

# Fuzzy H-continuous Mappings and Fuzzy Strongly Closed Graphs

K.Hur <sup>a</sup>, Y.S.Ahn <sup>b</sup>, and J.H. Ryou <sup>c</sup>

<sup>a</sup> Division of Mathematics and Informational Statistics  
Wonkwang University Iksan, Chonbuk, Korea 579-792  
E-mail:kulhur@wonkwang.ac.kr

<sup>b</sup> Dept. of Computer Information Science  
Dongkang College Kwangju, Korea 500-714  
E-mail:youngsin@dongkang.ac.kr

<sup>c</sup> Division of Mathematics and Informational Statistics  
Wonkwang University Iksan, Chonbuk, Korea 579-792  
E-mail:donggni@hanmail.net

## Abstract

We introduce the concepts of fuzzy H-continuity and fuzzy strongly closed graph, respectively and investigate some of their properties.

**Key words and phrases** : fuzzy almost continuous, fuzzy  $T_{2w}$ , fuzzy H-closed set, fuzzy H-continuous, fuzzy strongly closed graph, fuzzy closed graph.

## 1. Introduction and preliminaries

Since the introduction of fuzzy sets by Zadeh in his classic paper[17] of 1965 and fuzzy topological spaces by Chang[3] in 1968, certain mappings between topological spaces, weaker than usual open or continuous ones, have been generalized to and studied in fuzzy topological spaces by different authors[1, 2, 5, 6, 11, 13, 14, 16]. In this paper, we extend the notion of H-continuity introduced by P.E.Long and T.R. Hamlett[9] to fuzzy topological spaces.

In order to make the exposition self-contained as far as practicable, we list some definitions and results that will be used in the sequel. Let  $X$  be a non-empty(ordinary) set and let  $I$  the unit interval  $[0, 1]$ . A *fuzzy set*  $A$  in  $X$  is a mapping from  $X$  into  $I$ [17]. A *fuzzy point*  $x_\lambda$  in  $X$  is a fuzzy set in  $X$  defined by : for each  $y \in X$ ,

$$x_\lambda(y) = \begin{cases} \lambda, & \text{if } y = x, \\ 0, & \text{if } y \neq x, \end{cases}$$

where  $x \in X$  and  $\lambda \in (0, 1]$  are respectively called

the *support* and the *value* of  $x_\lambda$ [12,14]. A fuzzy point  $x_\lambda$  is said to *belong to* a fuzzy set  $A$  in  $X$  iff  $\lambda \leq A(x)$ [12]. A fuzzy set  $A$  in  $X$  is the union of all fuzzy points which belong to  $A$ [12]. A subfamily  $T$  of  $I^X$  is called a *fuzzy topology* on  $X$ [3] if (i)  $\emptyset, X \in T$ , (ii) for any  $\{U_\alpha\}_{\alpha \in \Lambda} \subset T$ ,  $\cup_{\alpha \in \Lambda} U_\alpha \in T$  and (iii) for any  $A, B \in T$ ,  $A \cap B \in T$ . In this case, each member of  $T$  is called a *fuzzy open*(in short, *F-open*) set in  $X$  and its complement a *fuzzy closed*(in short, *F-closed*) set in  $X$ . The pair  $(X, T)$  is called a *fuzzy topological space*(in short, *fts*). For a fts  $X$ ,  $FO(X)$  and  $FC(X)$  denote the collection of all F-open sets and F-closed sets in  $X$ , respectively. For a fuzzy set  $A$  in a fts  $X$ , the closure  $clA$  and the interior  $intA$  of  $A$  are defined respectively as  $clA = \cap \{V \in I^X : A \subset V \text{ and } V^c \in FO(X)\}$  and  $intA = \cup \{V \in FO(X) : V \subset A\}$ [12]. We will use the notion of fuzzy compactness in the sense of S. Ganguly and S. Saha[7].

**Definition 1.1**[17]. Let  $f$  be a mapping from a set  $X$  into a  $Y$ .  $A \in I^X$  and  $B \in I^Y$ . Then :

- (i) The *image* of  $A$  under  $f$ ,  $f(A)$  is a fuzzy set in  $Y$  defined by for each  $y \in Y$ ,  
 $[f(A)](y) = \sup_{x \in f^{-1}(y)} A(x)$  if  $f^{-1}(y) \neq \emptyset$ ,  
 $= 0$  otherwise,  
where  $f^{-1}(y) = \{x \in X : f(x) = y\}$ .
- (ii) The *inverse image* of  $B$  under  $f$ ,  $f^{-1}(B)$  is a fuzzy set in  $X$  denoted by for each  $x \in X$ ,

접수일자 : 2002년 3월 5일

완료일자 : 2002년 9월 16일

<sup>a</sup> Corresponding author

2000 Mathematics subject classification of AMS: 54A40, 54B20, 54C60

This paper was supported by Wonkwang University in 2002

$$f^{-1}(B)(x) = B(f(x)).$$

**Result 1.A[3,16].** Let  $f: X \rightarrow Y$  be a mapping. Then :

- (1)  $f^{-1}(B^c) = [f^{-1}(B)]^c$  for each  $B \in I^Y$ .
- (2)  $[f(A)]^c \subset f(A^c)$  for each  $A \in I^X$ .
- (3) If  $B_1 \subset B_2$ , then  $f^{-1}(B_1) \subset f^{-1}(B_2)$ ,  
where  $B_1, B_2 \in I^Y$ .
- (4) If  $A_1 \subset A_2$ , then  $f(A_1) \subset f(A_2)$ ,  
where  $A_1, A_2 \in I^X$ .
- (5)  $f(f^{-1}(B)) \subset B$  for each  $B \in I^Y$ .

In particular, if  $f$  is surjective, then

$$f(f^{-1}(B)) = B \text{ for each } B \in I^Y.$$

- (6)  $A \subset f^{-1}(f(A))$  for each  $A \in I^X$ .

In particular, if  $f$  is injective, then

$$f^{-1}(f(A)) = A \text{ for each } A \in I^X.$$

- (7) If  $\{B_\alpha\}_{\alpha \in \Lambda} \subset I^Y$ , then

$$f^{-1}(\cup_{\alpha \in \Lambda} B_\alpha) = \cup_{\alpha \in \Lambda} f^{-1}(B_\alpha)$$

and

$$f^{-1}(\cap_{\alpha \in \Lambda} B_\alpha) = \cap_{\alpha \in \Lambda} f^{-1}(B_\alpha).$$

- (8) If  $\{A_\alpha\}_{\alpha \in \Lambda} \subset I^X$ , then

$$f(\cup_{\alpha \in \Lambda} A_\alpha) = \cup_{\alpha \in \Lambda} f(A_\alpha).$$

- (9) Let  $g: Y \rightarrow Z$  be a mapping.

$$\text{Then } (g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$$

for each  $C \in I^Z$ .

**Result 1.B[4].** Let  $f: X \rightarrow Y$  be a mapping. Then :

- (1)  $f(x_\lambda) = [f(x)]_\lambda$  for each  $x_\lambda \in F_p(X)$ .
- (2) If  $A \in I^X$  and  $x_\lambda \in A$ , then  $f(x_\lambda) \in f(A)$ .
- (3) If  $A \in I^X$  and  $y_\lambda \in f(A)$ , then there exists  $x \in X$   
such that  $f(x) = y$  and  $x_\lambda \in A$ .
- (4) If  $B \in I^Y$ ,  $y \in f(X)$  and  $y_\lambda \in B$ , then for each  
 $x \in f^{-1}(y)$ ,  $x_\lambda \in f^{-1}(B)$ .
- (5) If  $B \in I^Y$  and  $x_\lambda \in f^{-1}(B)$ , then  $[f(x)]_\lambda \in B$ .

**Definition 1.2[12].** Let  $(X, T)$  be a fts and let  $Y$  a crisp subset of  $X$ . Then the family  $T_Y = \{A|_Y: A \in T\}$  is a fuzzy topology on  $Y$ . In this case,  $T_Y$  is called the *fuzzy relative topology* or *fuzzy subspace topology* of  $T$  to  $Y$  and the pair  $(Y, T_Y)$  is called a fuzzy subspace of  $(X, T)$ .

It is clear that  $A|_Y = A \cap Y$ .

**Definition 1.3[8].** Two fuzzy sets  $A$  and  $B$  in a set  $X$  are said to be *disjoint* if  $A \odot B = \emptyset$ , where  $(A \odot B)(x) = \max [0, A(x) + B(x) - 1]$  for each  $x \in X$ .

It is clear that  $A \odot B = \emptyset$  if and only if  $A \bar{q} B$ , i.e.,  $A \subset B^c$ .

**Definition 1.4[5].** A fts  $X$  is said to be *fuzzy  $T_{2w}$*  (in short,  $FT_{2w}$ ) if for any two distinct fuzzy points  $x_\lambda$  and  $y_\mu$  in  $X$ , there exist  $U, V \in FO(X)$  such that

$$x_\lambda \in U, y_\mu \in V \text{ and } U \odot V = \emptyset.$$

**Definition 1.5[12,14].** Let  $\{(X_\alpha, T_\alpha): \alpha \in \Lambda\}$  be a family of fts's, let  $X = \prod_{\alpha \in \Lambda} X_\alpha$  the usual Cartesian product of  $\{X_\alpha\}_{\alpha \in \Lambda}$  and let  $\pi_\alpha$  the projection form  $X$  onto  $X_\alpha$  for each  $\alpha \in \Lambda$ . Let  $\mathcal{P}_f(\Lambda)$  be the family of all finite subsets of  $\Lambda$ , let  $\mathcal{S} = \{\pi_\alpha^{-1}(B): B \in T_\alpha, \alpha \in \Lambda\}$  and let  $\mathcal{B} = \{\cap_{\alpha \in F} \pi_\alpha^{-1}(U_\alpha): U_\alpha \in T_\alpha, F \in \mathcal{P}_f(\Lambda)\}$ . Then there exists a unique fuzzy topology  $T$  on  $X$  for which  $\mathcal{B}$  is a base for  $T$  and  $\mathcal{S}$  is a subbase for  $T$ . In fact,  $T$  is the family of all unions of members of  $\mathcal{B}$ .

In this case,  $T$  is called the *fuzzy product topology* on  $X$  and the pair  $(X, T)$  is called the *fuzzy product topological space* (in short, *product fts*).

**Definition 1.6[2].** Let  $A$  be a fuzzy set in a fts  $X$ . Then :

- (1)  $A$  is called a *fuzzy regular open set* in  $X$  if  $A = \text{int}(\text{cl} A)$ .
- (2)  $A$  is called a *fuzzy regular closed set* in  $X$  if  $A = \text{cl}(\text{int} A)$ .

We denote the collection of all fuzzy regular open[resp. closed] set in  $X$  as  $FRO(X)$ [resp.  $FRC(X)$ ].

It is clear that  $FRO(X) \subset FO(X)$  and  $FRC(X) \subset FC(X)$ .

**Result 1.C[1, Lemma 3.1].** Let  $A$  be a fuzzy set in a fts  $X$ . Then :

- (1)  $\text{int}(\text{cl} A) \in FRO(X)$ .
- (2)  $\text{cl}(\text{int} A) \in FRC(X)$ .

**Definition 1.7[2].** Let  $X$  and  $Y$  be fts's. Then a mapping  $f: X \rightarrow Y$  is said to be *fuzzy almost continuous* (in short, *fal-continuous*) if for each  $V \in FRO(Y)$ ,  $f^{-1}(V) \in FO(X)$ .

It is clear that every F-continuous mapping is fal-continuous.

**Result 1.D[11, Theorem 3.3].** Let  $X$  and  $Y$  be fts's. Then a mapping  $f: X \rightarrow Y$  is fal-continuous if and only if for each  $x_\lambda \in F_p(X)$  and each  $V \in FO(Y)$  with  $f(x_\lambda) \in V$ , there exists  $U \in FO(X)$  such that  $x_\lambda \in U$  and  $f(U) \subset \text{int}(\text{cl} V)$ .

## 2. Fuzzy H-continuous mappings

From now on, we consider  $X, Y, Z$  as fts's.

**Definition 2.1.** Let  $A \in I^X$ . Then  $A$  is said to be *fuzzy H-closed relative to  $X$*  (in short, *fH-closed*) if for each F-open cover  $\{V_\alpha\}_{\alpha \in \Lambda}$  of  $A$  in  $X$ , there exists a finite subfamily  $A_0$  of  $\Lambda$  such that  $A \subset \cup_{\alpha \in A_0} (\text{cl} V_\alpha)$ . The fts  $X$  is said to be a *fH-closed space* if for each

F-open cover  $\{V_\alpha\}_{\alpha \in \Lambda}$  of  $X$ , there exists a finite subfamily  $\Lambda_0$  of  $\Lambda$  such that  $\bigcup_{\alpha \in \Lambda_0} (\text{cl } V_\alpha) = X$ .

**Lemma 2.2.** Let  $X$  be a  $FT_{2w}$ -space. If  $B$  is fH-closed in  $X$ , then  $B \in FC(X)$ .

**Proof.** Assume that  $\text{cl } B \not\subset B$ . Then there exists an  $x_\lambda \in F_p(X)$  such that  $x_\lambda \in \text{cl } B$  but  $x_\lambda \notin B$ . Let  $y_\mu \in B$ . Since  $X$  is  $FT_{2w}$ , there exist  $U_{y_\mu}$  and  $V_{y_\mu} \in FO(X)$  such that  $x_\lambda \in U_{y_\mu}$ ,  $y_\mu \in V_{y_\mu}$  and  $U_{y_\mu} \odot V_{y_\mu} = \emptyset$ . Let  $\mathcal{V} = \{V_{y_\mu} \in FO(X) : y_\mu \in B\}$ . Then clearly  $\mathcal{V}$  is a fuzzy open cover of  $B$ . Since  $B$  is fH-closed in  $X$ , there exist  $y_1, \dots, y_n \in B$  such that  $B \subset (\text{cl } V_{y_1} \cup \dots \cup \text{cl } V_{y_n}) = \text{cl}(\bigcup_{i=1}^n V_{y_i})$ , where  $\{V_{y_1}, \dots, V_{y_n}\} \subset \mathcal{V}$ . For each  $i = 1, \dots, n$ , let  $U_{y_i} \in FO(X)$  such that  $x_\lambda \in U_{y_i}$  and  $U_{y_i} \odot V_{y_i} = \emptyset$  i.e.,  $V_{y_i} \subset U_{y_i}^c$ . Then  $B \subset \text{cl}(\bigcup_{i=1}^n V_{y_i}) \subset \text{cl}(\bigcup_{i=1}^n U_{y_i}^c) = \text{cl}(\bigcap_{i=1}^n U_{y_i})^c$ . Thus  $\text{cl } B \subset \text{cl}(\bigcap_{i=1}^n U_{y_i})^c$ . Since  $x_\lambda \in \text{cl } B$ ,  $x_\lambda \in \text{cl}(\bigcap_{i=1}^n U_{y_i})^c$ . But  $x_\lambda \notin \text{cl}(\bigcap_{i=1}^n U_{y_i})^c$ . This is a contradiction. So  $\text{cl } B \subset B$ . Hence  $B \in FC(X)$ .

**Definition 2.3.** A mapping  $f: X \rightarrow Y$  is said to be *fuzzy H-continuous* (in short, *fH-continuous*) if for each  $x_\lambda \in F_p(X)$  and each  $V \in FO(Y)$  such that  $f(x_\lambda) \in V$  and  $V^c$  is fH-closed in  $Y$ , there exists  $U \in FO(X)$  such that  $x_\lambda \in U$  and  $f(U) \subset V$ .

**Theorem 2.4.** Let  $f: X \rightarrow Y$  be a mapping. Then the following are equivalent:

- (1)  $f$  is fH-continuous.
  - (2) If  $V \in FO(Y)$  and  $V^c$  is fH-closed in  $Y$ , then  $f^{-1}(V) \in FO(X)$ .
- These statements are implied by
- (3) If  $B$  is fH-closed in  $Y$ , then  $f^{-1}(B) \in FC(X)$ .

Furthermore, if  $Y$  is  $FT_{2w}$ , then all three statements are equivalent.

**Proof.** (1)  $\Rightarrow$  (2): Suppose  $f: X \rightarrow Y$  is fH-continuous. Let  $V \in FO(Y)$  and let  $V^c$  be fH-closed in  $Y$ . Let  $x_\lambda \in f^{-1}(V)$ . Then  $f(x_\lambda) \in V$ . By the hypothesis, there exists  $U \in FO(X)$  such that  $x_\lambda \in U$  and  $f(U) \subset V$ . Thus  $x_\lambda \in U \subset f^{-1}(V)$ . Hence, by Proposition 1.8 in [10],  $f^{-1}(V) \in FO(X)$ .

(2)  $\Rightarrow$  (1): Suppose the condition (2) holds. Let  $x_\lambda \in F_p(X)$  and let  $V \in FO(Y)$  such that  $f(x_\lambda) \in V$  and  $V^c$  is fH-closed in  $Y$ . Then, by the hypothesis,  $f^{-1}(V) \in FO(X)$  and  $x_\lambda \in f^{-1}(V)$ . Let  $U = f^{-1}(V)$ . Then clearly  $x_\lambda \in U \in FO(X)$  and  $f(U) \subset V$ . Hence  $f$  is fH-continuous.

(3)  $\Rightarrow$  (2): Suppose the condition (3) holds. Let  $V \in FO(Y)$  and let  $V^c$  be fH-closed in  $Y$ . Then, by the hypothesis,  $f^{-1}(V^c) = [f^{-1}(V)]^c \in FC(X)$ . Hence  $f^{-1}(V) \in FO(X)$ .

Now assume that  $Y$  is  $FT_{2w}$  and we show that (2) implies (3). Let  $B$  be any fH-closed set in  $Y$ . Since  $Y$  is  $FT_{2w}$ , by Lemma 2.2,  $B \in FC(Y)$ . Then  $B^c \in FO(Y)$ . Thus, by the condition (2),  $f^{-1}(B^c) = [f^{-1}(B)]^c \in FO(X)$ . Hence  $f^{-1}(B) \in FC(X)$ .

In general, the condition (2) does not imply the condition (3) as shown in Example.

**Example 2.5.** Let  $X = \{a, b, c\}$  and let  $Y = \{x, y\}$ . Consider the fuzzy topologies  $T_X$  and  $T_Y$  on  $X$  and  $Y$ , respectively defined by :

$$T_X = \{\emptyset, X, \{(a, 0.2), (b, 0.2), (c, 0.3)\}\}$$

and

$$T_Y = \{\emptyset, Y, \{(x, 0.2), (y, 0.3)\}\}.$$

Let  $f: (X, T_X) \rightarrow (Y, T_Y)$  be the mapping defined by  $f(a) = f(b) = x$  and  $f(c) = y$ . Let  $V = \{(x, 0.2), (y, 0.3)\}$ . Then clearly  $V \in FO(Y)$  and  $V^c$  is fH-closed in  $Y$ . Moreover,  $f^{-1}(V) \in FO(X)$ . So the condition (2) holds.

On the other hand, let  $B = \{(x, 0.2), (y, 0.3)\}$ . Then  $B$  is fH-closed in  $Y$  but  $f^{-1}(B) \notin FC(X)$ . Hence the condition (3) does not hold.

**Theorem 2.6.** A mapping  $f: X \rightarrow Y$  is fH-continuous if and only if for each F-closed fH-closed  $B$  in  $Y$ ,  $f^{-1}(B) \in FC(X)$ .

**Proof.** ( $\Rightarrow$ ): Suppose  $f$  is fH-continuous. Let  $B$  be any F-closed fH-closed set in  $Y$ . Then  $B^c \in FO(Y)$  and  $(B^c)^c = B$  is fH-closed in  $Y$ . By Theorem 2.4,  $f^{-1}(B^c) = [f^{-1}(B)]^c \in FO(X)$ . Hence  $f^{-1}(B) \in FC(X)$ .

( $\Leftarrow$ ): Suppose the necessary condition holds. Let  $V \in FO(Y)$  such that  $V^c$  is fH-closed in  $Y$ . Then clearly  $V^c$  is F-closed fH-closed in  $Y$ . By the hypothesis,  $f^{-1}(V^c) = [f^{-1}(V)]^c \in FC(X)$ . Thus  $f^{-1}(V) \in FO(X)$ . Hence, by Theorem 2.4,  $f$  is fH-continuous.

**Lemma 2.7.** If  $A$  and  $B$  are fH-closed in  $X$ , then  $A \cup B$  is fH-closed in  $X$ .

**Proof.** Let  $\{V_\alpha\}_{\alpha \in \Lambda}$  be a F-open cover of  $A \cup B$  in  $X$ . Since  $A \subset A \cup B$  and  $B \subset A \cup B$ ,  $\{V_\alpha\}_{\alpha \in \Lambda}$  is a F-open cover of  $A$  and  $B$  in  $X$ , respectively. Since  $A$  and  $B$  are fH-closed in  $X$ , there exist finite collections  $\Lambda_1$  and  $\Lambda_2$  of  $\Lambda$  such that  $A \subset \bigcup_{\alpha \in \Lambda_1} (\text{cl } V_\alpha)$  and  $B \subset \bigcup_{\beta \in \Lambda_2} (\text{cl } V_\beta)$ . Then  $A \cup B \subset \bigcup_{(\alpha, \beta) \in \Lambda_1 \times \Lambda_2} \text{cl}$

$(V_\alpha \cup V_\beta)$  and  $A_1 \times A_2$  is finite. Hence  $A \cup B$  is fH-closed in  $X$ .

**Lemma 2.8.** Let  $(X, T)$  be a fts and let  $\mathcal{B}^* = \{V \in T : V^c \text{ is fH-closed in } X\}$ . Then there exists a unique fuzzy topology  $T^*$  on  $X$  for which  $\mathcal{B}^*$  is a fuzzy base for  $T^*$ .

Furthermore  $T^* \subset T$  and  $(X, T^*)$  is always a fuzzy H-closed space. In this case,  $(X, T^*)$  is called the *induced H-closed fuzzy topological space* by  $(X, T)$  and will be denoted by  $X^*$ .

**Proof.** Clearly  $X \in \mathcal{B}^*$ . So  $\cup \mathcal{B}^* = X$ . Let  $B_1, B_2 \in \mathcal{B}^*$  and let  $x_\lambda \in B_1 \cap B_2$ . Then  $B_1, B_2 \in T$ ,  $B_1^c$  and  $B_2^c$  are fH-closed in  $X$ . Thus, by Lemma 2.7,  $B_1^c \cup B_2^c$  is fH-closed in  $X$ . Since  $B_1^c \cup B_2^c = (B_1 \cap B_2)^c$ ,  $(B_1 \cap B_2)^c$  is fH-closed in  $X$ . Moreover  $B_1 \cap B_2 \in T$ . Then  $B_1 \cap B_2 \in \mathcal{B}^*$ . Hence this completes the proof.

**Result 2.A[8, Lemma 3.7].** Let  $(X, T)$  be a fts and let  $\mathcal{B}_* = \{V \in T : V^c \text{ is F-compact in } X\}$ . Then there exists a unique fuzzy topology  $T_*$  on  $X$  for which  $\mathcal{B}_*$  is a fuzzy base for  $T_*$ .

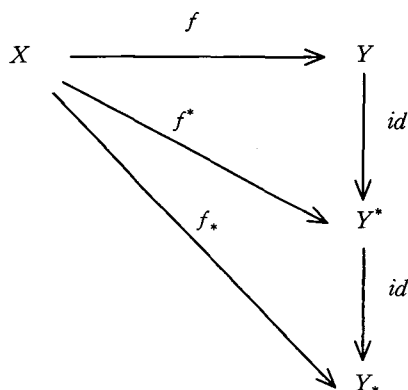
Furthermore  $T_* \subset T$  and  $(X, T_*)$  is always a compact fuzzy topological space. In this case,  $(X, T_*)$  is called the *induced compact fuzzy topological space* by  $(X, T)$  and will be denoted by  $X_*$ .

It is clear that  $T_* \subset T^* \subset T$ .

**Proposition 2.9.** Let  $f: X \rightarrow Y$  a mapping. Define the mappings  $f^*: X \rightarrow Y^*$  and  $f_*: X \rightarrow Y_*$  as follows, respectively : for each  $x \in X$ ,

$$f^*(x) = f(x) \text{ and } f_*(x) = f(x).$$

Consider the following diagram :



Then the following hold.

- (1)  $f$  is fH-continuous if and only if  $f^*$  is F-continuous.
- (2) [8, Proposition 3.8]  $f$  is fc-continuous if and only if  $f_*$  is F-continuous.

(3)  $id: Y \rightarrow Y^*$  and  $id: Y^* \rightarrow Y_*$  are F-continuous.

(4)  $id^{-1}: Y^* \rightarrow Y$  is fH-continuous.

(5)  $id^{-1}: Y_* \rightarrow Y^*$  is fc-continuous.

**Theorem 2.10.** Let  $f: X \rightarrow Y$  be fH-continuous. If  $f^*: X \rightarrow Y^*$  is F-closed (resp. F-open), then  $f$  is F-closed (resp. F-open).

**Proof.** Suppose  $f^*: X \rightarrow Y^*$  is F-closed (resp. F-open). Let  $F \in FC(X)$  (resp.  $F \in FO(X)$ ). Then, by the hypothesis,  $f^*(F)$  is F-closed (resp. F-open) in  $Y^*$ . By Proposition 2.9(3),  $id: Y \rightarrow Y^*$  is F-continuous. Then  $id^{-1}(f^*(F))$  is F-closed (resp. F-open) in  $Y$ . But  $id^{-1}(f^*(F)) = f(F)$ . So  $f(F)$  is F-closed (resp. F-open) in  $Y$ . Hence  $f$  is F-closed (resp. F-open).

**Theorem 2.11.** If  $f: X \rightarrow Y$  is fH-continuous and  $A \in P(X)$ , then  $f|_A: A \rightarrow Y$  is fH-continuous.

**Proof.** Let  $V \in FO(Y)$  such that  $V^c$  is fH-closed in  $Y$ . Then, by Theorem 2.4(2),  $f^{-1}(V) \in FO(X)$ . Thus  $(f|_A)^{-1}(V) = f^{-1}(V) \cap A \in FO(A)$ . Hence  $f|_A$  is fH-continuous.

**Theorem 2.12.** If  $f: X \rightarrow Y$  is F-continuous and  $g: Y \rightarrow Z$  is fH-continuous, then  $g \circ f: X \rightarrow Z$  is fH-continuous.

**Proof.** Let  $W \in FO(Z)$  such that  $W^c$  is fH-closed in  $Z$ . Then, by Theorem 2.4(2),  $g^{-1}(W) \in FO(Y)$ . Since  $f$  is F-continuous,  $f^{-1}(g^{-1}(W)) \in FO(X)$ . But  $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ . Thus  $(g \circ f)^{-1}(W) \in FO(X)$ . Hence, by Theorem 2.4(2),  $g \circ f$  is fH-continuous.

### 3. Fuzzy strongly closed graphs

Let  $f: X \rightarrow Y$  be a mapping. Then the subset  $G(f) = \{(x, f(x)) : x \in X\}$  of the Cartesian product  $X \times Y$  is called the *graph of  $f$* .

**Definition 3.1.** Let  $X$  and  $Y$  be fts's. Then a mapping  $f: X \rightarrow Y$  is said to *have a fuzzy strongly closed graph* (in short, *F-strongly closed graph*) or the graph  $G(f)$  is said to be *fuzzy strongly closed* (in short, *F-strongly closed*) in  $X \times Y$  if for each  $(x_\lambda, y_\mu) \notin F_p(G(f))$ , there exist  $U \in FO(X)$  and  $V \in FO(Y)$  such that  $x_\lambda \in U$ ,  $y_\mu \in V$  and  $(U \times cl V) \odot G(f) = \emptyset$ .

**Definition 3.2.** Let  $X$  and  $Y$  be fts's and let  $f: X \rightarrow Y$  be a mapping. Then the graph  $G(f)$  of  $f$  is said to *have an upper fuzzy point* in  $X \times Y$  provided that for each  $(x_\lambda, y_\mu) \notin F_p(G(f))$ , there exist  $U \in FO(X)$  and  $V \in FO(Y)$  such that  $x_\lambda \in U$ ,  $y_\mu \in V$  and if  $(U \times cl V) \odot G(f) \neq \emptyset$ , then there exists

$(a, b) \in G(f)$  such that  $(U \times \text{cl } V)(a, b) > \frac{1}{2}$ .

**Lemma 3.3.** Let  $X$  and  $Y$  be fts's, let  $f: X \rightarrow Y$  a mapping and let  $G(f)$  have an upper fuzzy point in  $X \times Y$ . Then  $f$  has a F-strongly closed graph if and only if for each  $x_\lambda \in F_p(X)$  and each  $y_\mu \in F_p(Y)$  such that  $y \neq f(x)$ , there exist  $U \in FO(X)$  and  $V \in FO(Y)$  such that  $x_\lambda \in U$ ,  $y_\mu \in V$  and  $f(U) \odot \text{cl } V = \emptyset$ .

**Proof.** ( $\Rightarrow$ ): Suppose  $f$  has a F-strongly closed graph. Let  $x_\lambda \in F_p(X)$  and let  $y_\mu \in F_p(Y)$  such that  $y \neq f(x)$ . Then clearly  $(x_\lambda, y_\mu) \notin F_p(G(f))$ . By the hypothesis, there exist  $U \in FO(X)$  and  $V \in FO(Y)$  such that  $x_\lambda \in U$ ,  $y_\mu \in V$  and  $(U \times \text{cl } V) \odot G(f) = \emptyset$ . Assume that  $f(U) \odot \text{cl } V \neq \emptyset$ . Then  $f(U) \cap \text{cl } V \neq \emptyset$ . Thus there exists  $b \in Y$  such that  $f(U)(b) + \text{cl } V(b) > 1$ , i.e.,  $\sup_{z \in f^{-1}(b)} U(z) + \text{cl } V(b) > 1$ . So there exists an  $a \in X$  such that  $b = f(a)$  and  $U(a) + \text{cl } V(b) > 1$ , i.e.,  $(U \times \text{cl } V)(a, b) > 0$ . Since  $(U \times \text{cl } V) \odot G(f) = \emptyset$  and  $(a, b) \in G(f)$ ,  $(U \times \text{cl } V)(a, b) = 0$ . This is a contradiction. Hence  $f(U) \odot \text{cl } V = \emptyset$ .

( $\Leftarrow$ ): Suppose the necessary condition holds. Let  $(x_\lambda, y_\mu) \notin F_p(G(f))$ . Then clearly

$x_\lambda \in F_p(X)$ ,  $y_\mu \in F_p(Y)$  and  $y \neq f(x)$ . By the hypothesis, there exist  $U \in FO(X)$  and  $V \in FO(Y)$  such that  $x_\lambda \in U$ ,  $y_\mu \in V$  and  $f(U) \odot \text{cl } V = \emptyset$ . Assume that  $(U \times \text{cl } V) \odot G(f) \neq \emptyset$ . Since  $G(f)$  has an upper fuzzy point in  $X \times Y$ , there exists  $(a, b) \in G(f)$  such that  $(U \times \text{cl } V)(a, b) > \frac{1}{2}$ , i.e.,

$\min [U(a), \text{cl } V(b)] > \frac{1}{2}$ . Then :

$$\begin{aligned} f(U)(b) + \text{cl } V(b) - 1 &= \sup_{z \in f^{-1}(b)} U(z) \\ &\quad + \text{cl } V(b) - 1 \\ &\geq U(a) + \text{cl } V(b) - 1 \\ &\geq 2 \min [U(a), \\ &\quad \text{cl } V(b)] - 1 \\ &> 0. \end{aligned}$$

Thus  $f(U) \odot \text{cl } V \neq \emptyset$ . This is a contradiction. So  $(U \times \text{cl } V) \odot G(f) = \emptyset$ . Hence  $f$  has a F-strongly closed graph.

**Remark 3.4.** If  $f: X \rightarrow Y$  has a F-strongly closed graph, then for each  $x_\lambda \in F_p(X)$  and each  $y_\mu \in F_p(Y)$  such that  $y \neq f(x)$ , there exist  $U \in FO(X)$  and  $V \in FO(Y)$  such that  $x_\lambda \in U$ ,  $y_\mu \in V$  and  $f(U) \odot \text{cl } V = \emptyset$ .

**Theorem 3.5.** Let  $f: X \rightarrow Y$  be fal-continuous and let  $G(f)$  have an upper fuzzy point in  $X \times Y$ . If  $Y$  is  $FT_{2w}$ , then  $f$  has a F-strongly closed graph.

**Proof.** Let  $x_\lambda \in F_p(X)$  and let  $y_\mu \in F_p(Y)$  such that  $y \neq f(x)$ . Then  $y_\mu \neq f(x_\lambda)$ . Since  $Y$  is  $FT_{2w}$ , there exists  $V \in FO(X)$  such that  $y_\mu \in V$  and

$f(x_\lambda) \in (\text{cl } V)^c$ . Since  $\text{cl } V \in FRC(Y)$ ,  $(\text{cl } V)^c \in FRO(Y)$ . Since  $f$  is fal-continuous, there exists  $U \in FO(X)$  such that  $x_\lambda \in U$  and  $f(U) \subset (\text{cl } V)^c$ . Then  $f(U) \odot \text{cl } V = \emptyset$ . Hence, by Lemma 3.3,  $f$  has a F-strongly closed graph.

**Corollary 3.5.** Let  $f: X \rightarrow Y$  be F-continuous and let  $G(f)$  have an upper fuzzy point in  $X \times Y$ . If  $Y$  is  $FT_{2w}$ , then  $f$  has a F-strongly closed graph.

**Theorem 3.6.** Let  $f: X \rightarrow Y$  be a surjection with F-strongly closed graph. Then  $Y$  is  $FT_{2w}$ .

**Proof.** Let  $y_\lambda$  and  $z_\mu$  be distinct fuzzy point in  $Y$ . Since  $f$  is surjective, there exists  $x \in X$  such that  $y_\lambda = f(x_\lambda)$ . Then  $(x_\lambda, z_\mu) \notin F_p(G(f))$ . Since  $f$  has a F-strongly closed graph, by Remark 3.4, there exist  $U \in FO(X)$  and  $V \in FO(Y)$  such that  $x_\lambda \in U$ ,  $z_\mu \in V$  and  $f(U) \odot \text{cl } V = \emptyset$ . Thus  $f(x_\lambda) = y_\lambda \in (\text{cl } V)^c$ . Hence  $Y$  is  $FT_{2w}$ .

The following is the immediate result of Theorem 3.6 and Corollary 3.5.

**Theorem 3.7.** A fts  $X$  is  $FT_{2w}$  if and only if the identity mapping  $id: X \rightarrow X$  has a F-strongly closed graph.

**Theorem 3.8.** If a mapping  $f: X \rightarrow Y$  has a F-strongly closed graph, then it is fH-continuous.

**Proof.** Let  $K$  be any fH-closed set in  $Y$  and let  $x_\lambda \in [f^{-1}(K)]^c$ . Let  $y_\mu \in K$ . Then clearly  $(x_\lambda, y_\mu) \notin F_p(G(f))$ . Since  $f$  has a F-strongly closed graph, by Lemma 3.3, there exist  $U_{y_\mu} \in FO(X)$  and  $V_{y_\mu} \in FO(Y)$  such that  $x_\lambda \in U_{y_\mu}$ ,  $y_\mu \in V_{y_\mu}$  and  $f(U_{y_\mu}) \odot \text{cl } V_{y_\mu} = \emptyset$ . Consider the family  $\{V_{y_\mu} \in FO(Y) : y_\mu \in K\}$ . Then clearly  $\{V_{y_\mu} \in FO(Y) : y_\mu \in K\}$  is an F-open cover of  $K$  in  $Y$ . Since  $K$  is fH-closed in  $Y$ , there exists a finite subset  $K_0$  of  $K$  such that  $K \subset \bigcup_{y_\mu \in K_0} (\text{cl } V_{y_\mu})$ . Let  $U = \bigcap \{U_{y_\mu} \in FO(X) : x_\lambda \in U_{y_\mu} \text{ and } y_\mu \in K_0\}$ . Then  $x_\lambda \in U \in FO(X)$  and  $U \odot f^{-1}(K) = \emptyset$ , i.e.,  $U \subset [f^{-1}(K)]^c$ . Thus  $[f^{-1}(K)]^c \in FO(X)$ . So  $f^{-1}(K) \in FC(X)$ . Hence, by Theorem 2.4,  $f$  is fH-continuous.

## References

- [1] N.Ajmal and S.K.Azad, Fuzzy almost continuity and its pointwise characterization by dual points and fuzzy nets, Fuzzy Sets and Systems, 34(1990), 81-101.
- [2] K.K.Azad, On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math.

저 자 소 개

- Anal. Appl., 82(1981), 14-32.
- [3] C.L.Chang, Fuzzy topological spaces , J. Math. Anal. Appl., 24(1968), 182-190.
- [4] Z.Deng, Fuzzy pseudo-metric spaces, J. Math. Anal. Appl., 86(1982), 74-95.
- [5] A.A. Fora, Fuzzy separation axioms and fuzzy continuity, Arab Gulf J. Scient. Res., Math. Phys. sci., A5(3)(1987), 307-318.
- [6] S. Ganguly and S.Saha, On separation axioms and  $T_i$ -fuzzy continuity, Fuzzy Sets and Systems, 16(1985), 265-275.
- [7] \_\_\_\_\_, A note on compactness in a fuzzy setting, Fuzzy Sets and Systems, 34(1990), 117-124.
- [8] K. Hur, J.R.Moon and J.H.Ryou, Fuzzy c-continuous mappings. International Journal of Fuzzy Logic and Intelligent Systems, Vol.1, no.1(2001), 50-55.
- [9] P.E. Long and T.R. Hamlett, H - continuous functions, Bollettino U.M.Z.(4)11 (1975), 552-558.
- [10] C.De Mitri and E.Pascali, Characterization of fuzzy topologies from neighborhoods of fuzzy points, J. Math. Anal. Appl., 93(1983), 1-14.
- [11] M.N.Mukherjee and S.P.Sina, On some weaker forms of fuzzy continuous and fuzzy open mappings on fuzzy topological spaces, Fuzzy Sets and Systems, 32(1989), 103-114.
- [12] Pu Pao-Ming and Liu Ying-Ming, Fuzzy topology I : neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., 76(1980), 571-599.
- [13] \_\_\_\_\_, Fuzzy topology II: product and quotient spaces, J. Math. Anal. Appl., 77(1980), 20-33.
- [14] C.K.Wong, Fuzzy topology, product and quotient theorems, J. Math. Anal. Appl., 45(1974), 512-521.
- [15] \_\_\_\_\_, Fuzzy points and local properties of fuzzy topology, J. Math. Anal. Appl., 46(1974), 316-328.
- [16] T. H. Yalvac, Fuzzy sets and functions on fuzzy spaces, J. Math. Anal. Appl., 126(1987), 409-423.
- [17] L. A. Zadeh, Fuzzy sets, Inform and Control, 8(1965), 338-353.

**K.Hur** <sup>a</sup>

제 8권 6호 참조

**Y.S.Ahn** <sup>b</sup>

제 11권 2호 참조

**J.H. Ryou** <sup>c</sup>

제 8권 6호 참조