International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 5, Issue 8, 2017, PP 8-14
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
DOI: http://dx.doi.org/10.20431/2347-3142.0508002
www.arcjournals.org

Fuzzy ideal of Partially Ordered Near-Ring

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Abstract: *In this paper we introduce the notion of fuzzy ideal of a partially ordered near-ring (PON), T-fuzzy ideal of PON, normal T-fuzzy ideal of PON and fuzzy magnified translation. Also we study the characterizations partially ordered near-rings.*

Keywords: *Fuzzy ideal of PON, T-fuzzy ideal of PON, normal T-fuzzy ideal of PON and f-invariant.*

1. Introduction

The important concept of fuzzy set has been introduced by L.A. Zadeh [14] in 1965. Since then many papers on fuzzy sets appeared showing its importance in the field of mathematics. The notion of a fuzzy subset was introduced by Zadeh [14] and then fuzzy subsets have been applied to various branches of mathematics. Near-rings are one of the generalized structures of rings. W.J. Liu [3] has studied fuzzy invariant subgroups and fuzzy ideals. In1998 S. D. Kim and H. S. kim [2] has been introduced analogue of fuzzy ideals of near-rings. G. Pilz [9] introduced near-rings and in 1971 Rosenfeld [11] initiated the study of fuzzy subgroups. In [4] M. Muralikrishna Rao studied T-fuzzy ideal in ordered T-semi rings. In [10] A. Radha Krishna and M. Bandari defined the partially ordered (P.O) near-ring. M. Akram[l] introduced the notion of fuzzy ideals in near-rings with respect to a t-norm T. In [12] Bh. Satyanarayana introduced Γ-near-rings. T. Srinivas and T. Nagaiah [13] introduced the notion of T-fuzzy ideals of Γ- near-rings and investigated some of their properties. After that T. Nagaiah et al. [5, 6, 7] studied fuzzy ideals of partially ordered Γ-semi groups and anti fuzzy ideals in near-ring. In [8] T. Nagaiah and L. Bhaskar introduced the notion of T-fuzzy ideal of PON. In this direction we study the fuzzy ideals of partially ordered near-rings.

2. Preliminaries

For the sake of continuity we recall the following definitions.

Definition 1.*A non-empty set* N *with two binary operations* "+" *and* "·" *is called a near-ring if*

(i) *(N, +) is a group (not necessarily abelian)*

(ii) *(N, ·) is a semi group*

(iii) $x\cdot (y+z) = x\cdot y + x\cdot z$ *for all* $x, y, z \in N$.

We will use the word "near-ring" to mean "left near-ring".

Definition 2[4]. A t-norm is a function T: [0, l] X [0, l] $\rightarrow$ [0, 1] that satisfies the following condition for all x, y, z $\in$ [0, 1]

(i) $T(x, 1) = x$

(ii) $T(x, y) = T(y, x)$ (commutativity)

(iii) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity)

(iv) $T(x, y) \le T(x, z)$ whenever $y \le z$ (monotonicity).

Note that a t-norm $T(0, 0) = 0$, $T(l, 1) = l$ and $T(x, y) \le \min(x, y)$.

Definition 3[10].*Let N be a near-ring. A near-ring N is called a PON if*

(i) $a \leq b$ *then* $a + g \leq b + g \ \forall a, b, g \in N$

(ii) $a \leq b$ *and* $c \geq 0$ *then* $ac \leq bc$ *and* $ca \leq cb \ \forall a, b, c \in N$.

Definition 4[8].*Let μ be a fuzzy subset of X and* $a \in [0, 1\text{-}\sup\{\mu(x)/x \in X\}]$, $b \in [0, 1]$. *The mappings*

$\mu_\alpha^T : X \to [0,1]$, $\mu_b^M : X \to [0,1]$ *and* $\mu_b^M{}_a^T : X \to [0,1]$ *are called fuzzy translation, fuzzy multiplication and fuzzy magnified translation of μ respectively* $\mu_a^T(x) = \mu(x) + a$, $\mu_b^M(x) = b\mu(x)$ *and* $\mu_b^M{}_a^T(x) = b\mu(x) + a$ *for all* $x \in X$ *respectively.*

3. Fuzzy ideals of partially ordered near-rings

Definition 5.Let N be a PON. A fuzzy subset μ of N is said to be a fuzzy sub near-ring of N if

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

(iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ *for all* $x, y \in N$

Definition 6.Let μ be a non-empty fuzzy subset of a PON N. Then μ is called a fuzzy ideal of N if

(i) $\mu(x-y) \geq \min\{\mu(x),\{\mu(y)\}$

(ii) $\mu(xy) \geq \mu(y)$

(iii) $\mu(x+z)y - xy) \geq \mu(z)$

(iv) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ *for all* $x, y \in N$

Note that μ *is fuzzy left ideal of N if it satisfies (i), (ii) and (iv) and*

μ is fuzzy right ideal of N if it satisfies (i), (iii) and (iv).

Definition7.A fuzzy subset μ of PON N is called T-fuzzy right (resp. left) ideal if

(i) $\mu(x - y) \geq T(\mu(x), \mu(y))$

(ii) $\mu((x + z)y - xy) \geq \mu(z)$ $(\mu(xy) \geq \mu(y))$

iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ *for all* $x, y, z \in N$.

If μ is a T-fuzzy left ideal and T-fuzzy right ideal of a PON then μ is called a T-fuzzy ideal of N.

Theorem (1): *If* $\{\mu_i : i \in I\}$ *is a family of T-fuzzy ideal of PON N, then* $\underset{i\in I}{V} \mu_i$ *is also a T-fuzzy ideal of N where* $\underset{i\in I}{V} \mu_i$ *is defined by* $\left(\underset{i\in I}{V} \mu_i\right)(x) = \sup\{\mu_i(x) : i \in I\}$ *for all* $x \in N$

Proof. *Let* $\{\mu_i : i \in I\}$ be a family of T-fuzzy ideal of a PON N. For any x, y, z$\in N$ then

$$(i)\ \left(\underset{i\in I}{V} \mu_i\right)(x-y) = \sup\{\mu_i(x-y) : i \in I\}$$
$$\geq \sup\{T(\mu_i(x), \mu_i(y)) : i \in I\}$$
$$= T\{\sup \mu_i(x) : i \in I, \sup \mu_i(y) : i \in I\}$$
$$= T\left(\left(\underset{i\in I}{V} \mu_i\right)(x), \left(\underset{i\in I}{V} \mu_i\right)(y)\right)$$

$$(ii)\quad \left(\underset{i\in I}{V}\mu_i\right)(xy)=\sup\{\mu_i(x\alpha y): i\in I\}$$
$$\geq \sup\{\mu_i(y): i\in I\}$$
$$=\left(\underset{i\in I}{V}\mu_i\right)(y)\qquad and$$
$$\left(\underset{i\in I}{V}\mu_i\right)((x+z)y-xy)=\sup[\mu_i(x+z)y-xy): i\in I|$$
$$\geq \sup\{\mu_i(z): i\in I\}$$
$$=\left(\underset{i\in I}{V}\mu_i\right)(z)$$
$$(iii)\quad x\leq y\Rightarrow\left(\underset{i\in I}{V}\mu_i\right)(x)=\sup\{\mu_i(x): i\in I\}$$
$$\geq \sup\{\mu_i(y): i\in I\}$$
$$=\underset{i\in I}{V}\mu_i(y)$$

Hence $\underset{i\in I}{V}\mu_i$ is a T-fuzzy ideal of N.

Theorem (2): *An epimorphic pre-image of a T-fuzzy ideal of a PON N is a T-fuzzy ideal.*

Proof. Let R and S be T-fuzzy ideals of a PON N. Let $f: R\rightarrow S$ be an epimorphism. Let v be a T-fuzzy ideal of S and μ be the pre-image of v under f. Then for any x, y, z$\in R$, we have

$$(i)\quad \mu(x-y)=(v\,o\,f)(x-y)$$
$$=v(f(x-y))=v(f(x)-f(y))$$
$$\geq T(v(f(x)),v(f(y)))$$
$$=T(v\,o\,f)(x),(v\,o\,f)(y))$$
$$=T(\mu(x),\mu(y))$$
$$(ii)\quad \mu(xy)\qquad =(v\,o\,f)(xy)$$
$$=v(f(xy))=v(f(x)f(y))$$
$$\geq \mu(f(y))$$
$$=(v\,o\,f)(y)$$
$$=\mu(y)\qquad and$$
$$\mu((x+z)y-xy)=(v\,o\,f)((x+z)y-xy)$$
$$=v(f((x+z)y-xy))$$
$$=v(f(yz))$$
$$=v(f(x)f(z))$$
$$\geq v(f(z))$$
$$=(v\,o\,f)(z)$$
$$=\mu(z).$$
$$(iii)\quad x\leq y. Then\ \mu(x)=(v\,o\,f)(x)$$
$$=v(f(x))$$
$$=v(f(y))$$
$$=(v\,o\,f)(y)$$
$$=\mu(y).$$

Hence μ is a T-fuzzy ideal of a PON N.

Theorem (3): *Let μ be a T-fuzzy ideal of PON N and μ^* be a fuzzy set in N defined by* $\mu^*(x)=\dfrac{\mu(x)}{\mu(1)}$ *for all $x\in N$. Then μ^* is normal T-fuzzy ideal of N containing μ.*

Proof. Let μ be a T-fuzzy ideal of a PON N. For any $x,y,z\in N$, then

$$(i)\quad \mu^*(x-y)=\frac{\mu(x-y)}{\mu(1)}$$
$$\geq\frac{1}{\mu(1)}T((\mu(x),(\mu(y)))$$
$$=T\left(\frac{1}{\mu(1)}\mu(x),\frac{1}{\mu(1)}\mu(y)\right)$$
$$=T\left(\mu^*(x),\mu^*(y)\right).$$

$$(ii)\quad \mu^*(xy)\qquad=\frac{\mu(xy)}{\mu(1)}$$
$$\geq\frac{1}{\mu(1)}(\mu(y))$$
$$=\mu^*(y)\qquad and$$
$$\mu^*((x+z)y-xy)=\frac{\mu((x+z)y-xy)}{\mu(1)}$$
$$\geq\frac{1}{\mu(1)}(\mu(z))$$
$$=\mu^*(z).$$

$$(iii)\quad x\leq y\Rightarrow\mu^*(x)=\frac{\mu(x)}{\mu(1)}$$
$$\geq\frac{\mu(y)}{\mu(1)}$$
$$=\mu^*(y).$$

Hence μ^* is a T-fuzzy ideal of N. Clearly $\mu^*(1)=\dfrac{1}{\mu(1)}\mu(1)=1$ and $\mu\subset\mu^*$.

Lemma (1):- *Let R and S be a PON'S and $f:R\to S$ is a homomorphism. Let μ be f-invariant fuzzy ideal of R. If x = f(a), then $f(\mu)(x)=\mu(a)$ for all $a\in R$.*

Theorem (4): *Let $f:R\to S$ be an epimorphism of a PON'S R and S. If μ is f-invariant T-fuzzy ideal of R, then $f(\mu)$ is a T-fuzzy ideal of S.*

Proof. Let $a,b,c\in S$. Then there exists $x,y,z\in R$ such that $f(x)=a,\ f(y)=b$ *and* $f(z)=c.$ suppose μ is f-invariant T-fuzzy ideal of R. Then we have

$$
\begin{aligned}
(i)\quad f(\mu)(a-b) &= f(\mu)(f(x)-f(y)) \\
&= f(\mu)f(x-y)) \\
&= \mu(x-y) \\
&\geq T(\mu(x),\mu(y)) \\
&= T(f(\mu)(a), f(\mu)(b)).
\end{aligned}
$$

$$
\begin{aligned}
(ii)\quad f(\mu)(ab) &= f(\mu)(f(x)f(y)) \\
&= f(\mu)f(xy)) \\
&= \mu(xy) \\
&\geq \mu(x) \\
&= f(\mu)(b) \qquad and
\end{aligned}
$$

$$
\begin{aligned}
f(\mu)((a+b)c-ab)) &= f(\mu)(f(x+y)z-f(xy)) \\
&= f(\mu)(f(x)+f(y))f(z)-f(x)f(y)) \\
&= f(\mu)(f(x+y)f(z)-f(x)f(y)) \\
&= \mu((x+y)z-xy) \\
&\geq \mu(z) \\
&= f(\mu)(c)
\end{aligned}
$$

$$
\begin{aligned}
(iii)\quad Let\ a\leq b \Rightarrow f(\mu)(a)& \\
&= f(\mu)(f(x)) \\
&= \mu(x) \\
&\geq \mu(y) \\
&= f(\mu)(b).
\end{aligned}
$$

Hence $f(\mu)$ is a T-fuzzy ideal of S.

Theorem (5): *Let μ be a T-fuzzy left ideal of PON N and $\mu^{+}(x)=\mu(x)+1-\mu(0)$ for all $x\in N$. Then μ^{+} is a normal T-fuzzy left ideal of N containing μ, provided t-norm holds for combined translation.*

Proof: Let μ be a T-fuzzy left ideal of PON N. We have $\mu^{+}(x)=\mu(x)+1-\mu(0)$ for all $x\in N$. Put $1-\mu(0)=a$ then $\mu^{+}(x)=\mu(x)+a$ and hence $\mu^{+}(x)=\mu_{\alpha}^{T}$. μ^{+} is a T-fuzzy left ideal of N. By definition of μ^{+}, $\mu\leq\mu^{+}$ *and* $\mu^{+}(0)=\mu(0)+1-\mu(0)$ and hence $\mu^{+}(0)=1$. Therefore μ^{+} is a normal T-fuzzy left ideal of N.

Theorem (6): Let ψ be an imaginable fuzzy subset of partially ordered near-ring N. Then ψ is a T-fuzzy left ideal of a partially ordered near-ring N if and only if the strongest fuzzy relation μ_{ψ} on N is an imaginable T-fuzzy left ideal of partially ordered near-ring $N\times N$.

Proof: Suppose that ψ is an imaginable T-fuzzy left ideal of PON N then obviously μ_{ψ} is a T-fuzzy left ideal of a PON $N\times N$, for any $(x_1,x_2),(y_1,y_2)\in N\times N$. Then

$$\mu_\psi(x_1,x_2) = T(\psi(x_1),\psi(x_2)) \ge T(T(\psi(x_1-y_1),\psi(y_1)),T(T(\psi(x_2-y_2),\psi(y_2)))$$
$$= T(\mu_\psi(x_1-y_1,x_2-y_2),\mu_\psi(y_1-y_2))$$
$$T(\mu_\psi(x_1,x_2),\mu_\psi(x_1,x_2) = T(T(\psi(x_1),\psi(x_2),T(\psi(x_1),\psi(x_2))$$
$$= T(T(\psi(x_1),\psi(x_1)),T(\psi(x_2),\psi(x_2))) = T(\psi(x_1),\psi(x_2)) = \mu_\psi(x_1,x_2)$$

Suppose $(x_1,x_2),(y_1,y_2)\in N\times N$ *and* $(x_1,x_2)\le(y_1,y_2)$ then $x_1\le y_1$ and $x_2\le y_2$. Therefore $T(\psi(x_1),\psi(x_2))\ge T(\psi(y_1),\psi(y_2))$.Hence $\mu_\psi(x_1,x_2)\ge\mu_\psi(y_1,y_2)$. Thus μ_ψ is an imaginable T-fuzzy left ideal of a partial ordered near-ring. Let $x,y\in N$. Then

$$(i)\quad \psi(x-y) = T(\psi(x-y),\psi(x-y)) = \mu_\psi(x-y,x-y) = \mu_\psi((x,x)-(y,y))$$
$$\ge T(\mu_\psi(x,x),(y,y)) = T(T(\psi(x),\psi(x)),T(\psi(y),\psi(y)))$$
$$= T(\mu_\psi((x,y),\mu_\psi(x,y)) = T(\mu_\psi(x,y)) = T(\psi(x),\psi(y))$$

$$(ii)\quad \psi(xy) = T(\psi(xy),\psi(xy)) = \mu_\psi(xy,xy)$$
$$= \mu_\psi((x,x)(y,y)) \ge T(\mu_\psi(y,y)) = T(\psi(y),\psi(y)) = \psi(y) \text{ and}$$
$$\psi(x) = T(\psi(x),\psi(x)) = \mu_\psi(x)$$
$$\ge T(\mu_\psi(x-y,x-y),\mu_\psi(y,y))$$
$$= T(T(\psi(x-y,\psi(x-y)),T(\psi(y),\psi(y))) = T(\psi(x-y),\psi(y))$$
$$(iii)\quad \text{Let } x,y\in N \text{ and } x\le y.\text{then } (x,x)\le(y,y)$$
$$\mu_\psi(x,x)\ge\mu_\psi(y,y)$$
$$T(\psi(x),\psi(x))\ge T(\psi(y),\psi(y))$$
$$\text{There fore } \psi(x)\ge\psi(y).$$

Hence ψ is a T-fuzzy left ideal of a PON N.

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Citation: *T. Nagaiah, L. Bhaskar, " Fuzzy ideal of Partially Ordered Near-Ring ", International Journal of Scientific and Innovative Mathematical Research, vol. 5, no. 8, p. 8-14, 2017., http://dx.doi.org/10.20431/ 2347-3142.0508002*