does not provide a unique solution, does not check the consistency of qualitative preference relations, and cannot be truly considered as a fully qualitative method. Bryson and colleagues[64] proposed a

Qualitative Discriminant Procedure (QDP) that involves qualitative scoring, imprecise pairwise comparisons between pairs of propositions, and an optimization algorithm to generate consistent imprecise quantitative belief function to combine. Parsons[65] proposed for the first time (to authors' knowledge) a qualitative Dempster-Shafer Theory (QET) by using techniques from qualitative reasoning.[66] Based on operation tables, he introduced a very simple arithmetic for qualitative addition $(+)$ and multiplication $(\times)$ operators. Due to impossibility to perform a qualitative normalization, Parsons used the un-normalized version of Dempster's rule by committing a *qualitative mass* to the empty set following the open-world approach of Smets.[67] This approach, however, cannot deal with truly closed-world problems since unfortunately there is no way to transfer the conflicting qualitative mass or to normalize the qualitative belief assignments in the spirit of DST. From 1998, Parsons started to develop Qualitative Probabilistic Reasoner (QPR).[68] In mid nineties, Zadeh proposed a new paradigm for computing with words (CW)[69] to combine qualitative/vague information expressed in natural language. CW is done essentially in three major steps: 1) translation of the qualitative information into fuzzy membership functions, 2) fuzzy combination of fuzzy membership functions; and 3) retranslation of the fuzzy (quantitative) result into natural language. All these steps cannot be uniquely accomplished since they depend on the chosen fuzzy operators. A possible approach for the third step is proposed by Yager.[70]

In this section, the authors propose a simple arithmetic with linguistic labels that enables the direct extension of the classical (quantitative) combination rules proposed in the DSmT framework into their qualitative counterparts. The qualitative belief assignments are well adapted for the manipulated information expressed in natural language and usually reported by human experts or AI-based expert systems. In other words, the authors propose here a new method for computing directly with words (CW) and combining qualitative information. Computing with words, more precisely computing with linguistic labels, is usually more vague, less precise than computing with numbers, but it is expected to provide higher robustness and flexibility for combining uncertain and conflicting human reports than computing with numbers owing to the fact that in most of the cases the human experts are less efficient in providing (and justifying) precise quantitative beliefs than qualitative beliefs. Prior to extending the quantitative DSmT-based combination rules to their qualitative counterparts, it will be necessary to define some new important operators on linguistic labels and the term qualitative belief assignment. Then, the authors will demonstrate by means of a few examples how the combination of qualitative beliefs can be accomplished in the DSmT framework.

Qualitative Operators

Let us define a finite set of linguistic labels $\tilde{L} = \{L_1, L_2, \ldots, L_m\}$, where $m \geq 2$ is an integer. $\tilde{L}$ is endowed with a total order relationship $\prec$, so that $L_1 \prec L_2 \prec \ldots \prec L_m$. In order to work on a closed linguistic set under linguistic addition and multiplication operators, we extend $\tilde{L}$ with two extreme values L_0 and L_{m+1}, where L_0 corresponds to the minimal qualitative value and L_{m+1} corresponds to the maximal qualitative value in such a way that

$$L_0 \prec L_1 \prec L_2 \prec \ldots \prec L_m \prec L_{m+1},$$

where $\prec$ means inferior to, or less (in quality) than, or smaller (in quality) than, etc.; hence, a relation of order from a qualitative point of view is established. However, if a correspondence between qualitative labels and quantitative values on the scale $[0,1]$ is considered, then $L_{\min} = L_0$ would correspond to the numerical value 0, while $L_{\max} = L_{m+1}$ would correspond to the numerical value 1, and each L_i would belong to $[0,1]$, i. e.

$$L_{\min} = L_0 < L_1 < L_2 < \ldots < L_m < L_{m+1} = L_{\max}.$$

Starting from here, we work on extended ordered set L of qualitative values:

$$L = \{L_0, \tilde{L}, L_{m+1}\} = \{L_0, L_1, L_2, \ldots, L_m, L_{m+1}\}.$$

The qualitative addition and multiplication operators are defined in the following way, respectively:

- Addition:
$$L_i + L_j = \begin{cases} L_{i+j}, & \text{if } i+j \leq m+1, \\ L_{m+1}, & \text{if } i+j > m+1. \end{cases} \tag{25}$$

- Multiplication:
$$L_i \times L_j = L_{\min\{i,j\}}. \tag{26}$$

These two operators are well-defined, commutative, associative, and unitary. Addition of labels is a unitary operation since $L_0 = L_{\min}$ is the unitary element, i.e. $L_i + L_0 = L_0 + L_i = L_{i+0} = L_i$ for all $0 \leq i \leq m+1$. Multiplication of labels is also a unitary operation since $L_{m+1} = L_{\max}$ is the unitary element, i.e. $L_i \times L_{m+1} = L_{m+1} \times L_i = L_{\min\{i,m+1\}} = L_i$ for $0 \leq i \leq m+1$. L_0 is the unit element for addition, while L_{m+1} is the unit element for multiplication. L is closed under $(+)$ and $(\times)$. The mathematical structure formed by $(L, +, \times)$ is a commutative bi-semigroup with different unitary elements for each operation. We recall that a bi-semigroup is a set S endowed with two associative binary operations such that S is closed under both operations.

If L is not an exhaustive set of qualitative labels, then other labels may exist in between the initial ones, so we can work with labels and numbers – since a refinement of L is possible. When mapping from L to crisp numbers or intervals, $L_0 = 0$ and $L_{m+1} = 1$, while $0 < L_i < 1$, for all i, as crisp numbers, or L_i included in $[0,1]$ as intervals/ subsets.

For example, L_1, L_2, L_3, and L_4 may represent the following qualitative values: $L_1 \triangleq$ very poor, $L_2 \triangleq$ poor, $L_3 \triangleq$ good, and $L_4 \triangleq$ very good, where the symbol $\triangleq$ means "by definition."

According to the authors, it is better to define the multiplication ($\times$) of $L_i \times L_j$ by $L_{\min\{i,j\}}$, since when multiplying two numbers a and b in $[0,1]$ one gets a result which is less than each of them; the product is not bigger than the two numbers as Bolanos and team did in their work[71] by approximating it as follows: $L_i \times L_j = L_{i+j} > \max\{L_i, L_j\}$. While, for addition it is just the opposite: adding two numbers in the interval $[0,1]$, the sum should be bigger than both of them, not smaller as Bolanos and colleagues[72] approximated it: $L_i + L_j = \min\{L_i, L_j\} < \max\{L_i, L_j\}$.

Qualitative Belief Assignment

We define a qualitative belief assignment (qba) and we call it *qualitative belief mass*, or *q-mass* for short, as a mapping function

$$qm(.) : G \mapsto L,$$

where G corresponds to the space of propositions generated with $\cap$ and $\cup$ operators and elements of Θ, taking into account the integrity constraints of the model. For example, if the Shafer's model is chosen for Θ, then G is nothing but the classical power set 2^{Θ},[73] whereas if the free DSm model is adopted, G will correspond to Dedekind's lattice (hyper-power set) D^{Θ}.[74] Note that in this qualitative framework there is no way to define normalized $qm(.)$, but qualitative quasi-normalization is still possible as shown later. Using the qualitative operations defined previously, we can easily extend the combination rules from quantitative to qualitative ones. In the sequel, we will consider $s \geq 2$ qualitative belief assignments $qm_1(.), \ldots, qm_s(.)$ defined over the same space G and provided by s independent sources $S_1, \ldots, S_s$ of evidence.

Important note: The addition and multiplication operators used in all qualitative fusion formulas in the following sections correspond to *qualitative addition* and *qualitative multiplication* operators defined in (25) and (26) and do not have to be confused with classical addition and multiplication operators for numbers.

Qualitative Conjunctive Rule (qCR)

The qualitative Conjunctive Rule (qCR) for $s \geq 2$ sources is defined similarly to the quantitative conjunctive consensus rule, i.e.

$$qm_{qCR}(X) = \sum_{\substack{X_1,\ldots,X_s \in G \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} qm_i(X_i) \tag{27}$$

The total qualitative conflicting mass is given by:

$$K_{1\dots s} = \sum_{\substack{X_1,\dots,X_s\in G\\ X_1\cap\dots\cap X_s=\emptyset}} \prod_{i=1}^{s} qm_i(X_i).$$

Qualitative DSm Classic Rule (q-DSmC)

The qualitative DSm Classic rule (qDSmC) for $s \geq 2$ is defined similarly to the DSm Classic fusion rule (DSmC) as follows: $qm_{qDSmC}(\emptyset) = L_0$ and for all $X \in D^\Theta \setminus \{\emptyset\}$,

$$qm_{qDSmC}(X) = \sum_{\substack{X_1,,\dots,X_s\in D^\Theta\\ X_1\cap\dots\cap X_s=X}} \prod_{i=1}^{s} qm_i(X_i) \qquad (28)$$

Qualitative DSm Hybrid Rule (q-DSmH)

The qualitative DSm Hybrid rule (qDSmH) is defined similarly to the quantitative DSm hybrid rule[75] as follows: $qm_{qDSmH}(\emptyset) = L_0$ and for all $X \in G \setminus \{\emptyset\}$

$$qm_{qDSmH}(X) \triangleq \phi(X) \cdot \Big[qS_1(X) + qS_2(X) + qS_3(X)\Big], \qquad (29)$$

where $\phi(X)$ is the *characteristic non-emptiness function* of a set X, i.e. $\phi(X) = L_{m+1}$ if $X \notin \boldsymbol{\emptyset}$ and $\phi(X) = L_0$ otherwise, where $\boldsymbol{\emptyset} \triangleq \{\boldsymbol{\emptyset}_\mathcal{M}, \emptyset\}$. $\boldsymbol{\emptyset}_\mathcal{M}$ is the set of all elements of D^Θ, which have been forced to be empty by the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical/ universal empty set. $qS_1(X) \equiv qm_{qDSmC}(X)$, $qS_2(X)$, and $qS_3(X)$ are defined by:

$$qS_1(X) \triangleq \sum_{\substack{X_1,X_2,\dots,X_s\in D^\Theta\\ (X_1\cap X_2\cap\dots\cap X_s)=X}} \prod_{i=1}^{s} qm_i(X_i) \qquad (30)$$

$$qS_2(X) \triangleq \sum_{\substack{X_1,X_2,\dots,X_s\in\boldsymbol{\emptyset}\\ [\mathcal{U}=X]\vee[(\mathcal{U}\in\boldsymbol{\emptyset})\wedge(X=I_t)]}} \prod_{i=1}^{s} qm_i(X_i) \qquad (31)$$

$$qS_3(X) \triangleq \sum_{\substack{X_1,X_2,\dots,X_k\in D^\Theta\\ X_1\cup X_2\cup\dots\cup X_s=X\\ X_1\cap X_2\cap\dots\cap X_s\in\boldsymbol{\emptyset}}} \prod_{i=1}^{s} qm_i(X_i) \qquad (32)$$

where all sets involved in the formulas are expressed in canonical form and $\mathcal{U} \triangleq u(X_1) \cup \dots \cup u(X_s)$, where $u(X)$ is the union of all θ_i that compose X, and $I_t \triangleq \theta_1 \cup \dots \cup \theta_n$ is the total ignorance. $qS_1(X)$ is nothing but the qDSmC rule for s independent sources based on $\mathcal{M}^f(\Theta)$; $qS_2(X)$ is the qualitative mass of all relatively and absolutely empty sets, which is transferred to the total or relative ignorances associated with non-existential constraints (if any, like in some dynamic problems); $qS_3(X)$ transfers the sum of relatively empty sets directly to the canonical disjunctive form of non-empty sets. qDSmH generalizes qDSmC so as to work for any model (free DSm, Shafer's, or any hybrid model) when manipulating qualitative belief assignments.

Qualitative PCR5 Rule (q-PCR5)

It has been proved in the classical/ quantitative DSmT framework that the Proportional Conflict Redistribution rule no. 5 (PCR5) provides very good and coherent results for combining (quantitative) belief masses.[76] When dealing with qualitative beliefs and using Dempster-Shafer Theory (DST), we unfortunately could not normalize since it is not possible to divide linguistic labels by linguistic labels. Other researchers have used the un-normalized Dempster's rule, which is actually equivalent to the Conjunctive Rule on the Shafer's model and to the DSm conjunctive rule on hybrid and free DSm models, respectively. Following the idea of the (quantitative) PCR5 fusion rule (9) we could, however, use a rough approximation for a qualitative version of PCR5 (denoted qPCR5) as will be presented in the next example. Unfortunately, the authors have not succeeded so far to derive a general formula for the qualitative PCR5 fusion rule (q-PCR5) due to the fact that the division of labels could not be defined.

Table 5: Addition Table.

$+$	L_0	L_1	L_2	L_3	L_4	L_5
L_0	L_0	L_1	L_2	L_3	L_4	L_5
L_1	L_1	L_2	L_3	L_4	L_5	L_5
L_2	L_2	L_3	L_4	L_5	L_5	L_5
L_3	L_3	L_4	L_5	L_5	L_5	L_5
L_4	L_4	L_5	L_5	L_5	L_5	L_5
L_5	L_5	L_5	L_5	L_5	L_5	L_5

Example

Let us consider the following set of ordered linguistic labels $L = \{L_0, L_1, L_2, L_3, L_4, L_5\}$ (for example, L_1, L_2, L_3, and L_4 may represent the values: $L_1 \triangleq$ *very poor*, $L_2 \triangleq$ *poor*, $L_3 \triangleq$ *good*, and $L_4 \triangleq$ *very good*, where the symbol $\triangleq$ means *by definition*), then the addition and multiplication tables are as given in Table 5 and Table 6, respectively.

Let us now consider a simple two-source case with a 2D-frame $\Theta = \{\theta_1, \theta_2\}$, Shafer's model for Θ, and qba expressed as follows:

$$qm_1(\theta_1) = L_1, \quad qm_1(\theta_2) = L_3, \quad qm_1(\theta_1 \cup \theta_2) = L_1$$

$$qm_2(\theta_1) = L_2, \quad qm_2(\theta_2) = L_1, \quad qm_2(\theta_1 \cup \theta_2) = L_2$$

Table 6: Multiplication Table.

$\times$	L_0	L_1	L_2	L_3	L_4	L_5
L_0	L_0	L_0	L_0	L_0	L_0	L_0
L_1	L_0	L_1	L_1	L_1	L_1	L_1
L_2	L_0	L_1	L_2	L_2	L_2	L_2
L_3	L_0	L_1	L_2	L_3	L_3	L_3
L_4	L_0	L_1	L_2	L_3	L_4	L_4
L_5	L_0	L_1	L_2	L_3	L_4	L_5

- *Fusion with qCR*: According to the qCR combination rule (27), the result shown in Table 7 will be obtained since:

$$\begin{aligned} qm_{qCR}(\theta_1) &= qm_1(\theta_1)qm_2(\theta_1) + qm_1(\theta_1)qm_2(\theta_1 \cup \theta_2) \\ &\quad + qm_2(\theta_1)qm_1(\theta_1 \cup \theta_2) \\ &= (L_1 \times L_2) + (L_1 \times L_2) + (L_2 \times L_1) \\ &= L_1 + L_1 + L_1 = L_{1+1+1} = L_3 \end{aligned}$$

$$\begin{aligned} qm_{qCR}(\theta_2) &= qm_1(\theta_2)qm_2(\theta_2) + qm_1(\theta_2)qm_2(\theta_1 \cup \theta_2) \\ &\quad + qm_2(\theta_2)qm_1(\theta_1 \cup \theta_2) \\ &= (L_3 \times L_1) + (L_3 \times L_2) + (L_1 \times L_1) \\ &= L_1 + L_2 + L_1 = L_{1+2+1} = L_4 \end{aligned}$$

$$qm_{qCR}(\theta_1 \cup \theta_2) = qm_1(\theta_1 \cup \theta_2)qm_2(\theta_1 \cup \theta_2) = L_1 \times L_2 = L_1$$

$$\begin{aligned} qm_{qCR}(\emptyset) \triangleq K_{12} &= qm_1(\theta_1)qm_2(\theta_2) + qm_1(\theta_2)qm_2(\theta_1) \\ &= (L_1 \times L_1) + (L_2 \times L_3) = L_1 + L_2 = L_3 \end{aligned}$$

Table 8 summarizes the results.

Table 7: Fusion with qCR.

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\emptyset$	$\theta_1 \cap \theta_2$
$qm_1(.)$	L_1	L_3	L_1		
$qm_2(.)$	L_2	L_1	L_2		
$qm_{qCR}(.)$	L_3	L_4	L_1	L_3	L_0

- *Fusion with qDSmC*: If we accept the free-DSm model instead of the Shafer's model, according to the qDSmC combination rule (28), the result given in Table 8 will be obtained.

Table 8: Fusion with qDSmC.

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\emptyset$	$\theta_1 \cap \theta_2$
$qm_1(.)$	L_1	L_3	L_1		
$qm_2(.)$	L_2	L_1	L_2		
$qm_{qDSmC}(.)$	L_3	L_4	L_1	L_0	L_3

- *Fusion with (qDSmH)*: Working with the Shafer's model for Θ, according to the qDSmH combination rule (29), one gets the result presented in Table 9, since $qm_{qDSmH}(\theta_1 \cup \theta_2) = L_1 + L_3 = L_4$.

Table 9: Fusion with qDSmH.

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\emptyset$	$\theta_1 \cap \theta_2$
$qm_1(.)$	L_1	L_3	L_1		
$qm_2(.)$	L_2	L_1	L_2		
$qm_{qDSmH}(.)$	L_3	L_4	L_4	L_0	L_0

- *Fusion with (qPCR5)*: Following the PCR5 method, we propose to transfer the qualitative partial masses:

  a) $qm_1(\theta_1)qm_2(\theta_2) = L_1 \times L_1 = L_1$ to θ_1 and θ_2 in equal parts (i.e., proportionally to L_1 and L_1, respectively, but $L_1 = L_1$); hence, $\frac{1}{2}L_1$ should go to each of them.

  b) $qm_2(\theta_1)qm_1(\theta_2) = L_2 \times L_3 = L_2$ to θ_1 and θ_2 proportionally to L_2 and L_3, respectively; however, since we are not able to do an exact proportionalization of labels, it is approximated through transferring $\frac{1}{3}L_2$ to θ_1 and $\frac{2}{3}L_2$ to θ_2.

  The transfer $1/3L_2$ to θ_1 and $2/3L_2$ to θ_2 is not arbitrary; it is an approximation since the transfer was performed proportionally to L_2 and L_3, and L_2 is smaller than L_3; the authors note that it is not possible to perfrom exact transfer. To the authors' knowledge, normalization of labels has not been done in the literature so far and the authors attempted a quasi-normalization here (i.e., an approximation).

  Summing a) and b) one gets: $\frac{1}{2}L_1 + \frac{1}{3}L_2 \approx L_1$, which represents the partial conflicting qualitative mass transferred to θ_1 and $\frac{1}{2}L_1 + \frac{2}{3}L_2 \approx L_2$, which represents the partial conflicting qualitative mass transferred to θ_2. Here, we have mixed qualitative and quantitative information.

  Therefore, the result presented in Table 10 is obtained finally.

Table 10: Fusion with qPCR5.

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\emptyset$	$\theta_1 \cap \theta_2$
$qm_1(.)$	L_1	L_3	L_1		
$qm_2(.)$	L_2	L_1	L_2		
$qm_{qPCR5}(.)$	L_4	L_5	L_1	L_0	L_0

For the reason that it is not possible to perform normalization (none of the previous researchers working on qualitative fusion were able), the authors propose for the first time *quasi-normalization* (which is an approximation of the normalization), i.e., instead of dividing each qualitative mass by a coefficient of normalization, the authors *subtract* from each qualitative mass a qualitative coefficient (label) of quasi-normalization in order to adjust the sum of masses.

Subtraction on L is defined in a manner similar to addition:

$$L_i - L_j = \begin{cases} L_{i-j}, & \text{if } i \geq j; \\ L_0, & \text{if } i < j; \end{cases} \tag{33}$$

L is closed under subtraction as well.

The subtraction can be used for quasi-normalization only, i.e., moving the final label result 1-2 steps/labels up or down. It is not used together with addition or multiplication.

The increment in the sum of fusioned qualitative masses is caused by the multiplication on L that is approximated by a bigger number due to the fact that multiplying any two numbers a and b in the interval $[0, 1]$ gives a product that is less than each of them, or we have approximated the product $a \times b = \min\{a, b\}$.

Using the quasi-normalization (subtracting L_1), one obtains with qDSmH and qPCR5 the *quasi-normalized* masses given in Table 11 (we use the $\star$ symbol to specify quasi-normalization):.

Table 11: Fusion with Quasi-Normalization.

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\emptyset$	$\theta_1 \cap \theta_2$
$qm_1(.)$	L_1	L_3	L_1		
$qm_2(.)$	L_2	L_1	L_2		
$qm^\star_{qDSmH}(.)$	L_2	L_3	L_3	L_0	L_0
$qm^\star_{qPCR5}(.)$	L_3	L_4	L_0	L_0	L_0

Conclusion

This article has presented the foundations of DSmT and its main combination rules for dealing with both quantitative and qualitative beliefs. The combination of qualitative beliefs presented here has resulted from very recent research investigations. DSmT, although not sufficiently known yet in the information fusion and artificial intelligence communities, as any new emerging theory, has however already been successfully applied in different fields such as multi-target tracking and classification and remote sensing applications. The authors hope that this special issue of *Information & Security: An International Journal* devoted to fusing uncertain, imprecise, and conflicting information will help readers involved in information fusion to become interested and hopefully more acquainted with authors' research work and their new ideas in data fusion. DSmT is a new promising paradigm shift for the combination of precise (and even imprecise), uncertain, and potentially highly conflicting quantitative or qualitative sources of information. It is important to emphasize that most of the methods, like the discounting techniques for example, developed to improve the management of quantitative beliefs in Dempster-Shafer Theory can also be directly applied in the DSmT framework.

Notes:

1. Florentin Smarandache and Jean Dezert, eds., *Advances and Applications of DSmT for Information Fusion* (Rehoboth: American Research Press, 2004), <http://www.gallup.unm.edu/~smarandache/DSmT-book1.pdf>.
2. Glenn Shafer, *A Mathematical Theory of Evidence* (Princeton, N.J.: Princeton University Press, 1976).
3. Lotfi A. Zadeh, *On the Validity of Dempster's Rule of Combination*, Memo M 79/24 (University of California, Berkeley, 1979); Frans Voorbraak, "On the Justification of Dempster's Rule of Combination," *Artificial Intelligence* 48, no. 2 (March 1991): 171–197; Lotfi

A. Zadeh, "A Simple View of the Dempster-Shafer Theory of Evidence and its Implication for the Rule of Combination," *AI Magazine* 7, no. 2 (1986): 85–90.

4. Didier Dubois and Henri Prade, "On the Unicity of Dempster Rule of Combination," *International Journal of Intelligent Systems* 1, no. 2 (1986): 133–142; Ronald R. Yager, "On the Dempster-Shafer Framework and New Combination Rules," *Information Sciences* 41 (1987): 93–138; Toshiyuki Inagaki, "Interdependence between Safety-control Policy and Multiple-sensor Schemes via Dempster-Shafer Theory," *IEEE Transactions on Reliability* 40, no. 2 (1991): 182–188; Eric Lefèvre, Olivier Colot, and Patrick Vannoorenberghe, "Belief Functions Combination and Conflict Management," *Information Fusion* 3, no. 2 (June 2002): 149–162; Kari Sentz and Scott Ferson, *Combination of Evidence in Dempster-Shafer Theory*, SANDIA Techical Report SAND2002-0835 (Sandia National Laboratories, Albuquerque, NM, April 2002), 96 pages; Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*.
5. Florentin Smarandache and Jean Dezert, "Proportional Conflict Redistribution Rules for Information Fusion," to appear in *Advances and Applications of DSmT for Information Fusion*, (Collected Works, Volume 2), ed. Florentin Smarandache and Jean Dezert (American Research Press, in preparation, July 2006 (preliminary version available at <http://arxiv.org/pdf/cs.AI/0408064>); Florentin Smarandache and Jean Dezert, "Information Fusion Based on New Proportional Conflict Redistribution Rules," in *Proceedings of the* 8^{th} *International Conference on Information Fusion - Fusion 2005*, volume 2 (Philadelphia, PA, 25-28 July 2005); Mihai Cristian Florea, Jean Dezert, Pierre Valin, Florentin Smarandache, and Anne-Laure Jouselme, "Adaptative Combination Rule and Proportional Conflict Redistribution Rule for Information Fusion," in *Proceedings of the Cogis 2006 Conference* (Paris, March 2006).
6. i.e. when the frame Θ and/or the model $\mathcal{M}$ is changing with time.
7. Jean Dezert and Florentin Smarandache, "A Short Overview on DSmT for Information Fusion," in *Proceedings of the* 10^{th} *International Conference on Fuzzy Theory and Technology FT&T 2005* (Salt Lake City, Utah, USA, 21-26 July 2005); Jean Dezert and Florentin Smarandache, "The DSmT Approach for Information Fusion and Some Open Challenging Problems" (paper presentaed at the NATO Advanced Study Institute, Albena, Bulgaria, 16-27 May 2005); Florentin Smarandache and Jean Dezert, "An introduction to DSm Theory of Plausible, Paradoxist, Uncertain, and Imprecise Reasoning for Information Fusion," in *Proceedings of the* 13^{th} *International Congress on Cybernetics and Systems* (Maribor, Slovenia, 6-10 July 2005); Jean Dezert and Florentin Smarandache, "DSmT: A New Paradigm Shift for Information Fusion" (paper presented at the Cogis'06 Conference) (Paris, France, 15-17 March 2006).
8. The authors do not assume here that elements θ_i have the same intrinsic nature and are necessarily exclusive. There is no other restriction on θ_i but the exhaustivity constraint, which is not a strong constraint since we can always introduce if necessary a closure element representing all missing hypotheses, say θ_0, in order to always work in a closed world.
9. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*.
10. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*.

11. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*.
12. Also called belief mass in the literature.
13. Shafer, *A Mathematical Theory of Evidence*.
14. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*.
15. Dubois and Prade, "On the Unicity of Dempster Rule of Combination."
16. Smarandache and Dezert, "Information Fusion Based on New Proportional Conflict Redistribution Rules;" Smarandache and Dezert, "Proportional Conflict Redistribution Rules for Information Fusion."
17. Shafer, *A Mathematical Theory of Evidence*; Smarandache and Dezert, eds., *Applications and Advances of DSmT for Information Fusion*.
18. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*; Smarandache and Dezert, "Information Fusion Based on New Proportional Conflict Redistribution Rules;" Smarandache and Dezert, "Proportional Conflict Redistribution Rules for Information Fusion;" Dezert and Smarandache, "DSmT: A New Paradigm Shift for Information Fusion;" Florea, Dezert, Valin, Smarandache, and Jouselme, "Adaptative Combination Rule and Proportional Conflict Redistribution Rule for Information Fusion."
19. While independence is a difficult concept to define in all theories managing epistemic uncertainty, we consider that two sources of evidence are independent (i.e. distinct and noninteracting) if each leaves one totally ignorant about the particular value the other will take.
20. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*.
21. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*.
22. Zadeh, *On the Validity of Dempster's Rule of Combination*; Zadeh, "A Simple View of the Dempster-Shafer Theory of Evidence and its Implication for the Rule of Combination."
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25. Gagnon, *Rapport Fusion de Données*.
26. Shafer, *A Mathematical Theory of Evidence*.
27. Rolf Haenni, "Shedding New Light on Zadeh's Criticism of Dempster's Rule of Combination," in *Proceedings of the International Conference on Information Fusion - Fusion 2005* (Philadelphia, PA, 26-29 July 2005).
28. Dezert and Smarandache, "DSmT: A New Paradigm Shift for Information Fusion."
29. Smarandache and Dezert, "Proportional Conflict Redistribution Rules for Information Fusion;" Smarandache and Dezert, "Information Fusion Based on New Proportional Conflict Redistribution Rules."

30. Smarandache and Dezert, "Proportional Conflict Redistribution Rules for Information Fusion;" Smarandache and Dezert, "Information Fusion Based on New Proportional Conflict Redistribution Rules."
31. Arnaud Martin and Christophe Osswald, "A New Generalization of the Proportional Conflict Redistribution Rule Stable in Terms of Decision," to appear in *Advances and Applications of DSmT for Information Fusion*, (Collected Works, Volume 2), ed. Florentin Smarandache and Jean Dezert (American Research Press, July 2006) (in preparation).
32. Smarandache and Dezert, "Proportional Conflict Redistribution Rules for Information Fusion."
33. Smarandache and Dezert, "Proportional Conflict Redistribution Rules for Information Fusion."
34. Smarandache and Dezert, "Proportional Conflict Redistribution Rules for Information Fusion."
35. A focal element is an element carrying strictly positive belief mass.
36. Florea, Dezert, Valin, Smarandache, and Jouselme, "Adaptative Combination Rule and Proportional Conflict Redistribution Rule for Information Fusion."
37. Inagaki, "Interdependence between Safety-control Policy and Multiple-sensor Schemes via Dempster-Shafer Theory."
38. Florea, Dezert, Valin, Smarandache, and Jouselme, "Adaptative Combination Rule and Proportional Conflict Redistribution Rule for Information Fusion."
39. Inagaki, "Interdependence between Safety-control Policy and Multiple-sensor Schemes via Dempster-Shafer Theory."
40. Zadeh, *On the Validity of Dempster's Rule of Combination*; Zadeh, "A Simple View of the Dempster-Shafer Theory of Evidence and its Implication for the Rule of Combination."
41. Zadeh, *On the Validity of Dempster's Rule of Combination*; Dubois and Prade, "On the Unicity of Dempster Rule of Combination;" Yager, "On the Dempster-Shafer Framework and New Combination Rules;" Voorbraak, "On the Justification of Dempster's Rule of Combination;" Smarandache and Dezert, eds., *Applications and Advances of DSmT for Information Fusion*.
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43. Ronald Yager suggested in his rule transfering the total conflicting mass to the total ignorance instead of using normalization as in Dempster's rule. Yager, "On the Dempster-Shafer Framework and New Combination Rules."
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45. Indeed, the DP rule has been developed for static fusion only, while the DSmH rule has been developed to take into account the possible dynamics of the frame itself and also its associated model.
46. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*; Smarandache and Dezert, "Proportional Conflict Redistribution Rules for Information Fusion."
47. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*.

48. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*, chapter 6.
49. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*.
50. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*, 139-140.
51. We consider now a dynamic/ temporal fusion problem.
52. A complete derivation of this result can be found in Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*, 139-140.
53. Smarandache and Dezert, eds., *Advances and Applications of DSmT for Information Fusion*; Jean Dezert, Florentin Smarandache, and Milan Daniel, "The Generalized Pignistic Transform," in *Proceedings of Fusion 2004 Conference - Applications and Advances of Plausible and Paradoxical Reasoning for Data Fusion* (Stockholm, Sweden, 28 June - 1 July 2004), 384–391.
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FUSION 2000, July 10-13, held in Paris. He has been involved in the Technical Program Committees of Fusion 2001-2004 International Conferences. He has been a member of the board of the International Society of Information Fusion (http://www.isif.org) and has served as secretary for ISIF since 2001. Dr. Dezert was executive vice-president of ISIF in 2004. In 2003, he organized together with Professor Smarandache the first special session devoted to plausible and paradoxical reasoning for information fusion at the International Conference on Information Fusion, Fusion 2003, Cairns, Australia, and also a panel discussion and a special session on DSmT at Fusion 2004, held in Stockholm in June 2004. Dr. Dezert taught several invited seminars and lectures on Data Fusion and Tracking in the last several years. He participated also as member of the Conference Technical Committee of Fusion 2005 and Fusion 2006 International Conferences on Information Fusion and Fuzzy Sets and Technology Conference, Salt Lake City, USA, in July 2005. He is also Associate Editor of the *Journal of Advances in Information Fusion* (JAIF). Most recent advances on DSmT can be found at: http://www.gallup.unm.edu/ smarandache/DSmT.htm.

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GENERALIZATION OF THE CLASSIC COMBINATION RULES TO DSm HYPER-POWER SETS

Milan DANIEL

Abstract: In this article, the author generalizes Dempster's rule, Yager's rule, and Dubois-Prade's rule for belief functions combination in order to be applicable to hyper-power sets according to the Dezert-Smarandache (DSm) Theory. A comparison of the rules with the DSm rule of combination is further presented.

Keywords: Dempster-Shafer Theory, Dempster's Rule, Yager's Rule, Dubois-Prade's Rule, DSm Theory, Hyper-power Set, DSm Model, DSm Rule of Combination.

Introduction

Belief functions are one of the widely-used formalisms for uncertainty representation and processing. Belief functions enable representation of incomplete and uncertain knowledge, belief updating, and combination of evidence. Belief functions were originally introduced as a principal notion in Dempster-Shafer Theory (DST) and in the Mathematical Theory of Evidence. [1]

Dempster's rule of combination is used for combination of beliefs in DST. Under strict probabilistic assumptions, the results of its implementation are correct and probabilistically interpretable for any couple of belief functions. However, these assumptions are rarely met in real applications. It is not uncommon to find examples where the assumptions are not met and where the results of Dempster's rule are counter-intuitive. [2] Thus, in such situations, a rule with more intuitive results is required.

For that reason, a series of modifications of Dempster's rule were suggested and alternative approaches presented. The classical ones are Dubois and Prade's rule [3] and Yager's belief combination rule. [4] Other approaches include a wide class of weighted operators, [5] the Transferable Belief Model (TBM) using the so-called non-normalized

Dempster's rule,[6] disjunctive (or dual Demspter's) rule of combination,[7] combination 'per elements' with its special case—minC combination[8]—and other combination rules. It is also necessary to mention the application of Dempster's rule in the case of partially reliable input beliefs.[9]

The Dezert-Smarandache (or Demspter-Shafer modified) theory (DSmT) with its DSm rule of combination presents a brand new approach. There are two main differences: 1) mutual exclusivity of elements of a frame of discernment is not assumed in general (mathematically it means that belief functions are not defined on the power set of the frame but on a so-called hyper-power set, i.e., on the Dedekind lattice defined by the frame); and 2) a new combination mechanism that overcomes problems with conflicts among the combined beliefs and also enables a dynamic fusion of beliefs.

Owing to the fact that the classical Shafer's frame of discernment could be considered the special case of a so-called hybrid DSm model, the DSm rule of combination has been compared with the classical rules of combination in the publications on DSmT.[10]

Unfortunately, none of the classical combination rules has been formally generalized to hyper-power sets, thus their comparison with the DSm rule is not fully objective until now.

This article presents a formal generalization of the classical Dempster's, Dubois-Prade's, and Yager's rules to hyper-power sets. These generalizations provide a solid theoretical background for a serious objective comparison of the DSm rule with the classical combination rules.

The next section briefly recalls the definitions of Dempster's, Dubois-Prade's, and Yager's combination rules followed by a section devoted to the basic notions of DSmT (Dedekind lattice, hyper-power set, DSm models, and DSm rule of belief combination).

A generalization of Dempster's rule is presented afterwards, followed by a section that generalizes Yager's rule. Both these classical rules are generalized in a straightforward manner as their ideas work on hyper-power sets simply without any problem.

More interesting and challenging is the case of Dubois-Prade's rule. The nature of this rule is closer to the DSm rule, but, on the other hand, the generalized Dubois-Prade's rule is not compatible with dynamic fusion in general. It works only for dynamic fusion without non-existential constraints, whereas a further extension of the generalized rule is necessary in the case of dynamic fusion with non-existential constraints.

A brief comparison of the rules and open problems for future research are given in a subsequent section, followed by a concluding section.

Classical Definitions

All the classical definitions assume an exhaustive finite *frame of discernment* $\Theta = \{\theta_1, ..., \theta_n\}$, whose elements are mutually exclusive.

A *basic belief assignment* (*bba*) is a mapping $m : \mathcal{P}(\Theta) \longrightarrow [0,1]$, such that $\sum_{A\subseteq\Theta} m(A) = 1$; the values of bba are called *basic belief masses (bbm)*. The value $m(A)$ is called the *basic belief mass (bbm) of* A.[11] A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Theta) \longrightarrow [0,1]$, $bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$; the belief function Bel uniquely corresponds to bba m and vice-versa. $\mathcal{P}(\Theta)$ is often denoted also by 2^Θ. A *focal element* is a subset X of the frame of discernment Θ, such that $m(X) > 0$. If a focal element is an one-element subset of Θ, we are referring to a *singleton*.

Let us start with the classic definition of Dempster's rule.

Dempster's (*conjunctive*) *rule of combination* $\oplus$ is given as

$$(m_1 \oplus m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y = A} K m_1(X) m_2(Y)$$

for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$.[12] Putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\textcircled{\cap}$.[13].

Yager's rule of combination $\textcircled{Y}$,[14] is given as

$$(m_1 \textcircled{Y} m_2)(A) = \textstyle\sum_{X,Y\subseteq\Theta,\, X\cap Y = A} m_1(X)m_2(Y)/] \text{ for } \emptyset \neq A \subset \Theta,$$
$$(m_1 \textcircled{Y} m_2)(\Theta) = m_1(\Theta)m_2(\Theta) + \textstyle\sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset} m_1(X)m_2(Y),$$

and $(m_1 \textcircled{Y} m_2)(\emptyset) = 0$.

Dubois-Prade's rule of combination $\textcircled{DP}$ is given as

$$(m_1 \textcircled{DP} m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y = A} m_1(X)m_2(Y) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset, X\cup Y = A} m_1(X)m_2(Y)$$

for $\emptyset \neq A \subseteq \Theta$, and $(m_1 \textcircled{DP} m_2)(\emptyset) = 0$.[15]

Introduction to DSm Theory

Due to the fact that the DSmT is a new theory, which is in a state of dynamic evolution, it has to be noted that the text here is related to DSmT's state described by the formulas and text presented in the basic publication on DSmT – the book "Advances and Applications of DSmT for Information Fusion."[16] Rapid development of the theory has been demonstrated around the announcement of the preparation of the second book on DSmT.

Dedekind Lattice, Basic DSm Notions

Dempster-Shafer modified Theory or Dezert-Smarandache Theory (DSmT) by Dezert and Smarandache[17] allows mutually overlapping elements of a frame of discernment. Thus, in DSmT, a frame of discernment is a finite exhaustive set of elements $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$, but not necessarily exclusive. As an example, we can introduce a three-element set of colours $\{Red, Green, Blue\}$ taken from the DSmT homepage.[18] DSmT allows an object to have two or three colours at the same time: e.g., it can be both red and blue, or red and green and blue at the same time, it corresponds to a composition of the colours made from the three basic ones.

DSmT uses basic belief assignments and belief functions defined analogically to the classic Dempster-Shafer theory (DST), however, they are defined on a so-called hyper-power set or Dedekind lattice instead of the classic power set of the frame of discernment. To be distinguishable from the classical definitions, they are called generalized basic belief assignments and generalized basic belief functions.

The *Dedekind lattice*, more often called *hyper-power set* D^Θ in DSmT, is defined as the set of all composite propositions built from elements of Θ with union and intersection operators $\cup$ and $\cap$, such that $\emptyset, \theta_1, \theta_2, ..., \theta_n \in D^\Theta$ and if $A, B \in D^\Theta$ then also $A \cup B \in D^\Theta$ and $A \cap B \in D^\Theta$; no other elements belong to D^Θ ($\theta_i \cap \theta_j \neq \emptyset$ in general, $\theta_i \cap \theta_j = \emptyset$ iff $\theta_i = \emptyset$ or $\theta_j = \emptyset$).

Thus, in general, the hyper-power set D^Θ of Θ is closed to $\cup$ and $\cap$ and $\theta_i \cap \theta_j \neq \emptyset$. Whereas the classic power set 2^Θ of Θ is closed to $\cup$, $\cap$ and complement, and $\theta_i \cap \theta_j = \emptyset$ for every $i \neq j$.

Some examples of hyper-power sets will be given. Let $\Theta = \{\theta_1, \theta_2\}$, we have $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$, i.e. $|D^\Theta| = 5$. Let now $\Theta = \{\theta_1, \theta_2, \theta_3\}$, we have $D^\Theta = \{\alpha_0, \alpha_1, ...\alpha_{18}\}$, where $\alpha_0 = \emptyset, \alpha_1 = \theta_1 \cap \theta_2 \cap \theta_3, \alpha_2 = \theta_1 \cap \theta_2, \alpha_3 = \theta_1 \cap \theta_3, ..., \alpha_{17} = \theta_2 \cup \theta_3, \alpha_{18} = \theta_1 \cup \theta_2 \cup \theta_3$, i.e., $|D^\Theta| = 19$ for $|\Theta| = 3$.

A *generalized basic belief assignment (gbba)* m is a mapping $m : D^\Theta \longrightarrow [0, 1]$, such that $\sum_{A \in D^\Theta} m(A) = 1$ and $m(\emptyset) = 0$. The quantity $m(A)$ is called the *generalized basic belief mass (gbbm) of* A. A *generalized belief function (gBF)* Bel is a mapping $Bel : D^\Theta \longrightarrow [0, 1]$, such that $Bel(A) = \sum_{X \subseteq A, X \in D^\Theta} m(X)$; the generalized belief function Bel uniquely corresponds to gbba m and vice-versa.

DSm Models

Let us assume a Dedekind lattice (a hyper-power set), given according to the definition described above, without any other assumptions, i.e., all the elements of an exhaustive frame of discernment can mutually overlap, then we refer to the *free DSm model* $\mathcal{M}^f(\Theta)$; i.e., we have a DSm model free of constraints.

In general, it is possible to add exclusivity or non-existential constraints into DSm models; in such cases, we speak about *hybrid DSm models*.

An exclusivity constraint $\theta_1 \cap \theta_2 \overset{\mathcal{M}_1}{\equiv} \emptyset$ means that the elements θ_1 and θ_2 are mutually exclusive in model $\mathcal{M}_1$, whereas both of them can overlap with θ_3. If we assume exclusivity constraints $\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_1 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, another exclusivity constraint follows them directly: $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$. In this case, all the elements of the 3-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3\}$ are mutually exclusive as in the classic Dempster-Shafer theory and we call such a hybrid DSm model a *Shafer's model* $\mathcal{M}^0(\Theta)$.

A non-existential constraint $\theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$ brings additional information about a frame of discernment saying that θ_3 is impossible; it forces all the gbbm of $X \subseteq \theta_3$ to be equal to zero for any gbba in model $\mathcal{M}_3$. It represents a certain meta-information with respect to generalized belief combination, which is used in dynamic fusion.

In the degenerated case of the *degenerated DSm model* $\mathcal{M}_\emptyset$ we always have $m(\emptyset) = 1$, $m(X) = 0$ for $X \neq \emptyset$. It is the only case where $m(\emptyset) > 0$ is allowed in DSmT.

The total ignorance on Θ is the union $I_t = \theta_1 \cup \theta_2 \cup ... \cup \theta_n$. $\boldsymbol{\emptyset} = \{\boldsymbol{\emptyset}_\mathcal{M}, \emptyset\}$, where $\boldsymbol{\emptyset}_\mathcal{M}$ is the set of all elements of D^Θ, which are forced to be empty through the constraints of the model $\mathcal{M}$, and $\emptyset$ is the classical empty set.[19] For a given DSm model we can define (in addition to Smarandache and Dezert[20]) $\Theta_\mathcal{M} = \{\theta_i | \theta_i \in \Theta, \theta_i \not\in \boldsymbol{\emptyset}_\mathcal{M}\}$, $\Theta_\mathcal{M} \overset{\mathcal{M}}{\equiv} \Theta$, and $I_\mathcal{M} = \bigcup_{\theta_i \in \Theta_\mathcal{M}} \theta_i$, i.e. $I_\mathcal{M} \overset{\mathcal{M}}{\equiv} I_t$, $I_\mathcal{M} = I_t \cap \Theta_\mathcal{M}$, $I_{\mathcal{M}_\emptyset} = \emptyset$.

The DSm Rule of Combination

The *classical DSm rule DSmC* is defined on the free DSm models as follows[21]:

$$m_{\mathcal{M}^f(\Theta)}(A) = (m_1 \circledast m_2)(A) = \sum_{X,Y \in D^\Theta,\, X \cap Y = A} m_1(X) m_2(Y).$$

Since D^Θ is closed under the operators $\cap$ and $\cup$ and all the $\cap$s are non-empty, the classical DSm rule guarantees that $(m_1 \circledast m_2)$ is a proper generalized basic belief assignment. The rule is commutative and associative. For n-ary version of the rule refer to the work of Smarandache and Dezert.[22]

When the free DSm model $\mathcal{M}^f(\Theta)$ cannot be applied due to the nature of the problem under consideration that requires some known integrity constraints to be taken into account, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. In such a case, the *hybrid DSm rule of combination DSmH* based on the hybrid model $\mathcal{M}(\Theta)$, $\mathcal{M}^f(\Theta) \neq \mathcal{M}(\Theta) \neq \mathcal{M}_\emptyset(\Theta)$, for $k \geq 2$ independent sources of information is defined as:

$$m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledast m_2 \circledast ... \circledast m_k)(A) = \phi(A)[S_1(A) + S_2(A) + S_3(A)],$$

where $\phi(A)$ is a *characteristic non-emptiness function* of the set A, i. e. $\phi(A) = 1$ if $A \notin \emptyset$ and $\phi(A) = 0$, otherwise. $S_1 \equiv m_{\mathcal{M}^f(\Theta)}$, $S_2(A)$, and $S_3(A)$ are defined for two sources (for n-ary versions see the work of Smarandache and Dezert[23]) as follows:

$$\begin{aligned}
S_1(A) &= \sum_{X,Y \in D^\Theta,\, X \cap Y = A} m_1(X)m_2(Y), \\
S_2(A) &= \sum_{X,Y \in \boldsymbol{\emptyset},\, [\mathcal{U}=A] \vee [(\mathcal{U} \in \boldsymbol{\emptyset}) \wedge (A = I_t)]} m_1(X)m_2(Y), \\
S_3(A) &= \sum_{X,Y \in D^\Theta,\; X \cup Y = A,\; X \cap Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y)
\end{aligned}$$

with $\mathcal{U} = u(X) \cup u(Y)$, where $u(X)$ is the union of all singletons θ_i that compose X and Y; all the sets A, X, Y are assumed to be in some canonical form, e.g. CNF. Unfortunately, the issue related to the canonical form is not discussed in the book of Smarandache and Dezert.[24] $S_1(A)$ corresponds to the classical DSm rule on the free DSm model $\mathcal{M}^f(\Theta)$; $S_2(A)$ represents the mass of all relatively and absolutely empty sets in both the input gbbas, which arises due to non-existential constraints and is transferred to the total or relative ignorance; and $S_3(A)$ transfers the sum of masses of relatively and absolutely empty sets, which arise as conflicts of the input gbbas, to the non-empty union of input sets.

The hybrid DSm rule generalizes the classical DSm rule so that it could be applicable to any DSm model. The hybrid DSm rule is commutative, but not associative. This is the reason to use the n-ary version of the rule in practical applications. For the n-ary version of $S_i(A)$, the reader could refer to the work of Smarandache and Dezert.[25]

Generalization of Dempster's Rule

Let us assume for the rest of this contribution that all elements X of D^Θ are in CNF, unless another form of X is explicitly specified. Let us also assume non-degenerated hybrid DSm models, i.e., $\Theta_{\mathcal{M}} \neq \emptyset$, $I_{\mathcal{M}} \notin \emptyset_{\mathcal{M}}$. Let us denote $\boldsymbol{\emptyset} = \emptyset_{\mathcal{M}} \cup \{\emptyset\}$, i.e. the set of all elements of D^{Θ}, which are forced to be empty trough the constraints of the DSm model $\mathcal{M}$, extended with classical empty set $\emptyset$; hence, we can write $X \in \boldsymbol{\emptyset}$ for all $X \overset{\mathcal{M}}{\equiv} \emptyset$ including $\emptyset$.

The classical Dempster's rule puts belief mass $m_1(X)m_2(Y)$ to $X \cap Y$ (the rule adds it to $(m_1 \oplus m_2)(X \cap Y)$) whenever it is non-empty, otherwise the mass is normalized. In the free DSm model, all the intersections of non-empty elements are always non-empty, thus no normalization is necessary and Dempster's rule generalized to the free DSm model $\mathcal{M}^f(\Theta)$ coincides with the classical DSm rule:

$$(m_1 \oplus m_2)(A) = \sum_{X,Y \in D^{\Theta},\, X \cap Y = A} m_1(X)m_2(Y) = (m_1 \circledcirc m_2)(A).$$

Therefore, Dempster's rule generalized to the free DSm model is defined for any couple of belief functions.

Empty intersections can appear in a general hybrid model due to the model's constraints, thus the normalization has to be used.

The *generalized Dempster's rule of combination* $\oplus$ is given as

$$(m_1 \oplus m_2)(A) = \sum_{X,Y \in D^{\Theta},\, X \cap Y \equiv A} K\, m_1(X)\, m_2(Y)$$

for $\emptyset \neq A \in D^{\Theta}_{\mathcal{M}}$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X,Y \in D^{\Theta},\, X \cap Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(A) = 0$, otherwise, i.e., for $A = \emptyset$ and for $A \notin D^{\Theta}_{\mathcal{M}}$.

Similarly to the classical case, the generalized Demspter's rule is not defined in fully contradicting cases[26] in hybrid DSm models, i.e. when $\kappa = 1$. In particular, the generalized Dempster's rule is not defined (and it cannot be defined) on the degenerated DSm model $\mathcal{M}_{\emptyset}$.

To be easily comparable with the DSm rule, we can rewrite the definition of the generalized Dempster's rule in the following equivalent form:

$$(m_1 \oplus m_2)(A) = \phi(A)[S_1^{\oplus}(A) + S_2^{\oplus}(A) + S_3^{\oplus}(A)],$$

where $\phi(A)$ is a *characteristic non-emptiness function* of the set A, i. e. $\phi(A) = 1$ if $A \notin \boldsymbol{\emptyset}$ and $\phi(A) = 0$, otherwise; $S_1^{\oplus}(A)$, $S_2^{\oplus}(A)$, and $S_3^{\oplus}(A)$ are defined by:

$$S_1^{\oplus}(A) = S_1(A) = \sum_{X,Y \in D^{\Theta},\, X\cap Y \equiv A} m_1(X)m_2(Y),$$

$$S_2^{\oplus}(A) = \frac{S_1(A)}{\sum_{Z\in D^{\Theta}\ Z\notin \emptyset} S_1(Z)} \sum_{X,Y \in \emptyset_{\mathcal{M}}} m_1(X)m_2(Y),$$

$$S_3^{\oplus}(A) = \frac{S_1(A)}{\sum_{Z\in D^{\Theta}\ Z\notin \emptyset} S_1(Z)} \sum_{\substack{X,Y\in D^{\Theta},\ X\cup Y\notin \emptyset \\ X\cap Y \in \emptyset_{\mathcal{M}}}} m_1(X)m_2(Y).$$

$S_1^{\oplus}(A)$ corresponds to a non-conflicting belief mass, $S_3^{\oplus}(A)$ includes all classic conflicting masses and the cases where one of X, Y is excluded by a non-existential constraint, and $S_2^{\oplus}(A)$ corresponds to the cases where both X and Y are excluded by (a) non-existential constraint(s).

It can be easily verified that the generalized Dempster's rule coincides with the classical one on the Shafer's model $\mathcal{M}^0$; for a proof the reader may refer to the work of Milan Daniel. [27] Hence, the definition of the generalized Dempster's rule given above is really a generalization of the classical Dempster's rule.

Generalization of Yager's Rule

The classical Yager's rule puts belief mass $m_1(X)m_2(Y)$ to $X \cap Y$ when it is non-empty; otherwise, the mass is added to $m(\Theta)$. As all the intersections are non-empty in the free DSm model, nothing should be added to $m_1(\Theta)m_2(\Theta)$ and Yager's rule generalized to the free DSm model $\mathcal{M}^f(\Theta)$ coincides also with the classical DSm rule.

$$(m_1 \circledY m_2)(A) = \sum_{X,Y\in D^{\Theta},\, X\cap Y = A} m_1(X)m_2(Y) = (m_1 \circledast m_2)(A).$$

The *generalized Yager's rule of combination* $\circledY$ for a general hybrid DSm model $\mathcal{M}$ is given as:

$$(m_1 \circledY m_2)(A) = \sum_{X,Y\in D^{\Theta},\, X\cap Y \equiv A} m_1(X)\, m_2(Y)$$

for $A \notin \emptyset,\ \Theta_{\mathcal{M}} \neq A \in D^{\Theta}_{\mathcal{M}}$,

$$(m_1 \circledY m_2)(\Theta_{\mathcal{M}}) = \sum_{\substack{X,Y\in D^{\Theta} \\ X\cap Y \equiv \Theta_{\mathcal{M}}}} m_1(X)\, m_2(Y) + \sum_{\substack{X,Y\in D^{\Theta} \\ X\cap Y \in \emptyset_{\mathcal{M}}}} m_1(X)\, m_2(Y)$$

and $(m_1 \circledY m_2)(A) = 0$, otherwise, i.e., for $A \in \emptyset$ and for $A \in (D^{\Theta} \setminus D^{\Theta}_{\mathcal{M}})$.

To be easily comparable with the DSm rule, we can rewrite the definition of the generalized Yager's rule in an equivalent form:

$$(m_1 \circledY m_2)(A) = \phi(A)[S_1^{\circledY}(A) + S_2^{\circledY}(A) + S_3^{\circledY}(A)],$$

where $S_1^{\circledY}(A)$, $S_2^{\circledY}(A)$, and $S_3^{\circledY}(A)$ are defined by:

$$\begin{aligned}
S_1^{\circledY}(A) &= S_1(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y\equiv A} m_1(X)m_2(Y),\\
S_2^{\circledY}(\Theta_{\mathcal{M}}) &= \sum_{X,Y\in \emptyset_{\mathcal{M}}} m_1(X)m_2(Y),\ S_2^{\circledY}(A) = 0 \text{ for } A \neq \Theta_{\mathcal{M}},\\
S_3^{\circledY}(\Theta_{\mathcal{M}}) &= \sum_{\substack{X,Y\in D^\Theta,\ X\cup Y\notin \emptyset\\ X\cap Y\in \emptyset_{\mathcal{M}}}} m_1(X)m_2(Y),\ S_3^{\circledY}(A)=0 \text{ for } A\neq\Theta_{\mathcal{M}}.
\end{aligned}$$

It is easy to verify that the generalized Yager's rule coincides with the classical one on the Shafer's model $\mathcal{M}^0$; for a proof the reader may refer to the work of Milan Daniel.[28] Hence, the definition of the generalized Yager's rule is really a generalization of the classical Yager's rule.

Generalization of Dubois-Prade's Rule

The classical Dubois-Prade's rule puts belief mass $m_1(X)m_2(Y)$ to $X \cap Y$ whenever it is non-empty; otherwise the mass $m_1(X)m_2(Y)$ is added to $X \cup Y$, which is always non-empty in the DST.

In the free DSm model, all the intersections of non-empty elements are always non-empty, thus nothing to be added to unions and Dubois-Prade's rule generalized to the free model $\mathcal{M}^f(\Theta)$ also coincides with the classical DSm rule.

$$(m_1 \circledD m_2)(A) = \sum_{X,Y\in D^\Theta, X\cap Y=A} m_1(X)m_2(Y) = (m_1 \circledast m_2)(A).$$

In the case of static fusion, only exclusivity constraints are used, thus all the unions of $X_i \in D^\Theta$, $X \notin \emptyset$ are also out of $\emptyset$. Thus, we can easily generalize Dubois-Prade's rule as:

$$(m_1 \circledD m_2)(A) = \sum_{\substack{X,Y\in D^\Theta\\ X\cap Y=A}} m_1(X)m_2(Y) + \sum_{\substack{X,Y\in D^\Theta,\ X\cap Y\in\emptyset_{\mathcal{M}}\\ X\cup Y=A}} m_1(X)m_2(Y)$$

for $A \in D^\Theta, A \notin \emptyset$, and $(m_1 \circledD m_2)(A) = 0$ for $A \in \emptyset$.

The situation is more complicated in the case of dynamic fusion where non-existential constraints are used. There are several sub-cases that may lead to $X \cap Y \in \boldsymbol{\emptyset}$; for details see another work of the author.[29]

So, we can now formulate a definition of the generalized Dubois-Prade's rule. Three cases of input generalized belief functions can be distinguished: (i) all inputs satisfy all the constraints of a hybrid DSm model used $\mathcal{M}(\Theta)$ (a static belief combination), (ii) inputs do not satisfy the constraints of $\mathcal{M}(\Theta)$ (a dynamic belief combination), but no non-existential constraint is used, (iii) completely general inputs, which do not satisfy the constraints, and non-existential constraints are allowed (a more general dynamic combination). According to these three cases, we can formulate three variants of the generalized Dubois-Prade's rule.

The *simple generalized Dubois-Prade's rule of combination* $\circledast$ is given as[30]:

$$(m_1 \circledast m_2)(A) \;=\; \sum_{X \cap Y \equiv A} m_1(X)\, m_2(Y) \;+\; \sum_{\substack{X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}} \\ X \cup Y \equiv A}} m_1(X)\, m_2(Y)$$

for $\emptyset \neq A \in D^{\Theta}_{\mathcal{M}}$, and $(m_1 \circledast m_2)(A) = 0$, otherwise, i.e., for $A = \emptyset$ and for $A \in (D^{\Theta} \setminus D^{\Theta}_{\mathcal{M}})$.

The *generalized Dubois-Prade rule of combination* $\circledast$ is given as

$$(m_1 \circledast m_2)(A) = \sum_{X \cap Y \equiv A} m_1(X) m_2(Y) + \sum_{\substack{X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}} \\ X \cup Y \equiv A}} m_1(X) m_2(Y) + \sum_{\substack{X \cup Y \in \boldsymbol{\emptyset}_{\mathcal{M}} \\ U_{X \cup Y} \equiv A}} m_1(X) m_2(Y)$$

for $\emptyset \neq A \in D^{\Theta}_{\mathcal{M}}$, and $(m_1 \circledast m_2)(A) = 0$, otherwise, i.e., for $A = \emptyset$ and for $A \in (D^{\Theta} \setminus D^{\Theta}_{\mathcal{M}})$, where $U_{X \cup Y}$ is disjunctive normal form of $X \cup Y$ with all $\cap$s substituted with $\cup$s.

The *extended generalized Dubois-Prade rule of combination* $\circledast$ is given as:

$$\begin{aligned}(m_1 \circledast m_2)(A) \;=\; & \sum_{X \cap Y \equiv A} m_1(X)\, m_2(Y) \;+\; \sum_{\substack{X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}} \\ X \cup Y \equiv A}} m_1(X)\, m_2(Y) \\ & + \sum_{\substack{X \cup Y \in \boldsymbol{\emptyset}_{\mathcal{M}} \\ U_{X \cup Y} \equiv A}} m_1(X)\, m_2(Y)\end{aligned}$$

for $\emptyset \neq A \neq \Theta_{\mathcal{M}},\; A \in D^{\Theta}_{\mathcal{M}}$,

$$\begin{aligned}(m_1 \circledast m_2)(\Theta_{\mathcal{M}}) \;=\; & \sum_{X \cap Y \equiv \Theta_{\mathcal{M}}} m_1(X)\, m_2(Y) \;+\; \sum_{\substack{X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}} \\ X \cup Y \equiv \Theta_{\mathcal{M}}}} m_1(X)\, m_2(Y) \\ & + \sum_{\substack{X \cup Y \in \boldsymbol{\emptyset}_{\mathcal{M}} \\ U_{X \cup Y} \equiv \Theta_{\mathcal{M}}}} m_1(X)\, m_2(Y) \;+\; \sum_{U_{X \cup Y} \in \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)\, m_2(Y),\end{aligned}$$

and $(m_1 \circledast m_2)(A) = 0$, otherwise, i.e., for $A \in \emptyset$ and for $A \in (D^\Theta \setminus D^\Theta_\mathcal{M})$, where $U_{X \cup Y}$ is disjunctive normal form of $X \cup Y$ with all $\cap$s substituted with $\cup$s.

It is easy to verify that the generalized Dubois-Prade rule coincides with the classical one in the Shafer's model $\mathcal{M}^0$; for a proof the reader may refer to another work of the author.[31]

The classical Dubois-Prade rule is not associative, neither the generalized one is. Similarly to the DSm approach, we can easily rewrite the definitions of the (generalized) Dubois-Prade rule for a combination of k sources.

To be easily comparable with the DSm rule, we can rewrite the definitions of the generalized Dubois-Prade rules to an equivalent form, similar to that of the DSm rule, as follows.

The generalized Dubois-Prade rule:

$$(m_1 \circledast m_2)(A) = \phi(A)[S_1^{\circledast}(A) + S_2^{\circledast}(A) + S_3^{\circledast}(A)], \text{ where}$$

$$\begin{aligned}
S_1^{\circledast}(A) &= S_1(A) = \sum_{X,Y \in D^\Theta,\, X \cap Y \equiv A} m_1(X) m_2(Y), \\
S_2^{\circledast}(A) &= \sum_{X,Y \in \emptyset_\mathcal{M},\, U_{X \cup Y} \equiv A} m_1(X) m_2(Y), \\
S_3^{\circledast}(A) &= \sum_{X,Y \in D^\Theta,\, X \cap Y \in \emptyset_\mathcal{M},\, (X \cup Y) \equiv A} m_1(X) m_2(Y).
\end{aligned}$$

The simple generalized Dubois-Prade rule:

$$(m_1 \circledast m_2)(A) = \phi(A)[S_1^{\circledast}(A) + S_3^{\circledast}(A)],$$

where $S_1^{\circledast}(A)$ and $S_3^{\circledast}(A)$ are given as above.

The extended generalized Dubois-Prade rule:

$$(m_1 \circledast m_2)(A) = \phi(A)[S_1^{\circledast}(A) + S_2^{\circledast}(A) + S_3^{\circledast}(A)],$$

where $S_1^{\circledast}(A)$ and $S_3^{\circledast}(A)$ are given as above, and

$$S_2^{\circledast}(A) = \sum_{X,Y \in \emptyset_\mathcal{M},\, [U_{X \cup Y} \equiv A] \vee [U_{X \cup Y} \in \emptyset \wedge A = \Theta_\mathcal{M}]} m_1(X) m_2(Y).$$

Proofs the reader may find in another publication of the author.[32]

A Brief Comparison of the Rules

As there are no conflicts in the free DSm model $\mathcal{M}^f(\Theta)$, all the presented rules coincide in the free DSm model $\mathcal{M}^f(\Theta)$. Thus the following statement holds:

Statement 1 *Dempster's rule, Yager's rule, Dubois-Prade's rule, the hybrid DSmH rule, and the classical DSmC rule are all mutually equivalent in the free DSm model* $\mathcal{M}^f(\Theta)$.

Similarly, the classical Dubois-Prade's rule is equivalent to the DSm rule for the Shafer's model. But, in general, all the generalized rules $\oplus, \circledY, \circledD$, and the DSm rule are different. A very slight difference comes in the case of Dubois-Prade's rule and the DSm rule. A difference appears only in the case of dynamic fusion, where some focal elements of the two (or all in an n-ary case) source basic belief assignments are equivalent to the empty set; an extension of the generalized Dubois-Prade's rule is necessary there.

Statement 2 *(i) If a hybrid DSm model* $\mathcal{M}(\Theta)$ *does not include any non-existential constraint or if all the input belief functions satisfy all the constraints of* $\mathcal{M}(\Theta)$*, then the generalized Dubois-Prade rule is equivalent to the DSm rule for model* $\mathcal{M}(\Theta)$.
(ii) The generalized Dubois-Prade's rule extended with addition of $m_1(X)m_2(Y)$ *(or* $\Pi_i\ m_i(X_i)$ *in an n-ary case) to* $m(\Theta)$ *for* $X, Y \in \emptyset_{\mathcal{M}}$ *(or for* $X_i \in \emptyset_{\mathcal{M}}$ *in an n-ary case) is totally equivalent to the hybrid DSmH rule on any hybrid DSm model.*

Proofs the reader may find in another publication of the author.[33]

Open Problems

The commutativity of transformation of the generalized belief functions to such that satisfy all the constraints of the used hybrid DSm model with the particular combination rules remains an open research question. Such commutativity feature may significantly simplify the functions S_2 and hence the entire definitions of the corresponding combination rules.

In the same way as it is used in this paper, we can also generalize the non-normalized conjunctive rule of combination. A generalization of the minC combination rule, whose computing mechanism (not a motivation nor an interpretation) has a relation to the conjunctive rules on the free DSm model $\mathcal{M}^f(\Theta)$ already in its classical case,[34] is currently under development.[35]

The question of a possible generalization of the conditionalization related to particular combination rules to the domain of DSm hyper-power sets has also to be considered.

Conclusion

The classical rules for combination of belief functions have been generalized in order to be applicable to the hyper-power sets used in DSm theory. The generalization provides a solid theoretical background for comprehensive and objective comparison of the characteristics of the classical rules with the characteristics of the DSm rule of combination. It also enables us to place the DSmT better among the other approaches for dealing with belief functions.

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Notes:

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11. $m(\emptyset) = 0$ is often assumed in accordance with the Shafer's definition (see Shafer, *A Mathematical Theory of Evidence*). A classical counter-example is Smets' Transferable Belief Model (TBM) which admits positive $m(\emptyset)$ as it assumes $m(\emptyset) \geq 0$.
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19. $\boldsymbol{\emptyset}$ should be $\boldsymbol{\emptyset}_{\mathcal{M}}$ extended with the classical empty set $\emptyset$, thus the expression $\boldsymbol{\emptyset} = \boldsymbol{\emptyset}_{\mathcal{M}} \cup \{\emptyset\}$ should be more correct.
20. Smarandache and Dezert, *Advances and Applications of DSmT for Information Fusion*.
21. To distinguish the DSm rule from Dempster's rule, we use $\circledast$ instead of $\oplus$ for the DSm rule in this text.
22. Smarandache and Dezert, *Advances and Applications of DSmT for Information Fusion*.
23. Smarandache and Dezert, *Advances and Applications of DSmT for Information Fusion*.
24. Smarandache and Dezert, *Advances and Applications of DSmT for Information Fusion*.
25. Smarandache and Dezert, *Advances and Applications of DSmT for Information Fusion*.
26. Note that in static combination it means a full conflict/contradiction between the input BFs. Whereas in the case of dynamic combination, it could be also a full conflict between mutually non-conflictling or partilally conflicting input BFs and the constraints of the used hybrid DSm model. E.g. $m_1(\theta_1 \cup \theta_2) = 1,\ m_2(\theta_2 \cup \theta_3) = 1$, where θ_2 is constrained in the used hybrid model.

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30. We present here three variants of the generalized Dubois-Prade rule; the formulas for all these rules include several summations over $X, Y \in D^{\Theta}$, where X, Y are more specified with other conditions. To simplify the formulas in order to increase their readability, we do not repeat the common condition $X, Y \in D^{\Theta}$ in the sums in all the formulas that follow for the generalized Dubois-Prade's rule.
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32. Daniel, "Classical Combination Rules Generalized to DSm Hyper-power Sets and Their Comparison with the Hybrid DSm Rule."
33. Daniel, "Classical Combination Rules Generalized to DSm Hyper-power Sets and Their Comparison with the Hybrid DSm Rule."
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35. Daniel, "A Generalization of the minC Combination to DSm Hyper-power Sets."

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A NEW CLASS OF FUSION RULES BASED ON T-CONORM AND T-NORM FUZZY OPERATORS

Albena TCHAMOVA, Jean DEZERT, and Florentin SMARANDACHE

Abstract: A new combination rule based on specified fuzzy T-Conorm/T-Norm operators is proposed and analyzed in this article - the TCN Rule of Combination. The rule does not belong to the general Weighted Operator Class. The advantages of the new rule could be defined as: very easy to implement, satisfying the impact of neutrality of Vacuous Belief Assignment, commutative, convergent to idempotence, reflecting majority opinion, and assuring an adequate data processing in case of total conflict. Several numerical examples and comparisons with the new advanced Proportional Conflict Redistribution Rules proposed recently by Florentin Smarandache and Jean Dezert within their theory of plausible and paradoxical reasoning are presented.

Keywords: Information Fusion, Combination Rules, Conjunctive Rule, Fuzzy Operators, Dezert-Smarandache Theory (DSmT), Proportional Conflict Redistribution Rules.

Introduction

There are many combination rules available for information fusion.[1] However, none of them could satisfy the whole range of requirements associated with all possible applications. The main requirements the combination rules have to meet in temporal multiple target tracking relate especially to the way of adequate conflict processing/redistribution, ease of implementation, satisfaction of the impact of neutrality of Vacuous Belief Assignment (VBA), reflection of majority opinion, etc. In this work, the authors propose to connect the combination rules for information fusion with particular fuzzy operators: the Conjunctive rule is replaced with fuzzy T-norm operator and the Disjunctive rule with T-conorm operator, respectively. These rules originate from the T-norm and T-conorm operators in fuzzy logic, where the AND logical operator corresponds to the conjunctive rule in information fusion and the OR logical operator corresponds to the disjunctive rule. While the logical operators deal with degrees of truth, the fusion rules deal with degrees of belief of hypotheses. In this

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work, the focus will be on the T-norm based conjunctive rule only as an analog of the ordinary conjunctive rule of combination. The reason is that the conjunctive rule is especially appropriate for identification problems, restricting the set of hypotheses under consideration.

Fuzzy Inference for Information Fusion

The main objective of information fusion is to produce a reasonably aggregated, refined and/or completed granule of data obtained from a single or multiple sources with a subsequent reasoning process. It means that the main problem here lies in the approach to aggregate correctly these sources of information, which, in general, are imprecise, uncertain, or/and conflicting. Actually, there is no single, unique rule to deal simultaneously with such kind of information specifics. Even more, there are a huge number of possible combination rules appropriate only for particular application conditions. Smarandache proposes an unification of fusion theories and a combination of fusion rules for solving different problems.[2] The most suitable model for each considered application is selected. In this article, the case with a given Shafer's model is considered.[3] Let $\Theta = \{\theta_1, \theta_2, ...\theta_n\}$ be the frame of discernment for the problem under consideration, where $\theta_1, \theta_2, ...\theta_n$ are a set of n exhaustive and exclusive hypotheses. Within the applied model, Dempster-Shafer's Power Set is described as: $2^\theta = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$. The basic belief assignment (bba) $m(.) : 2^\Theta \to [0, 1]$, associated with a given information granule is defined with:

$$m(\oslash) = 0; \quad \sum_{X \in 2^\Theta} m(X) = 1.$$

Having given two basic belief assignments $m_1(.)$ and $m_2(.)$ and Shafer's model, Dempster's rule of combination [4] appears to be the most frequently used combination rule. It is defined as:

$$m_{12}(X) = \frac{\sum_{\substack{X_i \cap X_j = X \\ X_i, X_j \in 2^\Theta}} m_1(X_i) \cdot m_2(X_j)}{1 - \sum_{\substack{X_i \cap X_j = \oslash \\ X_i, X_j \in 2^\Theta}} m_1(X_i) \cdot m_2(X_j)}$$

The term $k = \sum_{\substack{X_i \cap X_j = \oslash \\ X_i, X_j \in 2^\Theta}} m_1(X_i) \cdot m_2(X_j)$ describes the degree of conflict between the sources of information. The normalization step (i.e., the division by $(1 - k)$) in Dempster's rule is definitely the most sensitive and week point of the rule due to the fact that the fused result becomes a proper information granule only in the cases when $k < 1$. The new advanced Proportional Conflict Redistribution rules proposed recently by Smarandache and Dezert,[5] which are particular cases of the Weighted Operator, overcome successfully the main limitations of Dempster's rule.

In this work, the authors' objective is to propose a new, alternative combination rule, interpreting the fusion in terms of fuzzy operators, a rule that avoids the Dempster's rule limitation, possesses an adequate behavior in cases of total conflict, and has an easy implementation.

Fusion Interpretation

It is assumed that the relation between the two basic belief assignments (the information granules) $m_1(.)$ and $m_2(.)$ is considered a vague relation, characterized by the following two features:

- *The way of association* between the focal elements included in the basic belief assignments of the sources of information. It is a particular operation chosen among the operations union and intersection, respectively. These set operations correspond to the logic operations Conjunction and Disjunction.

- *The degree of association (interaction)* between the focal elements included in the basic belief assignments of the sources of information. It is obtained as a T-norm (for Conjunction) or T-conorm (for Disjunction) operators applied over the probability masses of the corresponding focal elements. There are multiple choices available to define T-norm and T-conorm operators.

In this article, as already mentioned, the authors will focus only on the T-norm based Conjunctive rule, more precisely the Minimum T-norm based Conjunctive rule as an analog of the ordinary conjunctive rule of combination. It will be demonstrated that it could give results very similar to the conjunctive rule, satisfying the principle of neutrality of VBA, reflecting the majority opinion, converging towards idempotence, and having adequate behavior in the presence of a total conflict. It is commutative, simple to apply, but not associative.

Main Features of the T-Norm Function

The $T - norm : [0,1]^2 \mapsto [0,1]$ is a function defined in fuzzy set/logic theory in order to represent the intersection between two particular fuzzy sets and the *AND* fuzzy logical operator, respectively. If one extends the T-norm to data fusion, it will be a substitute for the conjunctive rule. The T-norm has to satisfy the following conditions:

- Associativity: $Tnorm(Tnorm(x,y),z) = Tnorm(x,Tnorm(y,z))$;

- Commutativity: $Tnorm(x,y) = Tnorm(y,x)$;

- Monotonicity: if $(x \leq a)$ and $(y \leq b)$ then $Tnorm(x,y) \leq Tnorm(a,b)$;

- Boundary Conditions: $Tnorm(0,0) = 0$; $Tnorm(x,1) = x$.

Functions, Satisfying the T-Norm Conditions

There are many functions that satisfy the T-norm conditions:

- Zadeh's (default) min operator[6]: $m(X) = \min\{m_1(X_i), m_2(X_j)\}$;
- Algebraic product operator: $m(X) = m_1(X_i) \cdot m_2(X_j)$;
- Bounded product operator: $m(X) = \max\{[m_1(X_i) + m_2(X_j)], 0\}$.

A desirable characteristic is the chosen T-norm operator to satisfy the neutrality of VBA. From the functions described above, the default (min) and the algebraic product operators satisfy this condition. Considering this fact, the authors choose the default Minimum T-norm operator in order to define the degree of association between the focal elements of the information granules.

Proof of the Vague min Set Operator

The intersection $X_i \cap X_j$ for crisp (ordinary) subsets of the universe U includes all elements of X_i and X_j such that:

$$m(X) = 1, \quad if \;\; X \in X_i \; and \; X \in X_j$$

$$m(X) = 0, \quad if \;\; X \notin X_i \; or \; X \notin X_j.$$

Let X_i and X_j are some vague subsets of U. How do we define the conditions from above for the case of intersection $X_i \cap X_j$:

- First condition $X \in X_i$ *and* $X \in X_j$

  It means that the following case exists: $\{m(X \in X_i) = 1, m(X \in X_j) = 1\}$, for which: $\min\{m(X \in X_i), m(X \in X_j)\} = 1$;

- Second condition $X \notin X_i$ *or* $X \notin X_j$

  It means that one of the following cases exists:
  $\{m(X \in X_i) = 0, m(X \in X_j) = 0\}$ *or*
  $\{m(X \in X_i) = 1, m(X \in X_j) = 0\}$ *or*
  $\{m(X \in X_i) = 0, m(X \in X_j) = 1\}$
  for which: $\min\{m(X \in X_i), m(X \in X_j)\} = 0$.

From these expressions it follows that $\min\{m(X \in X_i), m(X \in X_j)\}$ provides the correct expression for intersection.

The T-conorm/T-norm (TCN) Combination Rule

Let us take a look at a general form of a fusion table, where the T-norm based interpretation of the ordinary conjunctive rule of combination for two given sources is considered (see Table 1). The frame of the fusion problem under consideration is $\Theta = \{\theta_1, \theta_2\}$ and the power set is: $2^\theta = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$. The two basic belief assignments (sources of information) $m_1(.)$ and $m_2(.)$ are defined over 2^Θ. It is assumed that $m_1(.)$ and $m_2(.)$ are normalized bbas ($m(\oslash) = 0$; $\sum_{X \in 2^\Theta} m(X) = 1$).

Step 1: Defining the min T-norm conjunctive consensus

The min T-norm conjunctive consensus is based on the default min T-norm function. The way of association between the focal elements of the given two sources of information is defined as $X = X_i \cap X_j$, and the degree of association is as follows:

$$\tilde{m}(X) = \min\{m_1(X_i), m_2(X_j)\},$$

where $\tilde{m}(X)$ represents[7] the mass of belief associated with the given proposition X by using T-Norm based conjunctive rule.

Table 1: Min T-norm based Interpretation of Conjunctive Rule.

	$m_2(\theta_1)$	$m_2(\theta_2)$	$m_2(\theta_1 \cup \theta_2)$
$m_1(\theta_1)$	$\theta_1 \cap \theta_1 = \theta_1$ $\tilde{m}(\theta_1) =$ $\min\{m_1(\theta_1), m_2(\theta_1)\}$	$\theta_1 \cap \theta_2$ $\tilde{m}(\theta_1 \cap \theta_2) =$ $\min\{m_1(\theta_1), m_2(\theta_2)\}$	$\theta_1 \cap (\theta_1 \cup \theta_2) = \theta_1$ $\tilde{m}(\theta_1) =$ $\min\{m_1(\theta_1), m_2(\theta_1 \cup \theta_2)\}$
$m_1(\theta_2)$	$(\theta_1 \cap \theta_2)$ $\tilde{m}(\theta_1 \cap \theta_2) =$ $\min\{m_1(\theta_2), m_2(\theta_1)\}$	$\theta_2 \cap \theta_2 = \theta_2$ $\tilde{m}(\theta_2) =$ $\min\{m_1(\theta_2), m_2(\theta_2)\}$	$\theta_2 \cap (\theta_1 \cup \theta_2) = \theta_2$ $\tilde{m}(\theta_2) =$ $\min\{m_1(\theta_2), m_2(\theta_1 \cup \theta_2)\}$
$m_1(\theta_1 \cup \theta_2)$	$(\theta_1 \cup \theta_2) \cap \theta_1 = \theta_1$ $\tilde{m}(\theta_1) =$ $\min\{m_1(\theta_1 \cup \theta_2), m_2(\theta_1)\}$	$(\theta_1 \cup \theta_2) \cap \theta_2 = \theta_2$ $\tilde{m}(\theta_2) =$ $\min\{m_1(\theta_1 \cup \theta_2), m_2(\theta_2)\}$	$(\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_2) = \theta_1 \cup \theta_2$ $\tilde{m}(\theta_1 \cup \theta_2) =$ $\min\{m_1(\theta_1 \cup \theta_2), m_2(\theta_1 \cup \theta_2)\}$

The proposed T-conorm/T-norm based Combination rule, called by the authors *TCN rule of combination*, is defined in the framework of Dempster-Shafer Theory for $\forall X \in 2^\Theta$ by the equation:

$$\tilde{m}(X) = \sum_{\substack{X_i \cap X_j = X \\ X_i, X_j \in 2^\Theta}} \min\{m_1(X_i), m_2(X_j)\}. \tag{1}$$

Step 2: Distribution of the mass, assigned to the conflict

The distribution of the mass assigned to the conflict follows to a certain degree the distribution of the conflicting mass in the DSmT Proportional Conflict Redistribution Rule 2,[8] but the procedure here is based on fuzzy operators. Let us denote the two bbas associated with the information sources in a matrix form as follows:

$$\begin{bmatrix} m_1(.) \\ m_2(.) \end{bmatrix} = \begin{bmatrix} m_1(\theta_1) & m_1(\theta_2) & m_1(\theta_1 \cup \theta_2) \\ m_2(\theta_1) & m_2(\theta_2) & m_2(\theta_1 \cup \theta_2) \end{bmatrix}.$$

The total conflicting mass is distributed proportionally among all non-empty sets with respect to the *maximum* (denoted here as $x_{12}(X)$) of the elements of the corresponding mass matrix's columns, associated with element X of the power set. It means that the bigger mass is redistributed towards the element, involved in the conflict and contributing to the conflict, with the maximum specified probability mass. The fuzzy operator *maximum* is used to interpret the summation of the corresponding mass matrix's columns associated with element X of the power set, as used in the DSmT Proportional Conflict Redistribution Rules.

$$\begin{aligned} x_{12}(\theta_1) &= \max(m_1(\theta_1), m_2(\theta_1)) \\ x_{12}(\theta_2) &= \max(m_1(\theta_2), m_2(\theta_2)) \end{aligned}$$

One denotes by $r(\theta_1)$ and $r(\theta_2)$ the part of the conflicting mass distributed to the propositions θ_1 and θ_2. Then, one gets:

$$\frac{r(\theta_1)}{x_{12}(\theta_1)} = \frac{r(\theta_2)}{x_{12}(\theta_2)} = \frac{r(\theta_1) + r(\theta_2)}{x_{12}(\theta_1) + x_{12}(\theta_2)} = \frac{k_{12}}{s_{12}}.$$

In turn, the conflicting masses that have to be redistributed are:

$$r(\theta_1) = x_{12}(\theta_1) \cdot \frac{k_{12}}{s_{12}}; \;\; r(\theta_2) = x_{12}(\theta_2) \cdot \frac{k_{12}}{s_{12}}.$$

Finally, the bba obtained as a result of the applied TCN rule with fuzzy-based Proportional Conflict Redistribution Rule 2, denoted here as $\tilde{m}_{PCR2}(.)$, becomes:

$$\begin{aligned} \tilde{m}_{PCR2}(\theta_1) &= \tilde{m}(\theta_1) + x_{12}(\theta_1) \cdot \frac{k_{12}}{s_{12}} \\ \tilde{m}_{PCR2}(\theta_2) &= \tilde{m}(\theta_2) + x_{12}(\theta_2) \cdot \frac{k_{12}}{s_{12}} \\ \tilde{m}_{PCR2}(\theta_1 \cup \theta_2) &= \tilde{m}(\theta_1 \cup \theta_2), \end{aligned}$$

where k_{12} is the total conflict; $x_{12}(X) = \max_{i=1,2}(m_i(X)) \neq 0$ and s_{12} is the sum of all non-zero maximum values of column's masses assigned to non-empty sets. The conflict mass is redistributed only among the propositions involved in the conflict.

Step 3: Normalization of the result

The final step of the TCN rule concerns the normalization procedure:

$$\tilde{m}_{PCR2}(X) = \frac{\tilde{m}_{PCR2}(X)}{\sum_{\substack{X \neq \emptyset \\ X \in 2^{\Theta}}} \tilde{m}_{PCR2}(X)}.$$

Implementation of the TCN Combination Rule

Example 1

Assume problem frame $\theta = \{\theta_1, \theta_2\}$ and two independent sources of information with basic belief assignments as follows:

$$\begin{array}{lll} m_1(\theta_1) = 0.6 & m_1(\theta_2) = 0.2 & m_1(\theta_1 \cup \theta_2) = 0.2 \\ m_2(\theta_1) = 0.4 & m_2(\theta_2) = 0.5 & m_5(\theta_1 \cup \theta_2) = 0.1 \end{array}$$

Applying the min T-norm based conjunctive consensus yields the results given in Table 2.

Table 2: Min T-norm based Interpretation of Conjunctive Rule.

	$m_2(\theta_1) = 0.4$	$m_2(\theta_2) = 0.5$	$m_2(\theta_1 \cup \theta_2) = 0.1$
$m_1(\theta_1) = 0.6$	$\tilde{m}(\theta_1) =$ $\min(0.6, 0.4) = 0.4$	$\tilde{m}(\theta_1 \cap \theta_2) =$ $\min(0.6, 0.5) = 0.5$	$\tilde{m}(\theta_1) =$ $\min(0.6, 0.1) = 0.1$
$m_1(\theta_2) = 0.2$	$\tilde{m}(\theta_1 \cap \theta_2) =$ $\min(0.2, 0.4) = 0.2$	$\tilde{m}(\theta_2) =$ $\min(0.2, 0.5) = 0.2$	$\tilde{m}(\theta_2) =$ $\min(0.2, 0.1) = 0.1$
$m_1(\theta_1 \cup \theta_2) = 0.2$	$\tilde{m}(\theta_1) =$ $\min(0.2, 0.4) = 0.2$	$\tilde{m}(\theta_2) =$ $\min(0.2, 0.5) = 0.2$	$\tilde{m}(\theta_1 \cup \theta_2) =$ $\min(0.2, 0.1) = 0.1$

Fusion with TCN Rule of Combination

Step 1: Obtaining min T-norm Conjunctive Consensus

Using Table 2 and applying Equation 1, the fusion result becomes:

$$\tilde{m}(\theta_1) = 0.4 + 0.2 + 0.1 = 0.7$$

$$\tilde{m}(\theta_2) = 0.2 + 0.1 + 0.2 = 0.5$$

$$\tilde{m}(\theta_1 \cap \theta_2) = 0.5 + 0.2 = 0.7$$

$$\tilde{m}(\theta_1 \cup \theta_2) = 0.1$$

Step 2: Redistribution of the conflict by using fuzzy-based PCR2

$$\frac{r(\theta_1)}{\max(m_1(\theta_1), (m_2(\theta_1))} = \frac{r(\theta_2)}{\max(m_1(\theta_2), (m_2(\theta_2))} =$$

$$\frac{r(\theta_1)}{\max(0.6, 0.4)} = \frac{r(\theta_2)}{\max(0.2, 0.5)} =$$

$$\frac{r(\theta_1) + r(\theta_2)}{\max(0.6, 0.4) + \max(0.2, 0.5)} = \frac{\tilde{m}(\theta_1 \cap \theta_2)}{0.6 + 0.5} = \frac{0.7}{1.1} = 0.636$$

$$r(\theta_1) = 0.6 \cdot 0.636 = 0.3816; \quad r(\theta_2) = 0.5 \cdot 0.636 = 0.318$$

Then, after conflict redistribution, the new masses become:

$$\begin{aligned}\tilde{m}_{PCR2}(.) = \{\, & \tilde{m}_{PCR2}(\theta_1) = 0.7 + 0.3816 = 1.0816;\\ & \tilde{m}_{PCR2}(\theta_2) = 0.5 + 0.318 = 0.818,\\ & \tilde{m}_{PCR2}(\theta_1 \cup \theta_2) = 0.1\}\,.\end{aligned}$$

Step 3: Normalization of the result

After the application of the normalization procedure, the final information granule is obtained as follows:

$$\tilde{m}_{PCR2}(.) = \{\, \tilde{m}_{PCR2}(\theta_1) = 0.54,\ \tilde{m}_{PCR2}(\theta_2) = 0.41,\ \tilde{m}_{PCR2}(\theta_1 \cup \theta_2) = 0.05\}\,.$$

Fusion with Ordinary Conjunctive Rule

The conjunctive consensus here is given by:

$$m(\theta_1) = 0.38,\ m(\theta_2) = 0.22,\ m(\theta_1 \cap \theta_2) = k = 0.38 \quad m(\theta_1 \cup \theta_2) = 0.02$$

The PCR2 rule[9] is used to redistribute the resulting conflict:

$$\frac{x}{0.4 + 0.6} = \frac{y}{0.5 + 0.2} = \frac{x + y}{1.7} = \frac{0.38}{1.7} = 0.224$$

Then, the final masses of belief become:

$$\begin{aligned} m_{PCR2}(\theta_1) &= 0.38 + 1.0 \cdot 0.224 = 0.604\\ m_{PCR2}(\theta_2) &= 0.22 + 0.7 \cdot 0.224 = 0.376\\ m_{PCR2}(\theta_1 \cup \theta_2) &= 0.02\end{aligned}$$

Table 3: Comparative Results.

Ordinary Conjunctive Rule with PCR2	TCN Rule with fuzzy based PCR2
$m_{PCR2}(\theta_1) = 0.604$	$\tilde{m}_{PCR2}(\theta_1) = 0.54$
$m_{PCR2}(\theta_2) = 0.376$	$\tilde{m}_{PCR2}(\theta_1) = 0.41$
$m_{PCR2}(\theta_1 \cup \theta_2) = 0.02$	$\tilde{m}_{PCR2}(\theta_1 \cup \theta_2) = 0.05$

Table 3 lists comparative results obtained using the ordinary conjunctive rule with PCR2 redistribution of conflicting mass and TCN rule with fuzzy based PCR2.

Zadeh's Example

Let us have $\theta = \{\theta_1, \theta_2, \theta_3\}$ and two independent sources of information with the corresponding bbas[10]:

$$\begin{array}{lll} m_1(\theta_1) = 0.99 & m_1(\theta_2) = 0.0 & m_1(\theta_3) = 0.01 \\ m_2(\theta_1) = 0.0 & m_2(\theta_2) = 0.99 & m_2(\theta_3) = 0.01 \end{array}$$

Fusion with TCN Rule of Combination

Here, the min T-norm based conjunctive consensus yields the following result:

$$\tilde{m}(\theta_3) = 0.01;\ \ \tilde{m}(\theta_1 \cap \theta_2) = 0.99;\ \ \tilde{m}(\theta_1 \cap \theta_3) = 0.01;\ \ \tilde{m}(\theta_2 \cap \theta_3) = 0.01.$$

The partial conflicting masses will be redistributed to corresponding non-empty sets contributing to the particular partial conflicts by using fuzzy-based PCR3. According to $\tilde{m}(\theta_1 \cap \theta_2) = 0.99$:

$$\frac{x_1}{\max(0, 0.99)} = \frac{y_1}{\max(0, 0.99)} = \frac{x_1 + y_1}{1.98} = \frac{0.99}{1.98} = 0.5$$

$$x_1 = 0.99 \cdot 0.5 = 0.495; \quad y_1 = 0.99 \cdot 0.5 = 0.495$$

Considering $\tilde{m}(\theta_1 \cap \theta_3) = 0.01$:

$$\frac{x_2}{\max(0, 0.99)} = \frac{z_1}{\max(0.01, 0.01)} = \frac{x_2 + z_1}{1.0} = \frac{0.01}{1.0} = 0.01$$

$$x_2 = 0.99 \cdot 0.01 = 0.0099; \quad z_1 = 0.01 \cdot 0.01 = 0.0001$$

Considering $\tilde{m}(\theta_2 \cap \theta_3) = 0.01$:

$$\frac{y_2}{\max(0, 0.99)} = \frac{z_2}{\max(0.01, 0.01)} = \frac{y_2 + z_2}{1.0} = \frac{0.01}{1.0} = 0.01$$

$$y_2 = 0.99 \cdot 0.01 = 0.0099; \quad z_2 = 0.01 \cdot 0.01 = 0.0001$$

After conflict redistribution using a fuzzy-based PCR3, the following result is obtained:

$$\tilde{m}_{PCR3}(\theta_1) = \tilde{m}(\theta_1) + x_1 + x_2 = 0 + 0.495 + 0.0099 = 0.5049$$

$$\tilde{m}_{PCR3}(\theta_2) = \tilde{m}(\theta_2) + y_1 + y_2 = 0 + 0.495 + 0.0099 = 0.5049$$

$$\tilde{m}_{PCR3}(\theta_3) = \tilde{m}(\theta_3) + z_1 + z_2 = 0.01 + 0.0001 + 0.0001 = 0.0102$$

Then, the normalization results in:

$$\tilde{m}_{PCR3}(.) = \{\tilde{m}_{PCR3}(\theta_1) = 0.495,\ \ \tilde{m}_{PCR3}(\theta_2) = 0.495,\ \ \tilde{m}_{PCR3}(\theta_3) = 0.01\}.$$

Fusion with Ordinary Conjunctive Rule

The conjunctive consensus is given by:

$$m(\theta_3) = 0.0001; \quad m(\theta_1 \cap \theta_2) = 0.98; \quad m(\theta_1 \cap \theta_3) = 0.0099; \quad m(\theta_2 \cap \theta_3) = 0.0099$$

Applying the PCR3 [11] rule to the partial conflicting masses, one gets as follows.

According to $m(\theta_1 \cap \theta_2) = 0.98$:

$$\frac{x_1}{0.99 + 0.0} = \frac{y_1}{0.99 + 0.0} = \frac{x_1 + y_1}{1.98} = \frac{0.98}{1.98} = 0.495$$

Considering $m(\theta_1 \cap \theta_3) = 0.0099$:

$$\frac{x_2}{0.99} = \frac{z_1}{0.02} = \frac{x_2 + z_1}{1.01} = \frac{0.0099}{1.01} = 0.0098$$

Considering $m(\theta_2 \cap \theta_3) = 0.0099$:

$$\frac{y_2}{0.99} = \frac{z_2}{0.02} = \frac{y_2 + z_2}{1.01} = \frac{0.0099}{1.01} = 0.0098$$

Finally, the result is given by:

$$m_{PCR3}(\theta_1) = 0 + (0.99 \cdot 0.495) + (0.99 \cdot 0.0098) = 0.49975$$

$$m_{PCR3}(\theta_2) = 0 + (0.99 \cdot 0.495) + (0.99 \cdot 0.0098) = 0.49975$$

$$m_{PCR3}(\theta_3) = 0.0001 + (0.02 \cdot 0.0098) + (0.02 \cdot 0.0098) = 0.0005$$

Table 4 lists the results of comparison.

Table 4: Comparative Results.

Ordinary Conjunctive Rule with PCR3	TCN Rule with fuzzy based PCR3
$m_{PCR3}(\theta_1) = 0.49975$	$\tilde{m}_{PCR3}(\theta_1) = 0.495$
$m_{PCR3}(\theta_2) = 0.49975$	$\tilde{m}_{PCR3}(\theta_2) = 0.495$
$m_{PCR3}(\theta_3) = 0.0005$	$\tilde{m}_{PCR3}(\theta_3) = 0.01$

Total Conflict Example

Let us consider a case with a problem frame $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and two independent sources of information:

$$
\begin{array}{llll}
m_1(\theta_1) = 0.3 & m_1(\theta_2) = 0.0 & m_1(\theta_3) = 0.7 & m_1(\theta_4) = 0.0 \\
m_2(\theta_1) = 0.0 & m_2(\theta_2) = 0.4 & m_2(\theta_3) = 0.0 & m_2(\theta_4) = 0.6
\end{array}
$$

Fusion with TCN Rule of Combination

In this case, the min T-norm conjunctive consensus yields the following result:

$$\tilde{m}(\theta_1 \cap \theta_2) = 0.3;\;\; \tilde{m}(\theta_1 \cap \theta_4) = 0.3;\;\; \tilde{m}(\theta_2 \cap \theta_3) = 0.4;\;\; \tilde{m}(\theta_3 \cap \theta_4) = 0.6$$

Here one obtains the partial conflicting masses that will be redistributed using fuzzy-based PCR3.
According to the partial conflict $\tilde{m}(\theta_1 \cap \theta_2) = 0.3$:

$$\frac{x_1}{\max(0, 0.3)} = \frac{y_1}{\max(0, 0.4)} = \frac{x_1 + y_1}{0.7} = \frac{0.3}{0.7} = 0.4285$$

According to the partial conflict $\tilde{m}(\theta_1 \cap \theta_4) = 0.3$:

$$\frac{x_2}{\max(0, 0.3)} = \frac{h_1}{\max(0, 0.6)} = \frac{x_2 + h_1}{0.9} = \frac{0.3}{0.9} = 0.3333$$

According to the partial conflict $\tilde{m}(\theta_2 \cap \theta_3) = 0.4$:

$$\frac{y_2}{\max(0, 0.4)} = \frac{z_1}{\max(0, 0.7)} = \frac{y_2 + z_1}{1.1} = \frac{0.4}{1.1} = 0.3636$$

According to the partial conflict $\tilde{m}(\theta_3 \cap \theta_4) = 0.6$:

$$\frac{z_2}{\max(0, 0.7)} = \frac{h_2}{\max(0, 0.6)} = \frac{z_2 + h_2}{1.3} = \frac{0.6}{1.3} = 0.4615$$

After conflict redistribution, the result is given by:

$$\tilde{m}_{PCR3}(\theta_1) = 0.2275;\;\; \tilde{m}_{PCR3}(\theta_2) = 0.3168;$$
$$\tilde{m}_{PCR3}(\theta_3) = 0.5775;\;\; \tilde{m}_{PCR3}(\theta_4) = 0.4768.$$

And finally, the normalization procedure yields the following result:

$$\tilde{m}_{PCR3}(\theta_1) = 0.1423,\;\; \tilde{m}_{PCR3}(\theta_2) = 0.1982,$$
$$\tilde{m}_{PCR3}(\theta_3) = 0.3612,\;\; \tilde{m}_{PCR3}(\theta_4) = 0.2983.$$

Fusion with Ordinary Conjunctive Rule

The conjunctive consensus is given by:

$$m(\theta_1 \cap \theta_2) = 0.12; \quad m(\theta_1 \cap \theta_4) = 0.18; \quad m(\theta_2 \cap \theta_3) = 0.28; \quad m(\theta_3 \cap \theta_4) = 0.42$$

After applying the PCR3 rule to the partial conflicting masses one finally gets:

$$m_{PCR3}(\theta_1) = 0.111; \quad m_{PCR3}(\theta_2) = 0.171;$$
$$m_{PCR3}(\theta_3) = 0.404; \quad m_{PCR3}(\theta_4) = 0.314.$$

The comparative results are given in Table 5.

Table 5: Comparative Results.

Ordinary Conjunctive Rule with PCR3	TCN Rule with fuzzy based PCR3
$m_{PCR3}(\theta_1) = 0.111$	$\tilde{m}_{PCR3}(\theta_1) = 0.1423$
$m_{PCR3}(\theta_2) = 0.171$	$\tilde{m}_{PCR3}(\theta_2) = 0.1982$
$m_{PCR3}(\theta_3) = 0.404$	$\tilde{m}_{PCR3}(\theta_3) = 0.3612$
$m_{PCR3}(\theta_4) = 0.314$	$\tilde{m}_{PCR3}(\theta_4) = 0.2983$

Example 5 (Convergence to Idempotence)

Let us consider a case with a problem frame $\theta = \{\theta_1, \theta_2\}$ and two independent sources of information:

$$m_1(.) = \{m_1(\theta_1) = 0.7; \quad m_1(\theta_2) = 0.3\}$$
$$m_2(.) = \{m_2(\theta_1) = 0.7; \quad m_2(\theta_2) = 0.3\}$$

Fusion with TCN Rule of Combination

Here the min T-norm conjunctive consensus yields the following result:

$$\tilde{m}(.) = \{\tilde{m}(\theta_1) = 0.7; \quad \tilde{m}(\theta_2) = 0.3; \quad \tilde{m}(\theta_1 \cap \theta_2) = 0.6\}.$$

After conflict redistribution using fuzzy-based PCR2 one gets:

$$\frac{x}{\max(0.7, 0.7)} = \frac{y}{\max(0.3, 0.3)} = \frac{0.6}{1.0} = 0.6$$

$$\tilde{m}_{PCR2}(\theta_1) = 1.12; \quad \tilde{m}_{PCR2}(\theta_2) = 0.48$$

After normalization the final fused result becomes:

$$\tilde{m}_{PCR2}(.) = \{\tilde{m}_{PCR2}(\theta_1) = 0.7; \quad \tilde{m}_{PCR2}(\theta_2) = 0.3\}.$$

Fusion with Ordinary Conjunctive Rule

The conjunctive consensus is given by:

$$m(\theta_1) = 0.49; \; m(\theta_2) = 0.09; \; m(\theta_1 \cap \theta_2) = 0.42$$

Finally, the vector of belief masses after applying the PCR2 rule to the partial conflicting mass becomes:

$$m_{PCR2}(\theta_1) = 0.784; \; m_{PCR2}(\theta_2) = 0.216$$

The comparative results are given in Table 6.

Table 6: Comparative Results.

Ordinary Conjunctive Rule with PCR2	TCN Rule with fuzzy based PCR2
$m_{PCR2}(\theta_1) = 0.784$	$\tilde{m}_{PCR2}(\theta_1) = 0.7$
$m_{PCR2}(\theta_2) = 0.216$	$\tilde{m}_{PCR2}(\theta_2) = 0.3$

It is obvious that the fusion results obtained using the TCN rule of combination converge strongly towards idempotence.

Example 6 (Majority Opinion)

Let us consider a case with a problem frame $\theta = \{\theta_1, \theta_2\}$ and two independent sources of information:

$$m_1(.) = \{m_1(\theta_1) = 0.8; \quad m_1(\theta_2) = 0.2\}$$
$$m_2(.) = \{m_2(\theta_1) = 0.3; \quad m_2(\theta_2) = 0.7\}$$

Assume that in the next time moment a third source of information is introduced with the following bba:

$$m_3(.) = \{m_3(\theta_1) = 0.3; \quad m_3(\theta_2) = 0.7\}.$$

Fusion with TCN Rule of Combination

The TCN rule with fuzzy-based PCR2 yields the following normalized fusion result:

$$\tilde{m}_{12PCR2}(\theta_1) = 0.557; \quad \tilde{m}_{12PCR2}(\theta_2) = 0.443$$

Let us now combine $\tilde{m}_{12PCR2}(.)$ with the gbba of the third source $m_3(.)$. Then the final fused result is obtained as:

$$\tilde{m}_{12,3PCR2}(\theta_1) = 0.417; \quad \tilde{m}_{12,3PCR2}(\theta_2) = 0.583$$

It is evident from this result that the final bba $\tilde{m}_{12,3PCR2}(.) = [0.417 \;\; 0.583]$ starts to reflect the majority opinion; it means that $\tilde{m}_{12,3PCR2}(\theta_1) < \tilde{m}_{12,3PCR2}(\theta_2)$. If a fourth source is considered with a probability mass vector supporting the majority opinion, i.e. $m_4(.) = \{m_4(\theta_1) = 0.3; \;\; m_4(\theta_2) = 0.7\}$, then the final probability mass vector becomes:

$$\tilde{m}_{(12,3),4PCR2}(\theta_1) = 0.348; \quad \tilde{m}_{(12,3),4PCR2}(\theta_2) = 0.652$$

The new fused vector $\tilde{m}_{(12,3),4PCR2}(.) = [0.348 \;\; 0.652]$ reflects again the majority opinion since $\tilde{m}_{(12,3),4PCR2}(\theta_1)$ decreases more and more and, at the same time, $\tilde{m}_{(12,3),4PCR2}(\theta_2)$ increases in the same manner.

Fusion with Ordinary Conjunctive Rule

The conjunctive consensus between sources 1 and 2 is given by:

$$m_{12}(.) = \{m_{12}(\theta_1) = 0.24; \quad m_{12}(\theta_2) = 0.14; \quad m_{12}(\theta_1 \cap \theta_2) = 0.62\}.$$

After applying the PCR2 rule to the partial conflicting mass $m_{12}(\theta_1 \cap \theta_2) = 0.62$, the final probability mass vector becomes:

$$m_{12PCR2}(\theta_1) = 0.58; \quad m_{12PCR2}(\theta_2) = 0.42;$$

Let us now combine $m_{12PCR2}(.)$ with the bba of the third source $m_3(.)$.

Then, after applying PCR2 to the obtained conjunctive consensus, the final probability mass vector becomes:

$$m_{12,3PCR2}(\theta_1) = 0.408; \quad m_{12,3PCR2}(\theta_2) = 0.592.$$

It is evident from this result that the final bba $m_{12,3PCR2}(.) = [0.408 \;\; 0.592]$ starts to reflect the majority opinion; it means that $m_{12,3PCR2}(\theta_1) < m_{12,3PCR2}(\theta_2)$. If a fourth source is considered with a probability mass vector supporting the majority opinion, i.e. $m_4(.) = \{m_4(\theta_1) = 0.3; \;\; m_4(\theta_2) = 0.7\}$, the final probability mass vector becomes:

$$m_{(12,3),4PCR2}(\theta_1) = 0.286; \quad m_{(12,3),4PCR2}(\theta_2) = 0.714$$

The new fused vector $m_{(12,3),4PCR2}(.) = [0.286 \;\; 0.714]$ reflects the majority opinion since $m_{(12,3),4PCR2}(\theta_1)$ decreases more and more and, at the same time, $m_{(12,3),4PCR2}(\theta_2)$ increases in the same manner.

The comparative results are given in Table 7.

Table 7: Comparative Results.

Ordinary Conjunctive Rule with PCR2	TCN Rule with fuzzy based PCR2
$m_{(12,3),4PCR2}(\theta_1) = 0.286$	$\tilde{m}_{(12,3),4PCR2}(\theta_1) = 0.348$
$m_{(12,3),4PCR2}(\theta_2) = 0.714$	$\tilde{m}_{(12,3),4PCR2}(\theta_2) = 0.652$

The new TCN combination rule with fuzzy-based PCR2 reflects the majority opinion slower than the PCR2.

Example 7 (Neutrality of VBA)

Let us consider a case with a problem frame $\theta = \{\theta_1, \theta_2\}$ and two independent sources of information:

$$m_1(.) = \{m_1(\theta_1) = 0.4; \quad m_1(\theta_2) = 0.5 \quad m_1(\theta_1 \cup \theta_2) = 0.1\}$$
$$m_2(.) = \{m_2(\theta_1) = 0.0; \quad m_2(\theta_2) = 0.0 \quad m_2(\theta_1 \cup \theta_2) = 1.0\}$$

The second source is characterized with vacuous gbba.

The TCN rule yields the following result:

$$\tilde{m}(.) = \{\tilde{m}(\theta_1) = 0.4; \quad \tilde{m}(\theta_2) = 0.5; \quad \tilde{m}(\theta_1 \cup \theta_2) = 0.1\}.$$

From the obtained result it is evident that TCN rule satisfies the principle of neutrality of the vacuous belief assignment (VBA). The min T-norm operator will always give a result that is equal to the non-vacuous bba $m_1(.)$, because in either case the probability masses assigned to their corresponding propositions will always be lower or equal to the probability mass assigned to the full ignorance in $m_2(.) = m_2(\theta_1 \cup \theta_2) = 1.0$. It means that according to the way of obtaining the degree of association between the focal elements in $m_1(.)$, and $m_2(.)$, ($\tilde{m}(X) = \min\{m_1(X_i), m_2(X_j)\}$), the resulting bba will become equal to the non-vacuous $m_1(.)$.

Main Characteristics of the TCN Combination Rule

Although the TCN rule is not associative (like most of the fusion rules except Dempster's rule and the conjunctive rule on free-DSm model), it presents the following advantages:

- The rule is simple and very easy to implement;
- It reflects the majority opinion;
- The rule is convergent toward idempotence in cases when there are no intersections and unions between the elementary hypotheses;
- It reflects the effect of neutrality of vacuous belief assignment;
- It leads to adequate solutions in case of a total conflict between the sources of information.

Conclusions

In this article, a new combination rule (the TCN combination rule) based on fuzzy T-conorm/T-norm operators is proposed and analyzed. It does not belong to the general Weighted Operator Class. It overcomes the main limitations of Dempster's rule related to the normalization in cases of high conflict and the counter-intuitive fusion results. The advantages of the new rule could be summarized as: very easy to implement, satisfying the impact of neutral Vacuous Belief Assignment, commutative, convergent to idempotence, reflecting majority opinion, and assuring adequate data processing in case of a partial or total conflict between the information granules. It is suitable for the requirements of temporal multiple target tracking. The main drawback of this rule is related to the lack of associativity, which is not a major issue in temporal data fusion applications such as those involved in target type tracking[12] and classification.

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Notes:

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ORDERED DSmT AND ITS APPLICATION TO THE DEFINITION OF CONTINUOUS DSm MODELS

Frédéric DAMBREVILLE

Abstract: A difficulty may arise during the implementation of DSmT from the possible high dimension of the hyperpower sets, which are in fact free structures. However, it is possible to reduce the dimension of these structures by imposing logical constraints. In this article, the logical constraints are related to a predefined order over the logical propositions. Using such orders and the resulting logical constraints ensure a great reduction of model complexity. These results are applied to the definition of continuous DSm models. In particular, a simplified description of the continuous imprecision is considered based on the imprecision intervals of the sensors. From this point of view, it is possible to control the contradictions between continuous sensors in a DSmT manner, while the complexity of the model remains under control.

Keywords: Evidence Theory, Continuous DSmT, Probability, Boolean Algebra, Hyperpower Set.

Introduction

Recent advances [1] in Dezert-Smarandache Theory (DSmT) have shown that this theory is able to handle the contradiction between propositions in a quite flexible way. This new theory has been already applied in different domains, such as:

- Data association in target tracking; [2]
- Environmental prediction. [3]

Although free DSm models are defined over hyperpower sets, whose sizes evolve exponentially with the number of *atomic* propositions, it appears that the manipulation of the fusion rule is still manageable for practical reasonably well-shaped problems. Moreover, the hybrid DSm models are of lower complexity.

While DSmT works well for discrete spaces, the manipulation of continuous DSm

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models is still not possible. The first question that arises is: *What could play the role of a hyperpower set for a continuous DSm model?* This issue does not appear so dramatically in Dempster-Shafer Theory (DST) or in Transfer Belief Models.[4] In DST, a continuous proposition could only be a measurable subset. On the other hand, a free DSm model defined over a hyperpower set will imply that any pair of propositions will have a non-empty intersection. This is disappointing since the notion of a *point* (a minimal non empty proposition) does not exist anymore in a hyperpower set.

However, even if it is possible to define a continuous propositional model, the manipulation of continuous basic belief assignments (bba) is still a challenging issue.[5] Ristic and Smets have proposed a restriction of the bba to intervals of $\mathbb{R}$.[6] This has made it possible then to derive a mathematical relation between a continuous bba density and its Bel function.

This article proposes the formation of continuous DSm models. This is based on a constrained model, where the logical constraints are implied by the definition of an order relation over the propositions.

An one-dimensional DSm model has been implemented in this work, where the definition of the basic belief assignment relies on a *generalized notion of intervals*. Although building this model has been realized on a different ground, it shares some amazing similarities with the view of Ristic and Smets. As in their work,[7] the bba is viewed as density defined over a two-dimensional measurable space. As will be demonstrated later, it is possible to derive the Belief function from the basic belief assignment by applying an integral computation. And finally, the conjunctive fusion operator $\oplus$ is derived by a rather simple integral computation.

This article is organized as follows. The next section provides a short introduction to the Dezert-Smarandache Theory. The section that follows presents ordered DSm models. A continuous DSm model is defined afterwards. This method is restricted to only one dimension. The details of the related computational methods are also provided. The implementation of the algorithm is described and an example computation is given. The paper is then concluded.

Short Introduction to DSmT

Background

The theory and its rationale are extensively covered in a book edited by Smarandache and Dezert.[8]

Dezert-Smarandache Theory belongs to the family of *Evidence Theories*. Similarly to the *Dempster-Shafer Theory*[9] and the *Transferable Belief Models*,[10] DSmT is a framework for fusing belief information originating from independent sensors. Free DSm models are defined over hyper-power sets, which are *fully open-world extensions* of sets. It is possible to restrict the hypothesis of an entire open-world by adding propositional constraints, resulting in the definition of *hybrid Dezert-Smarandache model*.

Hyperpower Set. *Let $\Phi = \{\phi_i / i \in I\}$ be a set of propositions (finite or infinite). The hyperpower set $< \Phi >$ is the boolean pre-algebra freely generated by Φ and the boolean operators $\wedge$ (AND) and $\vee$ (OR)*. It does not contain the negation $\neg$.

Example:

$$\begin{aligned}< a, b, c >= \big\{ & a, b, c, a \wedge b \wedge c, a \vee b \vee c, a \wedge b, b \wedge c, c \wedge a, \\ & a \vee b, b \vee c, c \vee a, (a \wedge b) \vee c, (b \wedge c) \vee a, (c \wedge a) \vee b, \\ & (a \vee b) \wedge c, (b \vee c) \wedge a, (c \vee a) \wedge b, (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \big\}\end{aligned}$$

It is easy to verify that this set will be left unchanged by any application of the operators $\wedge$ and $\vee$. For example:

$$(a \wedge b) \wedge \big((b \wedge c) \vee a \big) = (a \wedge b \wedge b \wedge c) \vee (a \wedge b \wedge a) = a \wedge b \;.$$

Definition. The relation $\subset$ is defined over $< \Phi >$ by:

$$\forall \phi, \psi \in < \Phi > , \quad \phi \subset \psi \overset{\Delta}{\iff} \phi \wedge \psi = \phi \;.$$

Dezert Smarandache Model. Assume that Φ is a finite set. A Dezert Smarandache model (DSmm) is a pair (Φ, m), where Φ is a set of propositions and the *basic belief assignment* m is a non negatively valued function defined over $< \Phi >$, such that:

$$\sum_{\phi \in < \Phi >} m(\phi) = 1 \;.$$

Belief Function. Assume that Φ is a finite set. The belief function Bel related to a bba m is defined by:

$$\forall \phi \in < \Phi >, \, \mathrm{Bel}(\phi) = \sum_{\psi \in < \Phi > : \psi \subset \phi} m(\psi) \;. \tag{1}$$

Equation (1) is invertible:

$$\forall \phi \in < \Phi >, \, m(\phi) = \mathrm{Bel}(\phi) - \sum_{\psi \in < \Phi > : \psi \subsetneq \phi} m(\psi) \;.$$

Fusion Rule. Assume that Φ is a finite set. For a given universe Φ and two basic belief assignments m_1 and m_2 associated with independent sensors, the fused basic belief assignment is $m_1 \oplus m_2$, defined by:

$$m_1 \oplus m_2(\phi) = \sum_{\psi_1,\psi_2 \in <\Phi>:\psi_1 \wedge \psi_2 = \phi} m_1(\psi_1) m_2(\psi_2) \;. \tag{2}$$

Some Extensions

Between Sets and Hyperpower Sets. Sets and hyperpower sets are tightly related structures. First, the set (with $\cap, \cup$ operators and complement) is a boolean algebra, while the hyperpower set is a free boolean pre-algebra. Certainly, a free boolean pre-algebra could be completed to a boolean algebra, so that a hyperpower set could be seen as a substructure of set. More precisely, $< \Phi > \subset \mathcal{B}(\Phi)$, where $\mathcal{B}(\Phi)$ is the free boolean algebra generated by Φ. In particular, when Φ is finite, the boolean algebra $\mathcal{B}(\Phi)$ is generated by:

$$\left\{ \bigwedge_{i \in I} \epsilon_i \middle/ \forall i \in I ,\, \epsilon_i \in \{\phi_i, \neg\phi_i\} \right\} \;.$$

Therefore, $\mathcal{B}(\Phi)$ is isomorphic to a set structure of $2^{\mathrm{card}(I)}$ elements in the finite case.

On the other hand, a set could be interpreted as a *constrained pre-algebra*, i.e. a "constrained hyperpower set." More precisely, when Φ is a finite set,[11] this set is isomorphic to the boolean pre-algebra generated by Φ, $\wedge$ and $\vee$, and verifying the logical constraints:

$$\forall i, j \in I, i \neq j \Rightarrow \phi_i \wedge \phi_j = \bot \;.$$

In this construction, the empty proposition $\bot$ has been implicitly defined. However, it is possible (see below) to build constrained pre-algebra without the adjunction of the empty proposition $\bot$.

One advantage of the constrained boolean pre-algebra is that it is less complex and "fractalized" than the simple hyperpower set. Jean Dezert and Florentin Smarandache have extended the DSmT fusion operator so that to involve any kind of pre-algebra (hybrid DSmT).[13] In this work, the author focuses only on pre-algebra constrained without adjunction of $\bot$ and in this case the fusion operator of the free DSmT is kept unchanged.

Partially Open World without $\bot$. Let $\Gamma \subset < \Phi > \times < \Phi >$ and let us define the pre-algebra $< \Phi >_\Gamma$ generated by Φ, $\wedge$, $\vee$ and constrained by:

$$\forall (\phi, \psi) \in \Gamma \,,\; \phi = \psi \;.$$

Example. Let us again consider the case $\Phi = \{a, b, c\}$. However, let us now introduce the constraints $a \wedge b = a \wedge c = b \wedge c$, which implies (for example) using a set $\Gamma = \big\{(a \wedge b, a \wedge c), (a \wedge c, b \wedge c)\big\}$. Then we have $a \wedge b = a \wedge c = b \wedge c = a \wedge b \wedge c$. It follows that $(a \wedge b) \vee c = c$, $(b \wedge c) \vee a = a$, and $(c \wedge a) \vee b = b$. We also have $(a \vee b) \wedge c = (b \vee c) \wedge a = (c \vee a) \wedge b = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = a \wedge b \wedge c$. Discarding these cases from the free hyperpower set $< a, b, c >$, it follows that:

$$< a, b, c >_\Gamma = \big\{a, b, c, a \wedge b \wedge c, a \vee b \vee c, a \vee b, b \vee c, c \vee a\big\}.$$

It can be seen that $< \Phi >_\Gamma$ is left unchanged by any application of the operators $\wedge$ and $\vee$ (and does not contain the external proposition $\perp$). Thus, when Φ is finite, the definition of bba m, belief Bel, and fusion $\oplus$ remains unchanged.

- The *basic belief assignment* m is a non-negatively valued function defined over $< \Phi >_\Gamma$ such that:
$$\sum_{\phi \in < \Phi >_\Gamma} m(\phi) = 1 \; .$$
- The belief function Bel related to a bba m is defined by:
$$\forall \phi \in < \Phi >_\Gamma, \; \mathrm{Bel}(\phi) = \sum_{\psi \in < \Phi >_\Gamma : \psi \subset \phi} m(\psi) \; .$$
- Being given two basic belief assignments m_1 and m_2, the fused basic belief assignment $m_1 \oplus m_2$ is defined by:
$$m_1 \oplus m_2(\phi) = \sum_{\psi_1, \psi_2 \in < \Phi >_\Gamma : \psi_1 \wedge \psi_2 = \phi} m_1(\psi_1) m_2(\psi_2) \; .$$

These extended definitions will be applied subsequently.

Ordered DSm Model

In order to reduce the complexity of the free DSm model, it is necessary to introduce logical constraints that will lower the size of the pre-algebra. Such type of constraints may appear clearly in the hypotheses of the problem. In this case, constraints come naturally and approximations may not be required. However, when the model is too complex and there are no explicit constraints for reducing complexity, it is necessary to approximate the model by introducing some new constraints. Two rules should be applied then:

- Only weaken pieces of information[12] do not produce information from nothing;

- Minimize the weakening of information.

The first rule guarantees that the approximation does not introduce false information. But some significant pieces of information (e.g., contradictions) are possibly missed. This drawback has to be overcome by the second rule.

In order to build a good approximation policy, some external knowledge such as distance or order relations among the propositions could be used. Behind these relations some kind of distance between the informations will be assumed : *the more are the informations distant, the more is their conjunctive combination valuable.*

Ordered Atomic Propositions

Let $(\Phi, \leq)$ be an ordered set of propositions. This order relation is assumed to describe the relative distance between the information. For example, the relation $\phi \leq \psi \leq \eta$ implies that ϕ and ψ are closer informations than ϕ and η . Thus, the information contained in $\phi \wedge \eta$ is stronger than the information contained in $\phi \wedge \psi$. Certainly, this comparison is meaningless when all the information is preserved; however, when approximations are necessary, it will be useful to be able to choose the best information.

Illustrative Example. Assume that three independent sensors provide three measures of a continuous parameter – x, y, and z. The parameters x, y, z are assumed to be real values, not of the *set* $\mathbb{R}$ but of its “pre-algebraic” extension (theoretical issues will be clarified later).[14] The fused information could be formalized by the proposition $x \wedge y \wedge z$ (in a DSmT viewpoint). What will happen if we would like to reduce the information by removing a proposition? Do we keep $x \wedge y$, $y \wedge z$, or $x \wedge z$? This is obviously information weakening. But it is possible that a piece of information is better than another. At this point, the order between the values x, y, z will be involved. Assume, for example, that $x \leq y < z$. It is clear that the proposition $x \wedge z$ indicates a greater contradiction than $x \wedge y$ or $y \wedge z$. Thus, the proposition $x \wedge z$ is the one that should be kept! The discarding constraint $x \leq y \leq z \Rightarrow x \wedge y \wedge z = x \wedge z$ is implied then.

Associated Pre-Algebra and Complexity

In regard to the previous example, the pre-algebra associated with the ordered propositions $(\Phi, \leq)$ is $< \Phi >_\Gamma$, where Γ is defined as follows:

$$\Gamma = \left\{ (\phi \wedge \psi \wedge \eta, \phi \wedge \eta) \big/ \phi, \psi, \eta \in \Phi \text{ and } \phi \leq \psi \leq \eta \right\} .$$

The following characteristics gives an approximate bound on the size of $< \Phi >_\Gamma$ in the case of a total order.

Proposition 1 *Assume that* $(\Phi, \leq)$ *is totally ordered. Then,* $< \Phi >_\Gamma$ *is a substructure of the set* Φ^2 *.*

Proof: First, due to the fact that the order is total, the added constraints are:

$$\forall \phi, \psi, \eta \in \Phi \,,\ \phi \wedge \psi \wedge \eta = \min\{\phi, \psi, \eta\} \wedge \max\{\phi, \psi, \eta\} \ .$$

Now, for any $\phi \in \Phi$, let us define $\breve{\phi}$ by:

$$\underline{\breve{\phi} = \left\{(\varphi_1, \varphi_2) \in \Phi^2 \,\big/\, \varphi_1 \leq \phi \leq \varphi_2\right\}}$$

It is noteworthy that:

$$\breve{\phi} \cap \breve{\psi} = \left\{(\varphi_1, \varphi_2) \in \Phi^2 \,\big/\, \varphi_1 \leq \min\{\phi, \psi\} \text{ and } \max\{\phi, \psi\} \leq \varphi_2\right\}$$

and

$$\breve{\phi} \cap \breve{\psi} \cap \breve{\eta} = \left\{(\varphi_1, \varphi_2) \in \Phi^2 \,\big/\, \varphi_1 \leq \min\{\phi, \psi, \eta\} \text{ and } \max\{\phi, \psi, \eta\} \leq \varphi_2\right\} \ .$$

By defining $m = \min\{\phi, \psi, \eta\}$ and $M = \max\{\phi, \psi, \eta\}$, it is inferred that:

$$\breve{\phi} \cap \breve{\psi} \cap \breve{\eta} = \breve{m} \cap \breve{M} \ . \tag{3}$$

Let $\mathcal{A} \subset \mathcal{P}(\Phi^2)$ be generated by $\breve{\phi}|_{\phi \in \Phi}$ using $\cap$ and $\cup$, i.e.,

$$\mathcal{A} = \bigcup_{n \geq 0} \left\{ \bigcup_{k=1}^{n} (\breve{\phi}_k \cap \breve{\psi}_k) \Big/ \forall k \,,\ \breve{\phi}_k, \breve{\psi}_k \in \Phi \right\} \ .$$

Then, the following mapping is defined by (3):

$$\boxed{\smile : \begin{cases} < \Phi >_\Gamma \longrightarrow \mathcal{A} \\ \displaystyle\bigvee_{k=1}^{n} \bigwedge_{l=1}^{n_k} \phi_{k,l} \longmapsto \bigcup_{k=1}^{n} \bigcap_{l=1}^{n_k} \breve{\phi}_{k,l} \end{cases} \quad , \quad \text{where } \phi_{k,l} \in \Phi}$$

which is an onto-morphism of pre-algebra.

Lemma 1 *Assume:*

$$\bigcup_{k=1}^{n} (\breve{\phi}_k^1 \cap \breve{\phi}_k^2) \subset \bigcup_{l=1}^{m} (\breve{\psi}_l^1 \cap \breve{\psi}_l^2) \quad , \quad \textit{where } \phi_k^j \,,\ \psi_l^j \in \Phi \ .$$

Then:

$$\forall k \,,\ \exists l \,,\ \min\{\phi_k^1, \phi_k^2\} \leq \min\{\psi_l^1, \psi_l^2\} \textit{ and } \max\{\phi_k^1, \phi_k^2\} \geq \max\{\psi_l^1, \psi_l^2\}$$

and

$$\forall k \,,\ \exists l \,,\ \breve{\phi}_k^1 \cap \breve{\phi}_k^2 \subset \breve{\psi}_l^1 \cap \breve{\psi}_l^2 \ .$$

Proof of Lemma: Let $k \in [\![1, n]\!]$.
Define $m = \min\{\phi_k^1, \phi_k^2\}$ and $M = \max\{\phi_k^1, \phi_k^2\}$.
Then $(m, M) \in \breve{\phi}_k^1 \cap \breve{\phi}_k^2$ holds, implying $(m, M) \in \bigcup_{l=1}^m (\breve{\psi}_l^1 \cap \breve{\psi}_l^2)$.
Let l be such that $(m, M) \in \breve{\psi}_l^1 \cap \breve{\psi}_l^2$.
Then $m \leq \min\{\psi_l^1, \psi_l^2\}$ and $M \geq \max\{\psi_l^1, \psi_l^2\}$.
Therefore, $\breve{\phi}_k^1 \cap \breve{\phi}_k^2 \subset \breve{\psi}_l^1 \cap \breve{\psi}_l^2$.

Considering that $\min\{\phi_k^1, \phi_k^2\} \leq \min\{\psi_l^1, \psi_l^2\}$ and $\max\{\phi_k^1, \phi_k^2\} \geq \max\{\psi_l^1, \psi_l^2\}$ it can also be inferred that $(\phi_k^1 \wedge \phi_k^2) \wedge (\psi_l^1 \wedge \psi_l^2) = \phi_k^1 \wedge \phi_k^2$ (definition of Γ). This fact just implies that $\phi_k^1 \wedge \phi_k^2 \subset \psi_l^1 \wedge \psi_l^2$. Finally, it is inferred:

Lemma 2 *Assume:*

$$\bigcup_{k=1}^{n} (\breve{\phi}_k^1 \cap \breve{\phi}_k^2) \subset \bigcup_{l=1}^{m} (\breve{\psi}_l^1 \cap \breve{\psi}_l^2) \quad , \quad \textit{where } \phi_k^j \ , \ \psi_l^j \in \Phi \ .$$

Then:

$$\bigvee_{k=1}^{n} (\phi_k^1 \wedge \phi_k^2) \subset \bigvee_{l=1}^{m} (\psi_l^1 \wedge \psi_l^2) \ .$$

It can be concluded from this lemma that $\smile$ is one to one.
And finally, $\smile$ *is an isomorphism of pre-algebra*, and $< \Phi >_\Gamma$ is a substructure of Φ^2.

General Properties of the Model

In the next section, the previous construction will be extended to the continuous case, i.e. $(\mathbb{R}, \leq)$. However, a strict logical manipulation of the propositions is not sufficient and instead a measurable generalization of the model will be used. It has been seen that a proposition of $< \Phi >_\Gamma$ could be described as a subset of Φ^2. In this subsection, the proposition model will be precisely characterized. Namely this characterization will be used and extended in the next section to the continuous case.

Proposition 2 *Let* $\phi \in < \Phi >_\Gamma$.
Then $\smile(\phi) \subset \mathcal{T}$, *where* $\mathcal{T} = \left\{(\phi, \psi) \in \Phi^2 / \phi \leq \psi\right\}$.

Proof: The proof is obvious since $\forall \phi \in \Phi$, $\breve{\phi} \subset \mathcal{T}$.

Definition 1 *A subset* $\theta \subset \Phi^2$ *is* increasing *if and only if:*

$$\forall (\phi, \psi) \in \theta \ , \ \forall \eta \leq \phi \ , \ \forall \zeta \geq \psi \ , \ (\eta, \zeta) \in \theta \ .$$

Let $\mathcal{U} = \left\{\theta \subset \mathcal{T} \middle/ \theta \text{ is increasing}\right\}$ be the set of increasing subsets of $\mathcal{T}$. Note that the intersection or the union of increasing subsets is increasing subset too, so that $(\mathcal{U}, \cap, \cup)$ is a pre-algebra.

Proposition 3 *For any choice of* Φ , $\left\{\smile(\phi) \middle/ \phi \in < \Phi >_\Gamma\right\} \subset \mathcal{U}$.
When Φ *is finite,* $\mathcal{U} = \left\{\smile(\phi) \middle/ \phi \in < \Phi >_\Gamma\right\}$.

Proof of $\supset$ **.** The proof is obvious since $\breve{\phi}$ is increasing for any $\phi \in \Phi$.

Proof of $\subset$ **.** Let $\theta \in \mathcal{U}$ and let $(a, b) \in \theta$.
Since $\breve{a} \cap \breve{b} = \left\{(\alpha, \beta) \in \Phi^2 \middle/ \alpha \leq a \text{ and } \beta \geq b\right\}$ and θ is increasing, it follows that $\breve{a} \cap \breve{b} \subset \theta$.
Finally, $\theta = \bigcup_{(a,b)\in\theta} \breve{a} \cap \breve{b} = \smile\left(\bigvee_{(a,b)\in\theta} a \wedge b\right)$.
Note that $\bigvee_{(a,b)\in\theta} a \wedge b$ is actually defined because θ is finite when Φ is finite.

When infinite $\vee$-ing is allowed, note that $\mathcal{U}$ may be considered as a model for $< \Phi >_\Gamma$ even if Φ is infinite. In the next section, the *continuous* pre-algebra related to $(\mathbb{R}, \leq)$ will be modelled by the *measurable increasing subsets* of $\left\{(x, y) \in \mathbb{R}^2 \middle/ x \leq y\right\}$.

Continuous DSm Model

The case $\Phi = \mathbb{R}$ is considered in this section.

Typically, in a continuous model, it will be necessary to manipulate any measurable proposition and intervals, for example. It appears that most intervals could not be obtained by a finite logical combination of the atomic propositions, but rather by infinite combination. For example, considering the set formalism, $[a, b] = \bigcup_{x\in[a,b]}\{x\}$ is obtained, which suggests the definition of the infinite disjunction "$\bigvee_{x\in[a,b]} x$." It is known that infinite disjunctions are difficult to handle in a logic. It is better to manipulate the models directly. The pre-algebra to be constructed should verify the property $x \leq y \leq z \Rightarrow x \wedge y \wedge z = x \wedge z$. As discussed previously and due to the fact that infinite disjunctions are allowed, a model for such algebra could be the measurable increasing subsets.

Measurable Increasing Subsets

A measurable subset $A \subset \mathbb{R}^2$ is a measurable increasing subset if:

$$\begin{cases} \forall\,(x, y) \in A\,,\ x \leq y\ , \\ \forall\,(x, y) \in A\,,\ \forall a \leq x\,,\ \forall b \geq y\,,\ (a, b) \in A\ . \end{cases}$$

The set of measurable increasing subsets is denoted $\mathcal{U}$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing measurable mapping such that $f(x) \geq x$ for any $x \in \mathbb{R}$. The set $\{(x, y) \in \mathbb{R}^2 / f(x) \leq y\}$ is a measurable increasing subset.

"Points." For any $x \in \mathbb{R}$, the measurable increasing subset $\breve{x}$ is defined by:

$$\breve{x} = \left\{(a, b) \in \mathbb{R}^2 \;/\; a \leq x \leq b\right\} .$$

The set $\breve{x}$ is naturally a model for the point $x \in \mathbb{R}$ within the pre-algebra (refer to the section devoted to ordered DSm models).

Generalized Intervals. A particular class of increasing subsets, the generalized intervals, will be considered in what follows.

For any $x \in \mathbb{R}$, the measurable sets $\grave{x}$ and $\acute{x}$ are then defined by:

$$\begin{cases} \grave{x} = \left\{(a, b) \in \mathbb{R}^2 \;/\; a \leq b \text{ and } x \leq b\right\} , \\ \acute{x} = \left\{(a, b) \in \mathbb{R}^2 \;/\; a \leq b \text{ and } a \leq x\right\} . \end{cases}$$

The following characteristics are then derived:

$$\breve{x} = \grave{x} \cap \acute{x} \;,\; \grave{x} = \bigcup_{z \in [x, +\infty[} \breve{z} \quad \text{and} \quad \acute{x} = \bigcup_{z \in]-\infty, x]} \breve{z}$$

Moreover, for any x, y such that $x \leq y$, it happens that:

$$\grave{x} \cap \acute{y} = \bigcup_{z \in [x, y]} \breve{z} .$$

In summary, the set $\grave{x}$, $\acute{x}$, and $\grave{x} \cap \acute{y}$ (with $x \leq y$) are the respective models for the intervals $[x, +\infty]$, $[-\infty, x]$ and $[x, y]$ within the pre-algebra. Naturally, the accents ` and ´ are used for opening and closing the intervals, respectively.

Finally, the set $\grave{x} \cap \acute{y}$, where $x, y \in \mathbb{R}$ are not constrained, constitutes a generalized definition of the notion of interval. In the case $x \leq y$, it works like the "classical" interval, but in the case $x > y$, a new class of intervals is obtained with negative width. In any case, $\grave{x} \cap \acute{y}$ is not empty and may have a non-zero measure.

The width $\delta = \frac{y-x}{2}$ of the interval $\grave{x} \cap \acute{y}$ could be considered as a measure of contradiction associated with this proposition, while its center $\mu = \frac{x+y}{2}$ should be considered as its median value. The interpretation of the measure of contradiction is left to the user. Typically, a possible interpretation could be:

- $\delta < 0$ means contradictory informations,
- $\delta = 0$ means exact informations,
- $\delta > 0$ means imprecise informations.

It is worth mentioning also that the set of generalized intervals

$$\underline{\mathcal{I} = \{\grave{x} \cap \acute{y} / x, y \in \mathbb{R}\}}$$

is left unchanged by the operator $\cap$, as seen in the following proposition.

Proposition 4 (Stability) Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
Define $x = \max\{x_1, x_2\}$ and $y = \min\{y_1, y_2\}$.
Then $\left(\grave{x}_1 \cap \acute{y}_1\right) \cap \left(\grave{x}_2 \cap \acute{y}_2\right) = \grave{x} \cap \acute{y}$.

The proof of this proposition is obvious.

The last feature makes possible the definition of basic belief assignment only over generalized intervals. This assumption is obviously necessary in order to reduce the complexity of evidence modeling. Behind this assumption lies the idea that a continuous measure is described by imprecision/contradiction around the sensored value. Such a hypothesis has been made by Ristic and Smets.[15] From this point on, all the defined bba will be zeroed outside $\mathcal{I}$. Now, since $\mathcal{I}$ is invariant in $\cap$, it is implied that all the bba which will be manipulated, from sensors or after fusion, will be zeroed outside $\mathcal{I}$. This makes the basic belief assignments equivalent to a density over the two-dimensional space $\mathbb{R}^2$.

Definition and Manipulation of Belief

The definitions of bba, belief function, and fusion rule result directly from those given in the section introducing DSmT, however, naturally, the bba becomes density and the summations are replaced by integrations.

Basic Belief Assignment. As discussed previously, it is hypothesized that the measures are characterized by a precision interval around the sensored values. In addition, there is uncertainty around the measure, which is translated into a basic belief assignment over the precision intervals.

According to this hypotheses, a bba will be a non-negatively valued function m defined over $\mathcal{U}$, zeroed outside $\mathcal{I}$ (set of generalized intervals), and such that:

$$\int_{x,y \in \mathbb{R}} m\left(\grave{x} \cap \acute{y}\right) dx dy = 1 \;.$$

Belief Function. The function of belief, Bel, is defined for any measurable proposition $\phi \in \mathcal{U}$ by:

$$\mathrm{Bel}\,(\phi) = \int_{\grave{x}\cap\acute{y}\subset\phi} m\big(\grave{x}\cap\acute{y}\big)dxdy \ .$$

In particular, for a generalized interval $\grave{x}\cap\acute{y}$:

$$\mathrm{Bel}\,\big(\grave{x}\cap\acute{y}\big) = \int_{u=x}^{+\infty}\int_{v=-\infty}^{y} m\big(\grave{u}\cap\acute{v}\big)dudv \ .$$

Fusion Rule. Being given two basic belief assignments m_1 and m_2, the fused basic belief assignment $m_1 \oplus m_2$ is defined by the following integral:

$$m_1 \oplus m_2\big(\grave{x}\cap\acute{y}\big) = \int_{\mathcal{C}=\{(\phi,\psi)/\phi\cap\psi=\grave{x}\cap\acute{y}\}} m_1(\phi)m_2(\psi)\,d\mathcal{C} \ .$$

Now, it is hypothesized that m_i is positive only for intervals of the form $\grave{x}_i\cap\acute{y}_i$. Proposition 4 implies:

$$\grave{x}_1\cap\acute{y}_1\cap\grave{x}_2\cap\acute{y}_2 = \grave{x}\cap\acute{y} \text{ where } \begin{cases} x = \max\{x_1,x_2\}\ , \\ y = \min\{y_1,y_2\}\ . \end{cases}$$

It is then inferred:

$$\begin{aligned} m_1 \oplus m_2\big(\grave{x}\cap\acute{y}\big) = &\int_{-\infty}^{x}\int_{y}^{+\infty} m_1\big(\grave{x}\cap\acute{y}\big)m_2\big(\grave{x}_2\cap\acute{y}_2\big)dx_2dy_2 \\ &+ \int_{-\infty}^{x}\int_{y}^{+\infty} m_1\big(\grave{x}_1\cap\acute{y}_1\big)m_2\big(\grave{x}\cap\acute{y}\big)dx_1dy_1 \\ &+ \int_{-\infty}^{x}\int_{y}^{+\infty} m_1\big(\grave{x}_1\cap\acute{y}\big)m_2\big(\grave{x}\cap\acute{y}_2\big)dx_1dy_2 \\ &+ \int_{-\infty}^{x}\int_{y}^{+\infty} m_1\big(\grave{x}\cap\acute{y}_1\big)m_2\big(\grave{x}_2\cap\acute{y}\big)dx_2dy_1 \ . \end{aligned}$$

In particular, it is now justified that a bba, from sensors or fused, will always be zeroed outside $\mathcal{I}$.

Implementation of the Continuous Model

Setting. In this implementation, the study has been restricted only to the interval $[-1,1]$ instead of $\mathbb{R}$. The previous results still hold by truncating over $[-1,1]$. In particular, any bba m is zeroed outside $\mathcal{I}_{-1}^{1} = \{\grave{x} \cap \acute{y} / x, y \in [-1,1]\}$ and its related belief function is defined by:

$$\text{Bel}\big(\grave{x} \cap \acute{y}\big) = \int_{u=x}^{1} \int_{v=-1}^{y} m\big(\grave{u} \cap \acute{v}\big) du dv \ ,$$

for any generalized interval of $\mathcal{I}_{-1}^{1}$. The bba resulting from the fusion of two bbas m_1 and m_2 is defined by:

$$\begin{aligned} m_1 \oplus m_2\big(\grave{x} \cap \acute{y}\big) &= \int_{-1}^{x} \int_{y}^{1} m_1\big(\grave{x} \cap \acute{y}\big) m_2\big(\grave{x}_2 \cap \acute{y}_2\big) dx_2 dy_2 \\ &+ \int_{-1}^{x} \int_{y}^{1} m_1\big(\grave{x}_1 \cap \acute{y}_1\big) m_2\big(\grave{x} \cap \acute{y}\big) dx_1 dy_1 \\ &+ \int_{-1}^{x} \int_{y}^{1} m_1\big(\grave{x}_1 \cap \acute{y}\big) m_2\big(\grave{x} \cap \acute{y}_2\big) dx_1 dy_2 \\ &+ \int_{-1}^{x} \int_{y}^{1} m_1\big(\grave{x} \cap \acute{y}_1\big) m_2\big(\grave{x}_2 \cap \acute{y}\big) dx_2 dy_1 \ . \end{aligned}$$

Method. The theoretical computation of these integrals is not easy. An approximation of the densities and of the integrals has been considered. More precisely, the densities have been approximated by means of two-dimensional *Chebyshev polynomials*, which show several good characteristics:

- The approximation grows quickly with the degree of the polynomial, without oscillation phenomena;
- The Chebyshev transform is very much related to the Fourier transform, which makes the parameters of the polynomials very quickly computable by means of the fast Fourier transform;
- The integration can be easily computed.

In the experiments, the author has chosen Chebyshev approximation of degree 128×128, which is more than sufficient for an almost exact computation.

Example. Two bba m_1 and m_2 have been constructed by normalizing the following functions mm_1 and mm_2 defined over $[-1,1]^2$:

$$mm_1\big(\grave{x} \cap \acute{y}\big) = \exp\big(-(x+1)^2 - y^2\big)$$

and

$$mm_2\big(\grave{x} \cap \acute{y}\big) = \exp\big(-x^2 - (y-1)^2\big) \ .$$

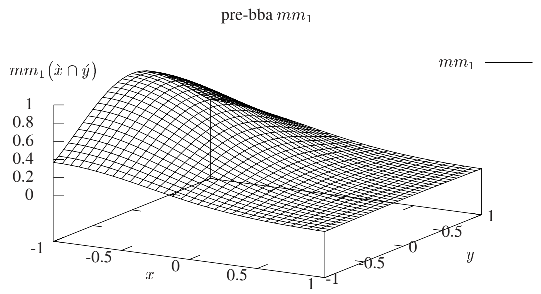

Figure 1: Non-Normalized bba mm_1.

The fused bba $m_1 \oplus m_2$ and the respective belief function $b_1, b_2, b_1 \oplus b_2$ have been computed. This computation has been instantaneous. All functions are presented in Figures 1 to 8.

Interpretation. The bba m_1 is density centered around the interval $[-1, 0]$, while m_2 is a density centered around $[0, 1]$. This explains why the belief b_1 increases faster from the interval $[-1, -1]$ to $[-1, 1]$ than from the interval $[1, 1]$ to $[-1, 1]$. And this characteristics is naturally inverted for b_2.

A comparison of the fused bba $m_1 \oplus m_2$ with the initial bbas m_1 and m_2 makes apparent a global forward move of the density. This just means that the fused bba is put on intervals with less imprecision and possibly on some intervals with negative width (i.e., associated with a degree of contradiction). Certainly, there is nothing surprising here since information fusion will reduce imprecision and produce some contradiction! It can also be observed that the fused bba is centered around the interval $[0, 0]$. This result coincides perfectly with the fact that m_1 and m_2, and their related sensors, put more belief over the interval $[-1, 0]$ and the interval $[0, 1]$, respectively; and of course $[-1, 0] \cap [0, 1] = [0, 0]$.

Conclusion

This article has looked at continuous information fusion and has proposed to solve it in terms of the DSmT paradigm. The proposed methodology is flexible and able to specify the typical various degrees of contradiction of a DSm model. It has been implemented efficiently for bounded continuous information. The work is still in

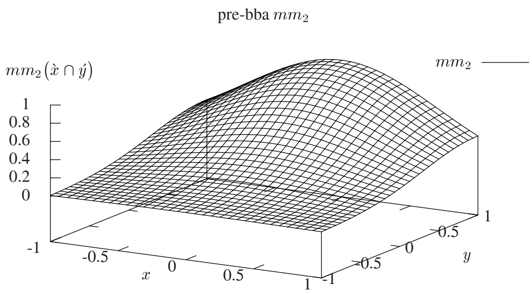

Figure 2: Non-Normalized bba mm_2.

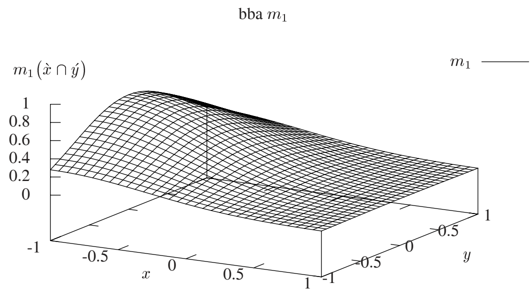

Figure 3: Basic Belief Assignment m_1.

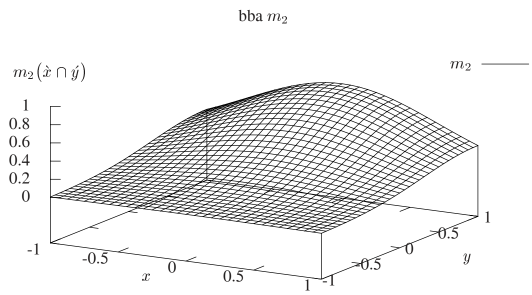

Figure 4: Basic Belief Assignment m_2.

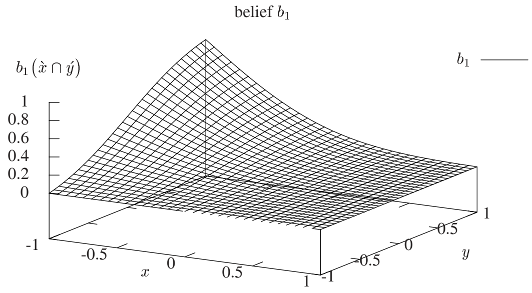

Figure 5: Belief Function b_1.

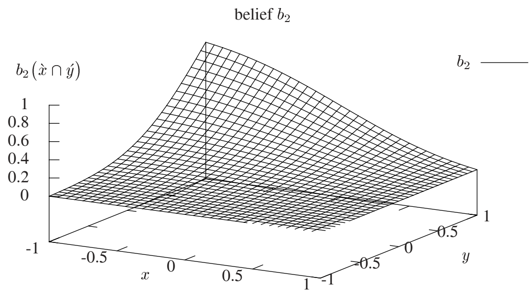

Figure 6: Belief Function b_2.

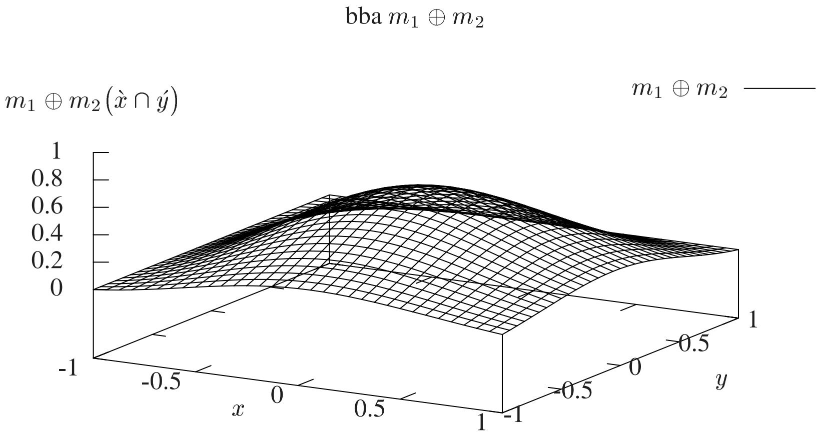

Figure 7: Fused bba $m_1 \oplus m_2$.

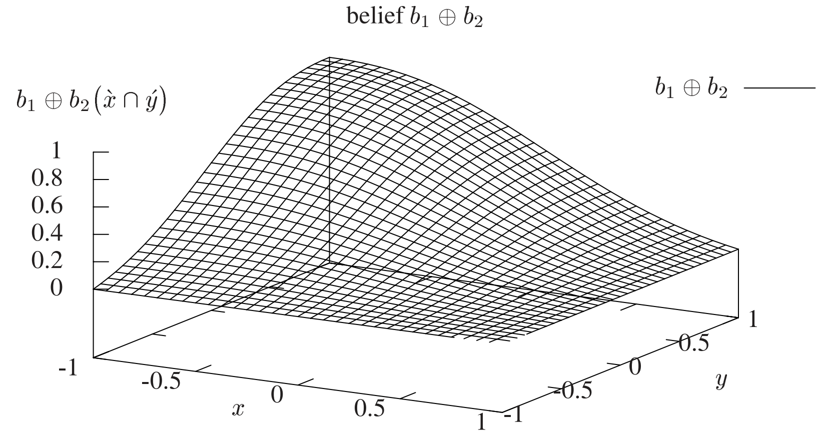

Figure 8: Fused bba $b_1 \oplus b_2$.

progress; however, applications should be sought in the future on localization problems. Presently, the methodology is restricted to one-dimensional information. However, some research has been Accomplished to extend the method to multi-dimensional domains.

Notes:

1. Florentin Smarandache and Jean Dezert, eds., *Advances and Applications of DSmT for Information Fusion* (Rehoboth: American Research Press, 2004), <http://www.gallup.unm.edu/~smarandache/DSmT-book1.pdf>.
2. Albena Tchamova, Jean Dezert, Tzvetan Semerdjiev, and Pavlina Konstantinova, "Target Tracking with Generalized Data Association based on the General DSm Rule of Combination," in *Proceedings of the Seventh International Conference on Information Fusion*, (Stockholm, Sweden, 28 June - 1 July 2004), ed. Per Svensson and Johan Schubert (Mountain View, CA: International Society of Information Fusion), 392–399.
3. Samuel Corgne, Laurence Hubert-Moy, Grégoire Mercier, and Jean Dezert, "Application of DSmT for Land Cover Change Prediction," in *Advances and Applications of DSmT for Information Fusion*, ed. Florentin Smarandache and Jean Dezert (Rehoboth: American Research Press, 2004), 371–382.
4. Philippe Smets and Robert Kennes, "The Transferable Belief Model," *Artificial Intelligence* 66, no. 2 (1994): 191–234.
5. Branko Ristic and Philippe Smets, "Belief Function Theory on the Continuous Space with an Application to Model-based Classification," in *Proceedings of the Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'2004)* (Perugia, Italy, 4-9 July 2004), 1119–1126; Thomas M. Strat, "Continuous Belief Functions for Evidential Reasoning," in *Proceedings of the National Conference on Artificial Intelligence*, ed. Ronald J. Brachman (Austin, TX, 6-10 August 1984), (AAAI Press, 1984), 308–313.
6. Ristic and Smets, "Belief Function Theory on the Continuous Space with an Application to Model-based Classification."
7. Ristic and Smets, "Belief Function Theory on the Continuous Space with an Application to Model-based Classification."
8. Smarandache and Dezert, *Advances and Applications of DSmT for Information Fusion*.

9. Arthur P. Dempster, "Upper and Lower Probabilities Induced by a Multiple Valued Mapping," *Annals of Mathematical Statistics* 38 (1967): 325–339; Glenn Shafer, *A Mathematical Theory of Evidence* (Princeton, N.J.: Princeton University Press, 1976).
10. Smets and Kennes, "The Transferable Belief Model."
11. When Φ is infinite, this result requires "infinite $\vee$-ing."
12. Typically, a constraint such as $\phi \wedge \psi \wedge \eta = \phi \wedge \psi$ will weaken the information by erasing η from $\phi \wedge \psi \wedge \eta$.
13. Smarandache and Dezert, *Advances and Applications of DSmT for Information Fusion*.
14. In particular, as we are working in a pre-algebra, $x \wedge y$ makes sense and it is possible that $x \wedge y \neq \bot$ even when $x \neq y$.
15. Ristic and Smets, "Belief Function Theory on the Continuous Space with an Application to Model-based Classification."

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ROBOT MAP BUILDING FROM SONAR SENSORS AND DSmT

Xinde LI, Xinhan HUANG, and Min WANG

Abstract: Knowledge acquisition in map building is characterized with uncertainty and imprecision. This uncertainty is especially severe in the course of building grid maps using sonar. Jean Dezert and Florentin Smarandache have recently proposed a new information fusion paradigm (DSmT), whose major advantage is that it deals with uncertainty and conflict of information. In this article, based on the Dezert-Smarandache Theory, the authors demonstrate how to fuse information from homogeneous or heterogeneous sensors differing in reliability. Then, they build the belief model of sonar grid map and construct the generalized basic belief assignment function (gbbaf). Pioneer II mobile robot has served as experimental platform and a 3D-Map has been built online based on DSmT. Finally, this work has established a firm foundation for a firm foundation for the simultaneous study of a dynamic unknown environment and multi-robots' map building.

Keywords: Uncertainty, DSmT, Grid Map, Information Fusion, Mobile Robots.

Introduction

The research on exploration of entirely unknown environments for intelligent mobile robots has been a popular and difficult subject for experts in the field of robotics for a long time. Robots are not aware of the environment around them; that is, they do not have practical knowledge about the environment, such as size, shape, layout, and also no knowledge about signs such as beacons, landmarks, allowing them to determine their location within the environment. Thus, the relation between self-localization and map building for mobile robots is like the *chicken and egg problem*.[1] Or, in other words, if the mobile robot builds map of the environment, it has to know its own exact position within the environment; at the same time, if the robot wants to know its own position, then it must have a referenced map of the environment. Though it is hard to answer this question, some intelligent sensors, such as odometer, electronic compass, sonar detector, laser range finder, and vision sensor have been installed on mobile robots as if a person has perceptive organs.

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How to manage and utilize the perceptive information acquired by the "organs" is a new subject in information fusion that will play an important role in this work, too. As far as the authors are aware, the experts have not yet provided a unified solution. As to the practical field or system, different control architectures have been proposed, such as hierarchical, centralized, distributed, and composite, and then, according to the given integrated hierarchy, the validity of all kinds of classical (Probability) and intelligent (Fuzzy Logic, Neural Networks (NN), Rough Sets theory, Dempster-Shafer theory (DST), etc.) computational techniques have been compared. Considering mobile robots, the most commonly used techniques for self-localization in unknown environment relying on interoceptive sensors (odometer, electronic compass) and exteroceptive sensors (sonar detector, laser range finder, and visual sensor) are Markov localization[2] or Monte Carlo localization.[3] The map of the environment is built using some computational technique, such as Probability theory, Fuzzy Sets theory, Neutrosophic theory, and NN. The information about the environment can be expressed as a grid map, geometrical feature or topological map, etc., where the grid map is the most common arithmetic expression.[4] This work adopts the Dezert-Smarandache Theory (DSmT), a theory proposed recently by Jean Dezert and Florentin Smarandache that is based on the Bayesian approach and Dempster-Shafer theory.[5] DSmT is a general, flexible, and valid arithmetic framework for fusion. Its greatest advantage is that it can deal with uncertain and imprecise information in an effective way, which in turn provides a powerful tool for dealing with the uncertain information acquired by a sonar detector in the process of building grid maps. In this article, the authors present a new application of DSmT that deals with unreliable sensors by means of the discounting method.[6]

Fusion of Unreliable Sources with DSmT

The Dezert-Smarandache theory is a new, general, and flexible arithmetic for fusion, which can solve the fusion problem on different tiers, including data-tier, feature-tier, and decision-tier. Even more, it could work not only with the static problem of fusion, but also with the dynamic one. Especially, the theory has a prominent advantage that it can deal with uncertain and highly conflicting information.[7]

Short Overview of DSmT

1. Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be the frame of discernment, which includes n finite focal elements θ_i, $(i = 1, \ldots, n)$. Owing to the fact that the focal elements are not precisely defined and separated, no refinement of Θ in a new larger set Θ_{ref} of disjoint elementary hypotheses is possible.

2. The hyper-power set D^Θ is defined as the set of all compositions built from elements of Θ by means of the $\cup$ and $\cap$ operators (Θ generates D^Θ under operators $\cup$ and $\cap$), such that

   (a) $\varnothing, \theta_1, \dots, \theta_n \in D^\Theta$;

   (b) If $A, B \in D^\Theta$, then $A \cap B$ and $A \cup B$ belong to D^Θ;

   (c) No other elements belong to D^Θ, except those obtained using rules (a) or (b).

3. *General Belief and Plausibility Functions*

   Let $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ be the general frame of discernment. For each evidential source S, let us define a set of maps $m(.) : D^\Theta \to [0,1]$ associated to it (abandoning Shafer's model) by assuming here the fuzzy/vague/relative nature of elements θ_i $(i = 1, \dots, n)$ that can be non-exclusive, as well as no refinement of Θ into a new finer exclusive frame of discernment Θ_{ref} is possible. The mapping $m(.)$ is called a generalized basic belief assignment function if it satisfies[8]:

   $$m(\varnothing) = 0 \qquad \text{and} \qquad \sum_{A \in D^\Theta} m(A) = 1,$$

   where $m(A)$ is called A's generalized basic belief assignment function (gbbaf). The general belief function and the plausibility function are defined in almost the same manner as in DST, i.e.:

   $$\text{Bel}(A) = \sum_{B \in D^\Theta, B \subseteq A} m(B) \tag{1}$$

   $$\text{Pl}(A) = \sum_{B \in D^\Theta, B \cap A \neq \emptyset} m(B) \tag{2}$$

4. *Classical (free) DSmT Rule of Combination*

   Let $\mathcal{M}^f(\Theta)$ be a free DSm model. The classical (free) DSm rule of combination, denoted (DSmC) for short, for $k \geq 2$ sources is given by $m_{\mathcal{M}^f(\Theta)}(\emptyset) = 0$ and $\forall A \neq \emptyset, A \in D^\Theta$ by

   $$\begin{aligned} m_{\mathcal{M}^f(\Theta)}(A) &\triangleq [m_1 \oplus \ldots \oplus m_k](A) \\ &= \sum_{\substack{X_1, \ldots, X_k \in D^\Theta \\ X1 \cap \ldots \cap X_k = A}} \prod_{i=1}^{k} m_i(X_i) \end{aligned} \tag{3}$$

Fusion of Unreliable Sources

On the Necessity of Discounting Sources

In real systems, the sources of information are actually unreliable due to the occurrence of sensors with different functions and practical utilization. For example, considering the mobile robots' sensors, the measurement precision and resolution of the laser range finder are higher than those of the sonar detector. Even more, if only sonar detectors are considered, they will have also different precision due to manufacturing and other factors. Under these conditions, if data from unreliable information sources is treated as data of reliable sources, then the result of fusion will be very unreliable and even worse. Thus, unreliable sources must be considered with great care. In this research, working in a DSmT framework and based on the discounting method,[9] the authors provide a valuable technique for dealing with unreliable sensors.

Principle of Discounting Method

Let us consider k evidential sources of information $(S_1, S_2, \ldots, S_k)$; here, the authors propose a uniform way for dealing with homogeneous and heterogeneous information sources. So, $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ is the frame of discernment and $m(.)$ is the basic belief assignment. Let $m_i(.) : D^\Theta \rightarrow [0, 1]$ be a set of maps, and let p_i represent the degree of reliability supported by S_i, $i = 1, 2, \ldots, k$; considering that $\sum_{A\in D^\Theta} m_i(A) = 1$, let $I_t = \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ be the total ignorance, and then let $m_i^g(I_t) = 1 - p_i + p_i m_i(I_t)$ represent the belief assignment of the total ignorance for the global system *after discounting*, and due to the occurrence of some malfunctioning, that is, $\sum_{A\in D^\Theta} m_i(A) = p_i$, the quantity $1 - p_i$ is again assigned to the total ignorance.

Thus, the rule of combination in terms of DSmT based on the discounting method with $k \geq 2$ evidential sources is given as shown in equation (3), i.e. the conjunctive consensus on the hyper-power set is given by $m^g_{\mathcal{M}^f(\Theta)}(\emptyset) = 0$ and $\forall A \neq \emptyset, A \in D^\Theta$,

$$\begin{aligned} m^g_{\mathcal{M}^f(\Theta)}(A) &\triangleq [m_1^g \oplus \ldots \oplus m_k^g](A) \\ &= \sum_{\substack{X_1,\ldots,X_k \in D^\Theta \\ X1\cap\ldots\cap X_k = A}} \prod_{i=1}^{k} p_i m_i(X_i) \end{aligned} \tag{4}$$

Figure 1: Pioneer II Mobile Robot.

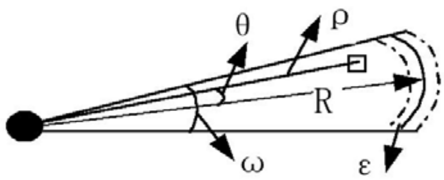

Figure 2: Illustration of the Principle of Work of the Sonar.

Modeling of Sonar Grid Map Building Based on DSmT

The mobile robot Pioneer II used in this work is shown in Figure 1. Here, the authors focus mainly on the sonar detector, whose principle of work (shown in Figure 2) is as follows: producing sheaves of cone-shaped wave and detecting the objects by receiving the reflected wave. Due to the reduced number of sonar physical characteristics, the measured data exhibits some uncertainty as follows:

1. Besides sonar's own errors due to manufacturing, the influence of the external environment is also very strong, for example, factors such as temperature, humidity, atmospheric pressure play a significant role.

2. Because the sound wave spreads outwards in the form of a loudspeaker and there exists a cone-shaped angle, the true position of the object detected could not be known along the fan-shaped area, with the increase of the distance between the sonar and the object.

3. The use of many sonar detectors will result in interference. For example, when the i-th sonar sends out a detecting wave towards an object of irregular shape, if the angle of incidence is too wide, the sonar wave might be reflected out of the receiving range of the i-th sonar detector or it might also be received by other sonar detectors.

4. Because sonar detectors utilize the principle of reflected sound waves, if the object absorbs a very heavy sound wave, the sonar detector might become saturated (giving wrong measurements).

Describing the characteristics of sonar measurement, the authors now construct a model of uncertain information acquired from grid map using sonar based on DSmT. Here, it is supposed that there are two focal elements in the system, that is, $\Theta = \{\theta_1, \theta_2\}$, where θ_1 means *the grid is empty*, and θ_2 means *the grid is occupied*; then, its hyper-power set $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$ is obtained. Every grid in the environment is scanned $k \geq 2$ times, each of which is viewed as a new source of evidence. Then, a set of maps assigned to each source of evidence could be defined and the general basic belief assignment functions (gbbaf) could be constructed as follows:

- $m(\theta_1)$ is defined as the gbbaf for grid-unoccupied (empty);
- $m(\theta_2)$ is defined as the gbbaf for grid-occupied;
- $m(\theta_1 \cap \theta_2)$ is defined as the gbbaf for grid-unoccupied and occupied simultaneously (a case of conflict);
- $m(\theta_1 \cup \theta_2)$ is defined as the gbbaf for grid-ignorance due to restriction of knowledge and present experience (here referring to the gbbaf for these grids still not scanned); it reflects the degree of ignorance about grid-unoccupied or occupied.

The gbbaf of the set of maps $m(.) : D^\Theta \to [0, 1]$ is constructed by the authors according to formulas (5)-(8) based on the sonar physical characteristics.

$$m(\theta_1) = E(\rho)E(\theta) = \begin{cases} (1 - (\rho/R)^2)\lambda & \text{if } \begin{cases} R_{min} \leq \rho \leq R \leq R_{max} \\ 0 \leq \theta \leq \omega/2 \end{cases} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$$m(\theta_2) = O(\rho)O(\theta) = \begin{cases} e^{-3\rho_v(\rho - R)^2}\lambda & \text{if } \begin{cases} R_{min} \leq \rho \leq R + \epsilon \leq R_{max} \\ 0 \leq \theta \leq \omega/2 \end{cases} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

$$m(\theta_1 \cap \theta_2) = \tag{7}$$
$$\begin{cases} [1 - [\frac{2(\rho - R + 2\epsilon)}{R}]^2]\lambda & \text{if} \quad \begin{cases} R_{min} \leq \rho \leq R \leq R_{max} \\ 0 \leq \theta \leq \omega/2 \end{cases} \\ 0, \quad \text{otherwise} \end{cases}$$

$$m(\theta_1 \cup \theta_2) = \tag{8}$$
$$\begin{cases} \tanh(2(\rho - R))\lambda & \text{if} \quad \begin{cases} R \leq \rho \leq R_{max} \\ 0 \leq \theta \leq \omega/2 \end{cases} \\ 0, \quad \text{otherwise} \end{cases}$$

where $\lambda = E(\theta) = O(\theta)$ is given by (see the work of Elfes and Moravec [10] for justification):

$$\lambda = \begin{cases} 1 - (2\theta/\omega)^2 & \text{if} \quad 0 \leq |\theta| \leq \omega/2 \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

and where ρ_v in Equation (6) is defined as an environment-adjusting variable, that is, the fewer the objects in the environment, the greater the value of the variable ρ_v is; it makes the function $m(\theta_2)$ more sensitive. Here, let ρ_v be one. The functions $E(.)$ and $O(.)$ are expressed as the *Effect Functions* of ρ and θ to the grid being empty or occupied. In order to guarantee that the sum of all masses is one, we have to renormalize it. The analyses of the characteristics of the gbbaf are presented in Figures 3-7, with $R = 1.5$ meters.

Considering Figure 3, one can observe that $m(\theta_1)$ has a falling tendency with respect to the increase of the distance between the grid and the sonar, and reaches its maximum at R_{min} and zero at R. From the point of view of the working principle of the sonar, the more the distance between them approaches the measured value, the higher the probability that the grid might be occupied. Thus, the probability that the grid indicated is empty is very low; of course, to the gbbaf of grid-unoccupied is given a low value.

Considering Figure 4, it could be seen that $m(\theta_2)$ takes on the distribution of a Gaussian function with the increase of the distance between the grid and the sonar and reaches its maximum at R, which reflects the characteristic of a sonar acquiring information.

Looking at Figure 5, $m(\theta_1 \cap \theta_2)$ takes on the distribution of a parabola function with respect to the increase of the distance between them. In fact, when $m(\theta_1)$ equals $m(\theta_2)$, $m(\theta_1 \cap \theta_2)$ reaches its maximum there. However, it is very difficult and unnecessary to find the point of intersection of the two functions. Generally, we let the position of $R - 2\epsilon$ replace the point of intersection. Experience has indicated that its approximate value is more rational.

Considering Figure 6, it can be seen that $m(\theta_1 \cup \theta_2)$ takes on the distribution of a hyperbola function with respect to the increase of the distance between the grid and the sonar, and is zero at R. This function reflects well the ignorance of the grid information for $R \leq \rho \leq R_{max}$.

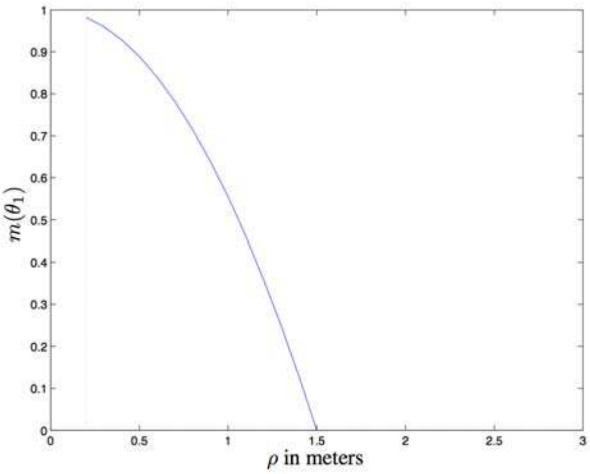

Figure 3: $m(\theta_1)$ as a Function of ρ Given by (5).

The relationship between θ and λ is illustrated in Figure 7, where the more the position of the grid approaches the central axis, the greater λ becomes, that is, the greater the contribution to the belief assignment. Otherwise, it is lower.

In short, the general basic belief assignment functions (gbbaf) entirely fit to the characteristics of sonar acquiring information. This fact provides a theoretical foundation for dealing with uncertain information in grid map building.

Experimental Results

Fusion of Sonar Information

The Pioneer II mobile robot has 16 sonar detectors. As can be seen from Figure 8, there are just 8 front sonar detectors and they are distributed asymmetrically. The flow

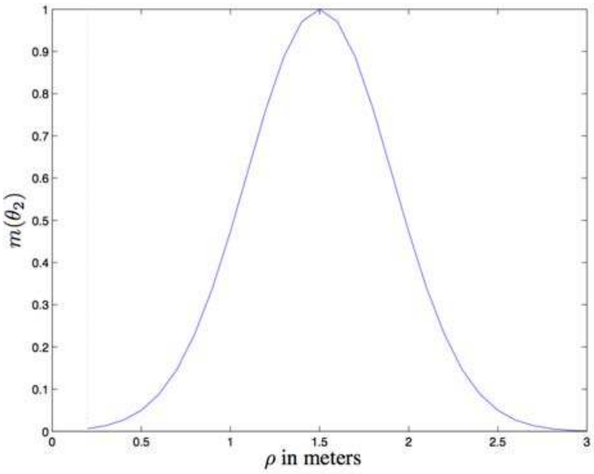

Figure 4: $m(\theta_2)$ as a Function of ρ Given by (6).

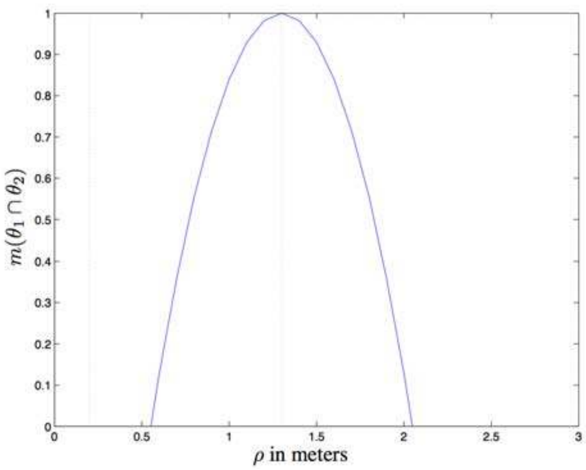

Figure 5: $m(\theta_1 \cap \theta_2)$ as a Function of ρ Given by (7).

chart of the procedure of robot sonar map building based on the DSmT is shown in Figure 13. The procedure includes the fusion steps described below.

Visual C++ 6.0 has served as development environment, running on Linux platform, and using TCP/IP protocol.

1. At the beginning of the procedure, it has been proposed that all grids are fully

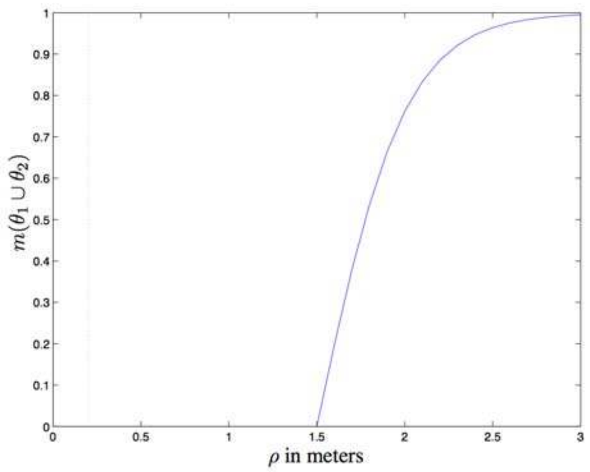

Figure 6: $m(\theta_1 \cup \theta_2)$ as a Function of ρ Given by (8).

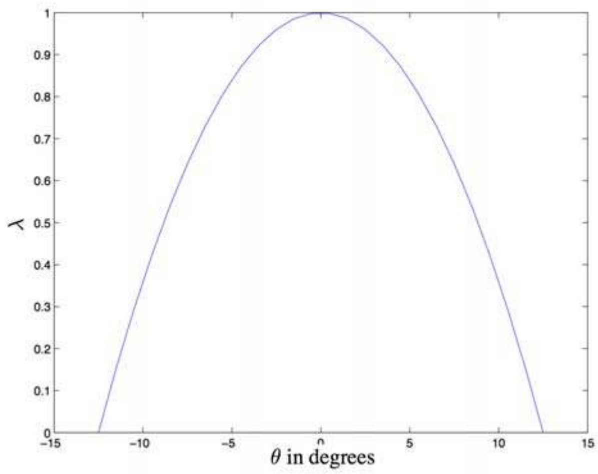

Figure 7: λ as a Function of θ Given by (9).

occupied, that is, $\text{Bel}(\theta_2) = 1$, and the mobile robot starts from the origin of the coordinate system shown in Figure 9. Therefore, the mobile robot may receive orders to go to some spots. Of course, mobile robot's path planning and avoiding obstacles issues have to be considered as well, but they are not important issues here. The robot may get the information from all sonar sensors at spot (i, j). For simplified calculations, the authors apply the arithmetic of restricted spreading, which only computes the grid information in the fan-shaped area that each sonar

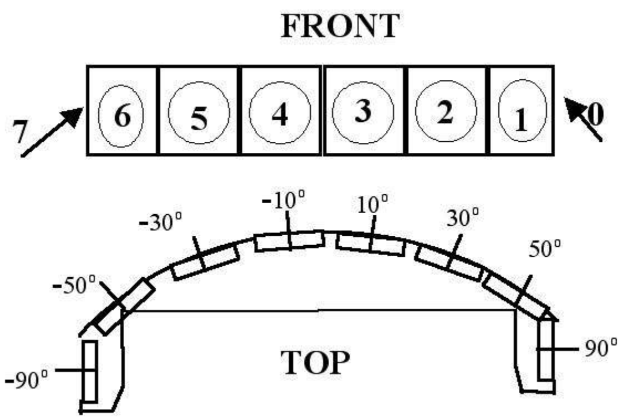

Figure 8: Schematic Illustration of the Layout of Sonars.

can scan (shown in Figure 2).[11] At the same time, it is also assumed that each sonar detector has a different degree of reliability, and the one of the i-th sonar is p_i ($i = 1, \ldots, 16$). Here, p_i is derived experimentally, by estimating sonar's occurrence of malfunctioning. Of course, there is a rule: if information from a different sonar is fused, the corresponding degrees of reliability have to be renormalized.

2. Applying the model of uncertainty belief presented above, through the equations (5)-(9), gbbaf (such as $m(\theta_1)$, $m(\theta_2)$, $m(\theta_1 \cap \theta_2)$, and $m(\theta_1 \cup \theta_2)$) are computed, respectively. If the sum of masses does not equal one, then it has to be renormalized.

3. Assess whether the information of every grid scanned by all sonar sensors is new or not. If Yes, then go to Step 1. Otherwise, go to the next step.

4. Evaluate whether the grid has been scanned before or not. If No, save the information of this grid. If Yes, go to Step 5.

5. Evaluate whether the fusion has been performed more than two times or not. If Yes, then stop fusing. Otherwise, go to Step 6.

6. Continue fusing and, at the same time, estimate further – whether the grid's information is fused for the first time or not. If it is not the first time, then compute and update the Bel function, then go to Step 7. If it is for first time, go to Step 8.

7. Check whether the Bel function of all grids needs to be updated. If Yes, then go to step 9. Otherwise, go to Step 1.

8. Update the grid's original mass with the new mass after fusion. Then, go to Step 1.

9. Finally, Stop and Exit the procedure.

Analysis on the Results of Fusion

Suppose that the environment (size: 5m×5m) is partitioned into 2500 equal discrete rectangular grids (squares 50×50). Objects in the rectangular grid map are presented in Figure 9. The mobile robot moves around and receives 79 location points for acquiring information (here, in order to simplify, only three key spots are marked in Figure 9). Owing to the fact that the environment is small, moreover, the robot runs less time, the precision for self-localization of the mobile robot is sufficient to realize the approach. Of course, it would have been better to consider a Markov or Monte Carlo positioning approach, especially, when the environment is large and complex, since this can improve the quality of map building and can solve the *chicken and egg problem*. In fact, here the authors do not consider the optimal trajectory; if the readers are interested, they may try to find it. Presently, the authors also conduct some research on it in order to economize the time and decrease the number of location points. To improve the precision of fusion and get rid of this under-proof data, let us assume that the mobile robot acquires information from three different directions for each location point and builds the map on-line according to the step of combination given in a previous section. The 3D-grid map is built based on the DSmT and shown in Figures 10-12 when the numbers of locations (n) is 23, 67, and 79, respectively.

The results could be analyzed as follows:

- Figures 10-12 show the process of online map building for the mobile robot Pioneer II. In Figures 10-12, the Z-axis represents the Belief of every grid occupied, zero means that the grid is entirely empty; one denotes that the grid is fully occupied. Though the discerning rate is very high in this process, with the increase of the complexity of the map, the discerning rate might decrease. However, the final discerning effect is very satisfactory, which can be seen from the comparison between Figure 10 and Figure 12. This facilitates very much the development of a human-computer interface for mobile robots exploring unknown, dangerous, and invisible areas.

- Low coupling. Though there are many objects in the grid map, the phenomenon of the apparently separate, but actually connected ones does not occur. Thus, it gives a powerful evidence for self-localization, path planning, and navigation for mobile robots.

- High validity of calculation. The discounting approach as preprocessing for the DSm fusion rule considering the restrained spreading arithmetic is adopted; it overcomes the shortcoming that the global grids in map have to be considered once for sonar scanning every time, and improves the validity of calculation.

- In this work, the authors simply apply the classical model of the DSmT to a static environment; if the size of the environment is very large and complex, then SLAM has to be considered. Aiming at dynamic environments, with moving objects and walking person therein, the hybrid model of DSmT has to be taken in consideration.[12]

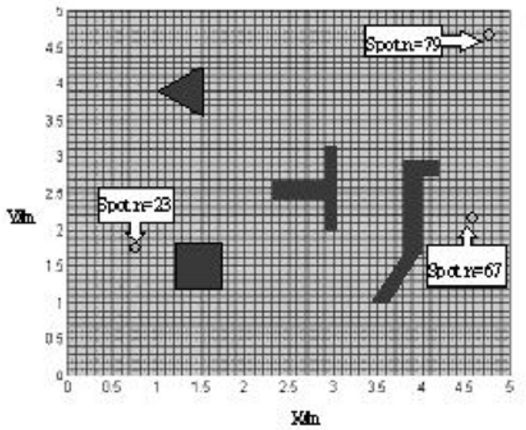

Figure 9: The Grid Map of the Original Environment.

Conclusions

In this work, the authors have applied the DSmT based on the classical discounting approach to mobile robot's map building in a small environment. Then, they have established the belief model for sonar grid map and constructed the generalized basic belief assignment function. The experimental work performed have demonstrated the high potential of the approach to robot's map building. Even more, DSmT has proved to be a very solid framework for solving such a problem. Naturally, if the size of the environment is very large and complex, or irregular, the robot's position could not

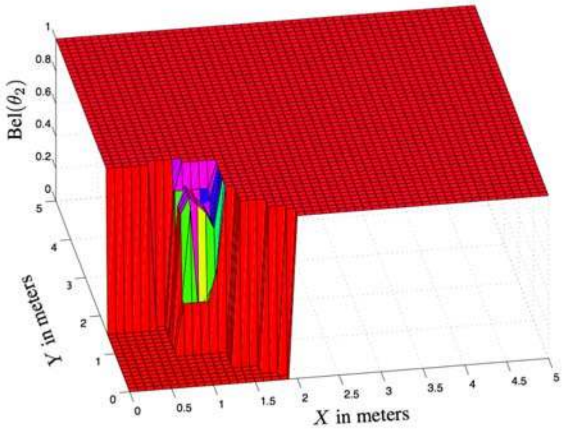

Figure 10: Estimation of the Grid Map when $n = 23$.

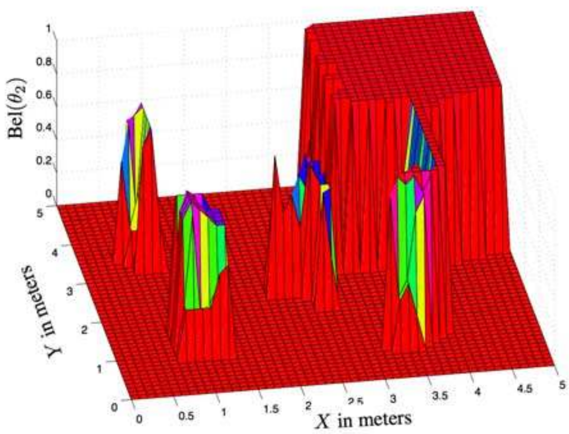

Figure 11: Estimation of the Grid Map when $n = 67$.

be neglected. Under this condition, robot self-localization has to be considered. In fact, the authors carry out some research on SLAM based on DSmT. They believe that it is very important to improve the robustness and practicability of the arithmetic. In short, this study has provided a convenient way for developing a human-computer interface for mobile robots exploring unknown environment and has established a firm foundation for the simultaneous study of a dynamic unknown environment and multi-robots' map building.

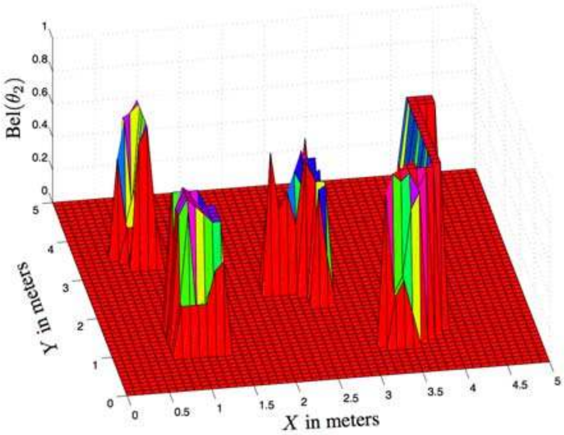

Figure 12: Estimation of the Grid Map when $n = 79$.

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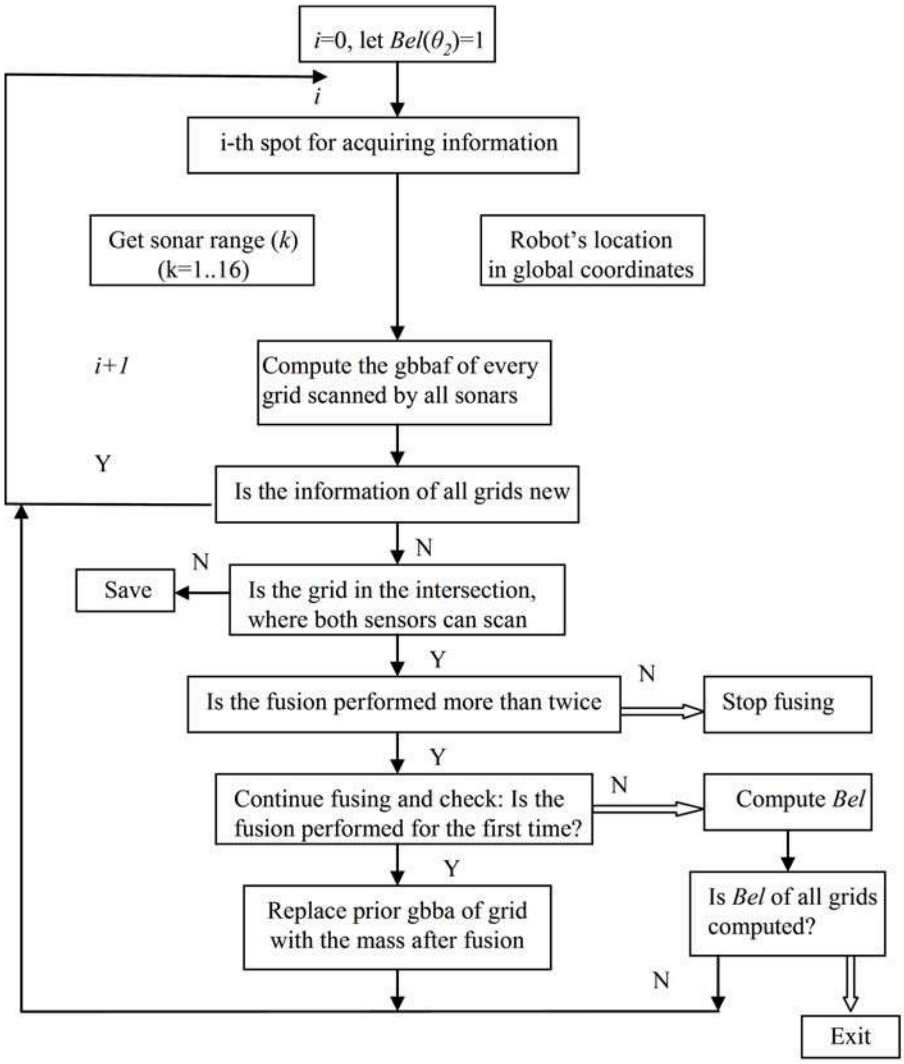

Figure 13: Flowchart of the Procedure of Sonar Map Building Based on DSmT.

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HUMAN EXPERT FUSION FOR IMAGE CLASSIFICATION

Arnaud MARTIN and Christophe OSSWALD

Abstract: In image classification, merging the opinion of several human experts is very important for solving different tasks such as evaluation or training. Indeed, the reality is rarely known before scene imaging. The authors propose in this article different models in order to fuse the information given by two or more experts. The considered unit for classification, a small tile of the image, can contain one or more of the considered classes given by the experts. A second problem that has to be considered, is the degree of certainty of the expert s/he puts in each pixel of the tile. In order to solve these problems, the authors define five models in the context of Dempster-Shafer and Dezert-Smarandache theories and investigate the possible solutions with these models.

Keywords: Expert Fusion, Dempster-Shafer Theory (DST), Dezert-Smarandache (DSmT), Image Classification.

Introduction

Fusing the opinion of several human experts, also known as expert fusion, is an important problem in the field of image classification and it is not well investigated. Indeed, the reality is rarely known before the scene has been imaged; consequently, human experts have to provide their perception of the images in order to train the classifiers (for supervised classifiers) and also to evaluate the classification of the images. In most of the real applications, the experts cannot provide the different classes on the images with certainty. Moreover, the difference in expert perception can be very large and also too many parts of the images contain conflicting information. Thereby, only one expert opinion (*reality*) is not reliable enough and expert fusion is required.

Image classification is generally performed on a local part of the image (at pixel level or most of the time on small tiles of 16×16 or 32×32 pixels, for example). The classification methods can usually be described as comprised of three steps. First, significant features are extracted from these tiles. Generally, a second step is necessary in order

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to reduce the number of features because they are usually too many. In the third step, these features are provided to classification algorithms. The reason to consider small tiles in image classification is that sometimes more than one class can co-exist on a tile.

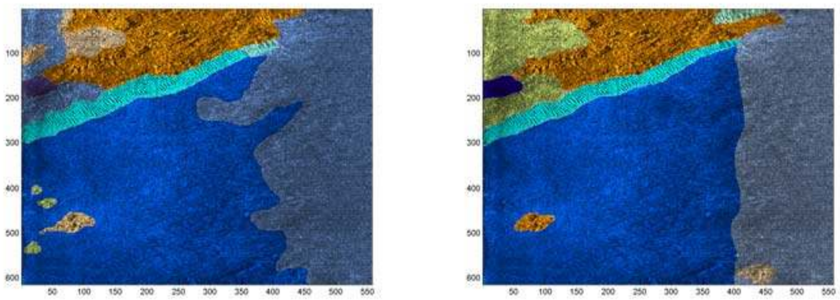

Figure 1: Segmentation Given by Two Experts.

Seabed characterization is often considered as an example for image classification. This process serves many useful purposes, such as helping the navigation of Autonomous Underwater Vehicles or providing data to sedimentologists. In such sonar applications, which will be considered as examples throughout this article, seabed images are obtained with many imperfections.[1] Indeed, in order to build such images, a large number of physical data (geometry of the device, ship coordinates, movements of the sonar, etc.) are taken into account, but these data often contain a large amount of noise due to instrumentation. In addition, some interferences might occur caused by the traveling of the signal on multiple paths (reflection on the bottom or surface) or speckle, and/or fauna and flora. Therefore, sonar images have a lot of imperfections, such as imprecision and uncertainty; thus sediment classification on sonar images is a difficult task. In this type of applications, the reality is unknown and different experts could propose different classifications of the image. Figure 1 demonstrates the difference between the interpretation and the degree of certainty of two sonar experts trying to differentiate the type of sediment (rock, cobbles, sand, ripple, and silt) or the background when an objects could not be seen. Each color corresponds to a different kind of sediment and the associated certainty of the expert for this sediment expressed in terms of certainty (‘sure’, ‘moderately sure’ and ‘not sure’). Thus, in order to train an automated classification algorithm, this difference and the uncertainty of each expert have to be taken into consideration. For example, how a tile of rock labeled as *not sure* has to be considered in the learning step of the classifier and how to interpret this tile if another expert says that it is sand? Another problem is: how to classify the tiles with more than one sediment?

Many fusion theories can be used for expert fusion in image classification, such as voting rules,[2] possibility theory,[3] and belief function theory.[4] In the case considered, the experts can express the certainty in their perception. As a result, probability theories such as the Bayesian theory or the belief function theory are better adapted and applicable. Indeed, the possibility theory is more suitable for work with imprecise data whereas probability-based theories are better adapted to fit uncertain data. Certainly, both possibility and probability-based theories could model imprecise and uncertain data at the same time, but not so easily. That is why, the authors have chosen the belief function theory, also known as Dempster-Shafer Theory (DST).[5] In general, the fusion approach could be divided into four steps: belief function model, parameter estimation depending on the model (not always necessary), combination, and decision. The most difficult step is presumably the first one: the belief function model that forms the basis for the other steps.

Moreover, in real applications of image classification, there could be a significant conflict between the opinion of the experts, and also the heterogeneity of the tiles has to be considered (more than one class can be present on a tile). Consequently, the Dezert-Smarandache Theory (DSmT),[6] an extension of the belief function theory, can better fit to the problem of image classification if there is a conflict. Indeed, the following space of discernment is considered $\Theta = \{C_1, C_2, \ldots, C_n\}$, where C_i is the hypothesis "the considered object belongs to the class i." In the classical belief function theory, the belief functions, also called the basic belief assignments, are defined by a mapping of the power set 2^Θ onto $[0, 1]$. The power set 2^Θ is closed under the $\cup$ operator, and $\emptyset \in 2^\Theta$. In the extension proposed in DSmT, generalized basic belief assignments are defined by a mapping of the hyper-power set D^Θ onto $[0, 1]$, where the hyper-power set D^Θ is closed under both $\cup$ and $\cap$ operators. In consequence, the conflict between the experts can be better managed and also tiles with more than one class can be considered.

In the next section, the authors present and discuss different belief function models based on the power set and the hyper-power set. These models attempt to address the problem under consideration. The models are also studied in the steps of combination and decision of information fusion. These models enable, demonstrated in another section, a general discussion of the difference between DSmT and DST in terms of capacity to represent the considered problem and decision. Finally, the authors provide an illustration of the proposed expert fusion on real sonar images, which represent a particularly uncertain environment.

The Proposed Models

In this section, the authors present five models taking into account various specifics of the application. First, the principles of DST and DSmT applied here are recalled. Then, a numerical example is provided, which illustrates the five proposed models presented afterwards. The first three models are given in the framework of DST, the fourth model in the framework of DSmT, and the fifth model in both frameworks.

Theoretical Background

Belief Function Models

The belief functions or basic belief assignments m are defined by the mapping of the power set 2^Θ onto $[0,1]$ in DST and by the mapping of the hyper-power set D^Θ onto $[0,1]$ in DSmT as follows:

$$m(\emptyset) = 0, \tag{1}$$

and

$$\sum_{X\in 2^\Theta} m(X) = 1, \tag{2}$$

in DST, and

$$\sum_{X\in D^\Theta} m(X) = 1, \tag{3}$$

in DSmT, where X is a given tile of the image.

Equation (1) makes it possible to assume a closed world.[7] We can define the belief function only as:

$$m(\emptyset) > 0, \tag{4}$$

and the world will be open.[8] In a closed world, by adding one element will be possible to propose an open world.

The simple conditions in equation (1) and (2) or (1) and (3) provide a room for a large body of definitions of the belief functions, which is one of the difficulties of the theory. Therefore, the belief functions have to be chosen according to the application considered.

In the case presented here, the space of discernment Θ represents the different kinds of sediment on sonar images, such as rock, sand, silt, cobble, ripple or shadow (that means there is no information about sediment). The experts provide their perception and belief according to their degree of certainty. For instance, an expert can be moderately sure of her/his choice when s/he labels one part of the image as belonging to a certain class, and absolutely uncertain about another part of the image. Moreover, on a considered tile, more than one type of sediment can be present.

Consequently, all these aspects and specifics of the application have to be taken into account. In order to make the explanation easier, the authors simplify the scenario and consider only two classes in what follows: rock is labeled A, and sand is labeled B. The proposed models could be easily extended to more classes, however they are better and easier understood when considering only two classes.

Therefore, on certain tiles, A and B can be present for one or more experts. The belief functions have to take into account the degree of certainty provided by the experts (referred respectively to as c_A and c_B, two numbers in $[0, 1]$), as well as the proportion of the kind of sediment in tile X (referred to as p_A and p_B, also two numbers in $[0, 1]$). There are two possible interpretations of "the expert believes A:" it can mean that the expert thinks that there is A on X and not B, or it can mean that the expert thinks that there is A on X and it can also have B, however s/he does not say anything about it. The first interpretation entails that the hypotheses A and B are exclusive and the second interpretation means that they are not exclusive. The authors consider only the first case: A and B are exclusive. But on a tile X, the expert can also provide the opinion that both A and B are present, and in this case the two propositions "the expert believes A" and "the expert believes A and B" are not exclusive.

Combination Rules

In the last decade, many combination rules have been proposed in the context of belief function theory.[9] In the DST framework, the combination rule most commonly used nowadays seems to be the conjunctive consensus rule given by Smets[10] for all $X \in 2^\Theta$ by:

$$m(X) = \sum_{Y_1 \cap \ldots \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j), \tag{5}$$

where $Y_j \in 2^\Theta$ is the opinion (answer) of expert j, and $m_j(Y_j)$ is the associated belief function.

In the context of DSmT, the conjunctive consensus rule can be used for all $X \in D^\Theta$ and $Y \in D^\Theta$. If we would like to make the decision considering only the elements of Θ, some rules propose to redistribute the conflict among these elements. The most comprehensive rule that provides this redistribution is the PCR5 rule given by Smarandache and Dezert[11] for two experts and for $X \in D^\Theta$, $X \neq \emptyset$ by:

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in D^\Theta, \\ c(X \cap Y) = \emptyset}} \left(\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)} \right), \tag{6}$$

where $m_{12}(.)$ is the conjunctive consensus rule given by Equation (5), $c(X \cap Y)$ is the conjunctive normal form of $X \cap Y$ and the denominators are not null. This rule can be easily generalized for M experts, for $X \in D^\Theta$, $X \neq \emptyset$:

$$m_{PCR6}(X) = m(X) + \sum_{i=1}^{M} m_i(X)^2 \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)}, \ldots, Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}} \left(\frac{\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X) + \sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right), \tag{7}$$

where σ_i counts from 1 to M avoiding i:

$$\begin{cases} \sigma_i(j) = j & \text{if } j < i, \\ \sigma_i(j) = j + 1 & \text{if } j \geq i, \end{cases} \tag{8}$$

$m_i(X) + \sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \neq 0$, and m is the conjunctive consensus rule given by Equation (5).

In this research, the comparison of all the combination rules is not an objective. Therefore, the authors use Equation (5) in the context of DST and Equation (7) in the context of DSmT.

Decision Rules

Making the right decision is a difficult task. No measures are able to provide the best decision for all cases. In general, the authors consider the maximum of one of the three functions: credibility, plausibility, and pignistic probability.

In the context of DST, the credibility function is given for all $X \in 2^\Theta$ by:

$$\text{bel}(X) = \sum_{Y \in 2^X, \, Y \neq \emptyset} m(Y). \tag{9}$$

The plausibility function is given for all $X \in 2^\Theta$ by:

$$\mathrm{pl}(X) = \sum_{Y \in 2^\Theta, Y \cap X \neq \emptyset} m(Y) = bel(\Theta) - bel(X^c), \tag{10}$$

where X^c is the complementary of X. The pignistic probability, introduced by Smets,[12] is here given for all $X \in 2^\Theta$, with $X \neq \emptyset$ by:

$$\mathrm{betP}(X) = \sum_{Y \in 2^\Theta, Y \neq \emptyset} \frac{|X \cap Y|}{|Y|} \frac{m(Y)}{1 - m(\emptyset)}. \tag{11}$$

Generally, the maximum of these functions is taken on the elements of Θ, but the authors will give the values on all the focal elements.

In the context of DSmT, the corresponding generalized functions have been proposed by Dezert, Smarandache, and Daniel.[13] The generalized credibility Bel is defined as follows:

$$\mathrm{Bel}(X) = \sum_{Y \in D^X} m(Y) \tag{12}$$

The generalized plausibility Pl is given by:

$$\mathrm{Pl}(X) = \sum_{Y \in D^\Theta, X \cap Y \neq \emptyset} m(Y) \tag{13}$$

The generalized pignistic probability is defined for all $X \in D^\Theta$, $X \neq \emptyset$, as follows:

$$\mathrm{GPT}(X) = \sum_{Y \in D^\Theta, Y \neq \emptyset} \frac{\mathcal{C}_\mathcal{M}(X \cap Y)}{\mathcal{C}_\mathcal{M}(Y)} m(Y), \tag{14}$$

where $\mathcal{C}_\mathcal{M}(X)$ is the DSm cardinality corresponding to the number of parts of X in the Venn diagram of the problem.[14]

If the credibility function provides a pessimistic decision, the plausibility function is often too optimistic. Thus, the pignistic probability is often taken as a compromise. The authors present the three functions for their models.

Numerical and Illustrative Example

Consider two experts giving their opinion about a tile X. The first expert says that on tile X there is a kind of rock A with a certainty equal to 0.6. Hence, for this first expert we have: $p_A = 1$, $p_B = 0$, and $c_A = 0.6$. The second expert thinks that there are 50% of rock and 50% of sand on the considered tile X with certainty of 0.6 and 0.4, respectively. Therefore, for the second expert we have: $p_A = 0.5$, $p_B = 0.5$, $c_A = 0.6$, and $c_B = 0.4$. This numerical example will serve as illustration of all the models proposed in this article.

Model M_1

If the space of discernment given by $\Theta = \{A, B\}$ is considered, the following belief function can be defined:

$$\begin{array}{l} \text{if the expert says } A: \\ \left\{ \begin{array}{l} m(A) = c_A, \\ m(A \cup B) = 1 - c_A, \end{array} \right. \\ \\ \text{if the expert says } B: \\ \left\{ \begin{array}{l} m(B) = c_B, \\ m(A \cup B) = 1 - c_B. \end{array} \right. \end{array} \qquad (15)$$

In this case, it is natural to distribute $1 - c_A$ and $1 - c_B$ on $A \cup B$, which represents the ignorance.

This model takes into account the degree of certainty of the experts, but the space of discernment does not consider the possible heterogeneity of the given tile X. Hence, we have to add another focal element meaning that there are two classes A and B on X. In the context of Dempster-Shafer theory, we can call this focal element C, the space of discernment is given by $\Theta = \{A, B, C\}$, and the power set is given by $2^\Theta = \{\emptyset, A, B, A \cup B, C, A \cup C, B \cup C, A \cup B \cup C\}$. Therefore, the first model M_1 for the considered application can be defined as follows:

$$\begin{array}{l} \text{if the expert says } A: \\ \left\{ \begin{array}{l} m(A) = c_A, \\ m(A \cup B \cup C) = 1 - c_A, \end{array} \right. \\ \\ \text{if the expert says } B: \\ \left\{ \begin{array}{l} m(B) = c_B, \\ m(A \cup B \cup C) = 1 - c_B, \end{array} \right. \\ \\ \text{if the expert says } C: \\ \left\{ \begin{array}{l} m(C) = p_A.c_A + p_B.c_B, \\ m(A \cup B \cup C) = 1 - (p_A.c_A + p_B.c_B). \end{array} \right. \end{array} \qquad (16)$$

On the given numerical example, the following results are obtained:

	A	B	C	$A \cup B \cup C$
m_1	0.6	0	0	0.4
m_2	0	0	0.5	0.5

Hence, for the consensus combination for model M_1, the belief function m_{12}, the credibility, the plausibility, and the pignistic probability are given by:

$element$	m_{12}	bel	pl	betP
$\emptyset$	0.3	0	0	–
A	0.3	0.3	0.5	0.5238
B	0	0	0.2	0.0952
$A \cup B$	0	0.3	0.5	0.6190
C	0.2	0.2	0.4	0.3810
$A \cup C$	0	0.5	0.7	0.9048
$B \cup C$	0	0.2	0.4	0.4762
$A \cup B \cup C$	0.2	0.7	0.7	1

where

$$m_{12}(\emptyset) = m_{12}(A \cap C) = 0.30. \tag{17}$$

This belief function provides ambiguity due to the fact that the same mass is put on A, the rock, and $\emptyset$, the conflict. This ambiguity is suppressed with the maximum of credibility, plausibility or pignistic probability since these functions do not consider the empty set.

Model M_2

In the first model M_1, the possible heterogeneity of the tile is taken into account. However, the ignorance is characterized by $A \cup B \cup C$ and not by $A \cup B$ anymore, and the class C represents the situation when the two classes A and B are present on X. Consequently, $A \cup B \cup C$ could be equal to $A \cup B$, and we can propose another model M_2 as follows:

$$\begin{array}{l} \text{if the expert says } A: \\ \begin{cases} m(A) = c_A, \\ m(A \cup B) = 1 - c_A, \end{cases} \\ \\ \text{if the expert says } B: \\ \begin{cases} m(B) = c_B, \\ m(A \cup B) = 1 - c_B, \end{cases} \\ \\ \text{if the expert says } C: \\ \begin{cases} m(C) = p_A.c_A + p_B.c_B, \\ m(A \cup B) = 1 - (p_A.c_A + p_B.c_B). \end{cases} \end{array} \tag{18}$$

On the considered numerical example, we have:

	A	B	C	$A \cup B$
m_1	0.6	0	0	0.4
m_2	0	0	0.5	0.5

In the model M_2, the ignorance is partial and the conjunctive consensus rule, the credibility, the plausibility, and the pignistic probability are given by:

element	m_{12}	bel	pl	betP
$\emptyset$	0.5	0	0	−
A	0.3	0.3	0.3	0.6
B	0.2	0.2	0.2	0.4
$A \cup B$	0	0.5	0.5	1
C	0	0	0	0
$A \cup C$	0	0.3	0.3	0.6
$B \cup C$	0	0.2	0.2	0.4
$A \cup B \cup C$	0	0.5	0.5	1

where

$$m_{12}(\emptyset) = m_{12}(A \cap C) + m_{12}(C \cap (A \cup B)) = 0.30 + 0.2 = 0.5. \qquad (19)$$

The previous ambiguity in M_1 between A (the rock) and $\emptyset$ (the conflict) is still present with a belief on $\emptyset$ higher than A. Moreover, in this model the mass on C is null!

These models M_1 and M_2 are different because in DST the classes A, B, and C are assumed to be exclusive. Indeed, the fact that the power set 2^Θ is not closed under $\cap$ operator leads to the exclusivity of the classes.

Model M_3

In the application presented here, A, B, and C cannot be considered exclusive on X. In order to propose a model following DST, only exclusive classes have to be studied. Hence, in this application, a space of discernment of three exclusive classes $\Theta = \{A \cap B^c, B \cap A^c, A \cap B\} = \{A', B', C'\}$ could be considered, following the notations given in Figure 2.

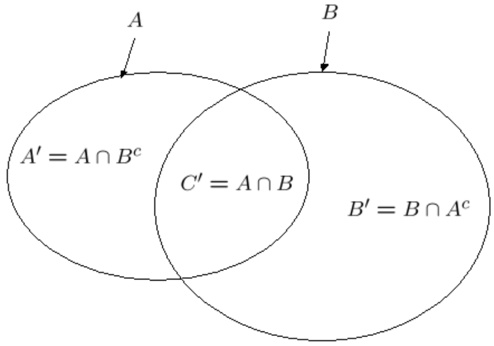

Figure 2: Notation of the Intersection of Two Classes A and B.

Hence, a new model M_3 could be proposed, given as follows:

$$
\begin{array}{l}
\text{if the expert says } A\text{:}\\
\left\{\begin{array}{l} m(A' \cup C') = c_A, \\ m(A' \cup B' \cup C') = 1 - c_A, \end{array}\right.\\
\\
\text{if the expert says } B\text{:}\\
\left\{\begin{array}{l} m(B' \cup C') = c_B, \\ m(A' \cup B' \cup C') = 1 - c_B, \end{array}\right.\\
\\
\text{if the expert says } C\text{:}\\
\left\{\begin{array}{l} m(C') = p_A.c_A + p_B.c_B, \\ m(A' \cup B' \cup C') = 1 - (p_A.c_A + p_B.c_B). \end{array}\right.
\end{array}
\tag{20}
$$

Note that $A' \cup B' \cup C' = A \cup B$. On the numerical example considered, one obtains:

	$A' \cup C'$	$B' \cup C'$	C'	$A' \cup B' \cup C'$
m_1	0.6	0	0	0.4
m_2	0	0	0.5	0.5

Therefore, the conjunctive consensus rule, the credibility, the plausibility, and the pignistic probability are given by:

element	m_{12}	bel	pl	betP
$\emptyset$	0	0	0	–
$A' = A \cap B^c$	0	0	0.5	0.2167
$B' = B \cap A^c$	0	0	0.2	0.0667
$A' \cup B' = (A \cap B^c) \cup (B \cap A^c)$	0	0	0.5	0.2833
$C' = A \cap B$	0.5	0.5	1	0.7167
$A' \cup C' = A$	0.3	0.8	1	0.9333
$B' \cup C' = B$	0	0.5	1	0.7833
$A' \cup B' \cup C' = A \cup B$	0.2	1	1	1

where

$$m_{12}(C') = m_{12}(A \cap B) = 0.2 + 0.3 = 0.5. \tag{21}$$

On this example, with model M_3 the decision will be A – the one with the maximum pignistic probability. However, a decision could *a priori* be made also about $C' = A \cap B$ because $m_{12}(C')$ attains the highest value. The authors show, however, later in the discussion section that it is not possible.

Model M_4

In the context of DSmT, the authors write $C = A \cap B$ and easily propose another model, M_4, not considering the exclusivity of the classes, given as follows:

$$\begin{array}{l}
\text{if the expert says } A\text{:}\\
\begin{cases} m(A) = c_A, \\ m(A \cup B) = 1 - c_A, \end{cases}\\[2ex]
\text{if the expert says } B\text{:}\\
\begin{cases} m(B) = c_B, \\ m(A \cup B) = 1 - c_B, \end{cases}\\[2ex]
\text{if the expert says } A \cap B\text{:}\\
\begin{cases} m(A \cap B) = p_A.c_A + p_B.c_B, \\ m(A \cup B) = 1 - (p_A.c_A + p_B.c_B). \end{cases}
\end{array} \tag{22}$$

The model M_4 makes it possible to represent the problem under investigation without adding an artificial class C. Thus, applying model M_4 based on DSmT, will give the following results:

	A	B	$A \cap B$	$A \cup B$
m_1	0.6	0	0	0.4
m_2	0	0	0.5	0.5

The mass m_{12} obtained by means of the conjunctive consensus rule is:

$$\begin{aligned} m_{12}(A) &= 0.30, \\ m_{12}(B) &= 0, \\ m_{12}(A \cap B) &= m_1(A)m_2(A \cap B) + m_1(A \cup B)m_2(A \cap B) \\ &= 0.30 + 0.20 = 0.5, \\ m_{12}(A \cup B) &= 0.20. \end{aligned} \tag{23}$$

These results are exactly the same as those for model M_3. These two models do not present ambiguity and show that the mass on $A \cap B$ (rock and sand) has the highest value.

The generalized credibility, the generalized plausibility, and the generalized pignistic probability are given below:

$element$	m_{12}	Bel	Pl	GPT
$\emptyset$	0	0	0	$-$
A	0.3	0.8	1	0.9333
B	0	0.5	0.7	0.7833
$A \cap B$	0.5	0.5	1	0.7167
$A \cup B$	0.2	1	1	1

Similarly to model M_3, on this example, the decision will be A considering the maximum of pignistic probability criteria. However, here the maximum of m_{12} is also attained at $A \cap B = C'$.

If only the kinds of possible sediments A and B are considered and not also their conjunctions, proportional conflict redistribution rules such as the PCR5 proposed by Smarandache and Dezert could be used.[15] In consequence, we have $x = 0.3.(0.5/0.3) = 0.5$ and $y = 0$, and the PCR5 rule provides:

$$\begin{aligned} m_{PCR5}(A) &= 0.30 + 0.5 = 0.8, \\ m_{PCR5}(B) &= 0, \\ m_{PCR5}(A \cup B) &= 0.20. \end{aligned} \tag{24}$$

The credibility, the plausibility, and the pignistic probability are given by:

$element$	m_{PCR5}	bel	pl	betP
$\emptyset$	0	0	0	$-$
A	0.8	0.8	1	0.9
B	0	0	0.2	0.1
$A \cup B$	0.2	1	1	1

On this numerical example, the decision will be the same as the one made by means of the consensus rule; here, the maximum of pignistic probability is reached for A (rock). In the next section, the authors show that it is not always the case.

Model M_5

Another model M_5, which can be used in the framework of both DST and DSmT, is given considering only one belief function according to the proportion as follows:

$$\begin{cases} m(A) = p_A.c_A, \\ m(B) = p_B.c_B, \\ m(A \cup B) = 1 - (p_A.c_A + p_B.c_B). \end{cases} \tag{25}$$

If for one expert the tile contains only A, $p_A = 1$, and $m(B) = 0$. If for another expert the tile contains both A and B, the certainty and proportion of the two sediments are taken into account but not only on one focal element. Therefore, we simply have:

	A	B	$A \cup B$
m_1	0.6	0	0.4
m_2	0.3	0.2	0.5

In the DST context, the consensus rule, the credibility, the plausibility, and the pignistic probability are given by:

element	m_{12}	bel	pl	betP
$\emptyset$	0.12	0	0	−
A	0.6	0.6	0.8	0.7955
B	0.08	0.08	0.28	0.2045
$A \cup B$	0.2	0.88	0.88	1

In this case, we do not have plausibility to decide on $A \cap B$, because the conflict is on $\emptyset$.

In the DSmT context, the consensus rule, the generalized credibility, the generalized plausibility, and the generalized pignistic probability are given as follows:

element	m_{12}	Bel	Pl	GPT
$\emptyset$	0	0	0	−
A	0.6	0.72	0.92	0.8933
B	0.08	0.2	0.4	0.6333
$A \cap B$	0.12	0.12	1	0.5267
$A \cup B$	0.2	1	1	1

The decision made considering the maximum of pignistic probability criteria is again A.

The PCR5 rule provides:

$element$	m_{PCR5}	bel	pl	betP
$\emptyset$	0	0	0	–
A	0.69	0.69	0.89	0.79
B	0.11	0.11	0.31	0.21
$A \cup B$	0.2	1	1	1

where

$$m_{PCR5}(A) = 0.60 + 0.09 = 0.69,$$
$$m_{PCR5}(B) = 0.08 + 0.03 = 0.11.$$

With this model and on this example, the decision made applying the PCR5 rule will also be A and there is no difference between the consensus rules in DST and DSmT.

Discussion

In the previous section, the authors have shown how to build models M_1, M_2, M_3, M_4, and M_5 in the framework of DSmT in order to take into account the decision considering also $A \cap B$ ("there is rock and sand on the tile"). In fact, only the models M_1 and M_2 will be able to do that. In the case of model M_2, both experts have to say $A \cap B$. These two models assume that A, B, and $A \cap B$ are exclusive. Of course, this assumption is not true. For the models M_3, M_4, and M_5, the decision has to be made on the credibilities, plausibilities, or pignistic probabilities, but these three functions for $A \cap B$ cannot be higher than A or B (or for C', than $A' \cup C'$ and $B' \cup C'$ with the notations of model M_3). Indeed, for all $x \in A \cap B$, $x \in A$, and $x \in B$, so for all $X \subseteq Y$:

$$\begin{aligned}
&\text{bel}(X) \leq \text{bel}(Y),\\
&\text{pl}(X) \leq \text{pl}(Y),\\
&\text{betP}(X) \leq \text{betP}(Y),\\
&\text{Bel}(X) \leq \text{Bel}(Y),\\
&\text{Pl}(X) \leq \text{Pl}(Y),\\
&\text{GPT}(X) \leq \text{GPT}(Y).
\end{aligned}$$

Therefore, the first problem considered has not been solved: it will never be possible to choose $A \cap B$ with the maximum of credibility, plausibility, or pignistic probability. If the two experts think that the considered tile contains both rock and sand ($A \cap B$), then the pignistic probabilities are equal. However, the belief on $A \cap B$ can have the highest value (refer to the examples presenting models M_3 and M_4). The limits of the decision rules are reached in this case.

It has been demonstrated that it is possible to describe the problem under consideration in both DST and DSmT context. The DSmT is better adapted to model the belief on $A \cap B$, for example with model M_4, but model M_3 in the DST framework can provide exactly the same belief on A, B, and $A \cap B$. Consequently, the only difference that can be expected on the decision comes from the combination rules. On the presented numerical example, the decisions are the same: A is always chosen.

An Example of Decision Instability

Let us consider another example with the last model M_5. The first expert provides: $p_A = 0.5$, $p_B = 0.5$, $c_A = 0.6$, and $c_B = 0.4$, and the second expert provides: $p_A = 0.5$, $p_B = 0.5$, $c_A = 0.86$, and $c_B = 1$. The objective is to make a decision only about A or B. Therefore, we have:

	A	B	$A \cup B$
m_1	0.3	0.2	0.5
m_2	0.43	0.5	0.07

For model M_5 in DST context:

element	m_{12}	bel	pl	betP
$\emptyset$	0.236	0	0	–
A	0.365	0.365	0.4	0.5007
B	0.364	0.364	0.399	0.4993
$A \cup B$	0.035	0.764	0.764	1

Model M_5 together with the PCR5 rule gives (the partial conflicts are $x_1 = 0.0562$, $y_1 = 0.0937$, $x_2 = 0.0587$, and $y_2 = 0.0937$):

element	m_{PCR5}	bel	pl	betP
$\emptyset$	0	0	0	–
A	0.479948	0.479	0.5149	0.4974
B	0.485052	0.485	0.5202	0.5026
$A \cup B$	0.035	1	1	1

The last example demonstrates that there is a difference between DST and DSmT; then, what is the best solution? Considering DST, A will be chosen, while in DSmT framework, the choice will be B. The authors could show that the decision will be the same in most of the cases (about 99.4%).

Stability of the Decision Process

The space where the experts can form their opinion on which n classes are present on a given tile is part of $[0,1]^n$: $\mathcal{E} = [0,1]^n \cap (\sum_{X \in \Theta} m(X) \leq 1)$. In order to study the different combination rules and the situations where they differ, the authors of this article apply a Monte Carlo method, considering the weights p_A, c_A, p_B, c_B, $\ldots$, as uniform variables and filtering them by the condition $\sum_{X \in \Theta} p_X c_X \leq 1$, for one expert.

Thus, the authors measure the proportion of situations where the decision differs between the consensus combination rule and the PCR5 rule, where the conflict is proportionally distributed.

$A \cap B$ cannot be chosen due to the fact that the measure of $A \cap B$ is always lower (or equal with probability 0) than the measure of A or B. In the case of two classes, $A \cup B$ is the ignorance and is usually excluded (as it always maximizes bel, pl, betP, Bel, Pl, and GPT). The authors restrict the possible choices to singletons: A, B, etc. Therefore, it is equivalent to tagging the tile with the most credible class (maximum for the bel function), the most plausible (maximum for pl), the most probable (maximum for betP) or the heaviest (maximum for m), as the only focal elements are singletons, Θ and $\emptyset$.

The only situation, where the total order induced by the masses m on singletons can be modified is when the conflict is distributed on the singletons, as is the case in the PCR5 method.

Thus, for two classes, the subspace where the decision is "rock" by the consensus rule is very similar to the subspace where the decision is "rock" by the PCR5 rule: only 0.6% of the volume differs. For a higher number of classes, the decision obtained fusing the two experts' opinion is much less stable:

number of classes	2	3	4	5	6	7
decision change	0.6%	5.5%	9.1%	12.1%	14.6%	16.4%

Therefore, the specifics of the PCR5 rule appear mostly in situations with more than two classes and the different combination rules are almost equivalent when the decision has to be taken within two possible classes.

The left part of Figure 3 shows the density of conflict within $\mathcal{E}$, for different number of classes: 2, 3, 6, and 7. The right part of the figure shows how this distribution changes if we restrict $\mathcal{E}$ to the cases where the decision changes between the consensus (dotted lines) and the PCR5 (plain lines) rules. Conflict is more important in this subspace, mostly because low conflict usually means a clearer decision: the measure on the best class is often very different from the measure on the second best class.

For the "two experts and two classes" case, it is difficult to characterize analytically the stability of the decision process. However, we can easily show that if $m_1(A) = m_2(B)$

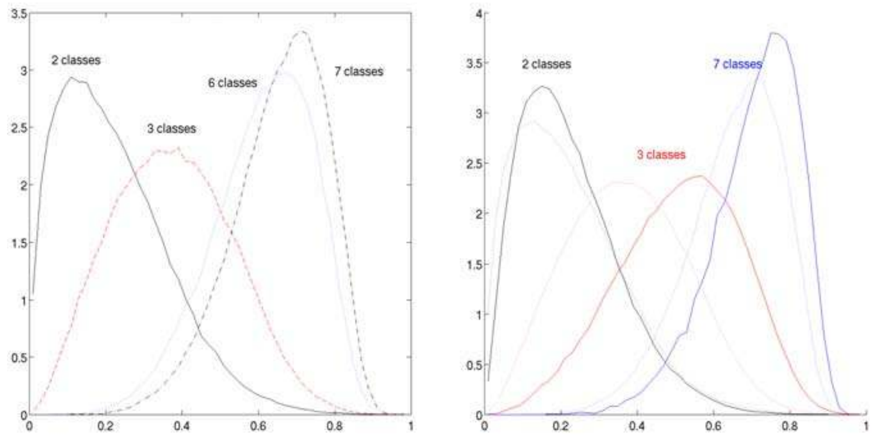

Figure 3: Density of Conflict for (left) Uniform Random Experts and (right) Data with Different Decision between Consensus and PCR5 Rules.

or if $m_1(A) = m_1(B)$, the final decision does not depend on the chosen combination rule.

Illustration

Database

The database in this work contains 40 sonar images provided by GESMA (Groupe d'Etudes Sous-Marines de l'Atlantique). These images were obtained with a Klein 5400 lateral sonar with a resolution of 20 to 30 cm in azimuth and 3 cm in range. The sea-bottom depth was between 15 and 40 meters.

Two experts have manually segmented these images giving their opinion on the kind of sediment (rock, cobble, sand, silt, ripple (horizontal, vertical, or at 45 degrees)), background or other (typically ships) objects on the images, being helped by the manual segmentation interface presented in Figure 4. All sediments are given with a certainty level ("sure", "moderately sure," or "not sure"). Hence, every pixel of an image is labeled as being either a certain type of sediment, or shadow (background), or other.

Results

The authors adopt the following notation: A = rock, B = cobble, C = sand, D = silt, E = ripple, F = shadow, and G = other; hence, we have seven classes and

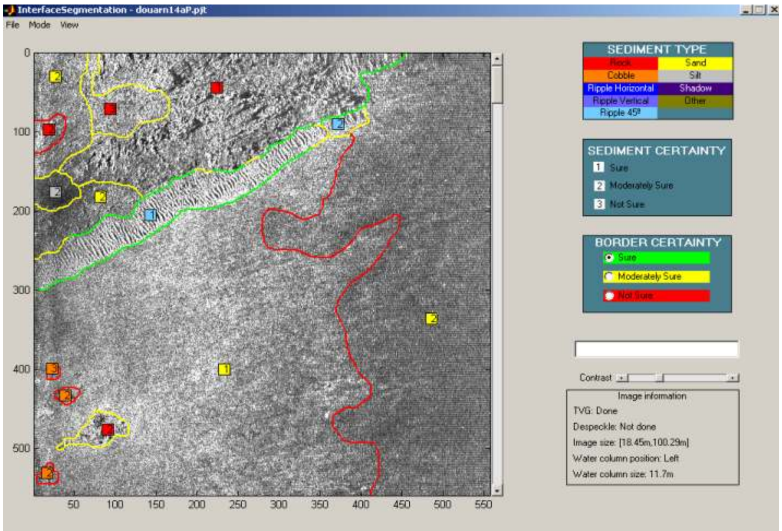

Figure 4: Manual Segmentation Interface.

$\Theta = \{A, B, C, D, E, F, G\}$. The generalized model M_5 has been applied on tiles of size 32×32, given by:

$$\begin{cases} m(A) = p_{A1}.c_1 + p_{A2}.c_2 + p_{A3}.c_3, \text{ for rock,} \\ m(B) = p_{B1}.c_1 + p_{B2}.c_2 + p_{B3}.c_3, \text{ for cobble,} \\ m(C) = p_{C1}.c_1 + p_{C2}.c_2 + p_{C3}.c_3, \text{ for ripple,} \\ m(D) = p_{D1}.c_1 + p_{D2}.c_2 + p_{D3}.c_3, \text{ for sand,} \\ m(E) = p_{E1}.c_1 + p_{E2}.c_2 + p_{E3}.c_3, \text{ for silt,} \\ m(F) = p_{F1}.c_1 + p_{F2}.c_2 + p_{F3}.c_3, \text{ for shadow,} \\ m(G) = p_{G1}.c_1 + p_{G2}.c_2 + p_{G3}.c_3, \text{ for other,} \\ m(\Theta) = 1 - m(A) - m(B) - m(C) - m(D) - m(E) - m(F) - m(G), \end{cases} \quad (26)$$

where c_1, c_2, and c_3 are the weights associated with the certainty, respectively: “sure”, “moderately sure”, and “not sure”. The weights chosen here are: $c_1 = 2/3$, $c_2 = 1/2$, and $c_3 = 1/3$. Indeed, the cases when the same kind of sediment (but with different certainties) is present on the same tile have to be considered. The proportion of each sediment in the tile associated to these weights is denoted, for instance for A as p_{A1}, p_{A2}, and p_{A3}. Table 1 gives the conflict matrix of the two experts. It has to be noted that most of the conflict comes from a difference in opinion between sand and silt. For example, expert 1 provides many tiles of sand, while expert 2 thinks that it is silt (conflict induced of 0.0524). This conflict is explained by the difficulty for the experts to differentiate sand and silt that differ only in intensity. Part of the conflict comes also from the fact that ripples are hard to distinguish from sand or silt. Ripples, that is, sand or silt in a special configuration, are sometimes difficult to be seen on the images and the ripples are most of the time visible in a global zone where sand or silt is present. Cobble also yields conflicts, especially, with sand, silt, and rock: cobble is described by some small rocks on sand or silt. The total conflict between the two experts is 0.1209. Therefore, the considered application does not produce a large conflict.

The authors have applied the consensus and the PCR5 rules with this model. The decision is given by the maximum of pignistic probability. In most of the cases, the decisions taken by the two rules are the same. Difference could be found only on 0.4657% of the tiles. Indeed, we are in the seven classes case with only 0.1209 of conflict. The simulation presented in Figure 3 shows that the probability that the decisions will differ is very low.

Conclusion

This article has proposed and presented five different models in order to address two classical problems in uncertain image classification (for training or evaluation): the heterogeneity of the considered tiles and the certainty level of the experts. These five models have been developed in the framework of DST and DSmT. Tile heterogeneity and the degree of certainty of the experts can be easily taken into account in the models. However, if we would like to have the plausibility (utility) of taking a decision on such a tile (with a conjunction $A \cap B$), the usual decision functions (credibility, plausibility, and pignistic probability) are not sufficient: they cannot enable such a decision. The decision can be made on $A \cap B$ only if we consider the belief function and if the model provides a belief on $A \cap B$.

Table 1: Matrix of Conflict ($\times 10^4$) between the Two Experts.

Expert 1 \ Expert 2	Rock	Cobble	Ripple	Sand	Silt	Shadow	Other
Rock	-	12.87	2.72	4.42	3.91	6.41	0.22
Cobble	5.59	-	0.85	18.44	3.85	0.04	0
Ripple	3.12	3.38	-	30.73	150.60	0.27	0.16
Sand	9.50	43.39	42.60	-	524.33	0.51	0.57
Silt	6.42	27.05	36.22	258.98	-	2.60	0.11
Shadow	3.82	0.15	2.13	1.38	0.50	-	0.41
Other	0	0.20	0.10	0.35	0.31	0.14	-

The authors have also studied the decision according to the conflict and combination rules: the conjunctive consensus rule and the PCR5 rule. The decision (taken according to the maximum of credibility, plausibility, or pignistic probability) is the same in most of the cases. For two experts, more classes lead to a higher conflict and to more cases yielding different decision with the different rules.

One of the proposed models has also been illustrated on real sonar images classified manually by two different experts. In this application, the total conflict between the two experts is 0.1209 and a difference in decision could only be seen on 0.4657% of the tiles.

The models proposed here can be easily generalized for three or more experts by using the generalized combination of the PCR5 rule given by Equation (7). Naturally, the conflict will be higher and the difference in decision have to be investigated.

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