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Fuzzy bi-ideals in semigroups

by

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The concept of a fuzzy set, introduced in Zadeh [5], was applied to the elementary theory of groupoids and groups in Rosenfeld [4] and of semigroups in the author [2]. In the present note we shall give a characterization of a semigroup which is a group, a union of groups and a semilattice of groups in terms of fuzzy bi-ideals.

A *fuzzy set* in a semigroup S is a function δ from S into the unit interval $[0, 1]$. A fuzzy set δ in S is called a *fuzzy subsemigroup* of S if

$$\delta(xy) \geqq \min\{\delta(x), \delta(y)\}$$

for all $x, y \in S$. A fuzzy subsemigroup δ of S is called a *fuzzy bi-ideal* of S if

$$\delta(xyz) \geqq \min\{\delta(x), \delta(z)\}$$

for all x, y and $z \in S$. A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subseteqq A$. The following theorem shows that the concept of a fuzzy bi-ideal in a semigroup is an extended one of a bi-ideal.

THEOREM 1. *For a non-empty subset A of a semigroup S the following conditions are equivalent.*

(1) *A is a bi-ideal of S.*

(2) *The characteristic function δ_A of A is a fuzzy bi-ideal of S.*

Proof. Assume that (1) holds. Let x, y and z be any elements of S. If $x \in A$ and $z \in A$, then we have

$$\delta_A(x) = \delta_A(z) = 1 .$$

Since A is a bi-ideal of S,

$$xyz \in ASA \subseteqq A .$$

Then we have

$$\delta_A(xyz) = 1 = \min\{\delta_A(x), \delta_A(z)\} .$$

If $x \notin A$ or $z \notin A$, then

$$\delta_A(x) = 0 \quad \text{or} \quad \delta_A(z) = 0 .$$

Then we have

$$\delta_A(xyz) \geqq 0 = \min\{\delta_A(x), \delta_A(z)\} .$$

Then it follows from Proposition 2.2 of [4] that the characteristic function δ_A is, if A is a subsemigroup of S, a fuzzy subsemigroup of S. Thus we obtain that δ_A is a fuzzy bi-ideal of S, and that (1) implies (2). Conversely, assume that (2) holds. Let $xyz(x, z \in A, y \in S)$ be any element of ASA. Since $x, z \in A$, we have

$$\delta_A(x) = \delta_A(z) = 1 .$$

Then, since δ_A is a fuzzy bi-ideal of S, we have

$$\delta_A(xyz) \geqq \min\{\delta_A(x), \delta_A(z)\} = 1 .$$

This implies that

$$\delta_A(xyz) = 1 ,$$

and so $xyz \in A$. Thus we obtain that $ASA \subseteqq A$. It follows from Proposition 2.2 of [4] that the non-empty subset A of S is, if δ_A is a fuzzy subsemigroup of S, a subsemigroup of S. Thus we obtain that A is a bi-ideal of S, and that (2) implies (1).

THEOREM 2. *For a semigroup S the following conditions are equivalent.*

(1) *S is a group.*

(2) *Every fuzzy bi-ideal of S is a constant function.*

Proof. First assume that S is a group. Let δ be any fuzzy bi-ideal of S, and a any element of S. We denote by e the identity of the group S. Then we have

$$\begin{aligned}\delta(a) &= \delta(eae) \geqq \min\{\delta(e), \delta(e)\} = \delta(e) \\ &= \delta(ee) = \delta\{(aa^{-1})(a^{-1}a)\} \\ &= \delta\{a(a^{-1}a^{-1})a\} \geqq \min\{\delta(a), \delta(a)\} \\ &= \delta(a) ,\end{aligned}$$

and so we have

$$\delta(a) = \delta(e) .$$

This means that δ is a constant fumction. Thus (1) implies (2). Conversely, assume that S is not a group. Then it follows from p. 84 of [1] that S contains a proper bi-ideal A of S. Since A is non-empty, the characteristic function δ_A of A is not a constant function. Then, since δ_A is a fuzzy bi-ideal of S by Theorem 1, (2) does not hold. Thus (2) implies (1). This completes the proof.

A semigroup S is called *completely regular* if, for each element a of S. there exists an element x in S such that

$$a=axa \quad \text{and} \quad ax=xa .$$

The following characterization of such a semigroup in due to p. 105 of [6].

LEMMA 3. *For a semigroup S the following conditions are equivalent.*

(1) *S is completely regular.*

(2) *S is a union of groups.*

(3) *$a \in a^2Sa^2$ for all $a \in S$.*

Now we give another characterization of a completely regular semigroup.

THEOREM 4. *For a semigroup S the following conditions are equivalent.*

(1) *S is completely regular.*

(2) *For every fuzzy bi-ideal δ of S,*

$$\delta(a)=\delta(a^2)$$

holds for all $a \in S$.

Proof. First assume that (1) holds. Let δ be any fuzzy bi-ideal of S and a any element of S. Then there exists an element x in S such that

$$a=a^2xa^2 .$$

Then, since δ is a fuzzy bi-ideal of S, we have

$$\begin{aligned}\delta(a)&=\delta(a^2xa^2)\geqq\min\{\delta(a^2), \delta(a^2)\}\\ &=\delta(a^2)\geqq\min\{\delta(a), \delta(a)\}=\delta(a) ,\end{aligned}$$

and so we have

$$\delta(a)=\delta(a^2) .$$

Thus (1) implies (2). Conversely, assume that (2) holds. We denote by $B[x]$ the principal bi-ideal of a semigroup S generated by x in S, that is,

$$B[x]=\{x\}\cup\{x^2\}\cup xSx .$$

Since the characteristic function $\delta_{B[a^2]}$ of the bi-ideal $B[a^2]$ is a fuzzy bi-ideal of S by Theorem 1, and since $a^2 \in B[a^2]$, we have

$$\delta_{B[a^2]}(a)=\delta_{B[a^2]}(a^2)=1 .$$

This implies that

$$a \in B[a^2]=\{a^2\}\cup\{a^4\}\cup a^2Sa^2 .$$

Then it is easily to see that S is completely regular. Thus (2) implies

(1). This completes the proof.

The following is due to Theorem 1 of [3].

LEMMA 5. *A semigroup S is a semilattice of groups if and only if the set of all bi-ideals of S is a semilattice under the multiplication of subsets.*

THEOREM 6. *For a semigroup S the following conditions are equivalent.*

(1) *S is a semilattice of groups.*

(2) *For every fuzzy bi-ideal δ of S,*

$$\delta(a)=\delta(a^2) \quad \text{and} \quad \delta(ab)=\delta(ba)$$

hold for $a, b \in S$.

Proof. Assume that (1) holds. Then S is a union of groups. Then it follows from Lemma 3 that S is completely regular. And so it follows from Theorem 4 that for every fuzzy bi-ideal δ of S

$$\delta(a)=\delta(a^2)$$

holds for all $a \in S$. Let a and b be any elements of S. Then by Lemma 5 we have

$$\begin{aligned}(ab)^3=(aba)(bab) &\in B[aba]B[bab]\\ &=B[bab](B[aba])^2 \subseteqq B[bab]SB[aba]\\ &\subseteqq babSaba \subseteqq baSba .\end{aligned}$$

This implies that there exists an element x in S such that

$$(ab)^3=(ba)x(ba) .$$

Then, for any fuzzy bi-ideal δ of S, we have

$$\begin{aligned}\delta(ab)&=\delta\{(ab)^3\}=\delta\{(ba)x(ba)\}\\ &\geqq \min\{\delta(ba), \delta(ba)\}=\delta(ba) .\end{aligned}$$

Similarly, we can prove that

$$\delta(ba) \geqq \delta(ab) .$$

Thus we obtain that

$$\delta(ab)=\delta(ba)$$

and that (1) implies (2). Conversely, assume that (2) holds. Then it follows from the first condition and from Theorem 4 that S is completely regular. Then it is easily proved that every bi-ideal of S is globally idempotent. Let A and B by any bi-ideals of S, and $ba (a \in A, b \in B)$ any element of BA. Then, since by Theorem 1 the characteristic function $\delta_{B[ab]}$ of the bi-ideal $B[ab]$ is a fuzzy bi-ideal of S,

$$\delta_{B[ab]}(ba)=\delta_{B[ab]}(ab)=1 .$$

This implies that

$$ba \in B[ab]=\{ab\} \cup \{abab\} \cup abSab .$$

Then

$$\text{(i)} \quad ba=ab \in AB , \qquad \text{(ii)} \quad ba=abab \in ABAB \subseteqq AB , \quad \text{or}$$

(iii) Since the product AB of the bi-ideals A and B of S is also a bi-ideal of S, we have

$$ba \in abSab \subseteqq (AB)S(AB) \subseteqq AB .$$

In any cases we have

$$BA \subseteqq AB .$$

Similarly we can see that the converse inclusion holds. Thus we obtain that

$$AB=BA ,$$

and that the set of all bi-ideals of S is a commutative idempotent semigroup. Therefore it follows from Lemma 5 that S is a semilattice of groups, and that (2) implies (1).

COROLLARY 7. *For an idempotent semigroup S the following conditions are equivalent.*

(1) *S is commutative.*

(2) *For every fuzzy bi-ideal δ of S,*

$$\delta(ab)=\delta(ba)$$

holds for all $a, b \in S$.

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