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Fuzzy covering-based rough set on two different universes and its application

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Abstract

In this paper, we propose a new type of fuzzy covering-based rough set model over two different universes by using Zadeh's extension principle. We mainly address the following issues in this paper. First, we present the definition of fuzzy β -neighborhood, which can be seen as a fuzzy mapping from a universe to the set of fuzzy sets on another universe and study its properties. Then we define a new type of fuzzy covering-based rough set model on two different universes and investigate the properties of this model. Meanwhile, we give a necessary and sufficient condition under which two fuzzy β -coverings to generate the same fuzzy covering lower approximation or the same fuzzy covering upper approximation. Moreover, matrix representations of the fuzzy covering lower and fuzzy covering upper approximation operators are investigated. Finally, we propose a new approach to a kind of multiple criteria decision making problem based on fuzzy covering-based rough set model over two universes. The proposed models not only enrich the theory of fuzzy covering-based rough set but also provide a new perspective for multiple criteria decision making with uncertainty.

Keywords: Fuzzy sets, Fuzzy β -covering, Fuzzy covering-based rough sets, Fuzzy β -neighborhood, Multiple criteria decision making

1 Introduction

This paper firstly studies fuzzy covering-based rough set on two different universes and gives an illustrated example of multiple criteria decision making problem, which is discussed by using fuzzy covering-based rough set. Now, the development of rough set and fuzzy covering-based rough set, the motivation of our research and the work of present paper are shown as follows:

1.1 A brief review on rough set

Rough set theory (RST) was originally proposed by Pawlak [40, 41] in 1982 as a useful mathematical tool for dealing with the vagueness and granularity in information systems and data analysis. This theory can approximately characterize an arbitrary subset of a universe by using two definable subsets called lower and upper approximations [8]. In the Pawlak's rough set model, the relationship of objects were built on equivalence relations. All equivalence classes form a partition of a universe of discourse. However, an equivalence relation imposes restrictions on various applications [11, 62]. Hence, many extensions have been made in recent years by replacing equivalence relation or partition by notions such as general binary relations [19, 49, 50], neighborhood systems and Boolean algebras [10, 63], and coverings of the universe of discourse [7, 43, 44]. Based on the notion of covering, Pomykala [43, 44] obtained two pairs of dual approximation operators. Yao [71] further examined these approximation operators by the concepts of neighborhood and granularity. Such undertakings have stimulated more researches in this area [9, 31, 72, 79, 88]. Over the past 40 years, RST has indeed become a topic of great interest to researchers and has been applied to various domains. This success is due in part to the following three aspects of the theory:

- (1) Only the facts hidden in data are analyzed.
- (2) No additional information about the data is required such as thresholds or expert knowledge.
- (3) A minimal knowledge representation can be attained.

1.2 A brief review on fuzzy covering-based rough set

However, RST is designed to process qualitative (discrete) data, and it faces serious limitations when dealing with real-valued data sets since the values of attributes in databases could be both symbolic and real-valued [26]. Fuzzy set theory (FST) [77], is very useful to overcome these limitations, as it can deal effectively with vague concepts and graded indiscernibility. Nowadays, rough set theory and fuzzy set theory are two main tools used to process uncertainty and incomplete information in the information systems. The two theories are related but distinct and complementary [42, 73]. In the past 30 years, the research on the connection between rough sets and fuzzy sets has attracted much attention. Intentions on combining rough set theory and fuzzy set theory can be found in different mathematical fields [39, 61, 73]. Dubois and Prade firstly proposed the concept of fuzzy rough sets [17], which combined these

two theories and influenced numerous authors who used different fuzzy logical connectives and fuzzy relations to define fuzzy rough set models. In the aspect of theory, D’eer et al. [13] have critically evaluated most relevant fuzzy rough set models proposed in the literature and established a formally correct and unified mathematical framework for them. In [20–22], Hu and Wong have constructed generalized interval-valued fuzzy rough sets and generalized interval-valued fuzzy variable precision rough sets by using fuzzy logical operators. Meanwhile, L -fuzzy rough set have been studied by constructive method and axiomatic method [33, 35, 36, 45–47, 59, 64, 65]. Yeung et al. [76] have proposed some definitions of upper and lower approximation operators of fuzzy sets by means of arbitrary fuzzy relations and studied the relationship among them from the viewpoint of constructive approach. In axiomatic approach, they have characterized different classes of generalized upper and lower approximation operators of fuzzy sets by different sets of axioms. In application, Hu et al. [24] have integrated kernel functions with fuzzy rough set models and propose two types of kernelized fuzzy rough sets and have applied these measures to evaluate and select features for classification problems. Sun et al. [51] have presented a new extension of the rough set theory by means of integrating the classical Pawlak rough set theory with the interval-valued fuzzy set theory and applied in decision making. The most common fuzzy rough set models are obtained by replacing the crisp binary relations and the crisp subsets with the fuzzy relations and the fuzzy subsets on the universe respectively.

As well as RST, some researchers have tried to generalize the fuzzy rough set based on fuzzy relation by using the concept of fuzzy covering. De Cock et al. [12] defined fuzzy rough sets based on the R -foresets of all objects in a universe of discourse with respect to (w.r.t.) a fuzzy binary relation. When R is a serial fuzzy relation, the family of all R -foresets forms a fuzzy covering of the universe of discourse. Analogously, Deng et al. [16] fuzzified the general rough approximation operators and presented a new approach to fuzzy rough sets through the use of techniques provided by lattice theory. The new fuzzy rough approximation operators are established based on a fuzzy covering, a binary fuzzy conjunction logical operator with lower semi-continuity in its second argument, and the adjunctional implication operator of the conjunction. Li and Ma [27], on the other hand, constructed two pairs of fuzzy rough approximation operators based on fuzzy coverings, the standard min operator \mathcal{T}_M , and the Kleene-Dienes implicator \mathcal{I}_{KD} . It should be noted that fuzzy coverings in the models proposed by Deng [16] and De Cock et al. [12] are induced from fuzzy relations. So, they are not fuzzy coverings in the general sense. Although fuzzy coverings are used by Li and Ma [27] in their models, they only employed two special logical operators i.e., the standard min operator and the Kleene-Dienes implicator. Thus, it is necessary to construct more general fuzzy rough set models based on fuzzy coverings. Recently, an excellent introduction to the topic of fuzzy covering-based rough set is due to some scholars. In [48], Šešelja have investigated lattice-valued (L -fuzzy) covering, or fuzzy neighboring relation arising from a given lattice-valued order. Based on

this work, Li et al. [28] have proposed two pairs of generalized lower and upper fuzzy rough approximation operators by means of an implicator \mathcal{I} and a triangular norm \mathcal{T} . In 2011, Yao et al. [70] have studied attribute reduction in fuzzy decision systems (as a fuzzy covering-based approximation space) based on generalized fuzzy evidence theory. Moreover, Feng et al. [18] have studied reduction of a fuzzy covering and fusion of multi-fuzzy covering systems based on the evidence theory and rough set theory. In 2016, Huang et al. [25] developed a new rough set model, which is a generalization of the β -neighborhood fuzzy covering rough sets and IF (intuitionistic fuzzy) rough sets. D'eer et al. [14] have introduced the notion of fuzzy neighborhood system of an object based on a given fuzzy covering, as well as the notion of the fuzzy minimal and maximal descriptions of an object. Moreover, they have extended the definition of four covering-based neighborhood operators as well as six derived coverings discussed by Yao and Yao to the fuzzy setting. The main work about fuzzy covering-based rough set can be regarded as a bridge linking covering-based rough set theory and fuzzy rough set theory. However, there exist limitations in some applications by using fuzzy covering-based rough set, which are defined based on original definition of fuzzy covering (see [18, 28]).

1.3 The motivation of our research

To overcome the limitations of original definition of fuzzy covering in some applications, Ma [32] generalized the fuzzy covering to fuzzy β -covering by replacing 1 with a parameter β ($0 < \beta \leq 1$). Furthermore, Ma defined two new types of fuzzy covering-based rough set model by introducing the new concept of fuzzy β -neighborhood. As shown in Figure. 1, Yang and Hu put forward some types of fuzzy coverings-based rough set model based on the concepts of fuzzy β -neighborhood in [67, 69]. Based on these work, some research focus on fuzzy neighborhood operators are addressed, such as [5, 6, 34]. Furthermore, a fuzzy covering-based rough set model was proposed in [66] by using fuzzy β -minimal description. Based on these research, some researchers combine decision-theoretic rough set with fuzzy covering-based rough set and apply these models in decision making problem [80, 85, 86]. In addition, fuzzy covering-based rough set models were also used in attribute reduction [58].

However, fuzzy covering-based rough sets, which were mentioned above, are all defined in fuzzy covering-based approximation (U, \mathbf{C}) . In other words, these work only investigate incomplete decision system (U, V, AT, D) under condition that $U = V$. Then it is necessary to study fuzzy covering-based rough sets in fuzzy covering-based approximation (U, V, \mathbf{C}) . Especially, in [57], Wang et al. used the concept of homomorphism as a basic tool to study the communication between fuzzy information systems. Meanwhile, Yang and Hu [68] discussed the communication between fuzzy information systems by using fuzzy covering mappings and fuzzy covering-based rough sets, respectively. In fact, there naturally exists two different universe of discourse for the issue of communication between fuzzy information systems. In [52], Sun and Ma proposed a type of fuzzy rough set model on two different universes and applied this model into

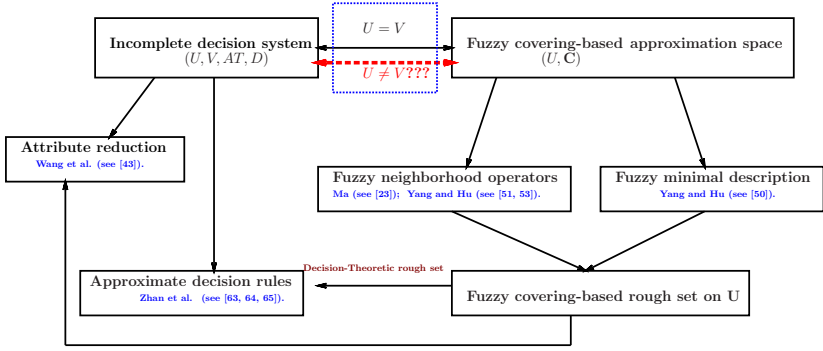


Fig. 1 Present research on fuzzy covering-based rough set

decision making. In [53], Sun et al. presented three types of multigranulation fuzzy rough set over two universes by the constructive approach, respectively and applied these models into multiple criteria decision making. With reference to the requirement of the applications in practice, as well as the complement of the theoretical aspect of fuzzy covering-based rough set, we mainly focus on rough approximation of a fuzzy concept in fuzzy covering approximation space over two universes, i.e., fuzzy covering-based rough set model over two universes.

1.4 The work of the present paper

In this paper, we propose a novel type of fuzzy covering-based rough set model on two different universes by using Zadeh's extension principle and apply this model to a kind of multiple criteria decision making problem. Main works in this paper are listed as follows.

- (1) The concept of fuzzy β -neighborhood on two different universes is defined and its properties are studied.
- (2) Based on the definition of fuzzy β -neighborhood, a type of fuzzy covering-based rough set model on two universes is investigated by constructive and axiomatic approach, respectively. Especially, by introducing the concept of β -reduct of a fuzzy β -covering, we explore the conditions under which two fuzzy β -coverings generate the same fuzzy covering lower or fuzzy covering upper approximations.
- (3) Matrix is an important mathematical concept, which also is an important tool for numerical calculation method. The matrix representations make it possible to calculate the fuzzy covering-based lower and upper approximation operators through the operations on matrices, which is algorithmic, and can easily be implemented through the computer. Then the matrix representations of the fuzzy covering approximation operators is a vital problem to investigate in this paper.
- (4) A new approach to a kind of multiple criteria decision making problem is proposed in this paper.

The remainder of this paper is organized as follows. In Section 2, some basic concepts and properties of fuzzy set theory and fuzzy covering-based rough set used in this paper are introduced. In Section 3, a new type of fuzzy covering-based rough set model on two different universes is investigated by constructive and axiomatic approach, respectively. Meanwhile, the matrix representations of the fuzzy covering lower and fuzzy covering upper approximation operators are proposed. In Section 5, an application to the multiple criteria decision making problem of the fuzzy covering-based rough set model is introduced. Section 6 concludes this paper.

2 Preliminaries

The theory of fuzzy sets, is an extension of set theory for the study of intelligent systems characterized by fuzzy information. In this section, we review some notions in fuzzy set theory and fuzzy covering-based rough set used in this paper.

Let U be a discourse of universe. A fuzzy set A , or rather a fuzzy subset A of U , is defined by a function assigning to each element x of U a value $A(x) \in [0, 1]$. We denote by $\mathcal{F}(U)$ the family of all fuzzy subsets of U , i.e., the set of all functions from U to $[0, 1]$, and call it the fuzzy power set of U .

For any $A, B \in \mathcal{F}(U)$, we say that A is contained in B , denoted by $A \subseteq B$, if $A(x) \leq B(x)$ for all $x \in U$, and we say that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

For any family $\alpha_i \in [0, 1], i \in I, I \subseteq \mathbb{N}^*$ (\mathbb{N}^* is the set of all positive integers), we write $\bigvee_{i \in I} \alpha_i$ (resp. $\bigwedge_{i \in I} \alpha_i$) as the supremum (resp. infimum) of $\{\alpha_i : i \in I\}$. Given $A, B \in \mathcal{F}(U)$, the union of A and B , denoted as $A \cup B$, is defined by $(A \cup B)(x) = A(x) \vee B(x)$ for all $x \in U$. The intersection of A and B , denoted as $A \cap B$, is given by $(A \cap B)(x) = A(x) \wedge B(x)$ for all $x \in U$. The complement of A , denoted as A^c , is defined by $A^c(x) = 1 - A(x)$ for all $x \in U$.

Let $X \in \mathcal{F}(U)$ and $\alpha \in [0, 1]$. X is said to be constant if $X(x) = \alpha$ for all $x \in U$, noted by α_U . X is called a fuzzy point if $X(x) = \begin{cases} \alpha, & x = y, \\ 0, & x \neq y, \end{cases}$ noted by y_α .

Definition 1 ([32]) Let U be an arbitrary universal set. For each $\beta \in (0, 1]$, we call $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$, with $C_i \in \mathcal{F}(U)$ ($i = 1, 2, \dots, m$), a fuzzy β -covering of U , if $(\bigcup_{i=1}^m C_i)(x) \geq \beta$ for each $x \in U$. We also call (U, \mathbf{C}) a fuzzy β -covering approximation space.

In fact, the concept of fuzzy covering in [18, 28] is a special case of fuzzy β -covering with $\beta = 1$.

Definition 2 ([32]) Let (U, \mathbf{C}) be a fuzzy β -covering approximation space and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$. For each $x \in U$, we define the fuzzy β -neighborhood \hat{N}_x^β of x as:

$$\hat{N}_x^\beta = \bigcap \{C_i \in \mathbf{C} : C_i(x) \geq \beta\}.$$

Example 1 Let $U = \{x_1, x_2, \dots, x_{10}\}$ and $\hat{\mathbf{C}} = \{C_1, C_2, \dots, C_8\}$ be a family of fuzzy sets of U . Where

$$\begin{aligned} C_1 &= \frac{0.12}{x_1} + \frac{0.27}{x_3} + \frac{0.46}{x_4} + \frac{0.25}{x_5} + \frac{0.6}{x_6} + \frac{0.79}{x_7} + \frac{0.01}{x_8} + \frac{0.8}{x_9} + \frac{0.42}{x_{10}}, \\ C_2 &= \frac{0.45}{x_1} + \frac{0.36}{x_2} + \frac{0.75}{x_3} + \frac{0.64}{x_4} + \frac{0.2}{x_5} + \frac{0.3}{x_6} + \frac{0.55}{x_7} + \frac{0.24}{x_8} + \frac{0.56}{x_9} + \frac{0.59}{x_{10}}, \\ C_3 &= \frac{0.3}{x_1} + \frac{0.42}{x_2} + \frac{0.5}{x_3} + \frac{0.44}{x_4} + \frac{0.68}{x_5} + \frac{0.4}{x_6} + \frac{0.56}{x_7} + \frac{0.65}{x_9} + \frac{0.87}{x_{10}}, \\ C_4 &= \frac{0.25}{x_1} + \frac{0.58}{x_2} + \frac{0.39}{x_3} + \frac{0.4}{x_4} + \frac{0.72}{x_5} + \frac{0.2}{x_7} + \frac{0.45}{x_8} + \frac{0.6}{x_9} + \frac{0.3}{x_{10}}, \\ C_5 &= \frac{0.15}{x_1} + \frac{0.5}{x_2} + \frac{0.66}{x_3} + \frac{0.32}{x_4} + \frac{0.45}{x_5} + \frac{0.09}{x_6} + \frac{0.28}{x_7} + \frac{0.78}{x_8} + \frac{0.33}{x_9} + \frac{0.24}{x_{10}}, \\ C_6 &= \frac{0.41}{x_1} + \frac{0.32}{x_2} + \frac{1}{x_7} + \frac{0.43}{x_8} + \frac{0.16}{x_9} + \frac{0.64}{x_{10}}, \\ C_7 &= \frac{0.54}{x_1} + \frac{0.06}{x_2} + \frac{0.45}{x_3} + \frac{0.67}{x_4} + \frac{0.45}{x_5} + \frac{0.5}{x_6} + \frac{0.65}{x_7} + \frac{0.22}{x_8} + \frac{0.36}{x_9} + \frac{0.17}{x_{10}}, \\ C_8 &= \frac{0.6}{x_1} + \frac{0.31}{x_2} + \frac{0.8}{x_3} + \frac{0.44}{x_4} + \frac{0.65}{x_5} + \frac{0.27}{x_6} + \frac{0.9}{x_7} + \frac{0.99}{x_8} + \frac{0.12}{x_9} + \frac{1}{x_{10}}. \end{aligned}$$

We can see that $\hat{\mathbf{C}}$ is a fuzzy β -covering of U ($0 < \beta \leq 0.58$). Let $\beta = 0.5$. We have fuzzy β -neighborhoods, for example,

$$\begin{aligned} \tilde{N}_{x_2}^{0.5} &= C_4 \cap C_5 = \frac{0.15}{x_1} + \frac{0.5}{x_2} + \frac{0.39}{x_3} + \frac{0.32}{x_4} + \frac{0.45}{x_5} + \frac{0.2}{x_7} + \frac{0.45}{x_8} + \frac{0.33}{x_9} + \frac{0.24}{x_{10}}, \\ \tilde{N}_{x_7}^{0.5} &= C_1 \cap C_2 \cap C_3 \cap C_6 \cap C_7 \cap C_8 = \frac{0.12}{x_1} + \frac{0.55}{x_7} + \frac{0.12}{x_9} + \frac{0.17}{x_{10}}. \end{aligned}$$

In [78], Zadeh proposed the extension principle, which has become an important tool in fuzzy set theory and its applications. In the following discussions, we denote $\mathbf{Map}(U, V)$ as the family of all mappings from U to V and $\mathbf{Sur}(U, V)$ as the family of all surjective mappings from U to V .

Let U, V be two universes, $A \in \mathcal{F}(U)$, $P \in \mathcal{F}(V)$ and $f \in \mathbf{Map}(U, V)$. Then f can induce a fuzzy mapping f^\rightarrow from $\mathcal{F}(U)$ to $\mathcal{F}(V)$, i.e., $f^\rightarrow(A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A(x), & y \in f(U), \\ 0, & y \notin f(U). \end{cases}$ and a fuzzy mapping f^\leftarrow from $\mathcal{F}(V)$ to $\mathcal{F}(U)$, i.e., $f^\leftarrow(P)(x) = P(f(x))$, $x \in U$. Under no confusion in subsequent discussion, we simply denote f^\rightarrow and f^\leftarrow by f and f^{-1} , respectively.

3 Constructive and axiomatic approaches to fuzzy covering-based approximation operators over two universes

In this section, by introducing the notion of fuzzy β -neighborhood, i.e., $\tilde{N}_x^\beta : U \rightarrow \mathcal{F}(V)$, we define a type of fuzzy covering-based rough set model over two

universes. We will address the following issues in this section. First, we present the definition of fuzzy β -neighborhood by using Zadeh's extension principle and study its properties. Then, we define a type of fuzzy covering-based rough set model over two universes and investigate the properties of this model. Moreover, the axiomizations of the fuzzy covering lower and fuzzy covering upper approximations are the vital problems we investigate in this section.

3.1 Some properties of fuzzy β -covering space over two universes

Based on Zadeh's extension principle, if \mathbf{C} is a fuzzy β -covering on U for some $\beta \in (0, 1]$, we cannot ensure that $f(\mathbf{C})$ is a fuzzy β -covering on V . Below we give an illustrative example.

Example 2 Let $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$ and $f \in \mathbf{Map}(U, V)$, $f(x_1) = f(x_2) = y_1$. Let $\mathbf{C} = \{A_1, A_2\}$, where

$$A_1 = \frac{0.8}{x_1} + \frac{0.4}{x_2}, A_2 = \frac{0.3}{x_1} + \frac{0.9}{x_2}.$$

Obviously, \mathbf{C} is a fuzzy β -covering of U , where $\beta \in (0, 0.8]$. By Zadeh's extension principle, we have $f(A_1)(y_1) = \bigvee_{x \in f^{-1}(y_1)} A_1(x) = A_1(x_1) \vee A_1(x_2) = 0.8$, $f(A_1)(y_2) = \bigvee_{x \in f^{-1}(y_2)} A_1(x) = 0$, $f(A_2)(y_1) = \bigvee_{x \in f^{-1}(y_1)} A_2(x) = A_2(x_1) \vee A_2(x_2) = 0.9$ and $f(A_2)(y_2) = \bigvee_{x \in f^{-1}(y_2)} A_2(x) = 0$. It is easy to see that $\{f(A_1), f(A_2)\}$ is not a fuzzy β -covering of V .

Based on the above example, there naturally arises an issue that under what condition $f(\mathbf{C})$ is a fuzzy β -covering of V for a fuzzy β -covering \mathbf{C} of U .

Proposition 1 ([68]) *Let U and V be two universes, $\beta \in (0, 1]$ and $f \in \mathbf{Sur}(U, V)$.*

- (1) *If \mathbf{C} is a fuzzy β -covering on U , then $f(\mathbf{C})$ is also a fuzzy β -covering on V .*
- (2) *\mathbf{P} is a fuzzy β -covering on V if and only if $f^{-1}(\mathbf{P})$ is a fuzzy β -covering on U .*

The converse of (1) of Proposition 1 is not hold. In the following, we can give a necessary condition for that converses of (1) in Proposition 1 holds.

Theorem 2 ([68]) *Let U and V be two universes, $f : U \rightarrow V$ be a bijection from U to V , $\beta \in (0, 1]$, and \mathbf{C} be a family of fuzzy sets on U . Then \mathbf{C} is a fuzzy β -covering on U if and only if $f(\mathbf{C})$ is also a fuzzy β -covering on V .*

Definition 3 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering for some $\beta \in (0, 1]$. For each $x \in U$, we define the fuzzy β -neighborhood \tilde{N}_x^β of x as:

$$\tilde{N}_x^\beta = \cap \{f(C_i) : C_i(x) \geq \beta\}.$$

In [32], fuzzy β -neighborhood of $x \in U$ was defined in (U, \mathbf{C}) . In the following discussions, this definition is defined in (U, V, \mathbf{C}) .

Example 3 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3, y_4\}$ and

$$f : U \rightarrow V, f(x) = \begin{cases} y_1, & x \in \{x_1, x_3\}, \\ y_2, & x \in \{x_2, x_5\}, \\ y_3, & x = x_4, \\ y_4, & x = x_6. \end{cases}$$

Let $\mathbf{C} = \{C_1, C_2, C_3, C_4\}$, where

$$\begin{aligned} C_1 &= \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{1}{x_5} + \frac{0.7}{x_6}, \\ C_2 &= \frac{0.2}{x_1} + \frac{0.9}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.3}{x_5} + \frac{0.5}{x_6}, \\ C_3 &= \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.7}{x_3} + \frac{1}{x_4} + \frac{0.2}{x_5} + \frac{0.7}{x_6}, \\ C_4 &= \frac{0.5}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{0.4}{x_5} + \frac{0.5}{x_6}. \end{aligned}$$

Then by Zadeh's extension principle, we have

$$\begin{aligned} f(C_1) &= \frac{0.4}{y_1} + \frac{1}{y_2} + \frac{0.1}{y_3} + \frac{0.7}{y_4}, f(C_2) = \frac{0.5}{y_1} + \frac{0.9}{y_2} + \frac{0.4}{y_3} + \frac{0.5}{y_4}, \\ f(C_3) &= \frac{0.7}{y_1} + \frac{0.6}{y_2} + \frac{1}{y_3} + \frac{0.7}{y_4}, f(C_4) = \frac{0.5}{y_1} + \frac{0.4}{y_2} + \frac{0.1}{y_3} + \frac{0.5}{y_4}. \end{aligned}$$

It is easy to see that \mathbf{C} and $f(\mathbf{C})$ are fuzzy β -coverings on U and V , respectively, where $\beta \in (0, 0.5]$. Let $\beta = 0.4$. Then

$$\begin{aligned} \tilde{N}_{x_1}^{0.4} &= \frac{0.5}{y_1} + \frac{0.4}{y_2} + \frac{0.1}{y_3} + \frac{0.5}{y_4}, \tilde{N}_{x_2}^{0.4} = \frac{0.4}{y_1} + \frac{0.4}{y_2} + \frac{0.1}{y_3} + \frac{0.5}{y_4}, \\ \tilde{N}_{x_3}^{0.4} &= \frac{0.4}{y_1} + \frac{0.4}{y_2} + \frac{0.1}{y_3} + \frac{0.5}{y_4}, \tilde{N}_{x_4}^{0.4} = \frac{0.5}{y_1} + \frac{0.6}{y_2} + \frac{0.4}{y_3} + \frac{0.5}{y_4}, \\ \tilde{N}_{x_5}^{0.4} &= \frac{0.4}{y_1} + \frac{0.4}{y_2} + \frac{0.1}{y_3} + \frac{0.5}{y_4}, \tilde{N}_{x_6}^{0.4} = \frac{0.4}{y_1} + \frac{0.4}{y_2} + \frac{0.1}{y_3} + \frac{0.5}{y_4}. \end{aligned}$$

Based on the Definition 3, we give the properties of fuzzy β -neighborhood.

Proposition 3 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering for some $\beta \in (0, 1]$. Then the following statements hold.

- (1) $\tilde{N}_x^\beta(f(x)) \geq \beta$ for each $x \in U$.
- (2) Let f be injective. $\forall x, y, z \in U$, if $\tilde{N}_x^\beta(f(y)) \geq \beta$ and $\tilde{N}_y^\beta(f(z)) \geq \beta$, then $\tilde{N}_x^\beta(f(z)) \geq \beta$.
- (3) For each $\beta \in (0, 1]$, there are

$$f(C_i) \supseteq \cup \{\tilde{N}_x^\beta : C_i(x) \geq \beta, x \in U\}, i \in \{1, 2, \dots, m\}.$$

- (4) If $0 < \beta_1 \leq \beta_2 \leq \beta$, then $\tilde{N}_x^{\beta_1} \subseteq \tilde{N}_x^{\beta_2}$ for all $x \in U$.

Proof (1) For each $x \in U$,

$$\begin{aligned}\tilde{N}_x^\beta(f(x)) &= \bigwedge_{C_i(x) \geq \beta} f(C_i)(f(x)) = \bigwedge_{C_i(x) \geq \beta} f(C_i)(f(x)) \\ &= \bigwedge_{C_i(x) \geq \beta} \bigvee_{x' \in f^{-1}(f(x))} C_i(x') \\ &\geq \bigwedge_{C_i(x) \geq \beta} C_i(x) \\ &\geq \beta.\end{aligned}$$

(2) Since $\tilde{N}_x^\beta(f(y)) = \bigwedge_{C_i(x) \geq \beta} C_i(y) \geq \beta$ and $\tilde{N}_y^\beta(f(z)) = \bigwedge_{C_i(z) \geq \beta} C_i(z) \geq \beta$.

Then for each $i \in \{1, 2, \dots, m\}$, $C_i(x) \geq \beta$ implies $C_i(y) \geq \beta$ and $C_i(y) \geq \beta$ implies $C_i(z) \geq \beta$. Hence, for each $i \in \{1, 2, \dots, m\}$, $C_i(x) \geq \beta$ implies $C_i(z) \geq \beta$, i.e., $\tilde{N}_x^\beta(f(z)) = \bigwedge_{C_i(x) \geq \beta} C_i(z) \geq \beta$.

(3) For each $i \in \{1, 2, \dots, m\}$, according to the definition of \tilde{N}_x^β , $C_i(x) \geq \beta$ means $\tilde{N}_x^\beta \subseteq f(C_i)$. Then we have $f(C_i) \supseteq \cup \{\tilde{N}_x^\beta : C_i(x) \geq \beta, x \in U\}$.

(4) For each $x \in U$, $\beta_1 \leq \beta_2$ implies that $\{f(C_i) : C_i(x) \geq \beta_1\} \supseteq \{f(C_i) : C_i(x) \geq \beta_2\}$. Then there is $\tilde{N}_x^{\beta_1} = \bigcap \{f(C_i) : C_i(x) \geq \beta_1\} \subseteq \{f(C_i) : C_i(x) \geq \beta_2\} = \tilde{N}_x^{\beta_2}$ for all $x \in U$. \square

It follows from $0 < \beta_1 \leq \beta_2 \leq \beta$ that \mathbf{C} is a fuzzy β_i -covering of U , $i = 1, 2$. The property (4) of the above theorem indicates the relationship between the fuzzy β_1 -neighborhood and the fuzzy β_2 -neighborhood of x . Furthermore, the relationship between two different fuzzy β -neighborhoods is presented by the following proposition.

Proposition 4 *Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$, f be injective and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering of U for some $\beta \in (0, 1]$. For all $x, y \in U$, $\tilde{N}_x^\beta(f(y)) \geq \beta$ if and only if $\tilde{N}_y^\beta \subseteq \tilde{N}_x^\beta$. So $\tilde{N}_x^\beta(f(y)) \geq \beta$ and $\tilde{N}_y^\beta(f(x)) \geq \beta$ if and only if $\tilde{N}_x^\beta = \tilde{N}_y^\beta$.*

Proof (\Rightarrow): Since $\tilde{N}_x^\beta(f(y)) = \bigwedge_{C_i(x) \geq \beta} f(C_i)(y) = \bigwedge_{C_i(x) \geq \beta} f(C_i)(f(y)) \geq \beta$, we have that $\{f(C_i) \in f(\mathbf{C}) : C_i(x) \geq \beta\} \subseteq \{f(C_i) \in f(\mathbf{C}) : C_i(y) \geq \beta\}$. So $\tilde{N}_x^\beta(f(z)) = \bigwedge_{C_i(x) \geq \beta} f(C_i)(f(z)) \geq \bigwedge_{C_i(y) \geq \beta} f(C_i)(f(z)) = \tilde{N}_y^\beta(f(z))$ holds for each $z \in U$. Therefore $\tilde{N}_y^\beta \subseteq \tilde{N}_x^\beta$.

(\Leftarrow): For all $x, y \in U$, if $\tilde{N}_y^\beta \subseteq \tilde{N}_x^\beta$, then $\beta \leq \tilde{N}_y^\beta(f(y)) \leq \tilde{N}_x^\beta(f(y))$. \square

Now, we define the β -neighborhood $\overline{N}_x^\beta : U \rightarrow V$ of $x \in U$.

Definition 4 *Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering for some $\beta \in (0, 1]$. For each $x \in U$, we define the β -neighborhood \overline{N}_x^β of x as:*

$$\overline{N}_x^\beta = \{y \in V : \tilde{N}_x^\beta(y) \geq \beta\}.$$

In [32], β -neighborhood of $x \in U$ was defined in (U, \mathbf{C}) . In the following discussions, this definition is defined in (U, V, \mathbf{C}) .

Example 4 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3, y_4\}$ and

$$f : U \rightarrow V, f(x) = \begin{cases} y_1, & x \in \{x_1, x_3\}, \\ y_2, & x \in \{x_2, x_5\}, \\ y_3, & x = x_4, \\ y_4, & x = x_6. \end{cases}$$

Let (U, \mathbf{C}) be the fuzzy β -covering approximation space in Example 3 and $\beta = 0.4$. Then

$$\overline{N}_{x_1}^{0.4} = \overline{N}_{x_2}^{0.4} = \overline{N}_{x_3}^{0.4} = \overline{N}_{x_5}^{0.4} = \overline{N}_{x_6}^{0.4} = \{y_1, y_2, y_4\}, \overline{N}_{x_4}^{0.4} = \{y_1, y_2, y_3, y_4\}.$$

We have the following properties of the β -neighborhoods in a fuzzy β -covering approximation space (U, \mathbf{C}) .

Proposition 5 *Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering for some $\beta \in (0, 1]$. Then the following statements hold.*

- (1) $f(x) \in \overline{N}_x^\beta$ for each $x \in U$.
- (2) Let f be injective. $\forall x, y \in U$, if $f(x) \in \overline{N}_y^\beta$, then $\overline{N}_x^\beta \subseteq \overline{N}_y^\beta$.

Proof (1) For each $x \in U$, it follows from $\overline{N}_x^\beta = \{y \in V : \tilde{N}_x^\beta(y) \geq \beta\}$ and $\tilde{N}_x^\beta(f(x)) \geq \beta$ that $f(x) \in \overline{N}_x^\beta$.

(2) For any $z \in \overline{N}_x^\beta$, we have $\tilde{N}_x^\beta(z) \geq \beta$. If $f(x) \in \overline{N}_y^\beta$, then $\tilde{N}_y^\beta(f(x)) \geq \beta$. According to Proposition 3, there is $\tilde{N}_y^\beta(z) \geq \beta$, then $z \in \overline{N}_y^\beta$. Hence $\forall x, y \in U$, if $f(x) \in \overline{N}_y^\beta$. \square

In fact, (2) of Proposition 5 can also be presented in the following remark.

Remark 1 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$, f be injective and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering for some $\beta \in (0, 1]$. For all $x, y \in U$, $f(x) \in \overline{N}_y^\beta$ if and only if $\overline{N}_x^\beta \subseteq \overline{N}_y^\beta$. So $f(x) \in \overline{N}_y^\beta$ and $f(y) \in \overline{N}_x^\beta$ if and only if $\overline{N}_x^\beta = \overline{N}_y^\beta$. The proof of this conclusion is straightforward.

Now, we give another property of the β -neighborhoods in the fuzzy β -covering approximation space.

Proposition 6 *Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$, f be injective and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering for some $\beta \in (0, 1]$. For all $x, y, z \in U$, if $f(x) \in \overline{N}_y^\beta$ and $f(y) \in \overline{N}_z^\beta$, then $f(x) \in \overline{N}_z^\beta$.*

Proof For all $x, y, z \in U$, $f(x) \in \overline{N}_y^\beta \Leftrightarrow \tilde{N}_y^\beta(f(x)) \geq \beta \Leftrightarrow \tilde{N}_x^\beta \subseteq \tilde{N}_y^\beta$ and $f(y) \in \overline{N}_z^\beta \Leftrightarrow \tilde{N}_z^\beta(f(y)) \geq \beta \Leftrightarrow \tilde{N}_y^\beta \subseteq \tilde{N}_z^\beta$, then $\tilde{N}_x^\beta \subseteq \tilde{N}_y^\beta \subseteq \tilde{N}_z^\beta$. Thus $\beta \leq \tilde{N}_x^\beta(f(x)) \leq \tilde{N}_z^\beta(f(x))$, i.e., $f(x) \in \overline{N}_z^\beta$. This completes the proof. \square

By Proposition 5 and Remark 1, the relationships between fuzzy β -neighborhood and β -neighborhood can be proposed as follows.

Proposition 7 *Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$, f be injective and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering for some $\beta \in (0, 1]$. For any $x, y \in U$, $\tilde{N}_x^\beta \subseteq \tilde{N}_y^\beta$ if and only if $\overline{N}_x^\beta \subseteq \overline{N}_y^\beta$.*

Proof For any $x, y \in U$, $\tilde{N}_x^\beta \subseteq \tilde{N}_y^\beta \Leftrightarrow \tilde{N}_y^\beta(f(x)) \geq \beta \Leftrightarrow f(x) \in \overline{N}_y^\beta \Leftrightarrow \overline{N}_x^\beta \subseteq \overline{N}_y^\beta$. \square

3.2 Constructions of fuzzy covering-based approximation operators over two universes

In this subsection, a new type of fuzzy covering-based rough set model over two universes for fuzzy subsets is defined and its properties are investigated.

Definition 5 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and \mathbf{C} be a fuzzy β -covering on U for some $\beta \in (0, 1]$. For each $X \in \mathcal{F}(V)$, we define the fuzzy covering lower approximation $\underline{\mathbf{C}}(X)$ and the upper approximation $\overline{\mathbf{C}}(X)$ of X as:

$$\begin{aligned}\underline{\mathbf{C}}(X)(x) &= \bigwedge_{y \in V} [(1 - \tilde{N}_x^\beta(y)) \vee X(y)], x \in U, \\ \overline{\mathbf{C}}(X)(x) &= \bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge X(y)], x \in U.\end{aligned}$$

If $\underline{\mathbf{C}}(X) \neq \overline{\mathbf{C}}(X)$, then X is called a fuzzy covering-based rough set.

Example 5 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3\}$ and

$$f : U \rightarrow V, f(x) = \begin{cases} y_1, & x \in \{x_1, x_2, x_3\}, \\ y_2, & x \in \{x_4, x_5\}, \\ y_3, & x = x_6. \end{cases}$$

Let $\mathbf{C} = \{C_1, C_2, C_3, C_4\}$, where

$$\begin{aligned}C_1 &= \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.5}{x_5} + \frac{0.5}{x_6}, \\ C_2 &= \frac{0.7}{x_1} + \frac{0.6}{x_2} + \frac{0.8}{x_3} + \frac{0.8}{x_4} + \frac{0.6}{x_5} + \frac{0.7}{x_6}, \\ C_3 &= \frac{0.3}{x_1} + \frac{0.4}{x_2} + \frac{0.1}{x_3} + \frac{0.7}{x_4} + \frac{0.9}{x_5} + \frac{0.7}{x_6}, \\ C_4 &= \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.5}{x_5} + \frac{0.3}{x_6}.\end{aligned}$$

Then by Zadeh's extension principle, we have

$$f(C_1) = \frac{0.5}{y_1} + \frac{0.7}{y_2} + \frac{0.5}{y_3}, f(C_2) = \frac{0.8}{y_1} + \frac{0.8}{y_2} + \frac{0.7}{y_3},$$

$$f(C_3) = \frac{0.4}{y_1} + \frac{0.9}{y_2} + \frac{0.7}{y_3}, f(C_4) = \frac{0.7}{y_1} + \frac{0.5}{y_2} + \frac{0.3}{y_3}.$$

It is easy to see that \mathbf{C} and $f(\mathbf{C})$ are fuzzy β -coverings on U and V , respectively, where $\beta \in (0, 0.7]$. Let $\beta = 0.5$. Then

$$\begin{aligned}\tilde{N}_{x_1}^{0.5} &= \frac{0.5}{y_1} + \frac{0.5}{y_2} + \frac{0.3}{y_3}, \tilde{N}_{x_2}^{0.5} = \frac{0.7}{y_1} + \frac{0.5}{y_2} + \frac{0.3}{y_3}, \tilde{N}_{x_3}^{0.5} = \frac{0.8}{y_1} + \frac{0.8}{y_2} + \frac{0.7}{y_3}, \\ \tilde{N}_{x_4}^{0.5} &= \frac{0.4}{y_1} + \frac{0.7}{y_2} + \frac{0.5}{y_3}, \tilde{N}_{x_5}^{0.5} = \frac{0.4}{y_1} + \frac{0.5}{y_2} + \frac{0.3}{y_3}, \tilde{N}_{x_6}^{0.5} = \frac{0.4}{y_1} + \frac{0.7}{y_2} + \frac{0.5}{y_3}.\end{aligned}$$

For $X = \frac{0.3}{y_1} + \frac{0.7}{y_2} + \frac{0.5}{y_3}$, we have

$$\begin{aligned}\underline{\mathbf{C}}(X) &= \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.5}{x_6}, \\ \overline{\mathbf{C}}(X) &= \frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{0.7}{x_3} + \frac{0.7}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}, \\ \underline{\mathbf{C}}(\emptyset) &= \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.2}{x_3} + \frac{0.3}{x_4} + \frac{0.5}{x_5} + \frac{0.3}{x_6}, \\ \overline{\mathbf{C}}(V) &= \frac{0.5}{x_1} + \frac{0.7}{x_2} + \frac{0.8}{x_3} + \frac{0.7}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}.\end{aligned}$$

Some properties of this fuzzy covering-based rough set model can be proposed through the following proposition.

Proposition 8 *Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering on U for some $\beta \in (0, 1]$. For each $X, Y \in \mathcal{F}(V)$, we have the following statements.*

- (1) $\underline{\mathbf{C}}(X^c) = (\overline{\mathbf{C}}(X))^c, \overline{\mathbf{C}}(X^c) = (\underline{\mathbf{C}}(X))^c$.
- (2) $\underline{\mathbf{C}}(V) = U, \overline{\mathbf{C}}(\emptyset) = \emptyset$.
- (3) $\underline{\mathbf{C}}(X \cap Y) = \underline{\mathbf{C}}(X) \cap \underline{\mathbf{C}}(Y), \overline{\mathbf{C}}(X \cup Y) = \overline{\mathbf{C}}(X) \cup \overline{\mathbf{C}}(Y)$.
- (4) If $X \subseteq Y$, then $\underline{\mathbf{C}}(X) \subseteq \underline{\mathbf{C}}(Y), \overline{\mathbf{C}}(X) \subseteq \overline{\mathbf{C}}(Y)$.
- (5) $\underline{\mathbf{C}}(X \cup Y) \supseteq \underline{\mathbf{C}}(X) \cup \underline{\mathbf{C}}(Y), \overline{\mathbf{C}}(X \cap Y) \subseteq \overline{\mathbf{C}}(X) \cap \overline{\mathbf{C}}(Y)$.
- (6) For each $x \in U$, if $1 - \tilde{N}_x^\beta(y) \leq X(y) \leq \tilde{N}_x^\beta(y)$ for all $y \in V$, then $\underline{\mathbf{C}}(X) \subseteq \overline{\mathbf{C}}(X)$.

Proof (1) For each $x \in U$,

$$\begin{aligned}\overline{\mathbf{C}}(X^c)(x) &= \bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge X^c(y)] = 1 - \bigwedge_{y \in V} [(1 - \tilde{N}_x^\beta(y)) \vee X(y)] \\ &= 1 - \underline{\mathbf{C}}(X)(x) \\ &= (\underline{\mathbf{C}}(X))^c(x).\end{aligned}$$

Then $\overline{\mathbf{C}}(X^c) = (\underline{\mathbf{C}}(X))^c$. Replacing X by X^c in this proof, we can obtain $\underline{\mathbf{C}}(X^c) = (\overline{\mathbf{C}}(X))^c$.

(2) It follows from $V(y) = 1$ and $\emptyset(y) = 0$ for all $y \in V$ that for each $x \in U$,

$$\begin{aligned}\underline{\mathbf{C}}(V)(x) &= \bigwedge_{y \in V} [(1 - \tilde{N}_x^\beta(y)) \vee V(y)] = 1 = U(x), \\ \overline{\mathbf{C}}(\emptyset)(x) &= \bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge \emptyset(y)] = 0 = \emptyset(x).\end{aligned}$$

Then $\underline{\mathbf{C}}(V) = U, \overline{\mathbf{C}}(\emptyset) = \emptyset$.

(3) For each $x \in U$, we have

$$\begin{aligned}\underline{\mathbf{C}}(X \cap Y)(x) &= \bigwedge_{y \in V} [(1 - \tilde{N}_x^\beta(y)) \vee (X \cap Y)(y)] \\ &= \bigwedge_{y \in V} [((1 - \tilde{N}_x^\beta(y)) \vee X(y)) \wedge ((1 - \tilde{N}_x^\beta(y)) \vee Y(y))] \\ &= \underline{\mathbf{C}}(X)(x) \cap \underline{\mathbf{C}}(Y)(x)\end{aligned}$$

and

$$\begin{aligned}\overline{\mathbf{C}}(X \cup Y)(x) &= \bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge (X \cup Y)(y)] \\ &= \bigvee_{y \in V} [(\tilde{N}_x^\beta(y) \wedge X(y)) \vee (\tilde{N}_x^\beta(y) \wedge Y(y))] \\ &= \overline{\mathbf{C}}(X)(x) \cup \overline{\mathbf{C}}(Y)(x).\end{aligned}$$

Then there are $\underline{\mathbf{C}}(X \cap Y) = \underline{\mathbf{C}}(X) \cap \underline{\mathbf{C}}(Y)$, $\overline{\mathbf{C}}(X \cup Y) = \overline{\mathbf{C}}(X) \cup \overline{\mathbf{C}}(Y)$.

(4) If $X \subseteq Y$, then $X(y) \leq Y(y)$ for each $y \in V$. For each $x \in U$, we have

$$\underline{\mathbf{C}}(X)(x) = \bigwedge_{y \in V} [(1 - \tilde{N}_x^\beta(y)) \vee X(y)] \leq \bigwedge_{y \in V} [(1 - \tilde{N}_x^\beta(y)) \vee Y(y)] = \underline{\mathbf{C}}(Y)(x)$$

and

$$\overline{\mathbf{C}}(X)(x) = \bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge X(y)] \leq \bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge Y(y)] = \overline{\mathbf{C}}(Y)(x).$$

Then there are $\underline{\mathbf{C}}(X) \subseteq \underline{\mathbf{C}}(Y)$ and $\overline{\mathbf{C}}(X) \subseteq \overline{\mathbf{C}}(Y)$.

(5) Since $X \subseteq X \cup Y$, $Y \subseteq X \cup Y$, $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, it follows from (4) that $\underline{\mathbf{C}}(X) \subseteq \underline{\mathbf{C}}(X \cup Y)$, $\underline{\mathbf{C}}(Y) \subseteq \underline{\mathbf{C}}(X \cup Y)$, $\overline{\mathbf{C}}(X \cap Y) \subseteq \overline{\mathbf{C}}(X)$ and $\overline{\mathbf{C}}(X \cap Y) \subseteq \overline{\mathbf{C}}(Y)$. Therefore $\underline{\mathbf{C}}(X \cup Y) \supseteq \underline{\mathbf{C}}(X) \cup \underline{\mathbf{C}}(Y)$, $\overline{\mathbf{C}}(X \cap Y) \subseteq \overline{\mathbf{C}}(X) \cap \overline{\mathbf{C}}(Y)$.

(6) For each $x \in U$, there is $1 - \tilde{N}_x^\beta(y) \leq X(y) \leq \tilde{N}_x^\beta(y)$ for all $y \in V$, then $X(y) = \tilde{N}_x^\beta(y) \wedge X(y) \leq \bigvee_{y' \in V} [\tilde{N}_x^\beta(y) \wedge X(y)] = \overline{\mathbf{C}}(X)(x)$ and $X(y) = (1 - \tilde{N}_x^\beta(y)) \vee X(y) \geq \bigwedge_{y' \in V} [(1 - \tilde{N}_x^\beta(y)) \vee X(y)] = \underline{\mathbf{C}}(X)(x)$. Therefore $\underline{\mathbf{C}}(X) \subseteq \overline{\mathbf{C}}(X)$. \square

Moreover, this fuzzy covering-based rough set model also has the following properties.

Proposition 9 *Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering on U for some $\beta \in (0, 1]$. For each $X \in \mathcal{F}(V)$ and $\alpha \in [0, 1]$, we have the following statements.*

(1) $\underline{\mathbf{C}}(X \cup \alpha_V) = \underline{\mathbf{C}}(X) \cup \alpha_U$.

(2) $\overline{\mathbf{C}}(X \cap \alpha_V) = \overline{\mathbf{C}}(X) \cap \alpha_U$.

Proof (1) For each $x \in U$,

$$\begin{aligned}\underline{\mathbf{C}}(X \cup \alpha_V)(x) &= \bigwedge_{y \in V} [(1 - \tilde{N}_x^\beta(y)) \vee (X \cup \alpha_V)(y)] \\ &= \bigwedge_{y \in V} [(1 - \tilde{N}_x^\beta(y)) \vee X(y) \vee \alpha] \\ &= (\bigwedge_{y \in V} [(1 - \tilde{N}_x^\beta(y)) \vee X(y)]) \vee (\bigwedge_{y \in V} [(1 - \tilde{N}_x^\beta(y)) \vee \alpha]) \\ &= \underline{\mathbf{C}}(X)(x) \vee \alpha.\end{aligned}$$

Then $\underline{\mathbf{C}}(X \cup \alpha_V) = \underline{\mathbf{C}}(X) \cup \alpha_U$.

(2) For each $x \in U$,

$$\begin{aligned} \overline{\mathbf{C}}(X \cap \alpha_V)(x) &= \bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge (X \cap \alpha_V)(y)] \\ &= \bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge X(y) \wedge \alpha] \\ &= (\bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge X(y)]) \wedge (\bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge \alpha]) \\ &= \overline{\mathbf{C}}(X)(x) \wedge \alpha. \end{aligned}$$

Then $\overline{\mathbf{C}}(X \cap \alpha_V) = \overline{\mathbf{C}}(X) \cap \alpha_U$. \square

Proposition 10 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering on U for some $\beta \in (0, 1]$. For any $x \in U, y \in V$ and $X \in \mathcal{F}(V)$, the following statements hold.

- (1) $\overline{\mathbf{C}}(1_y)(x) = \tilde{N}_x^\beta(y)$.
- (2) $\underline{\mathbf{C}}(1_{V-\{y\}})(x) = 1 - \tilde{N}_x^\beta(y)$.
- (3) $\overline{\mathbf{C}}(1_X)(x) = \bigvee_{y \in X} \tilde{N}_x^\beta(y)$.
- (4) $\underline{\mathbf{C}}(1_X)(x) = \bigwedge_{y \notin X} (1 - \tilde{N}_x^\beta(y))$.

Proof (1) For any $x', y \in V$, it follows from the definition of 1_y that $1_y(x') = 0$ for $x' \neq y$. Therefore

$$\overline{\mathbf{C}}(1_y)(x) = \bigvee_{z \in U} [\tilde{N}_x^\beta(z) \wedge 1_y(z)] = \tilde{N}_x^\beta(y).$$

(2) It follows immediately from (1) and the duality.

(3) For any $y \in V$ and $X \in \mathcal{F}(V)$, it follows from the definition of 1_X that $1_X(y) = 0$ if and only if $y \notin X$. Therefore

$$\begin{aligned} \overline{\mathbf{C}}(1_X)(x) &= \bigvee_{y \in U} [\tilde{N}_x^\beta(y) \wedge 1_X(y)] \\ &= \bigvee_{y \in X} [\tilde{N}_x^\beta(y) \wedge 1_X(y)] \vee (\bigvee_{y \notin X} [\tilde{N}_x^\beta(y) \wedge 1_X(y)]) \\ &= \bigvee_{y \in X} \tilde{N}_x^\beta(y). \end{aligned}$$

(4) It follows immediately from (3) and the duality. \square

By Example 5, it is easy to find that $\underline{\mathbf{C}}(\emptyset) \neq \emptyset$ and $\overline{\mathbf{C}}(V) \neq U$. In fact, by Definition 5, we can see that $\underline{\mathbf{C}}(\emptyset)(x) = \bigwedge_{y \in V} [1 - \tilde{N}_x(y)]$ and $\overline{\mathbf{C}}(V)(x) = \bigvee_{y \in V} \tilde{N}_x(y)$. Therefore there exists an issue that under what condition $\underline{\mathbf{C}}(\emptyset) = \emptyset$ and $\overline{\mathbf{C}}(V) = U$.

Proposition 11 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering on U for some $\beta \in (0, 1]$. $\underline{\mathbf{C}}(\emptyset) = \emptyset$ if and only if $\{y \in V : \forall C_i \in \mathbf{C} ((C_i(x) \geq \beta) \rightarrow (f(C_i)(y) = 1))\} \neq \emptyset$ holds for all $x \in U$.

Proof For each $x \in U$,

$$\begin{aligned}\underline{\mathbf{C}}(\emptyset)(x) = 0 &\Leftrightarrow \overline{\mathbf{C}}(V)(x) = 1 \\ &\Leftrightarrow \bigvee_{y \in V} \tilde{N}_x(y) = 1 \\ &\Leftrightarrow \{y \in V : \forall C_i \in \mathbf{C}((C_i(x) \geq \beta) \rightarrow (f(C_i)(y) = 1))\} \neq \emptyset.\end{aligned}$$

This completes the proof. \square

This proposition shows a necessary and sufficient condition that $\underline{\mathbf{C}}(\emptyset) = \emptyset$ and $\overline{\mathbf{C}}(V) = U$. Moreover, based on this proposition and the notion of β -neighborhood, another necessary and sufficient condition that $\underline{\mathbf{C}}(\emptyset) = \emptyset$ and $\overline{\mathbf{C}}(V) = U$ can be presented in the following corollary.

Corollary 1 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and \mathbf{C} be a fuzzy β -covering on U for some $\beta \in (0, 1]$. For each $x \in U$, $\overline{\mathbf{C}}(V)(x) = 1$ if and only if $\overline{N}_x^1 \neq \emptyset$.

To illustrate this corollary, let us see an example.

Example 6 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3\}$ and

$$f : U \rightarrow V, f(x) = \begin{cases} y_1, & x \in \{x_1, x_2, x_3\}, \\ y_2, & x \in \{x_4, x_5\}, \\ y_3, & x = x_6. \end{cases}$$

Let $\mathbf{C} = \{C_1, C_2, C_3, C_4\}$, where

$$\begin{aligned}C_1 &= \frac{0.7}{x_1} + \frac{0.4}{x_2} + \frac{1}{x_3} + \frac{0.3}{x_4} + \frac{0.6}{x_5} + \frac{1}{x_6}, C_2 = \frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{1}{x_6}, \\ C_3 &= \frac{0.8}{x_1} + \frac{0.2}{x_2} + \frac{1}{x_3} + \frac{0.5}{x_4} + \frac{1}{x_5} + \frac{1}{x_6}, C_4 = \frac{0.1}{x_1} + \frac{0.7}{x_2} + \frac{0.4}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{1}{x_6}.\end{aligned}$$

Then by Zadeh's extension principle, we have

$$\begin{aligned}f(C_1) &= \frac{1}{y_1} + \frac{0.6}{y_2} + \frac{1}{y_3}, f(C_2) = \frac{0.5}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}, \\ f(C_3) &= \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}, f(C_4) = \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}.\end{aligned}$$

It is easy to see that \mathbf{C} and $f(\mathbf{C})$ are fuzzy β -coverings on U and V , respectively, where $\beta \in (0, 0.7]$. Let $\beta = 0.7$. Then

$$\begin{aligned}\tilde{N}_{x_1}^{0.7} &= \frac{1}{y_1} + \frac{0.6}{y_2} + \frac{1}{y_3}, \tilde{N}_{x_2}^{0.7} = \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}, \tilde{N}_{x_3}^{0.7} = \frac{1}{y_1} + \frac{0.6}{y_2} + \frac{1}{y_3}, \\ \tilde{N}_{x_4}^{0.7} &= \frac{0.5}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}, \tilde{N}_{x_5}^{0.7} = \frac{0.5}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}, \tilde{N}_{x_6}^{0.7} = \frac{0.5}{y_1} + \frac{0.6}{y_2} + \frac{1}{y_3}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\overline{N}_{x_1}^1 &= \overline{N}_{x_3}^1 = \{y_1, y_3\}, \overline{N}_{x_2}^1 = \overline{N}_{x_4}^1 = \overline{N}_{x_5}^1 = \{y_2, y_3\}, \\ \overline{N}_{x_6}^1 &= \{y_3\}, \overline{\mathbf{C}}(V) = U, \underline{\mathbf{C}}(\emptyset) = \emptyset.\end{aligned}$$

3.3 Axiomatic characterizations of fuzzy covering-based approximation operators

Now, we need to know which are the characteristic properties for the fuzzy covering lower and upper approximation operations.

Definition 6 Let $L, H : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ be two operators. They are referred to as dual operators if for all $X \in \mathcal{F}(U)$,

$$(CL1) \quad L(X) = (H(X^c))^c,$$

$$(CH1) \quad H(X) = (L(X^c))^c.$$

Proposition 12 Suppose that $L, H : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ are two dual operators, $f \in \text{Sur}(U, V)$ and $\beta \in (0, 1]$. Then there exists a fuzzy β -covering \mathbf{C} of U such that for all $X \in \mathcal{F}(V)$, $L(X) = \underline{\mathbf{C}}(X)$ and $H(X) = \overline{\mathbf{C}}(X)$ if and only if L and H satisfy the axioms: for all $X, Y \in \mathcal{F}(V)$ and $\alpha \in [0, 1]$,

$$(CL2) \quad L(\alpha_V \cup X) = \alpha_U \cup L(X).$$

$$(CH2) \quad H(\alpha_V \cap X) = \alpha_U \cap H(X).$$

$$(CL3) \quad L(X \cap Y) = L(X) \cap L(Y).$$

$$(CH3) \quad H(X \cap Y) = H(X) \cup H(Y).$$

$$(CL4) \quad L(\emptyset)(x) \leq 1 - \beta \text{ for all } x \in U.$$

$$(CH4) \quad H(V)(x) \geq \beta \text{ for all } x \in U.$$

Proof (\Rightarrow): It follows immediately from Propositions 8, 9 and 10.

(\Leftarrow): Let $U = \{x_1, x_2, \dots, x_n\}$. Suppose that the operator H obeys the axioms (CH2), (CH3) and (CH4). Using H , we can define a fuzzy subset and a family of fuzzy subsets of U by:

$$C_i = \{H(1_y)(x_i) : y \in V\}, i = 1, 2, \dots, n \text{ and } \mathbf{C}' = \{C_i : i \in \{1, 2, \dots, n\}\}.$$

Then we have that $(\bigcup_{i=1}^n C_i)(y) \geq \beta$ for each $y \in U$ and $\mathbf{C}' = \{C_i : i \in \{1, 2, \dots, n\}\}$

must be a fuzzy β -covering of U by (CH4). Then $f^{-1}(\mathbf{C}')$ is the fuzzy β -covering on U . Let $\mathbf{C} = f^{-1}(\mathbf{C}')$. For each $X \in \mathcal{F}(V)$, there is $X = \bigcup_{y \in V} (1_y \cap (X(y))_V)$. For

each $x \in U$, by (CH2) and (CH3) we have that

$$\begin{aligned} \overline{\mathbf{C}}(X)(x) &= \overline{\mathbf{C}}(\bigcup_{y \in V} (1_y \cap (X(y))_V))(x) \\ &= [\bigcup_{y \in V} \overline{\mathbf{C}}(1_y \cap (X(y))_V)](x) \\ &= [\bigcup_{y \in V} \overline{\mathbf{C}}(1_y \cap (X(y))_V)](x) \\ &= \bigvee_{y \in V} [\overline{\mathbf{C}}(1_y)(x) \wedge X(y)] \\ &= \bigvee_{y \in V} [\tilde{N}_x^\beta(y) \wedge X(y)] \\ &= \bigvee_{y \in V} [C_i(y) \wedge X(y)], C_i = \tilde{N}_x^\beta, i \in \{1, 2, \dots, n\} \\ &= \bigvee_{y \in V} [H(1_y)(x) \wedge X(y)] \end{aligned}$$

$$\begin{aligned}
&= H[\bigcup_{y \in V} (1_y \cap (X(y))_V)](x) \\
&= H(X)(x),
\end{aligned}$$

which implies that $\overline{\mathbf{C}}(X) = H(X)$.

It follows immediately from the conclusion $\overline{\mathbf{C}}(X) = H(X)$ and the assumption that $\underline{\mathbf{C}}(X) = L(X)$ holds. This completes the proof of this proposition. \square

Finally, we propose another fuzzy covering-based rough set model for crisp subsets, which is defined as follows.

Definition 7 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and \mathbf{C} be a fuzzy β -covering on U for some $\beta \in (0, 1]$. For each $X \in \mathcal{P}(V)$, we define the lower approximation $\underline{\mathcal{C}}(X)$ and the upper approximation $\overline{\mathcal{C}}(X)$ of X as:

$$\begin{aligned}
\underline{\mathcal{C}}(X) &= \{x \in U : \overline{N}_x^\beta \subseteq X\}, \\
\overline{\mathcal{C}}(X) &= \{x \in U : \overline{N}_x^\beta \cap X \neq \emptyset\}.
\end{aligned}$$

If $\underline{\mathcal{C}}(X) \neq \overline{\mathcal{C}}(X)$, then X is called a fuzzy covering-based rough set.

Example 7 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3, y_4\}$ and

$$f : U \rightarrow V, f(x) = \begin{cases} y_1, & x \in \{x_1, x_3\}, \\ y_2, & x \in \{x_2, x_5\}, \\ y_3, & x = x_4, \\ y_4, & x = x_6. \end{cases}$$

Let (U, \mathbf{C}) be the fuzzy β -covering approximation space in Example 3 and $\beta = 0.4$. Then

$$\overline{N}_{x_1}^{0.4} = \overline{N}_{x_2}^{0.4} = \overline{N}_{x_3}^{0.4} = \overline{N}_{x_5}^{0.4} = \overline{N}_{x_6}^{0.4} = \{y_1, y_2, y_4\}, \overline{N}_{x_4}^{0.4} = \{y_1, y_2, y_3, y_4\}.$$

Let $X = \{y_3\}$ and $Y = \{y_1, y_2, y_4\}$. Then

$$\begin{aligned}
\underline{\mathcal{C}}(X) &= \emptyset, \overline{\mathcal{C}}(X) = \{x_4\}, \underline{\mathcal{C}}(Y) = \{x_1, x_2, x_3, x_5, x_6\}, \\
\overline{\mathcal{C}}(Y) &= \{x_1, x_2, x_3, x_4, x_5, x_6\}, \underline{\mathcal{C}}(\emptyset) = \overline{\mathcal{C}}(\emptyset) = \emptyset, \underline{\mathcal{C}}(V) = \overline{\mathcal{C}}(V) = U.
\end{aligned}$$

The following proposition shows that the relationship between two fuzzy covering-based rough set models which are defined in Definitions 5 and 7.

Proposition 13 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C}_1, \mathbf{C}_2$ be two fuzzy β -coverings of U for some $\beta \in (0, 1]$. For each $X \in \mathcal{P}(V)$, we have the following statements.

- (1) $\underline{\mathbf{C}}(1_X) = \underline{\mathcal{C}}(X)$.
- (2) $\overline{\mathbf{C}}(1_X) = \overline{\mathcal{C}}(X)$.

Proof It is straightforward. \square

4 Interdependency and matrix representations on fuzzy covering-based approximation operators

Furthermore, we explore the conditions under which two fuzzy β -coverings generate the same fuzzy covering lower or fuzzy covering upper approximations. Finally, matrix representations of the fuzzy covering lower and upper approximation operators are investigated.

4.1 Interdependency of fuzzy covering-based approximation operators

The focus of this subsection is on the condition under which two fuzzy β -coverings generate the same fuzzy covering-based lower approximation or the same fuzzy covering-based upper approximation of a fuzzy subset. To solve the issue of under what conditions two coverings generate the same covering-based lower approximation or the same covering-based upper approximation, Zhu and Wang firstly proposed the concept of reducible element in 2003 [88]. In order to obtain a condition under which two fuzzy β -coverings generate the same fuzzy covering-based lower and upper approximations, we also need to use this concept.

Definition 8 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$, \mathbf{C} be a fuzzy β -covering on U for some $\beta \in (0, 1]$, and $C \in \mathbf{C}$. If one of the following statements holds:

- (1) $C(x) < \beta$ for all $x \in U$,
- (2) for $x \in U$, $C(x) \geq \beta$ implies there exists $C' \in \mathbf{C} - \{C\}$ such that $C' \subseteq C$ and $C'(x) \geq \beta$,

then C is called a β -reducible element of \mathbf{C} , otherwise C is a β -irreducible element of \mathbf{C} .

Example 8 Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $\mathbf{C} = \{C_1, C_2, C_3, C_4, C_5, C_6\}$, where

$$\begin{aligned} C_1 &= \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.4}{x_3} + \frac{0.6}{x_4} + \frac{0.3}{x_5}, C_2 = \frac{0.4}{x_1} + \frac{0.8}{x_2} + \frac{0.4}{x_3} + \frac{0.6}{x_4} + \frac{0.1}{x_5}, \\ C_3 &= \frac{0.5}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0.7}{x_5}, C_4 = \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.1}{x_3} + \frac{0.3}{x_4} + \frac{0.3}{x_5}, \\ C_5 &= \frac{0.3}{x_1} + \frac{0.4}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{0.4}{x_5}, C_6 = \frac{0.4}{x_1} + \frac{0.9}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.3}{x_5}. \end{aligned}$$

It is easy to see that \mathbf{C} is a fuzzy β -coverings on U , where $\beta \in (0, 0.5]$. Let $\beta = 0.5$. It follows from $C_5(x) < 0.5$ for $x \in U$ that C_5 is a 0.5-reducible element of \mathbf{C} . Meanwhile, since $C_6(x_i) \geq 0.5, i = 2, 4$, $C_6(x_i) < 0.5, i = 1, 3, 5$, $C_2(x_i) \geq 0.5, i = 2, 4$ and $C_2 \subseteq C_6$, we also have that C_6 is a 0.5-reducible element of \mathbf{C} .

Based on the above example, we see that $\mathbf{C} - \{C_5, C_6\}$ is also a fuzzy β -covering of U , where $\beta \in (0, 0.5]$.

Proposition 14 *Let \mathbf{C} be a fuzzy β -covering of U for some $\beta \in (0, 1]$. If C is a β -reducible element of \mathbf{C} , then $\mathbf{C} - \{C\}$ is still a fuzzy β -covering of U .*

Proof Suppose $\mathbf{C} = \{C, C_1, C_2, \dots, C_m\}$, where $C, C_i \in \mathcal{F}(U)$ ($i = 1, 2, \dots, m$). If C is a β -reducible element of \mathbf{C} , then we have the following two cases:

Case (1). $C(x) < \beta$ for all $x \in U$ and

Case (2). For $x \in U$, if $C(x) \geq \beta$, then there exists $C_r \in \{C_1, C_2, \dots, C_m\}$ such that $C_r \subseteq C$ and $C_r(x) \geq \beta$.

For Case (1), we have that $((\bigcup_{j=1}^m C_j) \cup C)(x) = (\bigcup_{j=1}^m C_j)(x) \geq \beta$ for any $x \in U$.

For Case (2), for any $x \in U$, if $C(x) \geq \beta$, then $C_r(x) \geq \beta$, i.e., $(\bigcup_{j=1}^m C_j)(x) \geq C_r(x) \geq \beta$. If $C(x) < \beta$, then $((\bigcup_{j=1}^m C_j) \cup C)(x) = (\bigcup_{j=1}^m C_j)(x) \geq \beta$.

Therefore $\mathbf{C} - \{C\}$ is a fuzzy β -covering of U . \square

Proposition 15 *Let \mathbf{C} be a fuzzy β -covering of U for some $\beta \in (0, 1]$, C be a β -reducible element of \mathbf{C} , and $C_1 \in \mathbf{C} - \{C\}$. Then C_1 is a β -reducible element of \mathbf{C} if and only if it is a β -reducible element of $\mathbf{C} - \{C\}$.*

Proof (\Rightarrow): Suppose $\widehat{\mathbf{C}} = \{C, C_1, C_2, \dots, C_m\}$, where $C, C_i \in \mathcal{F}(U)$ ($i = 1, 2, \dots, m$). If C is a β -reducible element of $\widehat{\mathbf{C}}$, then we have the following two cases:

Case (1). $C(x) < \beta$ for all $x \in U$ and

Case (2). For $x \in U$, if $C(x) \geq \beta$, then there exists $C_r \in \{C_1, C_2, \dots, C_m\}$ such that $C_r \subseteq C$ and $C_r(x) \geq \beta$.

For Case (1), if C_1 is a β -reducible element of \mathbf{C} , then $C_1(y) < \beta$ for all $y \in U$ or for $y \in U$, if $C_1(y) \geq \beta$, then there exists $C' \in \mathbf{C}$ such that $C' \subseteq C_1$ and $C'(y) \geq \beta$. If $C_1(y) < \beta$ for all $y \in U$, it is easy to see that C_1 is a β -reducible element of $\widehat{\mathbf{C}} - \{C\}$. For $y \in U$, if $C(y) \geq \beta$, it is easy to find that $C' \neq C$. Therefore C_1 is a β -reducible element of $\mathbf{C} - \{C\}$.

For Case (2), if C_1 is a β -reducible element of \mathbf{C} , then $C_1(y) < \beta$ for all $y \in U$ or for $y \in U$, if $C_1(y) \geq \beta$, then there exists $C' \in \mathbf{C}$ such that $C' \subseteq C_1$ and $C'(y) \geq \beta$. If $C_1(y) < \beta$ for all $x \in U$, it is easy to see that C_1 is a β -reducible element of $\mathbf{C} - \{C\}$. For $y \in U$, if $C'(y) \geq \beta$ and $C' = C$, then there exists $C_r \in \{C_1, C_2, \dots, C_m\}$ such that $C_r \subseteq C' \subseteq C_1$ and $C_r(y) \geq \beta$, i.e., C_1 is a β -reducible element of $\mathbf{C} - \{C\}$. If $C' \neq C$, it is easy to have that C_1 is a β -reducible element of $\mathbf{C} - \{C\}$.

(\Leftarrow): It is straightforward.

This concludes the proof. \square

Proposition 14 guarantees that it is still a fuzzy β -covering after deleting a β -reducible element in a fuzzy β -covering. Whereas Proposition 15 shows that deleting a β -reducible element in a fuzzy β -covering will not generate any

new β -reducible elements or make other originally β -reducible elements into β -irreducible elements of the new fuzzy β -covering. So, we compute the β -reduct of a fuzzy β -covering of a universe U by deleting all β -reducible elements at the same time, or by deleting one β -reducible element in a step.

Definition 9 Let \mathbf{C} be a fuzzy β -covering of U for some $\beta \in (0, 1]$ and \mathbf{D} be a subset of \mathbf{C} . If $\mathbf{C} - \mathbf{D}$ is the set of all β -reducible elements of \mathbf{C} , then \mathbf{D} is called the β -reduct of \mathbf{C} , and is denoted as $\mathbf{R}^\beta(\mathbf{C})$.

Example 9 Let \mathbf{C} be the fuzzy β -covering in Example 8. Then $\mathbf{R}^{0.5}(\mathbf{C}) = \{C_1, C_2, C_3, C_4\}$.

Definition 10 Let \mathbf{C} be a fuzzy β -covering of U for some $\beta \in (0, 1]$. If every element of \mathbf{C} is a β -irreducible element, i.e., $\mathbf{R}^\beta(\mathbf{C}) = \mathbf{C}$, then \mathbf{C} is β -irreducible; otherwise \mathbf{C} is β -reducible.

In order to investigate the properties of fuzzy β -covering approximation space effectively, we define a new concept, namely fuzzy β -neighborhood family, based on the concept of fuzzy β -neighborhood.

Definition 11 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and \mathbf{C} be a fuzzy β -covering of U for some $\beta \in (0, 1]$. $\mathbf{Cov}^\beta(\mathbf{C}) = \{\tilde{N}_x^\beta : x \in U\}$ is called the fuzzy β -neighborhood family induced by \mathbf{C} .

Based on (1) of Proposition 3, we have that $\mathbf{Cov}^\beta(\mathbf{C}) = \{\tilde{N}_x^\beta : x \in U\}$ is also a fuzzy β -covering on V .

The following proposition indicates that deleting the reducible elements from the fuzzy β -covering has no influence on the fuzzy β -neighborhood family.

Proposition 16 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and \mathbf{C} be a fuzzy β -covering of U for some $\beta \in (0, 1]$. Then

$$\mathbf{Cov}^\beta(\mathbf{C}) = \mathbf{Cov}^\beta(\mathbf{R}^\beta(\mathbf{C})).$$

Proof Suppose $\mathbf{C} = \{C, C_1, C_2, \dots, C_m\}$, where $C, C_i \in \mathcal{F}(U)$ ($i = 1, 2, \dots, m$) and C is a β -reducible element of \mathbf{C} . It follows from Proposition 15 that $\mathbf{C} - \{C\}$ is still a fuzzy β -covering of U . For each $x \in U$, we denote the fuzzy β -neighborhood of x induced by fuzzy β -covering \mathbf{C} as \tilde{N}_x^β and denote the fuzzy β -neighborhood of x induced by fuzzy β -covering $\mathbf{C} - \{C\}$ as $\tilde{\mathbf{N}}_x^\beta$. Moreover, we have the following two cases:

Case (1). $C(x) < \beta$ for all $x \in U$ and

Case (2). For $x \in U$, if $C(x) \geq \beta$, then there exists $C_r \in \{C_1, C_2, \dots, C_m\}$ such that $C_r \subseteq C$ and $C_r(x) \geq \beta$.

For each $x \in U$, it follows from Case (1) that $\tilde{N}_x^\beta = \tilde{\mathbf{N}}_x^\beta$ holds obviously. If $C(x) \geq \beta$, by Case (2), there exists $C_r \in \{C_1, C_2, \dots, C_m\}$ such that $C_r \subseteq C$ and $C_r(x) \geq \beta$. Let $\mathbf{M} = \{C_{it} \in \mathbf{C} - \{C\} : C_{it}(x) \geq \beta, t = 1, 2, \dots, m\}$. It is clear that $C_r \in \mathbf{M}$. Therefore $\tilde{\mathbf{N}}_x^\beta = \bigcap f(\mathbf{M})$, $\tilde{N}_x^\beta = (\bigcap f(\mathbf{M})) \cap f(C)$. Since $C_r \in \mathbf{M}$, we have that $\bigcap f(\mathbf{M}) \subseteq f(C_r) \subseteq f(C)$, i.e., $(\bigcap f(\mathbf{M})) \cap f(C) = \bigcap f(\mathbf{M})$. Then $\tilde{N}_x^\beta = \tilde{\mathbf{N}}_x^\beta$. Taking into account the arbitrariness of x , we have that $\mathbf{Cov}^\beta(\mathbf{C}) = \mathbf{Cov}^\beta(\mathbf{C} - \{C\})$.

By Proposition 15, we have that $\mathbf{Cov}^\beta(\mathbf{C}) = \mathbf{Cov}^\beta(\mathbf{R}^\beta(\mathbf{C} - \{C\}))$ and $\mathbf{R}^\beta(\mathbf{C}) = \mathbf{R}^\beta(\mathbf{C} - \{C\})$. Therefore $\mathbf{Cov}^\beta(\mathbf{C}) = \mathbf{Cov}^\beta(\mathbf{R}^\beta(\mathbf{C}))$.

This completes the proof. \square

Proposition 17 *Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C}_1, \mathbf{C}_2$ be two fuzzy β -coverings of U for some $\beta \in (0, 1]$. $\mathbf{Cov}^\beta(\mathbf{C}_1) = \mathbf{Cov}^\beta(\mathbf{C}_2)$ if and only if $\mathbf{R}^\beta(\mathbf{C}_1) = \mathbf{R}^\beta(\mathbf{C}_2)$.*

Based on the above proposition, it is easy to find that two fuzzy β -neighborhood families of two different fuzzy β -coverings of U are equal if and only if their β -reducts are equal.

The focus of the following proposition is on the condition under which two fuzzy β -coverings to generate the same fuzzy covering-based lower approximation or the same fuzzy covering-based upper approximation of a fuzzy subset. By Definition 5 and Proposition 17, we have the following proposition.

Proposition 18 *Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C}_1, \mathbf{C}_2$ be two fuzzy β -coverings of U for some $\beta \in (0, 1]$. For any $X \in \mathcal{F}(V)$, the following statements are equivalent:*

- (1) $\mathbf{C}_1, \mathbf{C}_2$ generate the same fuzzy covering-based lower approximation of X .
- (2) $\mathbf{C}_1, \mathbf{C}_2$ generate the same fuzzy covering-based upper approximation of X .
- (3) $\mathbf{R}^\beta(\mathbf{C}_1) = \mathbf{R}^\beta(\mathbf{C}_2)$.

Proposition 18 shows that a necessary and sufficient condition for two fuzzy β -coverings to generate the same fuzzy covering-based lower approximation or the same fuzzy covering-based upper approximation of a fuzzy subset is that their β -reducts are equal. Moreover, it also shows that the fuzzy covering-based lower approximation and the fuzzy covering-based upper approximation determine each other.

4.2 Matrix representations of fuzzy covering-based approximation operators

In this subsection, we present matrix representations of the fuzzy covering-based lower and upper approximation operators defined in Definitions 5 and 7. The matrix representations of the approximation operators makes it possible

to calculate the lower and upper approximations of subsets through the operations on matrices, which is algorithmic, and can easily be implemented through the computer.

Definition 12 ([32]) Let $A = (a_{ik})_{n \times m}$ and $B = (b_{kj})_{m \times l}$ be two matrices. We define $C = A \odot B = (c_{ij})_{n \times l}$ and $D = A \diamond B = (d_{ij})_{n \times l}$ as follows:

$$c_{ij} = \bigvee_{k=1}^m (a_{ik} \wedge b_{kj}), i = 1, 2, \dots, n, j = 1, 2, \dots, l,$$

$$d_{ij} = \bigwedge_{k=1}^m ((1 - a_{ik}) \vee b_{kj}), i = 1, 2, \dots, n, j = 1, 2, \dots, l.$$

Definition 13 Let $U = \{x_1, x_2, \dots, x_n\}$, $V = \{y_1, y_2, \dots, y_m\}$ be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$. We call Boolean matrix $M_f = (t_{ij})_{n \times m}$ a representation of f , where

$$t_{ij} = \begin{cases} 1, & f(x_i) = y_j, \\ 0, & f(x_i) \neq y_j. \end{cases}$$

Definition 14 ([32]) Let $U = \{x_1, x_2, \dots, x_n\}$ be a finite universe and $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ be a fuzzy β -covering of U for some $\beta \in (0, 1]$. We call $M_{\mathbf{C}} = (C_j(x_i))_{n \times m}$ a matrix representation of \mathbf{C} , and call Boolean matrix $M_{\beta} = (s_{ij})_{n \times m}$ a β -matrix representation of \mathbf{C} , where

$$s_{ij} = \begin{cases} 1, & C_j(x_i) \geq \beta, \\ 0, & C_j(x_i) < \beta. \end{cases}$$

Example 10 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3, y_4\}$ and

$$f : U \rightarrow V, f(x) = \begin{cases} y_1, & x \in \{x_1, x_3\}, \\ y_2, & x \in \{x_2, x_5\}, \\ y_3, & x = x_4, \\ y_4, & x = x_6. \end{cases}$$

Let $\mathbf{C} = \{C_1, C_2, C_3, C_4\}$, where

$$C_1 = \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{1}{x_5} + \frac{0.7}{x_6},$$

$$C_2 = \frac{0.2}{x_1} + \frac{0.9}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.3}{x_5} + \frac{0.5}{x_6},$$

$$C_3 = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.7}{x_3} + \frac{1}{x_4} + \frac{0.2}{x_5} + \frac{0.7}{x_6},$$

$$C_4 = \frac{0.5}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{0.4}{x_5} + \frac{0.5}{x_6}.$$

It is easy to see that \mathbf{C} is a fuzzy β -covering on U , where $\beta \in (0, 0.5]$. Let $\beta = 0.4$. Then

$$\begin{aligned}
M_f = & \begin{matrix} & y_1 & y_2 & y_3 & y_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}, \quad M = \begin{matrix} & C_1 & C_2 & C_3 & C_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{bmatrix} 0.3 & 0.2 & 0.4 & 0.5 \\ 0.7 & 0.9 & 0.6 & 0.4 \\ 0.4 & 0.5 & 0.7 & 0.4 \\ 0.1 & 0.4 & 1 & 0.1 \\ 1 & 0.3 & 0.2 & 0.4 \\ 0.7 & 0.5 & 0.7 & 0.5 \end{bmatrix} \end{matrix}, \\
M_{0.4} = & \begin{matrix} & C_1 & C_2 & C_3 & C_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}.
\end{aligned}$$

Proposition 19 Let U, V be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and \mathbf{C} be a fuzzy β -covering on U for some $\beta \in (0, 1]$. Then $(M_f)^T \odot M_{\mathbf{C}} = M_{f(\mathbf{C})}$.

Proof Let $U = \{x_1, x_2, \dots, x_n\}$, $V = \{y_1, y_2, \dots, y_m\}$, $\mathbf{C} = \{C_1, C_2, \dots, C_k\}$, $(M_f)^T = (a_{ij})_{m \times n}$, $M_{\mathbf{C}} = (b_{ij})_{n \times k}$ and $M_{f(\mathbf{C})} = (c_{ij})_{m \times k}$. It follows from Definition 13 that $a_{ij} = 1 \Leftrightarrow f(x_j) = y_i$ and $a_{ij} = 0 \Leftrightarrow f(x_j) \neq y_i$. Then $c_{ij} = f(C_j)(y_i) = \bigvee_{x \in f^{-1}(y_i)} C_j(x) = \bigvee_{l=1}^n [C_j(x_l) \wedge a_{il}] = \bigvee_{l=1}^n (a_{il} \wedge b_{lj})$. Therefore $(M_f)^T \odot M_{\mathbf{C}} = M_{f(\mathbf{C})}$. \square

Example 11 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3, y_4\}$ and

$$f : U \rightarrow V, f(x) = \begin{cases} y_1, & x \in \{x_1, x_3\}, \\ y_2, & x \in \{x_2, x_5\}, \\ y_3, & x = x_4, \\ y_4, & x = x_6. \end{cases}$$

Let \mathbf{C} be the fuzzy β -covering in Example 10. Then

$$\begin{aligned}
M_{f(\mathbf{C})} &= (M_f)^T \odot M_{\mathbf{C}} \\
&= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0.3 & 0.2 & 0.4 & 0.5 \\ 0.7 & 0.9 & 0.6 & 0.4 \\ 0.4 & 0.5 & 0.7 & 0.4 \\ 0.1 & 0.4 & 1 & 0.1 \\ 1 & 0.3 & 0.2 & 0.4 \\ 0.7 & 0.5 & 0.7 & 0.5 \end{bmatrix} \\
&= \begin{bmatrix} 0.4 & 0.5 & 0.7 & 0.5 \\ 1 & 0.9 & 0.6 & 0.4 \\ 0.1 & 0.4 & 1 & 0.1 \\ 0.7 & 0.5 & 0.7 & 0.5 \end{bmatrix}.
\end{aligned}$$

Proposition 20 Let $U = \{x_1, x_2, \dots, x_n\}$, $V = \{y_1, y_2, \dots, y_m\}$ be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_k\}$ be a fuzzy β -covering on

U for some $\beta \in (0, 1]$. If M_β is a β -matrix representation of \mathbf{C} , M_f is a matrix representation of f and $M_{\mathbf{C}}$ is a matrix representation of \mathbf{C} , then

$$M_\beta \diamond ((M_f)^T \odot M_{\mathbf{C}})^T = (\tilde{N}_{x_i}^\beta(y_j))_{n \times m}.$$

Proof Since \mathbf{C} is a fuzzy β -covering of U , $M_\beta = (s_{ij})_{n \times k}$ is a β -matrix representation of \mathbf{C} , and $M_f = (t_{ij})_{n \times m}$ is a matrix representation of f , for each $i \in \{1, 2, \dots, n\}$, there is $l \in \{1, 2, \dots, k\}$ such that $s_{il} = 1$. It follows from Proposition 19 that $(M_f)^T \odot M_{\mathbf{C}} = M_{f(\mathbf{C})}$. Suppose that $M_{f(\mathbf{C})} = (b_{ij})_{m \times k}$ and $M_\beta \diamond ((M_f)^T \odot M_{\mathbf{C}})^T = (c_{ij})_{n \times m}$, then

$$\begin{aligned} c_{ij} &= \bigwedge_{l=1}^k ((1 - s_{il}) \vee b_{lj}) \\ &= \bigwedge_{s_{il}=1} ((1 - s_{il}) \vee f(C_l)(y_j)) \\ &= \bigwedge_{s_{il}=1} f(C_l)(y_j) \\ &= \bigwedge_{C_l(x_i) \geq \beta} f(C_l)(y_j) \\ &= [\bigcap_{C_l(x_i) \geq \beta} f(C_l)](y_j) \\ &= \tilde{N}_{x_i}^\beta(y_j), i = 1, 2, \dots, n, j = 1, 2, \dots, m. \end{aligned}$$

Then $M_\beta \diamond ((M_f)^T \odot M_{\mathbf{C}})^T = (\tilde{N}_{x_i}^\beta(y_j))_{n \times m}$. □

Example 12 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3, y_4\}$ and

$$f : U \rightarrow V, f(x) = \begin{cases} y_1, & x \in \{x_1, x_3\}, \\ y_2, & x \in \{x_2, x_5\}, \\ y_3, & x = x_4, \\ y_4, & x = x_6. \end{cases}$$

Let \mathbf{C} be the fuzzy β -covering in Example 10. Then

$$\begin{aligned} (\tilde{N}_{x_i}^{0.4}(y_j))_{n \times m} &= M_{0.4} \diamond ((M_f)^T \odot M_{\mathbf{C}})^T \\ &= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \diamond \begin{bmatrix} 0.4 & 1 & 0.1 & 0.7 \\ 0.5 & 0.9 & 0.4 & 0.5 \\ 0.7 & 0.6 & 1 & 0.7 \\ 0.5 & 0.4 & 0.1 & 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 & 0.4 & 0.1 & 0.5 \\ 0.4 & 0.4 & 0.1 & 0.5 \\ 0.4 & 0.4 & 0.1 & 0.5 \\ 0.5 & 0.6 & 0.4 & 0.5 \\ 0.4 & 0.4 & 0.1 & 0.5 \\ 0.4 & 0.4 & 0.1 & 0.5 \end{bmatrix}. \end{aligned}$$

Next we show that the calculations of the lower and upper approximations $\underline{\mathbf{C}}(X)$ and $\overline{\mathbf{C}}(X)$ of fuzzy set $X \in \mathcal{F}(V)$ can be converted to operations on matrices.

Proposition 21 Let $U = \{x_1, x_2, \dots, x_n\}$, $V = \{y_1, y_2, \dots, y_m\}$ be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_k\}$ be a fuzzy β -covering on U for some $\beta \in (0, 1]$. If M_β is a β -matrix representation of \mathbf{C} , M_f is a matrix representation of f and $M_{\mathbf{C}}$ is a matrix representation of \mathbf{C} , then for each $X \in \mathcal{F}(V)$, we have

$$\underline{\mathbf{C}}(X) = (M_\beta \diamond ((M_f)^T \odot M_{\mathbf{C}})^T) \diamond M_X,$$

$$\overline{\mathbf{C}}(X) = (M_\beta \diamond ((M_f)^T \odot M_{\mathbf{C}})^T) \odot M_X,$$

where $M_X = (X(y_i))_{1 \times m}$.

Proof For each $i \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} ((M_\beta \diamond ((M_f)^T \odot M_{\mathbf{C}})^T) \diamond M_X)(x_i) &= \bigwedge_{l=1}^m [(1 - \tilde{N}_{x_i}^\beta(y_l)) \vee X(y_l)] \\ &= \bigwedge_{y \in V} [(1 - \tilde{N}_{x_i}^\beta(y)) \vee X(y)] \\ &= \underline{\mathbf{C}}(X)(x_i), \\ ((M_\beta \diamond ((M_f)^T \odot M_{\mathbf{C}})^T) \odot M_X)(x_i) &= \bigvee_{l=1}^m [\tilde{N}_{x_i}^\beta(y_l) \wedge X(y_l)] \\ &= \bigvee_{y \in V} [\tilde{N}_{x_i}^\beta(y) \wedge X(y)] \\ &= \overline{\mathbf{C}}(X)(x_i). \end{aligned}$$

Then $\underline{\mathbf{C}}(X) = (M_\beta \diamond ((M_f)^T \odot M_{\mathbf{C}})^T) \diamond M_X$ and $\overline{\mathbf{C}}(X) = (M_\beta \diamond ((M_f)^T \odot M_{\mathbf{C}})^T) \odot M_X$ can be followed. \square

Example 13 The $\underline{\mathbf{C}}(X)$ and $\overline{\mathbf{C}}(X)$ in Example 5 can be calculated as follows.

$$\begin{aligned} \underline{\mathbf{C}}(X) &= (M_{0.5} \diamond ((M_f)^T \odot M_{\mathbf{C}})^T) \diamond M_X \\ &= \left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \diamond \left(\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0.5 & 0.7 & 0.3 & 0.6 \\ 0.3 & 0.6 & 0.4 & 0.7 \\ 0.4 & 0.8 & 0.1 & 0.4 \\ 0.7 & 0.8 & 0.7 & 0.4 \\ 0.5 & 0.6 & 0.9 & 0.5 \\ 0.5 & 0.7 & 0.7 & 0.3 \end{bmatrix}^T \right) \right) \diamond \begin{bmatrix} 0.3 \\ 0.7 \\ 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 \\ 0.3 \\ 0.3 \\ 0.5 \\ 0.6 \\ 0.5 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \overline{\mathbf{C}}(X) &= (M_{0.5} \diamond ((M_f)^T \odot M_{\mathbf{C}})^T) \odot M_X \\ &= \left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \diamond \left(\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0.5 & 0.7 & 0.3 & 0.6 \\ 0.3 & 0.6 & 0.4 & 0.7 \\ 0.4 & 0.8 & 0.1 & 0.4 \\ 0.7 & 0.8 & 0.7 & 0.4 \\ 0.5 & 0.6 & 0.9 & 0.5 \\ 0.5 & 0.7 & 0.7 & 0.3 \end{bmatrix}^T \right) \right) \odot \begin{bmatrix} 0.3 \\ 0.7 \\ 0.5 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 0.5 \\ 0.5 \\ 0.7 \\ 0.7 \\ 0.5 \\ 0.7 \end{bmatrix}.$$

In the following, we discuss the matrix representations of the lower and upper approximations $\underline{\mathcal{C}}(X)$ and $\overline{\mathcal{C}}(X)$ of crisp subset $X \in \mathcal{P}(V)$. In order to facilitate the expression, we use χ_X to denote the characteristic function of the crisp subset $X \in \mathcal{P}(V)$, where $\chi_X(x) = \begin{cases} 1, & x \in X, \\ 0, & x \notin X. \end{cases}$

Proposition 22 *Let $U = \{x_1, x_2, \dots, x_n\}$, $V = \{y_1, y_2, \dots, y_m\}$ be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_k\}$ be a fuzzy β -covering on U for some $\beta \in (0, 1]$. If M_β is a β -matrix representation of \mathbf{C} , M_f is a matrix representation of f and $M_{\mathbf{C}}$ is a matrix representation of \mathbf{C} , then $((M_f)^T \odot M_{\mathbf{C}})_\beta = (M_{f(\mathbf{C})})_\beta$.*

Proof It follows immediately from Proposition 19. □

Example 14 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3, y_4\}$ and

$$f : U \rightarrow V, f(x) = \begin{cases} y_1, & x \in \{x_1, x_3\}, \\ y_2, & x \in \{x_2, x_5\}, \\ y_3, & x = x_4, \\ y_4, & x = x_6. \end{cases}$$

Let \mathbf{C} be the fuzzy β -covering in Example 10. Then

$$(M_{f(\mathbf{C})})_{0.4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Proposition 23 *Let $U = \{x_1, x_2, \dots, x_n\}$, $V = \{y_1, y_2, \dots, y_m\}$ be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_k\}$ be a fuzzy β -covering on U for some $\beta \in (0, 1]$. If M_β is a β -matrix representation of \mathbf{C} , M_f is a matrix representation of f and $M_{\mathbf{C}}$ is a matrix representation of \mathbf{C} , then*

$$M_\beta \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_\beta = (\chi_{\overline{N}_{x_i}^\beta}(y_j))_{n \times m}.$$

Proof Denote $M_\beta \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_\beta = (d_{ij})_{n \times m}$, $M_\beta = (a_{ij})_{n \times k}$ and $((M_f)^T \odot M_{\mathbf{C}})^T)_\beta = (b_{ij})_{k \times m}$. If $d_{ij} = 1$, then $\bigwedge_{l=1}^k [(1 - a_{il}) \vee b_{lj}] = 1$. This implies that if $a_{il} = 1$ then $b_{lj} = 1$. Therefore $C_l(x_i) \geq \beta$ leads to $f(C_l)(y_j) \geq \beta$, and hence $y_j \in N_{x_i}^\beta$, i.e., $\chi_{\overline{N}_{x_i}^\beta}(y_j) = 1 = d_{ij}$. If $d_{ij} = 0$, then $\bigwedge_{l=1}^k [(1 - a_{il}) \vee b_{lj}] = 0$. This

implies that there exists l such that $b_{lj} = 0$ and $a_{il} = 1$. Therefore $C_l(x_i) \geq \beta$ and $f(C_l)(y_j) < \beta$. Then $y_j \notin \overline{N}_{x_i}^\beta$, i.e., $\chi_{\overline{N}_{x_i}^\beta}(y_j) = 0 = d_{ij}$. \square

Example 15 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3, y_4\}$ and

$$f : U \rightarrow V, f(x) = \begin{cases} y_1, & x \in \{x_1, x_3\}, \\ y_2, & x \in \{x_2, x_5\}, \\ y_3, & x = x_4, \\ y_4, & x = x_6. \end{cases}$$

Let (U, \mathbf{C}) be the fuzzy β -covering approximation space in Example 3 and $\beta = 0.4$. Then

$$\begin{aligned} (\chi_{\overline{N}_{x_i}^{0.4}}(y_j))_{6 \times 4} &= M_{0.4} \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_{0.4} \\ &= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \diamond \left(\left(\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0.3 & 0.2 & 0.4 & 0.5 \\ 0.7 & 0.9 & 0.6 & 0.4 \\ 0.4 & 0.5 & 0.7 & 0.4 \\ 0.1 & 0.4 & 1 & 0.1 \\ 1 & 0.3 & 0.2 & 0.4 \\ 0.7 & 0.5 & 0.7 & 0.5 \end{bmatrix} \right)^T \right)_{0.4} \\ &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Proposition 24 Let $U = \{x_1, x_2, \dots, x_n\}$, $V = \{y_1, y_2, \dots, y_m\}$ be two non-empty finite universes, $f \in \mathbf{Sur}(U, V)$ and $\mathbf{C} = \{C_1, C_2, \dots, C_k\}$ be a fuzzy β -covering on U for some $\beta \in (0, 1]$. If M_β is a β -matrix representation of \mathbf{C} , M_f is a matrix representation of f and $M_{\mathbf{C}}$ is a matrix representation of \mathbf{C} , then for each $X \in \mathcal{P}(V)$, we have

$$\begin{aligned} \chi_{\underline{\mathcal{C}}(X)} &= (M_\beta \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_\beta) \diamond \chi_X, \\ \chi_{\overline{\mathcal{C}}(X)} &= (M_\beta \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_\beta) \odot \chi_X. \end{aligned}$$

Proof For any $i \in \{1, 2, \dots, n\}$,

$$\begin{aligned} &[(M_\beta \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_\beta) \diamond \chi_X](x_i) = 1 \\ &\Leftrightarrow \bigwedge_{l=1}^k [(1 - \chi_{\overline{N}_{x_i}^\beta}(y_l)) \vee \chi_X(y_l)] = 1 \\ &\Leftrightarrow \chi_{\overline{N}_{x_i}^\beta}(y_l) = 1 \rightarrow \chi_X(y_l) = 1, l = 1, 2, \dots, k \\ &\Leftrightarrow y_l \in \overline{N}_{x_i}^\beta \rightarrow y_l \in X, l = 1, 2, \dots, k \\ &\Leftrightarrow \overline{N}_{x_i}^\beta \subseteq X \\ &\Leftrightarrow x_i \in \underline{\mathcal{C}}(X) \end{aligned}$$

$$\Leftrightarrow \chi_{\underline{\mathcal{C}}(X)}(x_i) = 1,$$

and

$$\begin{aligned} & [(M_\beta \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_\beta) \odot \chi_X](x_i) = 1 \\ & \Leftrightarrow \bigvee_{l=1}^k [\chi_{\overline{N}_{x_i}^\beta}(y_l) \wedge \chi_X(y_l)] = 1 \\ & \Leftrightarrow \exists l \in \{1, 2, \dots, k\}, \chi_{\overline{N}_{x_i}^\beta}(y_l) = \chi_X(y_l) = 1 \\ & \Leftrightarrow \exists l \in \{1, 2, \dots, k\}, y_l \in \overline{N}_{x_i}^\beta \cap X \\ & \Leftrightarrow \overline{N}_{x_i}^\beta \cap X \neq \emptyset \\ & \Leftrightarrow x_i \in \overline{\mathcal{C}}(X) \\ & \Leftrightarrow \chi_{\overline{\mathcal{C}}(X)}(x_i) = 1. \end{aligned}$$

Then $\chi_{\underline{\mathcal{C}}(X)} = (M_\beta \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_\beta) \diamond \chi_X$ and $\chi_{\overline{\mathcal{C}}(X)} = (M_\beta \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_\beta) \odot \chi_X$. \square

Example 16 The $\underline{\mathcal{C}}(Y)$ and $\overline{\mathcal{C}}(Y)$ in Example 7 can be calculated as follows.

$$\begin{aligned} \chi_{\underline{\mathcal{C}}(Y)} &= (M_{0.4} \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_{0.4}) \diamond \chi_Y \\ &= \left(\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \diamond \left(\left(\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0.3 & 0.2 & 0.4 & 0.5 \\ 0.7 & 0.9 & 0.6 & 0.4 \\ 0.4 & 0.5 & 0.7 & 0.4 \\ 0.1 & 0.4 & 1 & 0.1 \\ 1 & 0.3 & 0.2 & 0.4 \\ 0.7 & 0.5 & 0.7 & 0.5 \end{bmatrix} \right)^T \right)_{0.4} \right) \diamond \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, i.e., \underline{\mathcal{C}}(Y) = \{x_1, x_2, x_3, x_5, x_6\}. \end{aligned}$$

$$\begin{aligned} \chi_{\overline{\mathcal{C}}(Y)} &= (M_{0.4} \diamond (((M_f)^T \odot M_{\mathbf{C}})^T)_{0.4}) \odot \chi_Y \\ &= \left(\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \diamond \left(\left(\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0.3 & 0.2 & 0.4 & 0.5 \\ 0.7 & 0.9 & 0.6 & 0.4 \\ 0.4 & 0.5 & 0.7 & 0.4 \\ 0.1 & 0.4 & 1 & 0.1 \\ 1 & 0.3 & 0.2 & 0.4 \\ 0.7 & 0.5 & 0.7 & 0.5 \end{bmatrix} \right)^T \right)_{0.4} \right) \odot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, i.e., \overline{\mathcal{C}}(Y) = \{x_1, x_2, x_3, x_4, x_5, x_6\}. \end{aligned}$$

5 The application of fuzzy covering-based rough set model over two universes to multiple criteria decision making

One of the purposes of multiple criteria decision making is to find an optimal alternative from all feasible alternatives in a limited time according to certain criteria or attributes [15, 29, 30, 81, 82, 84]. An attribute is associated with a weight to indicate its relative importance. Multiple criteria decision making has been widely used in many fields such as [1, 54–56, 60, 75, 83]. A number of methods have been proposed to deal with multiple criteria decision making problems, including utility theory-based methods and outranking methods. These methods search for a ranking of alternatives toward a high quality decision-making. However, the traditional multiple criteria decision making methods only determine the optimal alternative according to the attribute values, but they ignore the costs/losses of alternatives in the procedure of decision-making.

Motivated by this, we try to establish a new approach to multiple criteria decision making problems based on fuzzy covering-based rough set over two different universes in this section. We present the basic description of a multiple criteria decision making problem under the framework of fuzzy β -neighborhood over two universes, and then give a general decision making methodology for multiple criteria decision making problem by using the fuzzy covering-based rough set theory over two universes.

5.1 Problem statement

We give the basic description of multiple criteria decision making problem considered in this section. We present the description by using a multiple criteria decision making problem in the case of graduate students recruiting.

Let $U = \{x_1, x_2, \dots, x_n\}$ be the finite set of n students, $V = \{y_1, y_2, \dots, y_m\}$ be the finite set of m research directions, e.g., y_1 denotes *applied mathematics*, y_2 denotes *biological mathematics*, y_3 denotes *cryptography*, \dots , and y_m denotes *computational mathematics*, and $\mathbf{C} = \{C_1, C_2, \dots, C_k\} \subseteq \mathcal{F}(U)$ be the criteria set which C_i ($i = 1, 2, \dots, k$) are k given criteria, e.g., $C_1(x_i) \in [0, 1]$ denotes x_i 's grade of *mathematical analysis*, $C_2(x_i) \in [0, 1]$ denotes x_i 's grade of *advanced algebra*, $C_3(x_i) \in [0, 1]$ denotes x_i 's grade of *analytic geometry*, \dots , $C_k(x_i) \in [0, 1]$ denotes x_i 's grade of *probability theory*, where $i \in \{1, 2, \dots, k\}$. Let $f \in \mathbf{Sur}(U, V)$, where $f(x_i) = y_j$ ($i \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, m\}$) denotes that x_i select y_j . It is easy to see that f partitions U into m classes. In general, in order to recruit enough students, the university sets a suitable score line according to the grade of all the students. In this problem, we set $\beta \in (0, \bigwedge_{i=1}^n (\bigvee_{j=1}^m C_j(x_i))$.

It is easy to see that \mathbf{C} is a fuzzy β -covering of U . Then we can consider this kind of multiple criteria decision making problem by using fuzzy covering-based rough set model. Meanwhile, by using Zadeh's extension principle,

$f(C_s)(y_j) = \bigvee_{(x_i \in U) \wedge (x_i \in f^{-1}(y_j))} C_s(x_i)$ denotes the important degree of the criteria C_s to the research direction y_j and $\tilde{N}_{x_i}^\beta(y_j) = \bigwedge_{C_l(x) \geq \beta} f(C_l)(y_j)$ denotes the possibility of the students x_i select the research direction y_j . For a given criterion X (described by a fuzzy set of universe V), we know that fuzzy covering-based lower approximation $\underline{\mathbf{C}}(X)$ of X denotes the subsethood degree of X and $(\tilde{N}_{x_i}^\beta)^c$, and fuzzy covering-based upper approximation $\overline{\mathbf{C}}(X)$ of X denotes the degree of intersection of X and $\tilde{N}_{x_i}^\beta$. Then $\underline{\mathbf{C}}(X)(x_i) < \beta$ and $\overline{\mathbf{C}}(X)(x_i) < \beta$ denote the student x_i does not satisfied with the criterion X . If $\underline{\mathbf{C}}(X)(x_i) \geq \beta$ or $\overline{\mathbf{C}}(X)(x_i) \geq \beta$, then the student x_i satisfies the criterion X .

In the following discussions, we give an approach to decision making for this kind of multiple criteria decision problem with the above described characteristic by using the fuzzy covering-based rough set on two universes.

5.2 Decision making methodology

Firstly, we construct the fuzzy information systems over two universes for the considered multiple criteria decision problem. From the description of the multiple criteria decision making problem in Section 5.1, we know that the grade of student, the tendentiousness of the student to research direction and the score line given by the university provide a fuzzy evaluation function from U to $\mathcal{F}(V)$. Then we can define a novel type of fuzzy covering-based rough set model over two different universes and apply this model to the multiple criteria decision making problem. Secondly, we calculate the fuzzy covering-based lower approximation $\underline{\mathbf{C}}(X)$ and fuzzy covering-based upper approximation $\overline{\mathbf{C}}(X)$ for the criterion X (described by a fuzzy set of universe V), which is given by university. Finally, we make the decision rule of the multiple criteria decision making problem in Section 5.1 as follows.

- (1) If $\underline{\mathbf{C}}(X)(x_i) \vee \overline{\mathbf{C}}(X)(x_i) < \beta$, then the student x_i ($i \in \{1, 2, \dots, n\}$) can not be recruited by the research direction y_j ($j \in \{1, 2, \dots, m\}$), where $y_j = f(x_i)$.
- (2) If $\underline{\mathbf{C}}(X)(x_i) \vee \overline{\mathbf{C}}(X)(x_i) \geq \beta$, then the student x_i ($i \in \{1, 2, \dots, n\}$) can be recruited by the research direction y_j ($j \in \{1, 2, \dots, m\}$), where $y_j = f(x_i)$.

Therefore, we establish an approach to multiple criteria decision making by using the fuzzy covering-based rough set model over two universes, which is defined in Definition 5.

Based on Proposition 18, we know that two fuzzy β -coverings generate the same fuzzy covering approximation operations of a fuzzy subset if and only if their β -reducts are equal. In other words, some criteria of the multiple criteria decision making problem in Section 5.1 can be deleted but the decision of this problem has no influence. In fact, if C is a β -reducible element of \mathbf{C} , then we have the following two cases:

Case (1). $C(x) < \beta$ for all $x \in U$ and

Case (2). For $x \in U$, if $C(x) \geq \beta$, then there exists $C_r \in \mathbf{C} - \{C\}$ such that $C_r \subseteq C$ and $C_r(x) \geq \beta$.

Case (1) indicates that the grade of course C of all the students in U is lower than β , we can not continuous consider C in the multiple criteria decision making problem in Section 5.1. Case (2) indicates that if the grade of course C of the student x is upper than β , and there exists course C' such that the grade of course C' of the student x is also upper than β and $C' \subseteq C$, then we can not continuous consider C in the multiple criteria decision making problem in Section 5.1. The definition of β -reduct can not only simplify the computational complexity, but has no influence in the multiple criteria decision making problem in Section 5.1.

In order to implement the decision-making process in this section, we give the following specific steps.

Input: Fuzzy information system (U, V, \mathbf{C}) over two universes for the considered multiple criteria decision problem; $f \in \mathbf{Sur}(U, V)$; threshold $\beta \in (0, 1]$.

Output: The ranking for all alternatives.

Step 1: Define a novel type of fuzzy covering-based rough set model over two different universes (see Definition 5).

Step 2: Calculate the fuzzy covering-based lower approximation $\underline{\mathbf{C}}(X)$ and fuzzy covering-based upper approximation $\overline{\mathbf{C}}(X)$ for the criterion X (described by a fuzzy set of universe V), which is given by university.

Step 3: If $\underline{\mathbf{C}}(X)(x_i) \vee \overline{\mathbf{C}}(X)(x_i) < \beta$, then the student x_i ($i \in \{1, 2, \dots, n\}$) can not be recruited by the research direction y_j ($j \in \{1, 2, \dots, m\}$), where $y_j = f(x_i)$.

Step 4: If $\underline{\mathbf{C}}(X)(x_i) \vee \overline{\mathbf{C}}(X)(x_i) \geq \beta$, then the student x_i ($i \in \{1, 2, \dots, n\}$) can be recruited by the research direction y_j ($j \in \{1, 2, \dots, m\}$), where $y_j = f(x_i)$.

Step 5: Rank the alternatives and select a top element of the ranking.

5.3 A test example

In this section, we consider a multiple criteria decision making problem in the case of graduate students recruiting to illustrate the decision method proposed in Section 5.2.

Let $U = \{x_1, x_2, \dots, x_{15}\}$ be 15 students, and $\mathbf{C} = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9\}$ be 9 courses, where $\mathbf{C} \subseteq \mathcal{F}(U)$ and $C_l(x_i)$ denotes the course C_l 's grade of the student x_i . C_1 denotes *mathematical analysis*, C_2 denotes *advanced algebra*, C_3 denotes *probability theory*, C_4 denotes *topology*, C_5 denotes *ordinary differential equations*, C_6 denotes *numerical analysis*, C_7 denotes *functional analysis*, C_8 denotes *english*, C_9 denotes *ideological politics*. The transcript of the students with respect to all courses can be listed in Table 1. By Table 1, we have $\bigwedge_{i=1}^{15} [\bigvee_{j=1}^9 C_j(x_i)] = 0.83$. Then $\mathbf{C} = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9\}$ is a fuzzy β -covering of U for $\beta \in (0, 0.81]$. Suppose that $\beta = 0.8$ given by the university. Since $C_j(x) < 0.8$ ($j = 3, 4, 8, 9$) for all $x \in U$, $C_5(x_5) \geq 0.8$, $C_7(x_5) \geq 0.8$ and $C_7 \subseteq C_5$, we have that C_3, C_4, C_5, C_8, C_9 are the 0.8-reducible elements of \mathbf{C} . Then we only need to consider $\mathbf{C}' = \mathbf{C} - \{C_3, C_4, C_5, C_8, C_9\} = \{C_1, C_2, C_6, C_7\}$ in this problem, which has no influence of the decision making.

Table 1 The transcript of the students

U	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
x_1	0.92	0.94	0.52	0.48	0.62	0.71	0.61	0.55	0.72
x_2	0.75	0.85	0.71	0.58	0.64	0.43	0.58	0.73	0.65
x_3	0.92	0.91	0.43	0.46	0.78	0.85	0.62	0.48	0.68
x_4	0.81	0.85	0.74	0.55	0.64	0.93	0.51	0.45	0.74
x_5	0.75	0.78	0.63	0.67	0.83	0.57	0.81	0.74	0.79
x_6	0.64	0.85	0.71	0.62	0.79	0.62	0.77	0.62	0.53
x_7	0.83	0.75	0.62	0.48	0.65	0.35	0.45	0.57	0.58
x_8	0.64	0.95	0.75	0.53	0.58	0.42	0.47	0.49	0.69
x_9	0.74	0.85	0.72	0.62	0.72	0.75	0.68	0.61	0.71
x_{10}	0.76	0.81	0.56	0.64	0.68	0.71	0.62	0.72	0.68
x_{11}	0.85	0.71	0.68	0.72	0.74	0.72	0.66	0.71	0.57
x_{12}	0.74	0.95	0.42	0.71	0.73	0.48	0.61	0.65	0.72
x_{13}	0.82	0.81	0.73	0.72	0.78	0.92	0.71	0.45	0.68
x_{14}	0.68	0.91	0.58	0.69	0.64	0.66	0.57	0.64	0.77
x_{15}	0.92	0.81	0.72	0.74	0.68	0.76	0.62	0.73	0.63

Let $V = \{y_1, y_2, y_3, y_4, y_5\}$ be 10 research directions, where y_1 denotes **fundamental mathematics**, y_2 denotes **probability theory**, y_3 denotes **computational mathematics**, y_4 denotes **applied mathematics**, y_5 denotes **statistics**, and the choice of the student x_i to research direction y_j can be represented as a mapping $f : U \rightarrow V$, $f(x_i) = y_j$, which is listed as follows:

$$f(x) = \begin{cases} y_1, & x \in \{x_1, x_2, x_3, x_4, x_5\}, \\ y_2, & x \in \{x_6, x_7, x_8\}, \\ y_3, & x \in \{x_9, x_{10}, x_{11}\}, \\ y_4, & x \in \{x_{12}, x_{13}\}, \\ y_5, & x \in \{x_{14}, x_{15}\}. \end{cases}$$

Then we have

$$M_{f(\mathbf{C}')} = (M_f)^T \odot M_{\mathbf{C}'} = \begin{bmatrix} 0.92 & 0.94 & 0.93 & 0.81 \\ 0.83 & 0.95 & 0.62 & 0.77 \\ 0.85 & 0.85 & 0.75 & 0.68 \\ 0.85 & 0.95 & 0.92 & 0.71 \\ 0.92 & 0.91 & 0.76 & 0.62 \end{bmatrix},$$

$$(\tilde{N}_{x_i}^{0.8}(y_j))_{15 \times 5} = (M_{C'})_{0.8} \diamond ((M_f)^T \odot M_{C'})^T = \begin{bmatrix} 0.92 & 0.83 & 0.85 & 0.85 & 0.91 \\ 0.94 & 0.95 & 0.85 & 0.95 & 0.91 \\ 0.92 & 0.62 & 0.75 & 0.85 & 0.76 \\ 0.92 & 0.62 & 0.75 & 0.85 & 0.76 \\ 0.81 & 0.77 & 0.68 & 0.71 & 0.62 \\ 0.94 & 0.95 & 0.85 & 0.95 & 0.91 \\ 0.92 & 0.83 & 0.85 & 0.85 & 0.92 \\ 0.94 & 0.95 & 0.85 & 0.95 & 0.91 \\ 0.94 & 0.95 & 0.85 & 0.95 & 0.91 \\ 0.94 & 0.95 & 0.85 & 0.95 & 0.91 \\ 0.92 & 0.83 & 0.85 & 0.85 & 0.92 \\ 0.94 & 0.95 & 0.85 & 0.95 & 0.91 \\ 0.92 & 0.62 & 0.75 & 0.85 & 0.76 \\ 0.94 & 0.95 & 0.85 & 0.95 & 0.91 \\ 0.92 & 0.83 & 0.85 & 0.85 & 0.91 \end{bmatrix}.$$

Let $X = \frac{0.75}{y_1} + \frac{0.35}{y_2} + \frac{0.85}{y_3} + \frac{0.65}{y_4} + \frac{0.85}{y_5}$ be the criterion given by university. Then $\underline{C}(X)$ and $\overline{C}(X)$ are listed in Table 2.

Table 2 Fuzzy covering lower approximation $\underline{C}(X)$ and upper approximation $\overline{C}(X)$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
$\underline{C}(X)$	0.35	0.35	0.38	0.38	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.38	0.35	0.35
$\overline{C}(X)$	0.85	0.85	0.76	0.76	0.75	0.85	0.85	0.85	0.85	0.85	0.85	0.85	0.76	0.85	0.85

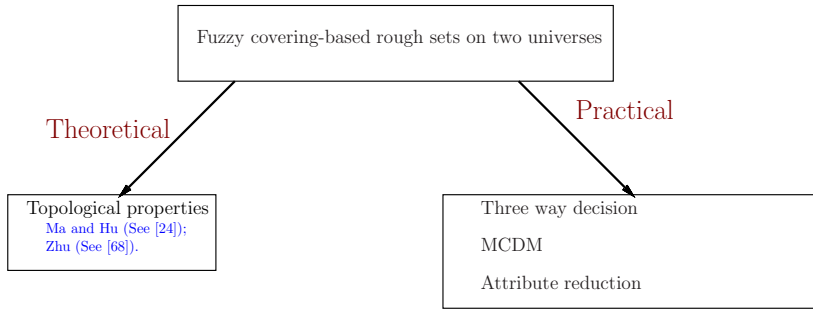
Then we can obtain the following results by analyzing $\underline{C}(X)$ and $\overline{C}(X)$ under the critical value $\beta = 0.8$.

- (1) Since $\underline{C}(X)(x_i) \vee \overline{C}(X)(x_i) < 0.8$ ($i = 3, 4, 5, 13$), we conclude that the students x_3, x_4, x_5 and x_{13} can not be recruited by the any research direction in V .
- (2) Since $\underline{C}(X)(x_i) \vee \overline{C}(X)(x_i) \geq 0.8$ ($i = 1, 2, 6, 7, 8, 9, 10, 11, 12, 14, 15$), we can conclude that the students x_1, x_2 can be recruited by the research direction y_1 , the students x_6, x_7, x_8 can be recruited by the research direction y_2 , the students x_9, x_{10}, x_{11} can be recruited by the research direction y_3 , the student x_{12} can be recruited by the research direction y_4 and the students x_{14}, x_{15} can be recruited by the research direction y_5 .

In this test example, we can see that the decision rule is depended on the score line β and the criterion $X \in \mathcal{F}(V)$. In other words, we can obtain different decisions by putting different β or different $X \in \mathcal{F}(V)$.

6 Conclusions

Fuzzy β -covering is a new notion defined by Ma in [32], which can build a bridge between covering-based rough set theory and fuzzy set theory. In this paper, a new type of fuzzy covering-based rough set model on two different universes was defined by using Zadeh's extension principle. Meanwhile, we proposed a

**Fig. 2** Further work

new approach for a kind of multiple criteria decision making problem by using fuzzy covering-based rough approximation operators. The core content and continuous work of this paper are summarized as follows:

- (1) Through a suitable blend of fuzzy β -neighborhood, we proposed a new type of fuzzy covering-based rough set model and studied some properties of it. Furthermore, the characteristic properties of fuzzy covering lower and upper approximation operations also have been proved.
- (2) We gave the concept of β -reduct of the fuzzy β -covering and a necessary and sufficient condition under which two fuzzy β -coverings generate the same fuzzy covering approximation operations have been proved, i.e., two fuzzy β -coverings generate the same fuzzy covering approximation operations of a fuzzy subset if and only if their β -reducts are equal. At the end of Section 3, we presented the matrix representations of the fuzzy covering lower and upper approximation operators.
- (3) We proposed a new approach to a kind of multiple criteria decision making problem based on the fuzzy covering-based rough set model over two universes. The decision rules and algorithm of the proposed method are given. Especially, this approach was illustrated by an example of handling multiple criteria decision making problem of graduate students recruiting.
- (4) There are several issues in fuzzy covering-based rough sets deserving further investigation. As shown in Figure. 2, topological of the fuzzy covering-based rough sets can be studied the same as [33, 89]. In practical viewpoint, three-way decisions [23, 74, 87] with fuzzy covering-based rough sets can be proposed. Moreover, application of the proposed approach for the multiple criteria decision making problems in this paper will be studied in further work. We also try to present methodology (algorithm) arising from the theory in our next research.
- (5) Moreover, the soft set theory has been conceived as another valuable mathematical method for tackling the uncertainty problem. In soft sets parameterization by attributes provide them an advantage over the other contemporary theories which deal with uncertainty. The authors of Atef et al. [2], Atef and Nada [3], Atef et al. [4], A. Mukherjee and S. Deb-nath [37, 38], successfully applied soft sets in the areas of decision-making

problems and soft linear equations. Based on these work, we try to to use the concept of soft set theory to generalize the proposed study in future.

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