Fusion Frames and G-Frames in Hilbert C^*-Modules

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Abstract

The notion of frame has some generalizations such as frames of subspaces, fusion frames and g-frames. In this paper we introduce frames of submodules, fusion frames and g-frames in Hilbert C^*-modules and we show that they share many useful properties with their corresponding notions in Hilbert space. We also generalize a perturbation result in frame theory to g-frames in Hilbert spaces.

1 Introduction

Frames for Hilbert spaces were introduced in 1952 by Duffin and Schaeffer. Many generalizations of frames were introduced, e.g. frames of subspaces [2, 1], Pseudo-frames [8], Oblique frames [4], G-frames [11], and fusion frames [3]. Meanwhile Frank and Larson presented a general approach to the frame theory in Hilbert C^*-modules [6] and some results on this subject can be found in [7, 12].

In this note we generalize the frame theory of subspaces, fusion frames and G-frames to Hilbert C^*-modules and we also extend some of the known results of these subjects to Hilbert C^*-modules.

The content of the present note is as follows: In Section 2, we state some of the definitions and basic properties of Hilbert C^*-modules and we introduce fusion frames and frames of submodules, which are generalizations of frames of subspaces. We also generalize some of the results about frames of subspaces and fusion frames and frames in Hilbert C^*-modules to frames of submodules.

In Section 3, we introduce g-frames in Hilbert C^*-modules and we generalize some of the results in [11] to Hilbert C^*-modules. Finally we generalize a perturbation result in frame theory to g-frames in Hilbert spaces.

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2 Frames of Submodules

The theory of Hilbert C^*-modules is a generalization of Hilbert spaces and we recall some of the basic definitions and properties of frames in Hilbert C^*-modules, for more details see [6].

Let A be a unital C^*-algebra with identity 1_A and for every $a \in A$, set $|a| = (a^*a)^{1/2}$. Let I and Jbe finite or countable index sets, let $\mathbb{C}$ be the complex field and let $\mathbb{N}$ be the set of natural numbers. Throughout this paper X and Y are countably or finitely generated Hilbert A-modules and $\{Y_i \mid i \in I\}$ is a sequence of closed Hilbert submodules of Y. For each $i \in I$, $End_A^*(X, Y_i)$ is the collection of all adjointable A-linear maps from X to Y_i and $End_A^*(X, X)$ is denoted by $End_A^*(X)$.

Definition 2.1. A pre-Hilbert A-module is a left A-module X equipped with an A-valued inner product $\langle .,. \rangle : X \times X \longrightarrow A$ such that

(i) $\langle x, x \rangle \geq 0$ for all x in X;

(ii) $\langle x, x \rangle = 0$ if and only if $x = 0$;

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for any $x, y \in X$;

(iv) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for any $a \in A$, $x, y, z \in X$.

We assume that the linear operations of A and X are compatible, i.e., $\lambda(ax) = (\lambda a)x$ for every $\lambda \in \mathbb{C}$, $a \in A$ and $x \in X$. For every $x \in X$ we define

$$\|x\| = \|\langle x, x \rangle\|^{1/2} \quad \text{and} \quad |x| = \langle x, x \rangle^{1/2}.$$

It is known that $\|.\|$ is a norm on X. If X is complete with respect to this norm, it is called a *Hilbert* A-module (or a Hilbert C^*-module over A). If X is a Hilbert A-module, it is a Banach A-module, i.e., $\|ax\| \leq \|a\|.\|x\|$ for all $a \in A$, $x \in X$, see [6].

Let X be a Hilbert A-module. We say X is algebrically finitely generated if there exists a finite subset $\{x_i \mid 1 \leq i \leq n\}$ of X such that X is the A-linear hull of $\{x_i \mid 1 \leq i \leq n\}$. X is countably generated if there exists a countable subset $\{x_i \mid i \in I\}$ of X such that X is the norm closure of the A-linear hull of $\{x_i \mid i \in I\}$.

By Kasparov Stablization Theorem [12], for any countably generated Hilbert A-module X, we have $X \oplus \ell_2(A) \cong \ell_2(A)$, where

$$\ell_2(A) = \{(a_i)_{i \in \mathbb{N}} : \sum a_i a_i^* \text{ converges in } \|.\|_A\}. \tag{1}$$

Definition 2.2. Let A be a unital C^*-algebra and X be a Hilbert A-module. A sequence $\{x_i : i \in I\}$ in X is called a frame for X if there exist real constants $C, D > 0$ such that for any $x \in X$,

$$C\langle x, x \rangle \leq \sum_{i \in I} \langle x, x_i \rangle \langle x_i, x \rangle \leq D\langle x, x \rangle. \tag{2}$$

The constants C and D are called a *lower and upper frame bound* for the frame. If $C = D = \lambda$, the frame is called a λ*-tight frame.* If $C = D = 1$,the frame is called *a normalized tight frame* or a *Parseval frame.* If the sum in middle of (2) converges in norm, the frame is standard. We say that $\{x_i : i \in I\}$ is a *frame sequence* if it is a frame for the closure of A-linear hull of $\{x_i : i \in I\}$. A closed submodule M of a Hilbert C^*-module X is *complemented* if for some closed submodule N of X we have $X = M \oplus N$ and $\pi_M : X \longrightarrow M$ is the orthogonal projection of M. We say M is *orthogonally complemented* if $X = M \oplus M^\perp$ and in this case $\pi_M \in End_A^*(X, M)$.

Definition 2.3. Let A be a unital C^*-algebra, X be a Hilbert A-module and let $\{v_i : i \in I\}$ be a family of weights in A, i.e., each v_i is a positive invertible element from the center of the C^*-algebra A. A sequence of closed submodules $\{M_i : i \in I\}$ is a *frame of submodules* if every M_i is orthogonally complemented and there exist real constants $0 < C \leq D < \infty$ such that

$$C\langle x, x\rangle \leq \sum_{i\in I} v_i^2 \langle \pi_{M_i}(x), \pi_{M_i}(x)\rangle \leq D\langle x, x\rangle \quad \text{for} \quad x \in X. \tag{3}$$

We call C and D the *lower* and *upper* bounds of the frame of submodules, and like [3] we call $\{(M_i, v_i) : i \in I\}$ a *fusion frame*. As usual if $C = D = \lambda$, the family $\{M_i;\ i \in I\}$ is called a λ*-tight frame* of submodules with respect to $\{v_i : i \in I\}$ and if $C = D = 1$, it is called a *Parseval* or a *normalized tight* frame of submodules.

If in (3) the sum converges in norm, it is called a standard frame of submodules. If $X = \oplus M_i$, the family $\{M_i : i \in I\}$ is called an *orthogonal basis* of *submodules*, and $\{(M_i, v_i) : i \in I\}$ is called an *orthonormal fusion frame*. If in (3) we only require to have the upper bound, then $\{M_i : i \in I\}$ is called a *Bessel sequence* and $\{(M_i, v_i) : i \in I\}$ is a Bessel fusion frame with Bessel bound D. We note that if $\{M_i : i \in I\}$ is a Bessel sequence, by corollary 15.3.9 of [12] each $\pi_{M_i} : X \longrightarrow M_i$ is adjointable.

Example 2.4. (a) Let $\{M_i : i \in I\}$ be a sequence of Hilbert A-modules and

$$X = \oplus_{i\in I} M_i = \{x = (x_i) : x_i \in M_i \text{ and } \sum_{i\in I}\langle x_i, x_i\rangle \text{ is norm convergent in } A\}.$$

Then X is a Hilbert A-module with A-valued inner product $\langle x, y\rangle = \sum_{i\in I}\langle x_i, y_i\rangle$, where $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$, pointwise operations and the norm defined by $\|a\| = \|\langle a, a\rangle\|^{1/2}$, see [7].

Plainly $\{M_i : i \in I\}$ is a standard Parseval frame of submodules of X with respect to $\{v_i : i \in I\}$, where $v_i = 1$ for each $i \in I$.

(b) If X is a Hilbert A-module, then by (a)

$$\ell_2(I, X) = \{x = (x_i)_{i\in I} : x_i \in X, \sum_{i\in I}\langle x_i, x_i\rangle \text{ is norm convergent in } A\}$$

is a Hilbert A-module.

A small modification in the proof of Theorem 3.2 in [2] gives the following result.

Theorem 2.5. Let $\{v_i : i \in I\}$ be a family of weights in A. Let for each $i \in I$, M_i be an orthogonally complemented submodule of X, and let $\{f_{ij} : j \in I_i\}$ be a frame for M_i with bounds C_i and D_i. Suppose $0 < C = \inf C_i \leq D = \sup_i D_i < \infty$. Then the following conditions are equivalent

(a) $\{v_i f_{ij} : i \in I,\ j \in I_i\}$ is a frame for X,

(b) $\{(M_i, v_i) : i \in I\}$ is a fusion frame for X.

Definition 2.6. Let X be a Hilbert A-module. (a) A sequence $\{f_i : i \in I\}$ in X is *complete* if the A-linear hull of $\{f_i : i \in I\}$ is dense in X, (b) a family of closed submodules $\{X_i : i \in I\}$ of X is *complete* if the A-linear hull of $\bigcup_{i\in I} X_i$ is dense in X.

Our next result is a generalization of Lemmas 3.4 and 3.5 in [2].

Lemma 2.7. Let X be a Hilbert A-module such that each closed submodule of X is orthogonally complemented, let $\{M_i : i \in I\}$ be a family of closed submodules of X and let $\{v_i : i \in I\}$ be a family of weights in A. Let for each $i \in I$, $\{f_{ij} : j \in I_i\}$ be a frame of M_i. Then we have

(a) if $\{(M_i, v_i) : i \in I\}$ is a fusion frame for X, then $\{M_i : i \in I\}$ is complete,

(b) $\{M_i : i \in I\}$ is complete if and only if $\{f_{ij} : i \in I, j \in I_j\}$ is complete.

Proof. It is enough to consider W, the closed A-linear hull of $\{M_i : i \in I\}$ and note that $X = W \oplus W^{\perp}$. $\square$

We note that if A is $K(H)$, the algebra of compact operators on a Hilbert space H, then all closed submodules of all Hilbert A-modules are orthogonally complemented [10].

Our next result is analogous to Lemma 3.9 in [2].

Lemma 2.8. Let X be a Hilbert A-module and let $\{(M_i, v_i) : i \in I\}$ be a Bessel fusion sequence for X with Bessel bound D. Then for each $x = (x_i)_{i\in I}$ in $\oplus_{i\in I} M_i$, $\sum_{i\in I} v_i x_i$ is convergent unconditionally and $\|\sum_{i\in I} v_i x_i\|^2 \leq D\|x\|^2$.

Proof. Let $x = (x_i)_{i\in I}$ be an element of $\oplus_{i\in I} M_i$. Assume that J is a finite subset of I and $y = \sum_{i\in J} v_i x_i$. Since π_{M_i} is self adjoint

$$\langle y, y\rangle = \langle y, \sum_{i\in J} v_i x_i\rangle = \sum_{i\in J} v_i \langle y, x_i\rangle = \sum_{i\in J} v_i \langle y, \pi_{M_i}(x_i)\rangle = \sum_{i\in J} \langle v_i \pi_{M_i}(y), x_i\rangle.$$

Now by [7, Prop. 1.1]

$$\|\langle y, y\rangle\|^2 \leq \|\sum_{i\in J} v_i^2 \langle \pi_{M_i}(y), \pi_{M_i}(y)\rangle\|.\|\sum_{i\in J}\langle x_i, x_i\rangle\|.$$

Since $\sum_{i\in J} v_i^2 \langle \pi_{M_i}(y), \pi_{M_i}(y)\rangle \leq \sum_{i\in I} v_i^2 \langle \pi_{M_i}(y), \pi_{M_i}(y)\rangle \leq D\langle y, y\rangle$ and these elements are positive we have

$$\|\sum_{i\in J} v_i^2 \langle \pi_{M_i}(y), \pi_{M_i}(y)\rangle\| \leq \|\sum_{i\in I} v_i^2 \langle \pi_{M_i}(y), \pi_{M_i}(y)\rangle\| \leq D\|\langle y, y\rangle\|.$$

Similarly $\|\sum_{i\in J}\langle x_i, x_i\rangle\| \leq \|\sum_{i\in I}\langle x_i, x_i\rangle\| = \|\langle x, x\rangle\| = \|x\|^2$. On the other hand by definition $\|y\|^2 = \|\langle y, y\rangle\|$. Hence $\|y\|^4 \leq D\|y\|^2 \|\sum_{i\in J}\langle x_i, x_i\rangle\| \leq D\|y\|^2.\|x\|^2$. Therefore $\|\sum_{i\in J} v_i x_i\|^2 \leq D\|\sum_{i\in J}\langle x_i, x_i\rangle\| \leq D\|x\|^2$, which shows that $\sum_{i\in I} v_i x_i$ is convergent unconditionally and $\|\sum_{i\in I} v_i x_i\|^2 \leq D\|x\|^2$. $\square$

We can also generalize Theorem 4.1 of [6] to fusion frames.

Theorem 2.9. (frame transform) Let A be a unital C^*-algebra and let X be a finitely or countably generated Hilbert A-module. Assume that $\{(M_i, v_i) : i \in I\}$ is a standard fusion frame for X with bounds C and D. Then the corresponding frame transform $\theta_{M,v} : X \longrightarrow \ell_2(I, X)$ defined by $\theta_{M,v}(x) = (v_i \pi_{M_i}(x))_{i\in I}$ for $x \in X$, is an isomorphic imbedding with closed range, and its adjoint operator $\theta^*_{M,v} : \ell_2(I, X) \longrightarrow X$ is surjective, bounded and defined by $\theta^*_{M,v}(y) = \sum_{i\in I} v_i \pi_{M_i}(y_i)$ for all $y = (y_i)_{i\in I}$ in $\ell_2(I, X)$.

Proof. Since $\{(M_i, v_i) : i \in I\}$ is a standard fusion frame, the frame transform is well-defined and for each $x \in X$,

$$C\langle x, x\rangle \leq \sum_{i\in I} v_i^2 \langle \pi_{M_i}(x), \pi_{M_i}(x)\rangle = \langle \theta_{M,v}(x), \theta_{M,v}(x)\rangle \leq D\langle x, x\rangle.$$

Now since $\langle x, x\rangle$ and $\langle \theta_{M,v}(x), \theta_{M,v}(x)\rangle$ are positive elements of C^*-algebra A, we obtain

$$\sqrt{C}\|x\| \leq \|\theta_{M,v}(x)\| \leq \sqrt{D}\|x\|.$$

Hence $\theta_{M,v}$ is one to one, bounded $\|\theta_{M,v}\| \leq \sqrt{D}$ and since X is complete, $\theta_{M,v}(X)$ is closed in $\ell_2(I, X)$. Therefore $\theta_{M,v} : X \longrightarrow \ell_2(I, X)$ is an isomorphic imbedding with norm closed range and by [7, Theorem 3.2], $\theta^*_{M,v}$ is surjective. To find the values of $\theta^*_{M,v}$, let $y=(y_i)\in \ell_2(I,X)$. For every $x\in X$,

$$\langle x,\theta^*_{M,v}(y)\rangle=\langle\theta_{M,v}(x),y\rangle=\sum_{i\in I}\langle v_i\pi_{M_i}(x),y_i\rangle=\sum_{i\in I}\langle x,v_i\pi_{M_i}(y)\rangle=\langle x,\sum_{i\in I}v_i\pi_{M_i}(y_i)\rangle. \tag{4}$$

Since the above relation holds for each $x\in X$, then $\theta^*_{M,v}(y)=\sum_{i\in I}v_i\pi_{M_i}(y_i)$ for all $y=(y_i)\in\ell_2(I,X)$, moreover $\theta^*_{M,v}$ is bounded, $\|\theta^*_{M,v}\|\leq\|\theta_{M,v}\|\leq\sqrt{D}$. $\square$

We note that if we regard $\theta_{M,v}:X\longrightarrow\theta_{M,v}(X)$, then $\theta_{M,v}$ is invertible and by [12, Theorem 15.3.8] or [7, Theorem 3.2], $\theta^*_{M,v}:\theta_{M,v}(X)\longrightarrow X$ is invertible. We also note that if $\{M_i:\ i\in I\}$ is a standard Parseval frame, then $\theta_{M,v}$ is an isometry.

Definition 2.10. Let $\{(M_i,v_i):\ i\in I\}$ be a standard fusion frame for X. Then the *fusion frame operator* $S_{M,v}$ for $\{(M_i,v_i):\ i\in I\}$ defined by

$$S_{M,v}(x)=\theta^*_{M,v}\theta_{M,v}(x)=\sum_{i\in I}v_i^2\pi_{M_i}(x),\qquad\qquad (x\in X).$$

Our next result is a generalization of [6, Theorem 6.1.] and [3, Prop. 4.1].

Theorem 2.11 (Reconstruction formula). Let $\{(M_i,v_i):\ i\in I\}$ be a standard fusion frame with bounds C and D. Then its associated frame operator $S_{M,v}$ is a positive invertible operator on X such that for all $x\in X$,

$$x=\sum_{i\in I}v_i^2 s^{-1}_{M,v}(\pi_{M_i}(x)).$$

Proof. For any $x,y\in X$ we have

$$\begin{aligned}\langle S_{M,v}(x),y\rangle &= \langle\sum_{i\in I}v_i^2\pi_{M_i}(x),y\rangle=\sum_{i\in I}v_i^2\langle\pi_{M_i}(x),y\rangle=\sum_{i\in I}v_i^2\langle\pi_{M_i}(x),\pi_{M_i}(y)\rangle\\ &= \langle x,\sum_{i}v_i^2\pi_{M_i}(y)\rangle=\langle x,S_{M,v}(y)\rangle,\end{aligned}$$

which shows that $S_{M,v}$ is a self-adjoint map and the above equality for $y=x$ shows that $S_{M,v}$ is positive. Since $\{(M_i,v_i):\ i\in I\}$ is a fusion frame with bounds C and D, for every $x\in X$ we have

$$C\langle x,x\rangle\leq\sum_{i\in I}v_i^2\langle\pi_{M_i}(x),\pi_{M_i}(x)\rangle=\langle S_{M,v}(x),S_{M,v}(x)\rangle\leq D\langle x,x\rangle.$$

As we mentioned earlier we can take $\theta_{M,v}$ and $\theta^*_{M,v}$ invertible, so $S_{M,v}:X\longrightarrow X$ is invertible and for every $x\in X$, $x=S^{-1}_{M,v}S_{M,v}(x)=\sum_{i\in I}v_i^2S^{-1}_{M,v}\pi_{M_i}(x)$.

Also by using the proof of Theorem 3.2 of [7] we can conclude that $S_{M,v}$ is invertible. $\square$

Our next result is a generalization of Prop. 3.11 of [2].

Theorem 2.12. Let $\{(M_i,v_i):\ i\in I\}$ be a standard fusion frame for X with bounds C and D and frame operator $S_{M,v}$. If $T\in End^*_A(X)$ is an invertible operator on X such that for each $i\in I$, $T^*T(M_i)\subseteq M_i$, then $\{(TM_i,v_i):\ i\in I\}$ is a standard fusion frame for X with frame operator $TS_{M,v}T^{-1}$.

Proof. Firstly for each $i\in I$, $T:M_i\longrightarrow TM_i$ is invertible, so each TM_i is a closed submodule of X. We show that $X=TM_i\oplus T(M_i^{\perp})$. Since $X=TX$, then for each

$x \in X$, there exists $y \in X$ such that $x = Ty$. On the other hand $y = u + v$, for some $u \in M_i$ and $v \in M_i^{\perp}$. Hence $X = Tu + Tv$, where $Tu \in T(M_i)$ and $Tv \in T(M_i^{\perp})$. Plainly $T(M_i) \cap T(M_i^{\perp}) = (0)$, therefore $X = T(M_i) \oplus T(M_i^{\perp})$. Now by assumption $T^* : TM_i \longrightarrow M_i$, therefore $T : M_i \longrightarrow TM_i$ is adjointable, see [7].

Hence for every $g \in M_i$, $h \in M_i^{\perp}$ we have $T^*Tg \in M_i$ and therefore $\langle Tg, Th\rangle = \langle T^*Tg, h\rangle = 0$, so $T(M_i^{\perp}) \subseteq (TM_i)^{\perp}$ and consequently $(TM_i^{\perp}) = (TM_i)^{\perp}$ which implies that TM_i is orthogonally complemented and $\pi_{TM_i} = T\pi_{M_i}T^{-1}$. Now by using Prop. 1.2 of [7] for T and T^{-1} very easily we conclude that for each $x \in X$,

$$\begin{aligned} C\|T^{-1}\|^{-2}.\|T\|^2\langle x, x\rangle &\leq \|T^{-1}\|^{-2}\sum v_i^2|\pi_{M_i}(T^{-1}x)|^2 \leq \sum v_i^2|\pi_{TM_i}(x)|^2 \\ &\leq \|T\|^2\sum v_i^2|\pi_{M_i}(T^{-1}x)|^2 \leq \|T\|^2.\|T^{-1}\|^2 D\langle x, x\rangle. \end{aligned}$$

Since $\{(M_i, v_i) : \ i \in I\}$ is standard, the above inequalities shows that $\{(TM_i, v_i) : \ i \in I\}$ is a standard fusion frame. Secondly for each $x \in X$ we have

$$S_{M,v}(T^{-1}x) = \sum v_i^2\pi_{M_i}(T^{-1}x) = T^{-1}(\sum v_i^2\pi_{TM_i}(x)) = T^{-1}S_{TM,v}(x).$$

Therefore $S_{TM,v} = TS_{M,v}T^{-1}$. $\square$

3 G-Frames

Sun in [11] introduced g-frames for Hilbert spaces, in this section we extend this notion to Hilbert C^*-modules. Let X and Y be Hilbert A-modules and for each $i \in I$, Y_i be a closed submodule of Y.

Definition 3.1. A sequence $\{\Lambda_i \in End_A^*(X, Y_i) : \ i \in I\}$ is called a *g-frame* or a *generalized frame* in X with respect to $\{Y_i : \ i \in I\}$ if there exist constants $C, D > 0$ such that for every $x \in X$,

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq D\langle x, x\rangle. \tag{5}$$

As usual C and D are g-frame bounds of $\{\Lambda_i : \ i \in I\}$. If $C = D = \lambda$, the g-frame is called λ-tight and if $C = D = 1$, it is called a Parseval or a normalized tight g-frame. The g-frame is *standard* if for every $x \in X$, the sum in (5) converges in norm. If for each i, $Y_i = Y$, we call it a g-frame of X with respect to Y.

Examples 3.2. (a) For every fusion frame $\{(M_i, v_i) : \ i \in I\}$ for X with respect to $\{v_i : \ i \in I\}$, the sequence $\{v_i\pi_{M_i} \in End_A^*(X, M_i) : \ i \in I\}$ is a g-frame of X with respect to $\{M_i : \ i \in I\}$.

(b) Let $\{x_i : \ i \in I\}$ be a frame of X. For each $i \in I$, we define $T_{x_i} : X \longrightarrow A$ by $T_{x_i}(x) = \langle x, x_i\rangle$ for all $x \in X$. Then T_{x_i} is adjointable with adjoint $T_{x_i}^*$ defined by $T_{x_i}^*(a) = ax_i$ for each $a \in A$, because for every $a \in A$, $x \in X$ we have $\langle\langle x, x_i\rangle, a\rangle_A = \langle x, x_i\rangle a^* = \langle x, ax_i\rangle$. Therefore $\{T_{x_i} \in End_A^*(X, A) : \ i \in I\}$ is a g-frame of X with respect to A.

Just like frames, for g-frames we can define the frame transform θ, the synthesis operator θ^* and the g-frame operator S as follows:

$$\theta : X \longrightarrow \oplus_{i\in I}Y_i, \qquad \theta(x) = (\Lambda_j x)_{j\in I},$$

$\theta^* : \oplus_{i\in I}Y_i \longrightarrow X,$ $\qquad \theta^*(y) = \sum \Lambda_j^*(y_j)$ for all $y = (y_j)$ in $\oplus_{j\in I}Y_j$, and $S = \theta^*\theta : X \longrightarrow X$ gives by $S(x) = \sum \Lambda_j^*\Lambda_j x$ for each $x \in X$. We know that $\|\theta^*\| \leq \sqrt{D}$ and

$\theta : X \longrightarrow \theta(X)$ is invertible with $\|\theta^{-1}\| \leq \frac{1}{\sqrt{C}}$. Moreover S is positive, self-adjoint and invertible with $\|S\| \leq D$ and $\|S^{-1}\| \leq \frac{1}{C}$.

Now we are able to generalize Theorem 2.5 and also Theorem 3.2 of [2] to g-frames.

Theorem 3.3. Let for every $i \in I$, $\Lambda_i \in End_A^*(X, Y_i)$ and $\{f_{ij} : j \in I_i\}$ be a frame for Y_i with frame bounds C_i, D_i and let $0 < C = \inf C_i \leq D = \sup D_i < \infty$. Then the following conditions are equivalent

(i) $\{\Lambda_i^* f_{i,j} : i \in I, j \in I_i\}$ is a frame for X,

(ii) $\{\Lambda_i : i \in I\}$ is a g-frame for X.

Proof. Let $i \in I$. Since $\sum_{i\in I}\sum_{j\in I_i} |\langle \Lambda_i x, f_{ij}\rangle|^2 = \sum_{i\in I}\sum_{j\in I_i} |\langle x, \Lambda_i^* f_{ij}\rangle)^2$ and $\{f_{ij} : j \in I_i\}$ is a frame for Y_i with bounds C_i, D_i we have

$$\begin{aligned} C\sum_{i\in I} |\langle \Lambda_i x, \Lambda_i x\rangle|^2 &\leq \sum_{i\in I} C_i |\langle \Lambda_i x, \Lambda_i x\rangle|^2 \leq \sum_{i\in I}\sum_{j\in I_i} |\langle \Lambda_i x, \Lambda_i x\rangle|^2 \\ &\leq \sum_{i\in I} D_i |\langle \Lambda_i x, \Lambda_i x\rangle|^2 \leq D\sum_{i\in I} |\langle \Lambda_i x, \Lambda_i x\rangle|^2. \end{aligned}$$

which shows that $\{\Lambda_i^*(f_{ij}) : i \in I, j \in I_i\}$ is a frame for X if and only if $\{\Lambda_i : i \in I\}$ is a g-frame for X. $\square$

Our next result is analog to Theorem 1 of [11].

Corollary 3.4. Let for each $i \in I$, $\Lambda_i \in End_A^*(X, Y_i)$ and $\{f_{ij} : j \in I_i\}$ be a Parseval frame of Y_i. Then we have

(i) $\{\Lambda_i : i \in I\}$ is a g-frame (resp. g-Bessel sequence, tight g-frame) for X if and only if $\{\Lambda_i^* f_{ij} : i \in I, j \in I_i\}$ is a frame (resp. Bessel sequence, tight frame) for X.

(ii) The g-frame operator of $\{\Lambda_i : i \in I\}$ is the frame operator of $\{\Lambda_i^* f_{ij} : i \in I, j \in I_i\}$.

Proof. (i) Follows from the above theorem.

(ii) For every $x \in X$ and $y \in Y_i$ we have

$$\begin{aligned} \langle x, \Lambda_i^* y\rangle &= \langle \Lambda_i x, y\rangle = \sum_{j\in I_i} \langle \Lambda_i x, f_{ij}\rangle\langle f_{ij}, y\rangle = \sum_{j\in I_i} \langle x, \Lambda_i^* f_{ij}\rangle\langle y, f_{ij}\rangle^* \\ &= \langle x, \sum_{j\in I_i} \langle y, f_{ij}\rangle \Lambda_i^* f_{ij}\rangle. \end{aligned}$$

Hence $\Lambda_i^* y = \sum_{j\in I_i} \langle y, f_{ij}\rangle \Lambda_i^* f_{ij}$. By using this result we can write

$$\begin{aligned} \sum_{i\in I} \Lambda_i^* \Lambda_i x &= \sum_{i\in I}\sum_{j\in I_i} \langle \Lambda_i x, f_{ij}\rangle \Lambda_i^*(f_{ij}) = \sum_{i\in I}\sum_{j\in I_i} \langle x, \Lambda_i^*(f_{ij})\rangle \Lambda_i^*(f_{ij}) \\ &= \sum_{i\in I}\sum_{j\in I_i} \langle x, \Lambda_i^* f_{ij}\rangle \Lambda_i^* f_{ij}. \end{aligned}$$

and we have the result. $\square$

Theorem 3.5. Let $\{\Lambda_j \in End_A^*(X, Y_j) : i \in I\}$ be a g-frame with bounds $C, D > 0$ and g-frame operator S, let M be a Hilbert A-module and let $T \in End_A^*(M, X)$ be invertible. Then $\{\Lambda_j T \in End_A^*(M, Y_j) : i \in I\}$ is a g-frame with g-frame operator T^*ST with bounds $C/\|T^{-1}\|^2$, $D\|T\|^2$.

Proof. A simple calculation shows that if $\{\Lambda_j \in End_A^*(X, Y_j) : j \in I\}$ is a g-frame with bounds C and D, then $\{\Lambda_j T \in End_A^*(M, Y_j); j \in I\}$ is a g-frame with respect to $\{Y_i : i \in I\}$ and for every $x \in M$, $C\langle Tx, Tx\rangle \leq \langle T^*STx, x\rangle \leq D\langle Tx, Tx\rangle$. Moreover for every $x \in M$,

$$T^*ST(x) = T^*(\sum_j \Lambda_j^* \Lambda_j Tx) = \sum_j T^* \Lambda_j^* \Lambda_j Tx = \sum_j (\Lambda_j T)^*(\Lambda_j T)(x).$$

Hence by [7, Prop. 1.2] for every $x \in M$,

$$\frac{C}{\|T^{-1}\|^2}\langle x,x\rangle \leq \langle T^*STx,x\rangle = \sum_j \langle \Lambda_j Tx, \Lambda_j Tx\rangle \leq D\|T\|^2\langle x,x\rangle.$$

Therefore $\{\Lambda_j T : \ j \in I\}$ is a g-frame of M with respect to $\{Y_j : \ j \in I\}$ with g-frame operator T^*ST. $\square$

Corollary 3.6. Let $\{\Lambda_j \in End_A^*(X,Y_j) : \ j \in I\}$ be a g-frame with bounds C, D and g-frame operator S. Then $\{\widehat{\Lambda}_j = \Lambda_j S^{-1} \in End_A^*(X,Y_j) : \ j \in I\}$ is a g-frame with bounds $1/D$, $1/C$, g-frame operator S^{-1} and for every $x \in X$, $x = \sum \widehat{\Lambda}_i \Lambda_i^* x = \sum (\widehat{\Lambda}_i)^* \Lambda_i(x)$.

Proof. If in the above lemma we take $M = X$ and $T = S^{-1}$, we conclude that $\{\widehat{\Lambda_j} = \Lambda_j S^{-1} \in End_A^*(X,Y_j) : \ j \in I\}$ is a g-frame with g-frame operator S^{-1} and for every $x \in X$ we have

$$\frac{1}{D}\langle x,x\rangle \leq \frac{1}{\|S\|}\langle x,x\rangle \leq \sum_j \langle \widehat{\Lambda_j}x, \widehat{\Lambda_j}x\rangle = \langle S^{-1}x,x\rangle \leq \|S^{-1}\|\langle x,x\rangle \leq \frac{1}{C}\langle x,x\rangle.$$

Moreover since for every $i \in I$, $(\widehat{\Lambda}_i)^* = S^{-1}\Lambda_i^*$ and for every $x \in X$, $x = S^{-1}Sx = SS^{-1}x$, then $x = \sum_i \widehat{\Lambda}_i \Lambda_i^* x = \sum_i (\widehat{\Lambda}_i)^* \Lambda_i(x)$. $\square$

The next result is a generalization of [1, Theorem 2.8] to g-frames in Hilbert spaces.

Theorem 3.7. Let H and K be Hilbert spaces and for each $i \in I$, K_i be a closed subspace of K. Let $\{\Lambda_j \in B(H,K_j) : \ j \in I\}$ be a g-frame for H with respect to $\{K_j : \ j \in I\}$. If $\{\mu_j \in B(H,K_j) : \ j \in I\}$ is a family of maps such that the map $V : H \longrightarrow H$ given by $V(f) = \sum(\mu_j^*\mu_j - \Lambda_j^*\Lambda_j)(f)$ is a compact operator and H_1 is the closed linear span of $\bigcup_{j\in I}\mu_j^*(K_j)$, then $\{\mu_j \in B(H_1,K_j) : \ j \in I\}$ is a g-frame with respect to $\{K_j : \ j \in I\}$.

Proof. Plainly V is a self-adjoint operator on H. Let S be the g-frame operator of $\{\Lambda_j \in B(H,K_j) : \ j \in I\}$. If we take $T = S + V : H \longrightarrow H$, then T is a bounded linear self-adjoint operator, with $\|T\| \leq D + \|V\|$, we also have

$$\|T\| = \sup_{\|f\|\leq 1} |\langle Tf,f\rangle| = \sup_{\|f\|\leq 1} \sum_j \|\mu_j f\|^2.$$

Hence for every $f \in H$,

$$\langle Tf,f\rangle = \sum_j \|\mu_j f\|^2 \leq \|T\|.\|f\|^2 \leq (D + \|V\|)\|f\|^2. \tag{6}$$

Now $S^{-1}V$ is compact and $S^{-1}T = I + S^{-1}V$, so a small modification of the proof of Theorem 2.8 in [1] shows that T has a closed range, and $N(T) = H_1^\perp$, because if $f \in N(T)$, then $T(f) = 0$ and therefore $\langle Tf,f\rangle = 0$, so for every $j \in I$, $\mu_j(f) = 0$, which implies that f is orthogonal to $\overline{span}(\cup_j \mu_j^*(K_j)) = H_1$, i.e., $f \in H_1^\perp$. Conversely if $f \in H_1^\perp$, then a simple calculation shows that $T(f) = 0$. Therefore $Range(T) = (N(T^*))^\perp = N(T)^\perp = H_1$, and T induces a bounded linear, self-adjoint operator $T_1 = T|_{H_1} : H_1 \longrightarrow H_1$ which is invertible. Now by Cauchy-Schwarz inequality and (6) for every $f \in H_1$,

$$\begin{aligned}\|T_1(f)\|^4 &= (\langle \sum_j \mu_j^*\mu_j f, T_1 f\rangle)^2 = (\sum_j \langle \mu_j f, \mu_j T_1 f\rangle)^2 &(7)\\ &\leq (\sum_j \|\mu_j f\|^2)(\sum_j \|\mu_j T_1 f\|^2) \leq (\sum_j \|\mu_j f\|^2)(D + \|V\|)\|T_1 f\|^2. &(8)\end{aligned}$$

Hence for every $f \in H_1$,

$$\sum_j \|\mu_j f\|^2 \geq \frac{1}{D + \|V\|} \cdot \frac{1}{\|T^{-1}\|^2} \cdot \|f\|^2,$$

and therefore we have the result. $\square$

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