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FUZZY SYSTEMS AND THEIR MATHEMATICS

Fuzzy incomplete linguistic preference relations

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Abstract

The effectiveness of preference relations in modeling decision-making processes makes it one of the most common representations of information use for solving decision-making problems. This research presents the fuzzy incomplete linguistic preference relations (Fuzzy InLinPreRa) approach as evaluated by decision-makers dealing with increasing complexity and uncertain economics, as well as social and managerial problems. By using Fuzzy InLinPreRa, the consistency measurements of decision-makers' evaluations will provide more accurate and reasonable solutions, allowing decision-makers to consider the objective weights of both the criteria and experts. An empirical example of the measurement of brand personality is included herein to demonstrate the feasibility of this method.

Keywords Fuzzy incomplete linguistic preference relations · Fuzzy InLinPreRa · Fuzzy preference relations · Multi-criteria decision-making

1 Introduction

Multi-Criteria Decision-Making (MCDM) methods are well-developed, feature a strong mathematical foundation, and are convenient for decision-making in the business domain. The analytic hierarchy process (AHP) is among the most discussed and popular of these methods (Franek and Kashi 2014; Kozłowska 2022), and used by both management practitioners and academics (Abastante et al. 2019; Goepel and Performance 2019; Hülle et al. 2011). In particular, AHP is the most common form of the MCDM model used in the strategic development of organizations, throughout the product design and development process (Khazaei Pool et al. 2018). Based on fuzzy set theory (Zadeh 1965) and fuzzy logic, Fuzzy AHP was developed to solve imprecise hierarchical problems (Laarhoven and Pedrycz 1983). Fuzzy AHP is the second-most widely used technique for solving decision-making problems in the fields of management and business, after hybrid fuzzy MCDM (Kubler et al. 2016; Mardani et al. 2015).

However, AHP suffers from inherent deficiencies stemming mainly from inconsistency problems caused by humans' pairwise comparison capability. Pairwise comparisons primarily involve evaluating and comparing the importance of several criteria, based on basic scales (Azhar et al. 2021). Asadabadi et al. (2019) explored the inconsistency problem of AHP, highlighting humans' inability to establish consistent pairwise comparisons once the number of criteria increased beyond three and their failing to provide rational rankings as a result. The ratios used in AHP are point estimates, while the comparison ratios used in the Fuzzy AHP method are given as fuzzy numbers; therefore, ratios given in fuzzy numbers are far more likely to be inconsistent (Wang and Chen 2008).

As one of the more effective tools, preference relations has received significant attention because it allows decision-makers to express preference opinions throughout the process of decision-making, with an emphasis on consistency. With the rapid development of social economies, uncertain and complex realistic decision-making issues are common. In such a decision-making environment, decision-makers tend to express their preferences through the use of qualitative preference opinions. Therefore, linguistic preference relations are proposed, where judgments of (complete) linguistic preference relations are represented

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by linguistic terms in a linguistic term set. However, sometimes the linguistic preference relations may be given by decision-makers with some values missing. In such cases, it could be that the decision-makers have no knowledge of the specific problem, or can not distinguish how much some alternatives are better than others (Li et al. 2022b). In this case, the linguistic preference relation is considered incomplete (Wu et al. 2020).

The problem of group decision-making with incomplete linguistic preference relations has been the focus of numerous researchers in recent years. To estimate unknown preference relations, Xu (2006) proposed a method for solving them based on the consistency of existing preference relations for the group decision-making problem (Wu et al. 2020; Zhao et al. 2017). Based on the Xu (2006) framework, Hsu and Wang (2011) proposed an alternative additive transitivity property-based estimation method called Incomplete Linguistic Preference Relations (InLinPreRa), which improved the consistency problem and number of pairwise comparisons by using horizontal, vertical, and oblique pairwise comparison algorithms. If there are n evaluation criteria present, AHP and FAHP need to compare $n(n-1)/2$ times, while InLinPreRa allows decision-makers to perform only $n-1$ pairwise comparisons, which can be faster and escape inconsistency problems. This method can make decision-based problems purer, simpler, and more effective; it also features complex flexibility, compatible subjective perception, coordination, and objective factors, and offers diversification and extensiveness (Hsu and Wang 2011; Kou et al. 2016).

In order to remedy the fact that the evaluations of decision-makers are always subjective and the process of decision-making imprecise, indefinite, and uncertain, this study proposes a method called Fuzzy Incomplete Linguistic Preference Relations (Fuzzy InLinPreRa), based on InLinPreRa. In addition to inheriting the advantages of InLinPreRa, Fuzzy InLinPreRa not only addresses the uncertainty and imprecision of fuzzy set theory, as fuzzy set theory provides the flexibility required for imprecise and ambiguous information stemming from a lack of knowledge (Kuo et al. 2007), it also solves the subjective judgments of decision-makers, taking into account their weights (with fuzzy set numbers) according to their positions and work experience, generating a more reasonable ranking of alternatives.

This study is organized as follows. Section 2 briefly introduces the basic conceptions of MCDM, the most commonly used MCDM methods, fuzzy preference relations, and InLinPreRa. Section 3 includes basic definitions and conceptions related to Fuzzy InLinPreRa, such as the formula for computing weights of criteria that also considers decision-making experts' positions and work experience. Section 4 provides an empirical example to illustrate the effectiveness and practicability of the proposed method. Section 5 offers conclusions and Sect. 6 lists the study's limitations and future research directions.

2 Literature review

MCDM and Fuzzy MCDM contain many decision-making alternatives and criteria, and thus represent critical topics in expert system and operations research. MCDM approaches can solve a wide range of engineering, economic, management, and social problems (Salih et al. 2019). Problems can have many solutions, and MCDM serves as both a quantitative and qualitative method for finding such solutions and making appropriate decisions among them (Bhole and Deshmukh 2018; Wang et al. 2021). In the recent years, numerous MCDM approaches have been applied to solve problems related to selection factors. Sotoudeh-Anvari (2022) concluded that AHP (and Fuzzy AHP), applied in 37.5% of academic studies at the time, was the favorite MCDM method for the COVID-19 problem, followed by the Technique for Order Preference by Similarity to the Ideal Solution (TOPSIS) and the Serbian VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR). Kozłowska (2022) argued that one of the most popular MCDM approaches is AHP. Other commonly used methods included Simple Additive Weighting (SAW), TOPSIS, VIKOR, Elimination and Choice Expressing the Reality (ELECTRE), and Preference Ranking Organization Method for Enrichment of Evaluations (PROMETHEE). These processes were developed to help decision-makers make appropriate choices, and each has advantages and limitations. These methods are illustrated briefly in Sect. 2.1 and summarized in Table 1.

2.1 Common MCDM methods

2.1.1 AHP

Pairwise comparison techniques have been used widely to address subjective and objective judgments of the qualitative and/or quantitative criteria used in MCDM (Kou et al. 2016). AHP is one of the most popular MCDM approaches when criteria are independent (Behzadian et al. 2012; İç et al. 2022; Kozłowska 2022). The method models subjective decision-making processes based on pairwise comparisons between criteria in a hierarchical system, giving relative weights to each (Li et al., 2022a; Saaty 1980; Tzeng and Huang 2011). AHP is applicable to most goals (i.e., planning, identification, selection, and evaluation) and uses a small number of criteria that can be individually based on each criterion and the hierarchical relations among them (Azhar et al. 2021) In AHP and

Table 1 Summary of MCDM methods: advantages, disadvantages, and applications

Method	Advantages	Disadvantages	Applications
AHP (*pairwise comparisons*)	1. *Easy to use and scalable* 2. *Hierarchical structure can easily adjust to fit various sizes of problems* 3. *Convenient and straightforward* 4. The ability to mix qualitative and quantitative criteria in the same decision-making framework	1. Interdependence amongst criteria and alternatives 2. The number of alternatives/criteria increasing may lead to inconsistency in pairwise comparisons 3. Some irregular rankings may occur 4. The use of additive aggregation causes some information to be lost	Applicable to most goals (planning, identification, selection, and evaluation) and different domains that have a small number of criteria that can be individually based on criteria and their hierarchical relationships
SAW (weight-based sum)	1. Ability to compensate between criteria 2. Intuitive for decision-makers 3. Simple calculations without complex computer programs	1. The revealed estimates do not always reflect the facts 2. The results may not be logical	The method is recommended for solving problems selected in multi-process decision-making systems and widely used in decision-making in cases with many attributes
TOPSIS (distance-based)	1. Has a simple process, so it is easy to use and program 2. The number of steps is the same, regardless of the numbers of criteria and attributes 3. Takes as input an unlimited number of criteria and attributes	1.The use Euclidean distance without considering the correlation of attributes causes difficulties with weighting and keeping judgment consistent 2. Uncertainty in weights is not considered	TOPSIS and VIKOR are mostly used for goals that need to be selected or evaluated, but cannot stand alone; should be combined with pairwise comparisons, fuzzing, genetic algorithms, or other methods that can handle inconsistency and uncertainty
VIKOR (distance-based)	1. Able to identify compromise solutions that reflect the attitudes of the majority of decision-makers 2. Provides the maximum group utility for the majority and minimum personal regret for the opponent	1. Subjective initial weighting is a challenge to validate 2. In the case of qualitative attributes, linguistic information processing may lead to information distortion	
ELECTRE (outranking)	1. Deals with qualitative and quantitative scales of criteria 2. Avoids compensation between criteria 3. The number of alternatives in a set of non-dominated alternatives is reduced sequentially	1. It is difficult for decision-makers to provide any justification for the parameters chosen for discordance and concordance thresholds 2. When the number of alternatives increases, the calculation complexity also increases	ELECTRE and PROMETHEE are suitable for goals that need to be selected or evaluated but given that there are many variants of both outranking methods, one needs to choose variants wisely. Some variants can stand alone but may perform better when combined. These variants are often combined with pairwise comparisons
PROMETHEE (outranking)	1. Deals with both qualitative and quantitative criteria 2. Expresses the criteria in its own units and requires less effort, reducing complexity and facilitating the use of this approach	1. Limited preference functions and requires more preference functions or improvements to existing functions for better results 2. Lack of a clear way to assign weights 3. When many criteria and options are available, the decision-makers may face difficulties in evaluating the results 3. Once a new alternative is introduced, a rank reversal problem may arise	

Fuzzy AHP, a list of criteria (both critical factors and sub-factors) is prioritized according to relative importance, a useful method for solving complex decision-making problems (Wu et al. 2009). However, once the number of elements increases beyond a certain point, humans cannot maintain consistent pairwise judgments (Asadabadi et al. 2019), and the resulting lack of consistency in decision-making leads to inconsistent conclusions (Herrera-Viedma et al. 2004). The disadvantages of AHP include irregularities in ranking, use of additive aggregation, and others. Therefore, important information may be lost and more pairwise wise comparisons required (Nallusamy et al. 2016).

2.1.2 Simple additive weighting

SAW is a simple weighted linear combination that is regarded as the most intuitive and simplest way of dealing with MCDM problems (Prasetiyo and Baroroh 2016; Rizka et al. 2018; Tzeng and Huang 2011). The concept is based on finding a weighted sum of the performance rating for each alternative for all attributes. The highest score reflects the best alternative (Ibrahim and Surya 2019; Prasetiyo and Baroroh 2016). It has the ability to compensate between criteria and is intuitive for decision-makers. The calculations are simple and can be performed without the help of complex computer programs. However, the estimates may not reflect the real-world situation. The results obtained may not logical, and the value of a particular criterion can be very different from the values of other criteria (Azhar et al. 2021). The method is recommended for solving problems selected in a multi-process decision-making system and used widely in decision-making scenarios with many attributes (Purba 2021).

2.1.3 TOPSIS

TOPSIS and VIKOR are based on an aggregating function that represents closeness to the ideal solution (Kozłowska 2022; Zhang and Wei 2013). The difference between TOPSIS and VIKOR is in the use of different types of normalization to eliminate the units of the criteria function. VIKOR uses linear normalization, while TOPSIS employs vector normalization (Azhar et al. 2021). The standard TOPSIS method selects the best alternative that simultaneously has the shortest distance from the positive ideal solution and furthest distance from the negative ideal solution (Behzadian et al. 2012). The process is simple, which facilitates its use and programming. The number of steps is the same regardless of the number of criteria and attributes. However, the use of Euclidean distance without consideration of the correlation of attributes results in difficulties with weighting and keeping judgment consistent, and uncertainties in weighting are not considered (Azhar et al. 2021; Velasquez and Hester 2013).

2.1.4 VIKOR

VIKOR ranks and selects from a group of alternatives in cases of conflicting criteria. The method ranks criteria based on the measure of closeness to the ideal solution and the agreement established by mutual concessions. VIKOR is often the preferred choice, due to its lower mathematical complexity (Azhar et al. 2021). The advantage is in its ability to handle MCDM problems with non-commensurable and even conflicting criteria and obtain an optimal compromise (Alfina et al. 2022; Fei et al. 2019). The compromise is acceptable to all decision-makers because it provides the maximum group utility to the majority and minimum personal regret to the opponent (Azhar et al. 2021; Chang 2014; Opricovic 1998; Tzeng and Huang 2011). The main disadvantage of the VIKOR method is its subjective initial weighting, which is challenging to validate (Wibawa et al. 2019). VIKOR experiences difficulties in the case of qualitative attributes because the linguistic information processing may lead to information distortion (Rahim et al. 2020).

TOPSIS and VIKOR are mostly used for goals that need to be selected or evaluated but cannot stand alone and may require combination with pairwise comparisons, fuzzing, genetic algorithms, or other methods, which is recommended for handling inconsistency and uncertainty (Azhar et al. 2021).

2.1.5 ELECTRE

Based on pairwise comparison rules, ELECTRE methods apply the concordance and discordance of criteria and threshold values to assess the scoring schemes between available alternatives (Kozłowska 2022). The major feature of the ELECTRE family includes the possibility of dealing with qualitative and quantitative scales of criteria. Such methods are able to handle qualitative performance scales and allow consideration of the original (either verbal or numeric) performance without any recoding. They can also handle heterogeneous scales. Regardless of the nature of the scale, each procedure can run with the preserved original performances of the actions, without the need for recoding. The multiple criteria aggregation procedures of ELECTRE methods do not allow performance compensation between criteria; performance degradation in some criteria cannot be compensated for by performance improvements in other criteria (Figueira et al. 2013). The operation of each ELECTRE version is different, as are the types of problems for which they can be used. For example, ELECTRE I, IV, and IS are applicable to the choice problem, where the goal is to select a smallest set of best alternatives. ELECTRE II, III, and IV were designed to establish rankings from best to worst (Govindan and Jepsen 2016). ELECTRE III has proven to be a practical and popular method for accomplishing multi-criteria (group) decision-making tasks and preventing cross-criteria compensation (Chen et al. 2021). Avoiding compensation between criteria is one of the main advantages of the ELECTRE method (Jahan and Zavadskas 2019; Nghiem and Chu 2021). The number of alternatives is a set of non-dominated alternatives that is reduced sequentially (Rahim et al. 2020). A disadvantage of the ELECTRE method is the set of parameters for discordance and concordance thresholds. It is difficult for decision-makers to provide any justification for the values chosen for these parameters (Keshavarz Ghorabaee et al. 2016). Also, when the number of alternatives increases, the calculation complexity also increases (Rahim et al. 2020).

2.1.6 PROMETHEE

Based on the dominance relationship principles and a generalization of the criterion notion, PROMETHEE belongs to the family of multi-criteria outranking methods that deal with both qualitative and quantitative criteria. The advantage of PROMETHEE is its ability to express these criteria in its own units, which requires less effort, reduces complexity, and facilitates the use of this approach. PROMETHEE also has limited preference functions and requires more preference functions or improvements in the existing function to obtain better results. The lack of a clear way to assign weights is one disadvantage. When many criteria and options are available, decision-makers may face difficulties in evaluating the results. Once a new alternative is introduced, a rank reversal problem may arise (Azhar et al. 2021).

ELECTRE and PROMETHEE are suitable for goals that need to be selected or evaluated, but given that there are many variants of both outranking methods, one must choose variants wisely. Some can stand alone but may perform better when combined. These variants are often combined with pairwise comparisons (Azhar et al. 2021).

From the above summaries of the various MCDM methods, it can be seen that AHP is not only one of the most commonly used MCDM tools with pairwise comparison techniques in various domains such as project management, enterprise resource planning system selection, risk assessment, and knowledge management tools evaluation (Chen et al. 2011; Slamaa et al. 2021), but it is also suitable for most goals. The main reasons for its popularity include its simple, flexible, intuitive appeal, and the ability to mix qualitative and quantitative criteria in the same decisions (Abdul et al. 2022; Ramanathan and Ganesh 1995). However, consistency and consensus are the basic requirements of comparison matrices to ensure meaningful results (Xu et al. 2022) and thus a topic of great concern to researchers using AHP analysis.

2.2 Conceptual methodologies associated with Fuzzy InLinPreRa

The basic conceptual methodologies associated with Fuzzy InLinPreRa originated from preference relations used by decision-makers to provide preference information in the decision-making process; thus, it has become a powerful and popular set of tools. Preference relations are constructed by pairwise comparisons across alternatives, where each value represents the preference intensity of one alternative over another (Xu 2007; Xu and Liao 2015). The most widely used pairwise comparison matrices are additive preference relations, also known as fuzzy preference relations (Rodríguez et al. 2021; Wu and Tu 2021). According to the operational laws of linguistic assessment scales and the acceptable incomplete LPR, Xu (2006) developed a method for constructing consistent complete linguistic preferences using additive transitive property relations. Extending incomplete LPR, Hsu and Wang (2011) proposed InLinPreRa, based on the algorithmic rules of three different pairwise comparison. The method allows decision-makers to express their preference intensity for all alternatives using a single crisp value with only $n-1$ pairwise comparisons needed.

As all the judgments in the three methods are crisp values that are hard to represent precisely in complex and uncertain cases, Fuzzy InLinPreRa was introduced. Fuzzy set theory was combined with MCDM methods to deal with problems emerging from uncertain environments (Zavadskas et al. 2014). The pioneering concept of fuzzy sets proposed by Zadeh (1965) to deal with the unavoidable uncertainty that arises in various real-world scenarios is one of the most well-known concepts (Chen et al. 2019).

2.2.1 Fuzzy preference relations

Fuzzy preference relations were proposed by Herrera-Viedma et al. (2004) to address the inconsistency in AHP caused by multiple decision-makers, multi-criteria, and multiple alternatives being presented (Hsu and Wang 2011). The effectiveness of preference relations in modeling decision-making processes makes it one of the most common representations of information used in solving decision-based problems. Linguistic preference information plays an important role in the decision-making process (Li et al. 2019; Wang and Chen 2008). In fuzzy preference relations, the expert associates the value of each pair of alternatives with a certain degree of preference, considered from the first alternative to the second (Capuano et al. 2018; Wang and Chen 2007). Decision-makers express their preferences using a single crisp value (Wang 2014). The method only requires $n-1$ to compare (n represents the number of criteria in the analysis). In addition, once the pairwise comparison is carried out, there is no need to use the consistency index to apply a round of consistency tests (Tang and Hsu 2018). Important decision-making models have been developed that use multiplicative preference relations and additive fuzzy preference relations (Herrera-Viedma et al. 2004; Hsu and Wang 2011) (see Appendix 1).

2.2.2 Incomplete linguistic preference relations

With regard to the fuzzy preference relations mentioned above, which are given as linguistic preference relations (Xia et al. 2014), Xu (2006) found that when comparing decision alternatives, decision-makers often used linguistic preference relations to document and express their preferences in situations in which each of the linguistic preference relations was required to complete all $n(n-1)/2$ judgments in the entire top triangular portion of the equation. Such linguistic preference relations are difficult to obtain, especially for higher-order linguistic preference relations, because decision-makers are forced to make these judgments under time constraints and with incomplete data. In this way, decision-makers may develop incomplete linguistic preference relations in which certain elements are not available. As a remedy, Xu (2006) proposed the incomplete linguistic preference relations method. In the process of pairwise comparisons, each decision-maker can choose any explicit item as a standard, based on their preference or recognition. They then carry out pairwise comparisons between the adjoining items to obtain the original preference matrix. An incomplete linguistic preference relation counters the fact that decision-makers can carry out the pairwise comparisons for all attributes through a preference matrix. When decision-makers use pairwise comparisons to compare raw preference values, the remaining unknown values are added to adjoin numbers equal to 0 through the corresponding opposite numbers, in order to obtain a complete matrix. The relevant definitions of incomplete linguistic preference relations can be found in Appendix 2 (Hsu and Wang 2011; Shih and Hsu 2016; Xu 2006; Zhao et al. 2016).

2.3 Extension of incomplete linguistic preference relations

HSU and Wang (2011) provided the general formula for decision-making related to basic preference relations, based on fuzzy preference relations and InLinPreRa. This method processes pairwise comparisons for decision-making by only taking $n-1$ times, which is markedly simpler and far more efficient than the $n(n-1)/2$ times required by AHP. Likewise, it avoids inconsistencies when considering criteria and evaluating the weight of criteria, major differences from the method provided by Xu (2006). Based on the above formula, the algorithmic rules for three different pairwise comparisons are applied to build the preference relation matrices. The general formula for the decision-making related to the basic preference relations is interpreted in the following section.

2.3.1 Construction of the decision-making matrix for InLinPreRa

In the evaluation process, it is assumed that there are n decision-makers denoted as E_e, where $e = 1, 2, \ldots, n$; C_r is denoted as the evaluation criteria, where $r = 1, 2, \ldots, k$; alternatives are denoted as A_i, where $i = 1, 2, \ldots, m$, the eth expert under the rth criterion; and the decision-making matrix ${}^rD^{(e)} = \left[{}^ra_{ij}^{(e)}\right]_{m\times m}$, which can be determined for m alternatives, is expressed as:

$$
{}^rD^{(e)} = \left[{}^ra_{ij}^{(e)}\right]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \cdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\ \begin{bmatrix} S_0 & {}^ra_{12}^{(e)} & {}^ra_{13}^{(e)} & {}^ra_{14}^{(e)} & \cdots & {}^ra_{1m}^{(e)} \\ {}^ra_{21}^{(e)} & S_0 & {}^ra_{23}^{(e)} & {}^ra_{24}^{(e)} & \cdots & {}^ra_{2m}^{(e)} \\ {}^ra_{31}^{(e)} & {}^ra_{32}^{(e)} & S_0 & {}^ra_{34}^{(e)} & \cdots & {}^ra_{3m}^{(e)} \\ {}^ra_{41}^{(e)} & {}^ra_{42}^{(e)} & {}^ra_{43}^{(e)} & S_0 & \cdots & {}^ra_{4m}^{(e)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ {}^ra_{m1}^{(e)} & {}^ra_{m2}^{(e)} & {}^ra_{m3}^{(e)} & {}^ra_{m4}^{(e)} & \cdots & S_0 \end{bmatrix}_{mxm} \end{array},
\quad e = 1, 2, \ldots, n \quad r = 1, 2, \ldots, k
$$

${}^rD$ is derived from the integration of the matrices of all decision-makers:

$$
{}^rD = \left[{}^ra_{ij}^{(e)}\right]_{m\times m} = \frac{1}{n}\left[{}^rD^{(1)} + {}^rD^{(2)} + {}^rD^{(3)} + \cdots + {}^rD^{(n)}\right] \tag{1}
$$

$w^{(e)}$ is represented as the expert's weights, and the weights of each expert are shown as follows:

$$
w^{(1)}, w^{(2)}, \ldots, w^{(n)}, \quad w^{(e)} \in [0, 1], \sum_{e=1}^{n} w^{(e)} = 1 \tag{2}
$$

${}^rw$ is represented as the criteria weights, and the weights of each criterion are shown as follows:

$$
{}^1w, {}^2w, \ldots, {}^kw, \quad {}^rw \in [0, 1], \sum_{r=1}^{k} {}^rw = 1 \tag{3}
$$

2.3.2 Algorithmic rules for the three different pairwise comparison decision matrices

A preference relation matrix is constructed according to the formula listed above. It can be obtained from only a few matrices for different known factors chosen by the decision-makers. The algorithmic rules for the three different decision matrices are as follows:

Type 1: Horizontal pairwise comparison

$$ {}^rD^{(e)} = \left[{}^ra_{ij}^{(e)}\right]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \cdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\ \begin{bmatrix} 0 & {}^ra_{12}^{(e)} & {}^ra_{13}^{(e)} & {}^ra_{14}^{(e)} & \ldots & {}^ra_{1m}^{(e)} \\ \times & 0 & \times & \times & \ldots & \times \\ \times & \times & 0 & \times & \ldots & \times \\ \times & \times & \times & 0 & \ldots & \times \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \times & \times & \times & \times & \ldots & 0 \end{bmatrix}_{mxm} \end{array}, $$

$$ r = 1,2,\ldots,k \quad e = 1,2,\ldots,n $$

$\times$ are the unknown variables. Assume the eth decision-maker sets ${}^rD^{(e)} = \left[{}^ra_{ij}^{(e)}\right]_{6\times 6}$ to indicate the reference and evaluation of the rth criterion for six alternatives ${}^rA^{(e)} \subset 6\times 6$. The pairwise comparison will then generate five original linguistic preference values, where:

$$ {}^ra_{12}^{(e)} = S_{-3}, \quad {}^ra_{13}^{(e)} = S_1, \quad {}^ra_{14}^{(e)} = S_{-1}, \quad {}^ra_{15}^{(e)} = S_{-2}, \quad {}^ra_{16}^{(e)} = S_1 $$

The mapping values are derived from equation $a_{ij} \in S, \quad a_{ij} \oplus a_{ji} = S_0, \quad a_{ii} = S_0$, shown as follows:

$$ {}^ra_{21}^{(e)} = -{}^ra_{12}^{(e)} = S_3, \quad {}^ra_{31}^{(e)} = -{}^ra_{13}^{(e)} = S_{-1}, $$
$$ {}^ra_{41}^{(e)} = -{}^ra_{14}^{(e)} = S_1, $$
$$ {}^ra_{51}^{(e)} = -{}^ra_{15}^{(e)} = S_2, \quad {}^ra_{61}^{(e)} = -{}^ra_{16}^{(e)} = S_{-1} $$

Then, all unknown variables $\times$ the upper half of the triangle are derived from equation $a_{ij} = a_{ik} \oplus a_{kj}$, shown as follows:

$$ {}^ra_{23}^{(e)} = {}^ra_{21}^{(e)} + {}^ra_{13}^{(e)} = S_3 + S_1 = S_4 $$
$$ {}^ra_{24}^{(e)} = {}^ra_{21}^{(e)} + {}^ra_{14}^{(e)} = S_3 + S_{-1} = S_2 $$
$$ {}^ra_{25}^{(e)} = {}^ra_{21}^{(e)} + {}^ra_{15}^{(e)} = S_3 + S_{-2} = S_1 $$
$$ {}^ra_{26}^{(e)} = {}^ra_{21}^{(e)} + {}^ra_{16}^{(e)} = S_3 + S_1 = S_4 $$
$$ {}^ra_{34}^{(e)} = {}^ra_{31}^{(e)} + {}^ra_{14}^{(e)} = S_{-1} + S_{-1} = S_{-2} $$
$$ {}^ra_{35}^{(e)} = {}^ra_{31}^{(e)} + {}^ra_{15}^{(e)} = S_{-1} + S_{-2} = S_{-3} $$
$$ {}^ra_{36}^{(e)} = {}^ra_{31}^{(e)} + {}^ra_{16}^{(e)} = S_{-1} + S_1 = S_0 $$
$$ {}^ra_{45}^{(e)} = {}^ra_{41}^{(e)} + {}^ra_{15}^{(e)} = S_1 + S_{-2} = S_{-1} $$
$$ {}^ra_{46}^{(e)} = {}^ra_{41}^{(e)} + {}^ra_{16}^{(e)} = S_1 + S_1 = S_2 $$
$$ {}^ra_{56}^{(e)} = {}^ra_{51}^{(e)} + {}^ra_{16}^{(e)} = S_2 + S_1 = S_3 $$

All unknown variables $\times$ the lower half of the triangle are derived from Eq. (9) $a_{ij} \oplus a_{ji} = S_0$, and the complete preference, the decision-making matrix, **is** shown as follows:

$$ {}^rD^{(e)} = \left[{}^ra_{ij}^{(e)}\right]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array} \\ \begin{bmatrix} 0 & S_{-3} & S_1 & S_{-1} & S_{-2} & S_1 \\ S_3 & 0 & S_4 & S_2 & S_1 & S_4 \\ S_{-1} & S_{-4} & 0 & S_{-2} & S_{-3} & S_0 \\ S_1 & S_{-2} & S_2 & 0 & S_{-1} & S_2 \\ S_2 & S_{-1} & S_3 & S_1 & 0 & S_3 \\ S_{-1} & S_{-4} & S_0 & S_{-2} & S_{-3} & 0 \end{bmatrix}_{6\times 6} \end{array}, $$

$$ r = 1,2,\ldots,k \quad e = 1,2,\ldots,n $$

Type 2 Vertical pairwise comparison

$$ {}^rD^{(e)} = \left[{}^ra_{ij}^{(e)}\right]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \cdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\ \begin{bmatrix} 0 & \times & {}^ra_{13}^{(e)} & \times & \ldots & \times \\ \times & 0 & {}^ra_{23}^{(e)} & \times & \ldots & \times \\ \times & \times & 0 & \times & \ldots & \times \\ \times & \times & {}^ra_{43}^{(e)} & 0 & \ldots & \times \\ \ldots & \ldots & \ldots & \ldots & 0 & \ldots \\ \times & \times & {}^ra_{m3}^{(e)} & \times & \ldots & 0 \end{bmatrix}_{m\times m} \end{array}, $$

$$ r = 1,2,\ldots,k \quad e = 1,2,\ldots,n $$

The complete matrix is derived based on the algorithmic rule of Type 1 $\times$ the unknown variables. Assuming the decision-makers' sets ${}^ra_{22}^{(e)}$ indicate the reference and evaluation of a criterion with six alternatives ${}^rA^{(e)} \subset 6\times 6$, the pairwise comparison will generate five original linguistic preference values, where:

$$ {}^ra_{13}^{(e)} = S_2, \quad {}^ra_{23}^{(e)} = S_{-1}, \quad {}^ra_{43}^{(e)} = S_3, \quad {}^ra_{53}^{(e)} = S_{-2}, \quad {}^ra_{63}^{(e)} = S_1 $$

The mapping values are derived from equation $a_{ij} \in S, \quad a_{ij} \oplus a_{ji} = S_0, \quad a_{ii} = S_0$, shown as follows:

$$ {}^ra_{31}^{(e)} = -{}^ra_{13}^{(e)} = S_{-2} \quad {}^ra_{32}^{(e)} = -{}^ra_{23}^{(e)} = S_1 $$
$$ {}^ra_{34}^{(e)} = -{}^ra_{43}^{(e)} = S_{-3} $$
$$ {}^ra_{53}^{(e)} = -{}^ra_{35}^{(e)} = S_2 \quad {}^ra_{36}^{(e)} = -{}^ra_{63}^{(e)} = S_{-1} $$

Then, all unknown variables $\times$ the upper half of the triangle are derived from equation $a_{ij} = a_{ik} \oplus a_{kj}$, shown as follows:

$$
\begin{aligned}
{}^{r}a_{12}^{(e)} &= {}^{r}a_{13}^{(e)} + {}^{r}a_{32}^{(e)} = S_2 + S_1 = S_3 \\
{}^{r}a_{14}^{(e)} &= {}^{r}a_{13}^{(e)} + {}^{r}a_{34}^{(e)} = S_2 + S_{-3} = S_{-1} \\
{}^{r}a_{15}^{(e)} &= {}^{r}a_{13}^{(e)} + {}^{r}a_{35}^{(e)} = S_2 + S_2 = S_4 \\
{}^{r}a_{16}^{(e)} &= {}^{r}a_{13}^{(e)} + {}^{r}a_{36}^{(e)} = S_2 + S_{-1} = S_1 \\
{}^{r}a_{21}^{(e)} &= {}^{r}a_{23}^{(e)} + {}^{r}a_{31}^{(e)} = S_{-1} + S_{-2} = S_{-3} \\
{}^{r}a_{22}^{(e)} &= {}^{r}a_{23}^{(e)} + {}^{r}a_{32}^{(e)} = S_{-1} + S_1 = S_0 \\
{}^{r}a_{25}^{(e)} &= {}^{r}a_{23}^{(e)} + {}^{r}a_{35}^{(e)} = S_{-1} + S_2 = S_1 \\
{}^{r}a_{26}^{(e)} &= {}^{r}a_{23}^{(e)} + {}^{r}a_{36}^{(e)} = S_{-1} + S_{-1} = S_2 \\
{}^{r}a_{45}^{(e)} &= {}^{r}a_{43}^{(e)} + {}^{r}a_{35}^{(e)} = S_3 + S_2 = S_5 \\
{}^{r}a_{46}^{(e)} &= {}^{r}a_{43}^{(e)} + {}^{r}a_{36}^{(e)} = S_3 + S_{-1} = S_2 \\
{}^{r}a_{56}^{(e)} &= {}^{r}a_{53}^{(e)} + {}^{r}a_{36}^{(e)} = S_{-2} + S_{-1} = S_{-3}
\end{aligned}
$$

All unknown variables × the lower half of the triangle are derived from Eq. (9) $a_{ij} + a_{ji} = S_0$ and the complete preference decision-making matrix is shown as follows:

$$
{}^{r}D^{(e)} = \left[{}^{r}a_{ij}^{(e)}\right]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array} \\ \left[\begin{array}{cccccc} 0 & S_3 & S_2 & S_{-1} & S_4 & S_1 \\ S_{-3} & 0 & S_{-1} & S_{-4} & S_1 & S_{-2} \\ S_{-2} & S_1 & 0 & S_{-3} & S_2 & S_{-1} \\ S_1 & S_4 & S_3 & 0 & S_5 & S_2 \\ S_{-4} & S_{-1} & S_{-2} & S_{-5} & 0 & S_{-3} \\ S_{-1} & S_2 & S_1 & S_{-2} & S_3 & 0 \end{array}\right]_{6\times 6} \end{array}
$$

Type 3 Oblique pairwise comparison

$$
{}^{r}D^{(e)} = \left[{}^{r}a_{ij}^{(e)}\right]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \cdots \\ A_6 \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\ \left[\begin{array}{cccccc} 0 & {}^{r}a_{12}^{(e)} & \times & \times & \ldots & \times \\ \times & 0 & {}^{r}a_{23}^{(e)} & \times & \ldots & \times \\ \times & \times & 0 & {}^{r}a_{34}^{(e)} & \ldots & \times \\ \times & \times & \times & 0 & \ddots & \times \\ \ldots & \ldots & \ldots & \ldots & \ddots & {}^{r}a_{m-1\ m}^{(e)} \\ \times & \times & \times & \times & \ldots & 0 \end{array}\right]_{m\times m} \end{array},
$$

$$r = 1, 2, \ldots, k$$
$$e = 1, 2, \ldots, n$$

The complete matrix is derived based on the algorithmic rule of Type 1 and × are the unknown variables. Assuming the eth decision-makers' sets ${}^{r}D^{(e)} = \left[{}^{r}a_{ij}^{(e)}\right]_{6\times 6}$ indicate the reference and evaluation of the rth criterion with six alternatives ${}^{r}A^{(e)} \subset 6 \times 6$, the pairwise comparison will generate five original linguistic preference values, where:

$$
{}^{r}a_{12}^{(e)} = S_2, \quad {}^{r}a_{23}^{(e)} = S_{-1}, \quad {}^{r}a_{34}^{(e)} = S_2, \quad {}^{r}a_{45}^{(e)} = S_1, \quad {}^{r}a_{56}^{(e)} = S_{-3}
$$

The mapping values are derived from Eq. (9) $a_{ij} \in S, \quad a_{ij} \oplus a_{ji} = S_0, \quad a_{ii} = S_0$, shown as follows:

$$
\begin{aligned}
{}^{r}a_{21}^{(e)} &= -{}^{r}a_{12}^{(e)} = S_2, \\
{}^{r}a_{54}^{(e)} &= -{}^{r}a_{45}^{(e)} = S_{-1}
\end{aligned}
$$

Then, all unknown variables × the upper half of the triangle are derived from Eq. (10) $a_{ij} = a_{ik} \oplus a_{kj}$, shown as follows:

$$
\begin{aligned}
{}^{r}a_{13}^{(e)} &= {}^{r}a_{12}^{(e)} + {}^{r}a_{23}^{(e)} = S_2 + S_{-1} = S_1 \\
{}^{r}a_{15}^{(e)} &= {}^{r}a_{14}^{(e)} + {}^{r}a_{45}^{(e)} = S_3 + S_1 = S_4 \\
{}^{r}a_{24}^{(e)} &= {}^{r}a_{23}^{(e)} + {}^{r}a_{34}^{(e)} = S_{-1} + S_2 = S_1 \\
{}^{r}a_{26}^{(e)} &= {}^{r}a_{25}^{(e)} + {}^{r}a_{56}^{(e)} = S_2 + S_3 = S_1 \\
{}^{r}a_{36}^{(e)} &= {}^{r}a_{33}^{(e)} + {}^{r}a_{56}^{(e)} = S_3 + S_{-3} = S_0
\end{aligned}
$$

All unknown variables × the lower half of the triangle are derived from Eq. (9) $a_{ij} \oplus a_{ji} = S_0$, and the complete preference decision-making matrix is shown as follows:

$$
{}^{r}D^{(e)} = \left[{}^{r}a_{ij}^{(e)}\right]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array} \\ \left[\begin{array}{cccccc} 0 & S_2 & S_1 & S_3 & S_4 & S_1 \\ S_{-2} & 0 & S_{-1} & S_1 & S_2 & S_{-1} \\ S_{-1} & S_1 & 0 & S_2 & S_3 & S_0 \\ S_{-3} & S_{-1} & S_{-2} & 0 & S_1 & S_{-2} \\ S_{-4} & S_{-2} & S_{-3} & S_{-1} & 0 & S_{-3} \\ S_{-1} & S_1 & S_0 & S_2 & S_3 & 0 \end{array}\right]_{6\times 6} \end{array}
$$

3 Fuzzy incomplete linguistic preference relations

This study proposes a new method called Fuzzy InLinPreRa to accommodate the vagueness and imprecision of information used in solving decision-making problems containing multi-criteria. By extending the advantages of InLinPreRa mentioned above, Fuzzy InLinPreRa will not only allow decision-makers to simply and efficiently carry out optimal alternative evaluations, it will also help with consideration of the weights of criteria and experts with respect to different pairwise comparisons located in the fuzzy environment.

3.1 Construct of the original decision-making matrix

Suppose there are n decision-makers, denoted as E_e; r evaluating criteria; m alternatives; and the ith alternative is denoted as A_i, where $i = 1, 2, \ldots, m$. The fuzzy evaluated values ${}^r\tilde{a}_{ij}^{(e)}$ construct the matrix ${}^r\tilde{D}^{(e)} = \left[{}^r\tilde{a}_{ij}^{(e)}\right]_{m\times m}$, which is under the rth criterion carried out by the eth decision-maker on alternative $A_1, A_2, \ldots A_m$, determined for the m alternative and expressed as:

$$
{}^r\tilde{D}^{(e)} = \left[{}^r\tilde{a}_{ij}^{(e)}\right]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \ldots \\ A_m \end{array}
\begin{array}{c}
\begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\
\begin{bmatrix}
\tilde{S}_0 & {}^r\tilde{a}_{12}^{(e)} & {}^r\tilde{a}_{13}^{(e)} & {}^r\tilde{a}_{14}^{(e)} & \ldots & {}^r\tilde{a}_{1m}^{(e)} \\
{}^r\tilde{a}_{21}^{(e)} & \tilde{S}_0 & {}^r\tilde{a}_{23}^{(e)} & {}^r\tilde{a}_{24}^{(e)} & \ldots & {}^r\tilde{a}_{2m}^{(e)} \\
{}^r\tilde{a}_{31}^{(e)} & {}^r\tilde{a}_{32}^{(e)} & \tilde{S}_0 & {}^r\tilde{a}_{34}^{(e)} & \ldots & {}^r\tilde{a}_{3m}^{(e)} \\
{}^r\tilde{a}_{14}^{(e)} & {}^r\tilde{a}_{42}^{(e)} & {}^r\tilde{a}_{43}^{(e)} & \tilde{S}_0 & \ldots & {}^r\tilde{a}_{4m}^{(e)} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
{}^r\tilde{a}_{m1}^{(e)} & {}^r\tilde{a}_{m2}^{(e)} & {}^r\tilde{a}_{m3}^{(e)} & {}^r\tilde{a}_{m4}^{(e)} & \ldots & \tilde{S}_0
\end{bmatrix}_{m\times m}
\end{array}, \quad \begin{array}{l} e = 1, 2, \ldots, n \\ r = 1, 2, \ldots, k \end{array}
$$

3.2 Weights of criteria and experts in Fuzzy InLinPreRa

In the case provided, there are n decision-makers with different weights $\tilde{w}^e$ (Luo et al. 2019), according to the importance of their positions or relative work experience, where $e = 1, 2, \ldots, n$ satisfies the fuzzy sets condition, represented as:

$$
\tilde{w}^{(1)}, \tilde{w}^{(2)}, \ldots, \tilde{w}^{(e)}, \ldots, \tilde{w}^{(n)}, \quad \text{where} \quad \tilde{w}^{(e)} = \left(a^{(e)}, b^{(e)}, c^{(e)}\right) \tag{4}
$$

The weight of each criterion is determined by n decision-makers and the eth expert evaluated the weight of the rth criterion, denoted as ${}^r\tilde{w}^{(e)}$, where $e = 1, 2, ..., n$. $r = 1, 2, ..., k$. For example, the weights of the first criterion are evaluated by all of the decision-makers and denoted as ${}^1\tilde{w}^{(1)}, \ldots, {}^1\tilde{w}^{(n)}$. The others can be expressed as follows:

$$
\begin{array}{lllll}
\text{1st criteria}: & {}^1\tilde{w}^{(1)}, & {}^1\tilde{w}^{(2)}, & \ldots & , {}^1\tilde{w}^{(n)} \\
\text{2st criteria}: & {}^2\tilde{w}^{(1)}, & {}^2\tilde{w}^{(2)}, & \ldots & , {}^2\tilde{w}^{(n)} \\
\ldots & \vdots & \vdots & \ldots & \vdots \\
k\text{th criteria}: & {}^k\tilde{w}^{(1)}, & {}^k\tilde{w}^{(2)}, & \ldots & , {}^k\tilde{w}^{(n)}
\end{array}
$$

To avoid the weights of criteria being subjectively determined by the decision-making experts and to approximate reality, the final weights of the criteria should consider the decision-making experts' position and work experience. Therefore, the final weights of the criteria are calculated according to Eq. (5), shown as follows:

$$
{}^r\tilde{w} = \frac{\sum_{e=1}^{n} \left({}^r\tilde{w}^{(e)} \otimes \tilde{w}^{(e)}\right)}{\sum_{r=1}^{k}\sum_{e=1}^{n} \left({}^r\tilde{w}^{(e)} \otimes \tilde{w}^{(e)}\right)}, \quad \text{where} \quad r = 1, 2, \ldots, k, \quad e = 1, 2, \ldots, n \tag{5}
$$

3.3 Basic definitions for fuzzy InLinPreRa

This study used a triangular membership function for fuzzification to set $\tilde{S}_0 = (L^{\tilde{S}_0}, M^{\tilde{S}_0}, R^{\tilde{S}_0})$ as the neutral fuzzy value (NFV). The right and left sides are $\tilde{S}_A =$

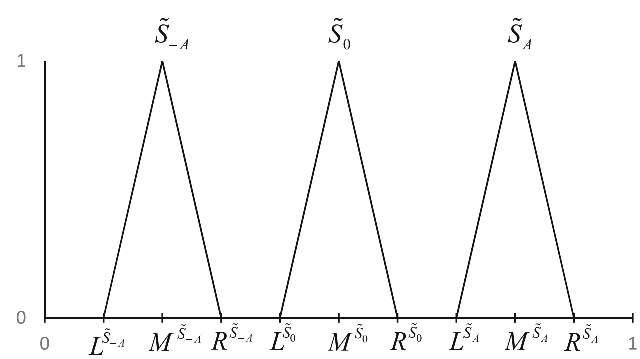

Fig. 1 $\tilde{S}_A$ and $\tilde{S}_{-A}$ are mirror reflections of an isosceles triangle

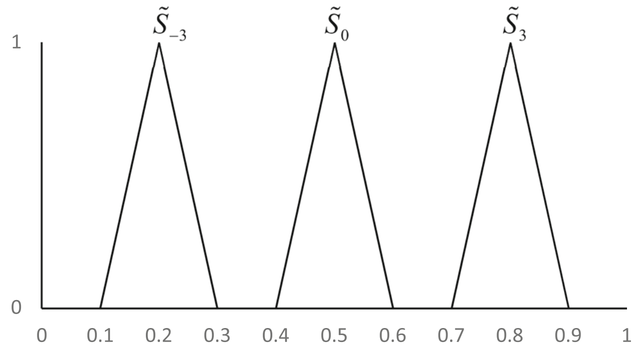

Fig. 2 $\tilde{S}_3$ and $\tilde{S}_{-3}$ are mapped based on NFV $\tilde{S}_0$

$(L^{\tilde{S}_A}, M^{\tilde{S}_A}, R^{\tilde{S}_A})$ and $\tilde{S}_{-A} = (L^{\tilde{S}_{-A}}, M^{\tilde{S}_{-A}}, R^{\tilde{S}_{-A}})$, respectively, which is a (mirror reflection) mapping of an isosceles triangle with the equal-length bases represented as in Fig. 1.

The fuzzy numbers of the isosceles triangles with bases of equal length have the following properties:

Properties

(1) $R^{\tilde{S}_{-A}} - L^{\tilde{S}_{-A}} = R^{\tilde{S}_0} - L^{\tilde{S}_0} = R^{\tilde{S}_A} - L^{\tilde{S}_A}$

(2) $M^{\tilde{S}_A} - M^{\tilde{S}_0} = M^{\tilde{S}_0} - M^{\tilde{S}_{-A}}$ or $M^{\tilde{S}_A} + M^{\tilde{S}_{-A}} = 2M^{\tilde{S}_0}$

(3) $L^{\tilde{S}_A} - M^{\tilde{S}_0} = M^{\tilde{S}_0} - R^{\tilde{S}_{-A}}$ or $L^{\tilde{S}_A} + L^{\tilde{S}_{-A}} = 2M^{\tilde{S}_0}$

(4) $R^{\tilde{S}_A} - M^{\tilde{S}_0} = M^{\tilde{S}_0} - L^{\tilde{S}_{-A}}$ or $R^{\tilde{S}_A} + L^{\tilde{S}_{-A}} = 2M^{\tilde{S}_0}$

(5) $L^{\tilde{S}_A} - L^{\tilde{S}_0} = L^{\tilde{S}_0} - L^{\tilde{S}_{-A}}$ or $L^{\tilde{S}_A} + L^{\tilde{S}_{-A}} = 2L^{\tilde{S}_0}$

(6) $R^{\tilde{S}_A} - R^{\tilde{S}_0} = R^{\tilde{S}_0} - R^{\tilde{S}_{-A}}$ or $R^{\tilde{S}_A} + R^{\tilde{S}_{-A}} = 2R^{\tilde{S}_0}$

(7) $L^{\tilde{S}_A} - R^{\tilde{S}_0} = L^{\tilde{S}_0} - R^{\tilde{S}_{-A}}$ or $L^{\tilde{S}_A} + R^{\tilde{S}_{-A}} = L^{\tilde{S}_0} + R^{\tilde{S}_0}$

(8) $R^{\tilde{S}_A} - L^{\tilde{S}_0} = R^{\tilde{S}_0} - L^{\tilde{S}_{-A}}$ or $R^{\tilde{S}_A} + L^{\tilde{S}_{-A}} = R^{\tilde{S}_0} + L^{\tilde{S}_0}$

(9) $M^{\tilde{S}_A} - R^{\tilde{S}_0} = M^{\tilde{S}_0} - R^{\tilde{S}_{-A}}$ or $M^{\tilde{S}_A} + R^{\tilde{S}_{-A}} = M^{\tilde{S}_0} + R^{\tilde{S}_0}$

(10) $M^{\tilde{S}_A} - L^{\tilde{S}_0} = M^{\tilde{S}_0} - L^{\tilde{S}_{-A}}$ or $M^{\tilde{S}_A} + L^{\tilde{S}_{-A}} = M^{\tilde{S}_0} + L^{\tilde{S}_0}$

Theorem *If the FNV $\tilde{S}_0$ is a set of fixed fuzzy numbers (a constant value), $\tilde{S}_A$ and $\tilde{S}_{-A}$ are mapped, and only if the fuzzy numbers $\tilde{S}_0$, $\tilde{S}_A$, and $\tilde{S}_{-A}$ are of isosceles triangles with bases of equal length, then $\tilde{S}_A \oplus \tilde{S}_{-A} = 2\tilde{S}_0$ (see* Fig. 2).

Proof Let

$\tilde{S}_0 = (L^{\tilde{S}_0}, M^{\tilde{S}_0}, R^{\tilde{S}_0})$

$\tilde{S}_A = (L^{\tilde{S}_A}, M^{\tilde{S}_A}, R^{\tilde{S}_A})$

$\tilde{S}_{-A} = (L^{\tilde{S}_{-A}}, M^{\tilde{S}_{-A}}, R^{\tilde{S}_{-A}})$

According to properties (1)–(10), then verify:

$$\begin{aligned} \tilde{S}_A \oplus \tilde{S}_{-A} &= (L^{\tilde{S}_A} + L^{\tilde{S}_{-A}}, M^{\tilde{S}_A} + M^{\tilde{S}_{-A}}, R^{\tilde{S}_A} + R^{\tilde{S}_{-A}}) \\ &= (2L^{\tilde{S}_0}, 2M^{\tilde{S}_0}, 2R^{\tilde{S}_0}) = 2\tilde{S}_0 \end{aligned} \quad (6)$$

In this case, a set of fixed fuzzy numbers $2\tilde{S}_0$ is represented as $\tilde{S}_{\text{const}}$, then:

$$\tilde{S}_A \oplus \tilde{S}_{-A} = 2\tilde{S}_0 = \tilde{S}_{\text{const}} \quad (7)$$

Definition 1 When alternative A_i is compared with A_j, the fuzzy evaluation number is expressed as $\tilde{a}_{ij}$; when A_j is compared with A_i, the fuzzy evaluation number is expressed as $\tilde{a}_{ji}$. Let $\tilde{a}_{ij}$ and $\tilde{a}_{ji}$ be mapped in the decision-making matrix, based on Equation $\tilde{S}_A \oplus \tilde{S}_{-A} = 2\tilde{S}_0 = \tilde{S}_{\text{const}}$; then,

$$\tilde{a}_{ij} \oplus \tilde{a}_{ji} = \tilde{S}_{\text{const}}, \quad i \neq j, \quad i,j = 1,2,\ldots,m \quad (8)$$

$\tilde{a}_{ii}$ are the elements on the diagonal in the decision-making matrices.

$$\tilde{a}_{ii} = \tilde{S}_0 \quad (9)$$

Numerical Example 1 Let NFV $\tilde{S}_0 = (0.4, 0.5, 0.6)$; the mapping of $\tilde{S}_3 = (0.7, 0.8, 0.9)$ in the decision-making matrix is $\tilde{S}_{-3} = (0.1, 0.2, 0.3)$, represented as in Fig. 2. The three fuzzy numbers $\tilde{S}_0$, $\tilde{S}_3$, and $\tilde{S}_{-3}$ satisfy properties (1)–(10) and Eq. (7) (i.e., $\tilde{S}_3 \oplus \tilde{S}_{-3} = (0.7, 0.8, 0.9) \oplus (0.1, 0.2, 0.3) = (0.8, 1.0, 1.2) = 2\tilde{S}_0 = \tilde{S}_{\text{const}}$).

Properties *If $\tilde{a}_{ij}$ is on the right side of $\tilde{S}_0$, then the mapping of the fuzzy number is on the left side of $\tilde{S}_0$ and represented as $\tilde{a}_{ji}$; the inverse is also true.*

Definition 2 Let $\tilde{D} = (\tilde{a}_{ij})_{m \times m}$ be a matrix of an incomplete linguistic preference relation; then $\tilde{D}$ is called a consistent incomplete linguistic preference relation if:

$$\tilde{a}_{ik} \oplus \tilde{a}_{kj} = \tilde{a}_{ij}, \quad \text{for all } i,j,k \quad (10)$$

Equation (10) is an additive transitivity relation. It represents the ideas as follows. The interpretation of $\tilde{a}_{ik}$ is on the right side of $\tilde{S}_0$ and $\tilde{a}_{ki}$ is on the left side of $\tilde{S}_0$. Then, the intensity of the preference for alternative x_i is over x_k. If $\tilde{a}_{kj}$ is on the right side of $\tilde{S}_0$, then the intensity of the preference for alternative x_k is over x_j. Therefore, the intensity of the preference for alternative x_i is over x_j, so that $\tilde{a}_{ij}$ should be on the right side of $\tilde{S}_0$ and $\tilde{a}_{ji}$ on the left side of $\tilde{S}_0$.

Properties *In the decision-making matrix, if $i < j$, $2 < j < m$, m is the number of alternatives; the element $\tilde{a}_{ij}$ is in the upper-right diagonal matrix and its mapping is $\tilde{a}_{ji}$ in the lower-left diagonal matrix. Then:*

$$\tilde{a}_{ij} \oplus \tilde{a}_{ji} = 2\tilde{S}_0 = \tilde{S}_{\text{const}} \tag{11}$$

3.4 Algorithmic rules for three different kinds of pairwise comparison decision-making matrices based on Fuzzy InLinPreRa

According to Hsu and Wang (2011), the algorithmic rules for three different kinds of pairwise comparison decision-making matrices for Fuzzy InLinPreRa are expressed as:

Type 1: Horizontal pairwise comparison

$$^r\tilde{D}^{(e)} = \left[{}^r\tilde{a}_{ij}^{(e)}\right]_{m\times m} = \begin{matrix} & A_1 & A_2 & A_3 & A_4 & \dots & A_m \\ A_1 & \tilde{S}_0 & {}^r\tilde{a}_{12}^{(e)} & {}^r\tilde{a}_{13}^{(e)} & {}^r\tilde{a}_{14}^{(e)} & \dots & {}^r\tilde{a}_{1m}^{(e)} \\ A_2 & \times & \tilde{S}_0 & \times & \times & \dots & \times \\ A_3 & \times & \times & \tilde{S}_0 & \times & \dots & \times \\ A_4 & \times & \times & \times & \tilde{S}_0 & \dots & \times \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_m & \times & \times & \times & \times & \dots & \tilde{S}_0 \end{matrix}_{mxm},$$
$$r = 1,2,\dots,k$$
$$e = 1,2,\dots,n$$

$\times$ are the unknown variables. Suppose that there are six alternatives; $^r\tilde{D}^{(e)} \subset 6\times 6$ sets $^r\tilde{a}_{ij}^{(e)}$ as the reference and evaluation of criteria by decision-makers. Then, the five original linguistic preference values are produced as follows:

$$^r\tilde{a}_{12}^{(e)} = \tilde{S}_{-3},\quad {}^r\tilde{a}_{13}^{(e)} = \tilde{S}_1,\quad {}^r\tilde{a}_{14}^{(e)} = \tilde{S}_{-1}, {}^r\tilde{a}_{16}^{(e)} = \tilde{S}_1$$

The original values of $^r\tilde{a}_{12}^{(e)}$, $^r\tilde{a}_{13}^{(e)}$, $^r\tilde{a}_{14}^{(e)}$, $^r\tilde{a}_{15}^{(e)}$, and $^r\tilde{a}_{16}^{(e)}$ are evaluated by the eth decision-maker under rth criteria. Their mappings are $^r\tilde{a}_{21}^{(e)}$, $^r\tilde{a}_{31}^{(e)}$, $^r\tilde{a}_{41}^{(e)}$, $^r\tilde{a}_{51}^{(e)}$, and $^r\tilde{a}_{61}^{(e)}$, produced according to Eq. (7). That is $^r\tilde{a}_{ij}^{(e)} \oplus {}^r\tilde{a}_{ji}^{(e)} = \tilde{S}_{\text{const}}$, and shown as follows:

$$^r\tilde{a}_{21}^{(e)} \oplus {}^r\tilde{a}_{12}^{(e)} = \tilde{S}_{\text{const}},\quad {}^r\tilde{a}_{13}^{(e)} \oplus {}^r\tilde{a}_{31}^{(e)} = \tilde{S}_{\text{const}},\quad {}^r\tilde{a}_{14}^{(e)} \oplus {}^r\tilde{a}_{41}^{(e)} = \tilde{S}_{\text{const}}$$
$$^r\tilde{a}_{15}^{(e)} \oplus {}^r\tilde{a}_{51}^{(e)} = \tilde{S}_{\text{const}},\quad {}^r\tilde{a}_{16}^{(e)} \oplus {}^r\tilde{a}_{61}^{(e)} = \tilde{S}_{\text{const}}$$

All of the unknown variables $\times$ of the upper half of the triangle are derived from Eq. (10):

$$^r\tilde{a}_{23}^{(e)} = {}^r\tilde{a}_{21}^{(e)} \oplus {}^r\tilde{a}_{13}^{(e)}\quad {}^r\tilde{a}_{25}^{(e)} = {}^r\tilde{a}_{21}^{(e)} \oplus {}^r\tilde{a}_{15}^{(e)}\quad {}^r\tilde{a}_{24}^{(e)} = {}^r\tilde{a}_{21}^{(e)} \oplus {}^r\tilde{a}_{14}^{(e)}$$
$$^r\tilde{a}_{26}^{(e)} = {}^r\tilde{a}_{21}^{(e)} \oplus {}^r\tilde{a}_{16}^{(e)}\quad {}^r\tilde{a}_{35}^{(e)} = {}^r\tilde{a}_{31}^{(e)} \oplus {}^r\tilde{a}_{15}^{(e)}\quad {}^r\tilde{a}_{34}^{(e)} = {}^r\tilde{a}_{31}^{(e)} \oplus {}^r\tilde{a}_{14}^{(e)}$$
$$^r\tilde{a}_{36}^{(e)} = {}^r\tilde{a}_{31}^{(e)} \oplus {}^r\tilde{a}_{16}^{(e)}\quad {}^r\tilde{a}_{45}^{(e)} = {}^r\tilde{a}_{41}^{(e)} \oplus {}^r\tilde{a}_{15}^{(e)}\quad {}^r\tilde{a}_{46}^{(e)} = {}^r\tilde{a}_{41}^{(e)} \oplus {}^r\tilde{a}_{16}^{(e)}$$
$$^r\tilde{a}_{56}^{(e)} = {}^r\tilde{a}_{51}^{(e)} \oplus {}^r\tilde{a}_{16}^{(e)}$$

All unknown variables $\times$ of the lower half of the triangle are derived from Eq. (28). The fuzzy complete preference decision-making matrix is represented as:

$$^r\tilde{D}^{(e)} = \left[{}^r\tilde{a}_{ij}^{(e)}\right]_{m\times m} = \begin{matrix} & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ A_1 & \tilde{S}_0 & \tilde{S}_{-3} & \tilde{S}_1 & \tilde{S}_{-1} & \tilde{S}_{-2} & \tilde{S}_1 \\ A_2 & \tilde{S}_3 & \tilde{S}_0 & {}^r\tilde{a}_{23}^{(e)} & {}^r\tilde{a}_{24}^{(e)} & {}^r\tilde{a}_{25}^{(e)} & {}^r\tilde{a}_{26}^{(e)} \\ A_3 & \tilde{S}_{-1} & {}^r\tilde{a}_{32}^{(e)} & \tilde{S}_0 & {}^r\tilde{a}_{34}^{(e)} & {}^r\tilde{a}_{35}^{(e)} & {}^r\tilde{a}_{36}^{(e)} \\ A_4 & \tilde{S}_1 & {}^r\tilde{a}_{42}^{(e)} & {}^r\tilde{a}_{43}^{(e)} & \tilde{S}_0 & {}^r\tilde{a}_{45}^{(e)} & {}^r\tilde{a}_{46}^{(e)} \\ A_5 & \tilde{S}_2 & {}^r\tilde{a}_{52}^{(e)} & {}^r\tilde{a}_{53}^{(e)} & {}^r\tilde{a}_{54}^{(e)} & \tilde{S}_0 & {}^r\tilde{a}_{56}^{(e)} \\ A_6 & \tilde{S}_{-1} & {}^r\tilde{a}_{62}^{(e)} & {}^r\tilde{a}_{63}^{(e)} & {}^r\tilde{a}_{64}^{(e)} & {}^r\tilde{a}_{65}^{(e)} & \tilde{S}_0 \end{matrix}_{6\times 6},$$
$$r = 1,2,\dots,k$$
$$e = 1,2,\dots,n$$

Type 2: Vertical pairwise comparison

$$^r\tilde{D}^{(e)} = \left[{}^r\tilde{a}_{ij}^{(e)}\right]_{m\times m} =, \begin{matrix} & A_1 & A_2 & A_3 & A_4 & \dots & A_m \\ A_1 & \tilde{S}_0 & \times & {}^r\tilde{a}_{13}^{(e)} & \times & \dots & \times \\ A_2 & \times & \tilde{S}_0 & {}^r\tilde{a}_{23}^{(e)} & \times & \dots & \times \\ A_3 & \times & \times & \tilde{S}_0 & \times & \dots & \times \\ A_4 & \times & \times & {}^r\tilde{a}_{43}^{(e)} & \tilde{S}_0 & \dots & \times \\ \dots & \dots & \dots & \dots & \dots & \dots & \times \\ A_m & \times & \times & {}^r\tilde{a}_{m3}^{(e)} & \times & \dots & \tilde{S}_0 \end{matrix}_{m\times m},$$
$$r = 1,2,\dots,k$$
$$e = 1,2,\dots,n,$$

$\times$ are the unknown variables. The fuzzy complete matrix is produced following the Type 1 algorithmic rules. Suppose that there are six alternatives; $^r\tilde{D}^{(e)} \subset 6\times 6$ is set $^r\tilde{a}_{ij}^{(e)}$ as the reference and evaluation of criteria by the decision-makers. Then, the five original linguistic preference values are produced as follows:

$$^r\tilde{a}_{13}^{(e)} = \tilde{S}_2,\quad {}^r\tilde{a}_{23}^{(e)} = \tilde{S}_{-1},\quad {}^r\tilde{a}_{43}^{(e)} = \tilde{S}_3,\quad {}^r\tilde{a}_{53}^{(e)} = \tilde{S}_{-2},$$
$$^r\tilde{a}_{63}^{(e)} = \tilde{S}_1$$

The mappings of the fuzzy numbers are derived from Eq. (11) and shown as follows:

$$^r\tilde{a}_{13}^{(e)} \oplus {}^r\tilde{a}_{31}^{(e)} = \tilde{S}_{\text{const}}\quad {}^r\tilde{a}_{23}^{(e)} \oplus {}^r\tilde{a}_{32}^{(e)} = \tilde{S}_{\text{const}}\quad {}^r\tilde{a}_{43}^{(e)} \oplus {}^r\tilde{a}_{34}^{(e)} = \tilde{S}_{\text{const}}$$
$$^r\tilde{a}_{53}^{(e)} \oplus {}^r\tilde{a}_{35}^{(e)} = \tilde{S}_{\text{const}}$$

All of the unknown variables $\times$ the upper half of the triangle are derived from Eq. (27):

$$^r\tilde{a}_{12}^{(e)} = {}^r\tilde{a}_{13}^{(e)} \oplus {}^r\tilde{a}_{32}^{(e)}\quad {}^r\tilde{a}_{14}^{(e)} = {}^r\tilde{a}_{13}^{(e)} \oplus {}^r\tilde{a}_{34}^{(e)}$$
$$^r\tilde{a}_{15}^{(e)} = {}^r\tilde{a}_{13}^{(e)} \oplus {}^r\tilde{a}_{35}^{(e)}\quad {}^r\tilde{a}_{16}^{(e)} = {}^r\tilde{a}_{13}^{(e)} \oplus {}^r\tilde{a}_{36}^{(e)}$$
$$^r\tilde{a}_{21}^{(e)} = {}^r\tilde{a}_{23}^{(e)} \oplus {}^r\tilde{a}_{31}^{(e)}\quad {}^r\tilde{a}_{22}^{(e)} = {}^r\tilde{a}_{23}^{(e)} \oplus {}^r\tilde{a}_{32}^{(e)}$$
$$^r\tilde{a}_{25}^{(e)} = {}^r\tilde{a}_{23}^{(e)} \oplus {}^r\tilde{a}_{35}^{(e)}\quad {}^r\tilde{a}_{26}^{(e)} = {}^r\tilde{a}_{23}^{(e)} \oplus {}^r\tilde{a}_{36}^{(e)}$$
$$^r\tilde{a}_{45}^{(e)} = {}^r\tilde{a}_{43}^{(e)} \oplus {}^r\tilde{a}_{35}^{(e)}\quad {}^r\tilde{a}_{46}^{(e)} = {}^r\tilde{a}_{43}^{(e)} \oplus {}^r\tilde{a}_{36}^{(e)}$$
$$^r\tilde{a}_{56}^{(e)} = {}^r\tilde{a}_{53}^{(e)} \oplus {}^r\tilde{a}_{36}^{(e)}$$

All of the unknown variables $\times$ of the lower half of the triangle are derived from Eq. (11) $\tilde{a}_{ij} \oplus \tilde{a}_{ji} = \tilde{S}_{\text{const}}$. The fuzzy complete preference decision-making matrix is represented as:

The fuzzy complete preference decision-making matrix is represented as:

$$
{}^r\tilde{D}^{(e)} = \left[{}^r\tilde{a}_{ij}^{(e)}\right]_{m\times m}
= \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array}
\begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array} \\
\begin{bmatrix}
\tilde{S}_0 & {}^r\tilde{a}_{12}^{(e)} & \tilde{S}_2 & {}^r\tilde{a}_{14}^{(e)} & {}^r\tilde{a}_{15}^{(e)} & {}^r\tilde{a}_{16}^{(e)} \\
{}^r\tilde{a}_{21}^{(e)} & \tilde{S}_0 & \tilde{S}_{-1} & {}^r\tilde{a}_{24}^{(e)} & {}^r\tilde{a}_{25}^{(e)} & {}^r\tilde{a}_{26}^{(e)} \\
\tilde{S}_{-2} & \tilde{S}_1 & \tilde{S}_0 & \tilde{S}_{-3} & \tilde{S}_2 & \tilde{S}_{-1} \\
{}^r\tilde{a}_{41}^{(e)} & {}^r\tilde{a}_{42}^{(e)} & \tilde{S}_3 & \tilde{S}_0 & {}^r\tilde{a}_{45}^{(e)} & {}^r\tilde{a}_{46}^{(e)} \\
{}^r\tilde{a}_{51}^{(e)} & {}^r\tilde{a}_{52}^{(e)} & \tilde{S}_{-2} & {}^r\tilde{a}_{54}^{(e)} & \tilde{S}_0 & {}^r\tilde{a}_{56}^{(e)} \\
{}^r\tilde{a}_{61}^{(e)} & {}^r\tilde{a}_{62}^{(e)} & \tilde{S}_1 & {}^r\tilde{a}_{64}^{(e)} & {}^r\tilde{a}_{65}^{(e)} & \tilde{S}_0
\end{bmatrix}_{6\times 6} \end{array}
$$

Type 3: Oblique pairwise comparison

$$
{}^r\tilde{D}^{(e)} = \left[{}^r\tilde{a}_{ij}^{(e)}\right]_{m\times m}
= \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \dots \\ A_m \end{array}
\begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \dots & A_m \end{array} \\
\begin{bmatrix}
\tilde{S}_0 & {}^r\tilde{a}_{12}^{(e)} & \times & \times & \dots & \times \\
\times & \tilde{S}_0 & {}^r\tilde{a}_{23}^{(e)} & \times & \dots & \times \\
\times & \times & \tilde{S}_0 & {}^r\tilde{a}_{34}^{(e)} & \dots & \times \\
\times & \times & \times & \tilde{S}_0 & \ddots & \times \\
\vdots\dots & \dots & \dots & \dots & \ddots & {}^r\tilde{a}_{m-1\,m}^{(e)} \\
\times & \times & \times & \times & \dots & \tilde{S}_0
\end{bmatrix}_{m\times m} \end{array},
$$

$$r = 1, 2, \ldots, k$$
$$e = 1, 2, \ldots, n$$

$\times$ are the unknown variables. The fuzzy complete matrix is formed following the Type 1 algorithmic rules. Suppose there are six alternatives; ${}^rA^{(e)} \subset 6\times 6$ sets ${}^r\tilde{a}_{ij}^{(e)}$ as the reference and evaluation of criteria by decision-makers. Then, the five original linguistic preference values are produced as follows:

$$
{}^r\tilde{a}_{12}^{(e)} = \tilde{S}_2, \quad {}^r\tilde{a}_{23}^{(e)} = \tilde{S}_{-1}, \quad {}^r\tilde{a}_{34}^{(e)} = \tilde{S}_2, \quad {}^r\tilde{a}_{45}^{(e)} = \tilde{S}_1, \quad {}^r\tilde{a}_{56}^{(e)} = \tilde{S}_{-3}
$$

The mappings of the fuzzy numbers are derived from Eq. (11) and shown as follows:

$$
{}^r\tilde{a}_{12}^{(e)} \oplus {}^r\tilde{a}_{21}^{(e)} = \tilde{S}_{\text{const}} \quad {}^r\tilde{a}_{23}^{(e)} \oplus {}^r\tilde{a}_{32}^{(e)} = \tilde{S}_{\text{const}} \quad {}^r\tilde{a}_{34}^{(e)} \oplus \tilde{a}_{43}^{(e)} = \tilde{S}_{\text{const}}
$$
$$
{}^r\tilde{a}_{45}^{(e)} \oplus {}^r\tilde{a}_{54}^{(e)} = \tilde{S}_{\text{const}} \quad {}^r\tilde{a}_{56}^{(e)} \oplus {}^r\tilde{a}_{65}^{(e)} = \tilde{S}_{\text{const}}
$$

All of the unknown variables $\times$ the upper half of the triangle are derived from Eq. (10):

$$
{}^r\tilde{a}_{13}^{(e)} = {}^r\tilde{a}_{12}^{(e)} \oplus {}^r\tilde{a}_{23}^{(e)} \quad {}^r\tilde{a}_{14}^{(e)} = {}^r\tilde{a}_{13}^{(e)} \oplus {}^r\tilde{a}_{34}^{(e)}
$$
$$
{}^r\tilde{a}_{16}^{(e)} = {}^r\tilde{a}_{15}^{(e)} \oplus {}^r\tilde{a}_{56}^{(e)} \quad {}^r\tilde{a}_{15}^{(e)} = {}^r\tilde{a}_{14}^{(e)} \oplus {}^r\tilde{a}_{45}^{(e)}
$$
$$
{}^r\tilde{a}_{24}^{(e)} = {}^r\tilde{a}_{23}^{(e)} \oplus {}^r\tilde{a}_{34}^{(e)} \quad {}^r\tilde{a}_{25}^{(e)} = {}^r\tilde{a}_{24}^{(e)} \oplus {}^r\tilde{a}_{45}^{(e)}
$$
$$
{}^r\tilde{a}_{26}^{(e)} = {}^r\tilde{a}_{25}^{(e)} \oplus {}^r\tilde{a}_{56}^{(e)} \quad {}^r\tilde{a}_{35}^{(e)} = {}^r\tilde{a}_{34}^{(e)} \oplus {}^r\tilde{a}_{45}^{(e)}
$$
$$
{}^r\tilde{a}_{36}^{(e)} = {}^r\tilde{a}_{33}^{(e)} \oplus {}^r\tilde{a}_{56}^{(e)} \quad {}^r\tilde{a}_{46}^{(e)} = {}^r\tilde{a}_{45}^{(e)} \oplus {}^r\tilde{a}_{56}^{(e)}
$$

All of the unknown variables $\times$ of the lower half of the triangle are derived from Eq. (11). The fuzzy complete preference decision-making matrix is represented as:

$$
{}^r\tilde{D}^{(e)} = \left[{}^r\tilde{a}_{ij}^{(e)}\right]_{m\times m}
= \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array}
\begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array} \\
\begin{bmatrix}
\tilde{S}_0 & \tilde{S}_3 & {}^r\tilde{a}_{13}^{(e)} & {}^r\tilde{a}_{14}^{(e)} & {}^r\tilde{a}_{15}^{(e)} & {}^r\tilde{a}_{16}^{(e)} \\
\tilde{S}_{-3} & \tilde{S}_0 & \tilde{S}_{-1} & {}^r\tilde{a}_{24}^{(e)} & {}^r\tilde{a}_{25}^{(e)} & {}^r\tilde{a}_{26}^{(e)} \\
{}^r\tilde{a}_{31}^{(e)} & \tilde{S}_1 & \tilde{S}_0 & \tilde{S}_{-3} & {}^r\tilde{a}_{35}^{(e)} & {}^r\tilde{a}_{36}^{(e)} \\
{}^r\tilde{a}_{41}^{(e)} & {}^r\tilde{a}_{42}^{(e)} & \tilde{S}_3 & \tilde{S}_0 & \tilde{S}_5 & {}^r\tilde{a}_{46}^{(e)} \\
{}^r\tilde{a}_{51}^{(e)} & {}^r\tilde{a}_{52}^{(e)} & {}^r\tilde{a}_{53}^{(e)} & \tilde{S}_{-5} & \tilde{S}_0 & \tilde{S}_{-3} \\
{}^r\tilde{a}_{61}^{(e)} & {}^r\tilde{a}_{62}^{(e)} & {}^r\tilde{a}_{63}^{(e)} & {}^r\tilde{a}_{64}^{(e)} & \tilde{S}_3 & \tilde{S}_0
\end{bmatrix}_{6\times 6} \end{array},
$$

$$r = 1, 2, \ldots, k$$
$$e = 1, 2, \ldots, n$$

3.5 Definition of the fuzzy numbers for the appropriate linguistic variables

In this study, the linguistic values were characterized by the triangular fuzzy number defined as [0, 1] (Chou and Chen 2020). The definition of the fuzzy number is different from that which was given by Xu (2006). Xu (2006) defined the corresponding values of $S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_0, S_1, S_2, S_3, S_4$ as − 4, − 3, − 2, − 1, 0, 1, 2, 3, 4; the neutral value is S_0, where $S_0 = 0$. In this study, the definition of the corresponding fuzzy numbers of $\tilde{S}_{-4}, \tilde{S}_{-3}, \tilde{S}_{-2}, \tilde{S}_{-1}, \tilde{S}_0, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4$ is between $\tilde{0} = (0.0, 0.0, 0.0)$ and $\tilde{1} = (1.0, 1.0, 1.0)$. The NFV is $\tilde{S}_0$, where $\tilde{S}_0 = (0.4, 0.5, 0.6)$.

Let $\tilde{S}_0 = (0.4, 0.5, 0.6)$ be the NFV. The fuzzy numbers are mapped on both sides. On its left side, the fuzzy numbers are called $\tilde{S}_{-4}, \tilde{S}_{-3}, \tilde{S}_{-2}, \tilde{S}_{-1}$, whereas the values of the fuzzy numbers are defined between $\tilde{S}_{-4} = (0.0, 0.1, 0.2)$ and $\tilde{S}_0 = (0.4, 0.5, 0.6)$. On the right side, the fuzzy numbers are called $\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4$, whereas the values of the fuzzy numbers are defined between $\tilde{S}_0 = (0.4, 0.5, 0.6)$ and $\tilde{S}_4 = (0.8, 0.9, 1.0)$. Here, the use of a triangular fuzzy number is one of the major components of fuzzy set theory (Wang and Chen 2008). The triangular

Table 2 Fuzzy linguistic variables and triangular fuzzy numbers

	Linguistic variables	Triangular fuzzy numbers
$\tilde{S}_{-4}$	Extremely not preferred	(0.0, 0.1, 0.2)
$\tilde{S}_{-3}$	Not preferred	(0.1, 0.2, 0.3)
$\tilde{S}_{-2}$	Moderately not preferred	(0.2, 0.3, 0.4)
$\tilde{S}_{-1}$	Slightly not preferred	(0.3, 0.4, 0.5)
$\tilde{S}_{0}$	Indifferent	(0.4, 0.5, 0.6)
$\tilde{S}_{1}$	Slightly preferred	(0.5, 0.6, 0.7)
$\tilde{S}_{2}$	Moderately preferred	(0.6, 0.7, 0.8)
$\tilde{S}_{3}$	Moderately preferred	(0.7, 0.8, 0.9)
$\tilde{S}_{4}$	Extremely preferred	(0.8, 0.9, 1.0)

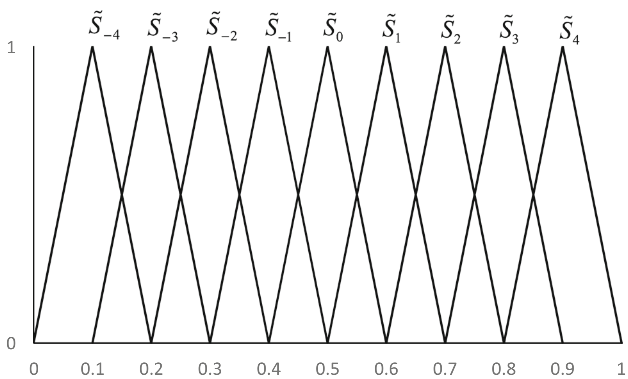

Fig. 3 Triangular fuzzy numbers for fuzzy linguistic variables

fuzzy numbers are shown in Table 2 as isosceles triangular fuzzy numbers with bases of equal lengths (see Fig. 3).

Xu (2006) considered $S_0 = 0$ to be the neutral value. On the right side are $S_1,\ S_2,\ S_3,\ S_4$ and the mapped values are $S_{-1},\ S_{-2},\ S_{-3},\ S_{-4}$. For example, when comparing A_2 to A_3, the linguistic term of evaluation is "strongly preferred." The element in the decision-making matrix is represented as $a_{23} = S_2 = 2$ and its mapping is $a_{32} = S_{-2} = -2$; then, $S_2 + S_{-2} = S_0 = 0$.

In this study, the FNV is a set of fuzzy numbers (i.e., $\tilde{S}_0 = (0.4, 0.5, 0.6)$). On the right side are $\tilde{S}_1,\ \tilde{S}_2,\ \tilde{S}_3,\ \tilde{S}_4$. Then, the mapping is $\tilde{S}_{-1},\ \tilde{S}_{-2},\ \tilde{S}_{-3},\ \tilde{S}_{-4}$, respectively. For example, when comparing A_2 to A_3, the linguistic term of evaluation is "strongly preferred," so that the element in the decision-making matrix is represented as $\tilde{a}_{23} = \tilde{S}_2 = (0.6, 0.7, 0.8)$, and its mapping is $\tilde{a}_{32} = \tilde{S}_{-2} = (0.2, 0.3, 0.4)$. Then, $\tilde{a}_{23} \oplus \tilde{a}_{32} = \tilde{S}_2 \oplus \tilde{S}_{-2} = (0.8, 1.0, 1.2)$.

3.6 Construction of the decision-making matrix for Fuzzy InLinPreRa

The processes of constructing a consistent decision-making matrix are represented as follows.

Step 1. Formation of the original decision-making matrix

In this study, the original decision-making matrix followed a Type 1 horizontal pairwise comparison, represented as follows:

$$
{}^{r}\tilde{D}^{(e)} = \left[{}^{r}\tilde{a}_{ij}^{(e)}\right]_{m\times m}
= \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \cdots \\ A_m \end{array}
\begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \cdots & A_m \end{array} \\
\begin{bmatrix}
\tilde{S}_0 & {}^{r}\tilde{a}_{12}^{(e)} & {}^{r}\tilde{a}_{13}^{(e)} & {}^{r}\tilde{a}_{14}^{(e)} & \cdots & {}^{r}\tilde{a}_{1m}^{(e)} \\
\times & \tilde{S}_0 & \times & \times & \cdots & \times \\
\times & \times & \tilde{S}_0 & \times & \cdots & \times \\
\times & \times & \times & \tilde{S}_0 & \cdots & \times \\
\cdots & \cdots & \cdots & \cdots & \cdots & \times \\
\times & \times & \times & \times & \cdots & \tilde{S}_0
\end{bmatrix}_{m\times m} \end{array},
\quad r = 1, 2, \ldots, k \quad e = 1, 2, \ldots, n
$$

The first-row values were evaluated by the eth decision-maker. Then, the values in the first column were obtained from the rule of mapping based on Eq. (28), $\tilde{a}_{ij} \oplus \tilde{a}_{ji} = 2\tilde{S}_0$. All unknown variables $\times$ the upper half of the triangle were calculated according to Eq. (27), $\tilde{a}_{ik} \oplus \tilde{a}_{kj} = \tilde{a}_{ij}$. The decision-making matrix is represented as follows:

$$
{}^{r}\tilde{D}^{(e)} = \left[{}^{r}\tilde{a}_{ij}^{(e)}\right]_{m\times m}
= \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \cdots \\ A_m \end{array}
\begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \cdots & A_m \end{array} \\
\begin{bmatrix}
\tilde{S}_0 & {}^{r}\tilde{a}_{12}^{(e)} & {}^{r}\tilde{a}_{13}^{(e)} & {}^{r}\tilde{a}_{14}^{(e)} & \cdots & {}^{r}\tilde{a}_{1m}^{(e)} \\
{}^{r}\tilde{a}_{21}^{(e)} & \tilde{S}_0 & {}^{r}\tilde{a}_{23}^{(e)} & {}^{r}\tilde{a}_{24}^{(e)} & \cdots & {}^{r}\tilde{a}_{2m}^{(e)} \\
{}^{r}\tilde{a}_{31}^{(e)} & \times & \tilde{S}_0 & {}^{r}\tilde{a}_{34}^{(e)} & \cdots & {}^{r}\tilde{a}_{3m}^{(e)} \\
{}^{r}\tilde{a}_{41}^{(e)} & \times & \times & \tilde{S}_0 & \cdots & {}^{r}\tilde{a}_{4m}^{(e)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
{}^{r}\tilde{a}_{m1}^{(e)} & \times & \times & \times & \cdots & \tilde{S}_0
\end{bmatrix}_{m\times m} \end{array},
\quad r = 1, 2, \ldots, k \quad e = 1, 2, \ldots, n
$$

Step 2. Conversion of the fuzzy number within the boundary $\tilde{S}_4 = (0.8, 0.9, 1.0)$.

In fuzzy set theory, each element is mapped to [0, 1] by a membership function (Radhika and Parvathi 2016). In this case, suppose the fuzzy number $\tilde{a}_{ik} = (L, M, R)$ in the upper half of the triangle of the decision matrix was outside the boundary $\tilde{S}_4 = (0.8, 0.9, 1.0)$; all of the fuzzy numbers must then be converted into the boundary's parameters.

Table 3 Experts' backgrounds

Sample no	Gender	Age	Educational level	Position	Tenure
1	Female	31–40	University	Sales specialist	6
2	Female	51–60	University	Sales	2
3	Male	21–30	Universitystudent	Sales	2
4	Male	21–30	University	Sales	2
5	Female	21–30	University	Sales	2
6	Male	31–40	University	Sales	8
7	Male	31–40	University	Sales	5
8	Male	21–30	University	Sales	4
9	Male	51–60	Master	Sales director	15
10	Female	31–40	University	Sales	10
11	Male	41–50	High school	Sales manager	6

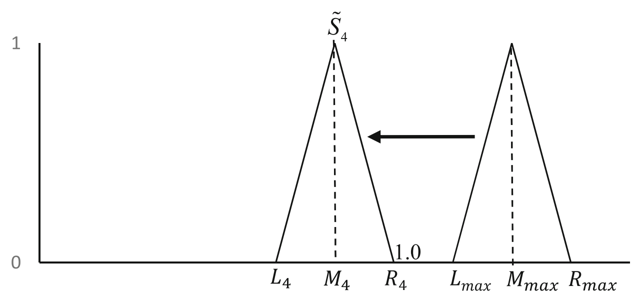

Fig. 4 Conversion of the fuzzy numbers for $L_{\max}$, $M_{\max}$, and $R_{\max}$

The conversion method is different here, with the formula $f(x) = \frac{x+a}{1+2a}$ provided by Herrera-Viedma et al. (2004). The first step in the conversion is to search out $L_{\max}$, $M_{\max}$, and $R_{\max}$ from the matrix:

$$L_{\max} = \max\{\tilde{a}_{ik} = (L, \times, \times) | \forall i, k, \quad i < k, \quad 2 < k < m\}$$
$$M_{\max} = \max\{\tilde{a}_{ik} = (\times, M, \times) | \forall i, k, \quad i < k, \quad 2 < k < m\}$$
$$R_{\max} = \max\{\tilde{a}_{ik} = (\times, \times, R) | \forall i, k, \quad i < k, \quad 2 < k < m\}$$

$\times$ is unknown. Then, the new converted fuzzy numbers are calculated according to Eq. (12).

$$\tilde{a}_{ik}^{\text{new}} = \left(\frac{L}{L_{\max}} \times L_4, \frac{M}{M_{\max}} \times M_4, \frac{R}{R_{\max}} \times R_4\right) = (L_{\text{new}}, M_{\text{new}}, R_{\text{new}}) \quad (12)$$

Let the fuzzy number be $\tilde{a}_{ik} = (L, M, R)$, where $L_{\text{new}} \le M_{\text{new}} \le R_{\text{new}}$. When the value of the fuzzy numbers $L_{\max}, M_{\max}, R_{\max}$ in the upper-right diagonal of the decision-making matrix are on the right side of $\tilde{S}_4 = (L_4, M_4, R_4) = (0.8, 0.9, 1.0)$(to exceed the values $(0.8, 0.9, 1.0)$), then the values of the fuzzy numbers should be converted according to Eq. (12), as follows (see Fig. 4).

$$\tilde{a}_{ik}^{\text{new}} = \left(\frac{L}{L_{\max}} \times 0.8, \frac{M}{M_{\max}} \times 0.9, \frac{R}{R_{\max}} \times 1.0\right) \quad (13)$$

The values of the fuzzy numbers in the lower-left diagonal of the decision-making matrix are calculated according to Eq. (11). The converted decision-making matrix ${}^{r}\tilde{C}^{(e)}$ is represented as follows:

$$ {}^{r}\tilde{C}^{(e)} = \left[{}^{r}\tilde{c}_{ij}^{(e)}\right]_{m \times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \cdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\ \begin{bmatrix} \tilde{S}_0 & {}^{r}\tilde{c}_{12}^{(e)} & {}^{r}\tilde{c}_{13}^{(e)} & {}^{r}\tilde{c}_{14}^{(e)} & \ldots & {}^{r}\tilde{c}_{1m}^{(e)} \\ {}^{r}\tilde{c}_{21}^{(e)} & \tilde{S}_0 & {}^{r}\tilde{c}_{23}^{(e)} & {}^{r}\tilde{c}_{24}^{(e)} & \ldots & {}^{r}\tilde{c}_{2m}^{(e)} \\ {}^{r}\tilde{c}_{31}^{(e)} & {}^{r}\tilde{c}_{32}^{(e)} & \tilde{S}_0 & {}^{r}\tilde{c}_{34}^{(e)} & \ldots & {}^{r}\tilde{c}_{3m}^{(e)} \\ {}^{r}\tilde{c}_{14}^{(e)} & {}^{r}\tilde{c}_{42}^{(e)} & {}^{r}\tilde{c}_{43}^{(e)} & \tilde{S}_0 & \ldots & {}^{r}\tilde{c}_{4m}^{(e)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ {}^{r}\tilde{c}_{m1}^{(e)} & {}^{r}\tilde{c}_{m2}^{(e)} & {}^{r}\tilde{c}_{m3}^{(e)} & {}^{r}\tilde{c}_{m4}^{(e)} & \ldots & \tilde{S}_0 \end{bmatrix}_{m \times m} \end{array},$$
$$e = 1, 2, \ldots, n$$
$$r = 1, 2, \ldots, k$$

Step 3. Multiplication by the fuzzy weight of each criterion

In general, the converted decision matrix is multiplied by the weight of the ${}^{1}\tilde{w}, {}^{2}\tilde{w}, \ldots, {}^{k}\tilde{w}$ criteria. In this case, this converted decision-making matrix is multiplied by the fuzzy weight of the rth criterion ${}^{r}\tilde{w}$, which is calculated according to Eq. (6) and represented as follows:

$$
{}^{r}\tilde{C}^{(e)} \times {}^{r}\tilde{w} = \left[{}^{r}\tilde{c}_{ij}^{(e)} \otimes {}^{r}\tilde{w} \right]_{m\times m}
= \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \ldots \\ A_m \end{array}
\begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\
\begin{bmatrix}
\tilde{S}_0 \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{12}^{(e)} \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{13}^{(e)} \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{14}^{(e)} \otimes {}^{r}\tilde{w} & \ldots & {}^{r}\tilde{c}_{1m}^{(e)} \otimes {}^{r}\tilde{w} \\
{}^{r}\tilde{c}_{21}^{(e)} \otimes {}^{r}\tilde{w} & \tilde{S}_0 \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{23}^{(e)} \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{24}^{(e)} \otimes {}^{r}\tilde{w} & \ldots & {}^{r}\tilde{c}_{2m}^{(e)} \otimes {}^{r}\tilde{w} \\
{}^{r}\tilde{c}_{31}^{(e)} \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{32}^{(e)} \otimes {}^{r}\tilde{w} & \tilde{S}_0 \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{34}^{(e)} \otimes {}^{r}\tilde{w} & \ldots & {}^{r}\tilde{c}_{3m}^{(e)} \otimes {}^{r}\tilde{w} \\
{}^{r}\tilde{c}_{14}^{(e)} \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{42}^{(e)} \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{43}^{(e)} \otimes {}^{r}\tilde{w} & \tilde{S}_0 \otimes {}^{r}\tilde{w} & \ldots & {}^{r}\tilde{c}_{4m}^{(e)} \otimes {}^{r}\tilde{w} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
{}^{r}\tilde{c}_{m1}^{(e)} \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{m2}^{(e)} \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{m3}^{(e)} \otimes {}^{r}\tilde{w} & {}^{r}\tilde{c}_{m4}^{(e)} \otimes {}^{r}\tilde{w} & \ldots & \tilde{S}_0 \otimes {}^{r}\tilde{w}
\end{bmatrix}_{m\times m} \end{array}
$$

where $e = 1, 2, \ldots, n,\ r = 1, 2, \ldots, k.$

Step 4. Integration of the decision matrix for all criteria respective to the individual expert

Integration of the decision matrix for k criteria must be accomplished respective to the individual expert, according to the equation of fuzzy number addition. Then, it is divided by the number of criteria. The criteria-integrated matrix is represented as follows:

$$
\widetilde{\mathrm{CIM}}^{(e)} = \frac{1}{k}\sum_{r=1}^{k} {}^{r}\tilde{c}_{ij}^{(e)} \oplus^{r} \tilde{w} = \left[\widetilde{\mathrm{cim}}^{(e)} \right]_{m\times m}
= \frac{1}{k} \times \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \ldots \\ A_m \end{array}
\begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\
\begin{bmatrix}
\tilde{S}_0 \oplus^{r} \tilde{w} & {}^{r}\tilde{c}_{12}^{(e)} \otimes^{r} \tilde{w} & {}^{r}\tilde{c}_{13}^{(e)} \otimes^{r} \tilde{w} & {}^{r}\tilde{c}_{14}^{(e)} \otimes^{r} \tilde{w} & \ldots & {}^{r}\tilde{c}_{1m}^{(e)} \otimes^{r} \tilde{w} \\
{}^{r}\tilde{c}_{21}^{(e)} \otimes^{r} \tilde{w} & \tilde{S}_0 \oplus^{r} \tilde{w} & {}^{r}\tilde{c}_{23}^{(e)} \otimes^{r} \tilde{w} & {}^{r}\tilde{c}_{24}^{(e)} \otimes^{r} \tilde{w} & \ldots & {}^{r}\tilde{c}_{2m}^{(e)} \otimes^{r} \tilde{w} \\
{}^{r}\tilde{c}_{31}^{(e)} \otimes^{r} \tilde{w} & {}^{r}\tilde{c}_{32}^{(e)} \otimes^{r} \tilde{w} & \tilde{S}_0 \oplus^{r} \tilde{w} & {}^{r}\tilde{c}_{34}^{(e)} \otimes^{r} \tilde{w} & \ldots & {}^{r}\tilde{c}_{3m}^{(e)} \otimes^{r} \tilde{w} \\
{}^{r}\tilde{c}_{14}^{(e)} \otimes^{r} \tilde{w} & {}^{r}\tilde{c}_{42}^{(e)} \otimes^{r} \tilde{w} & {}^{r}\tilde{c}_{43}^{(e)} \otimes^{r} \tilde{w} & \tilde{S}_0 \oplus^{r} \tilde{w} & \ldots & {}^{r}\tilde{c}_{4m}^{(e)} \otimes^{r} \tilde{w} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
{}^{r}\tilde{c}_{m1}^{(e)} \otimes^{r} \tilde{w} & {}^{r}\tilde{c}_{m2}^{(e)} \otimes^{r} \tilde{w} & {}^{r}\tilde{c}_{m3}^{(e)} \otimes^{r} \tilde{w} & {}^{r}\tilde{c}_{m4}^{(e)} \otimes^{r} \tilde{w} & \ldots & \tilde{S}_0 \oplus^{r} \tilde{w}
\end{bmatrix}_{m\times m} \end{array}
$$

$$
= \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \ldots \\ A_m \end{array}
\begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\
\begin{bmatrix}
\widetilde{\mathrm{cim}}_{11}^{(e)} & \widetilde{\mathrm{cim}}_{12}^{(e)} & \widetilde{\mathrm{cim}}_{13}^{(e)} & \widetilde{\mathrm{cim}}_{14}^{(e)} & \ldots & \widetilde{\mathrm{cim}}_{1m}^{(e)} \\
\widetilde{\mathrm{cim}}_{21}^{(e)} & \widetilde{\mathrm{cim}}_{22}^{(e)} & \widetilde{\mathrm{cim}}_{23}^{(e)} & \widetilde{\mathrm{cim}}_{24}^{(e)} & \ldots & \widetilde{\mathrm{cim}}_{2m}^{(e)} \\
\widetilde{\mathrm{cim}}_{31}^{(e)} & \widetilde{\mathrm{cim}}_{32}^{(e)} & \widetilde{\mathrm{cim}}_{33}^{(e)} & \widetilde{\mathrm{cim}}_{34}^{(e)} & \ldots & \widetilde{\mathrm{cim}}_{3m}^{(e)} \ldots \\
\widetilde{\mathrm{cim}}_{41}^{(e)} & \widetilde{\mathrm{cim}}_{42}^{(e)} & \widetilde{\mathrm{cim}}_{43}^{(e)} & \widetilde{\mathrm{cim}}_{44}^{(e)} & \ldots & \widetilde{\mathrm{cim}}_{4m}^{(e)} \\
\widetilde{\ldots} & \widetilde{\ldots} & \widetilde{\ldots} & \widetilde{\ldots} & \ldots & \widetilde{\ldots} \\
\widetilde{\mathrm{cim}}_{m1}^{(e)} & \widetilde{\mathrm{cim}}_{m4}^{(e)} & \widetilde{\mathrm{cim}}_{m3}^{(e)} & \widetilde{\mathrm{cim}}_{m4}^{(e)} & \ldots & \widetilde{\mathrm{cim}}_{mm}^{(e)}
\end{bmatrix}_{m\times m} \end{array}
$$

where $e = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, k.$

Step 5. Multiplication by the fuzzy weight of the individual decision-maker

Generally, the integrated decision matrix is multiplied by the weight of the individual decision-maker $\tilde{w}^{(1)}, \tilde{w}^{(2)}, \ldots, \tilde{w}^{(n)}$. In this case, this integrated decision-making matrix is multiplied by the weight of the eth decision-maker, $\tilde{w}^{(e)}$. The **experts' integrated matrix** is represented as follows:

$$\widetilde{\mathrm{EIM}}^{(e)} = \widetilde{\mathrm{EIM}}^{(e)} \otimes \tilde{w}^{(e)} = \left[\widetilde{\mathrm{eim}}^{(e)}\right]_{m\times m}$$

$$= \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \ldots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\ \begin{bmatrix} \widetilde{\mathrm{cim}}_{11}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{12}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{13}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{14}^{(e)} \otimes \tilde{w}^{(e)} & \ldots & \widetilde{\mathrm{cim}}_{1m}^{(e)} \otimes \tilde{w}^{(e)} \\ \widetilde{\mathrm{cim}}_{21}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{22}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{23}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{24}^{(e)} \otimes \tilde{w}^{(e)} & \ldots & \widetilde{\mathrm{cim}}_{2m}^{(e)} \otimes \tilde{w}^{(e)} \\ \widetilde{\mathrm{cim}}_{31}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{32}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{33}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{34}^{(e)} \otimes \tilde{w}^{(e)} & \ldots & \widetilde{\mathrm{cim}}_{3m}^{(e)} \otimes \tilde{w}^{(e)} \\ \widetilde{\mathrm{cim}}_{41}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{42}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{43}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{44}^{(e)} \otimes \tilde{w}^{(e)} & \ldots & \widetilde{\mathrm{cim}}_{4m}^{(e)} \otimes \tilde{w}^{(e)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \widetilde{\mathrm{cim}}_{m1}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{m4}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{m3}^{(e)} \otimes \tilde{w}^{(e)} & \widetilde{\mathrm{cim}}_{m4}^{(e)} \otimes \tilde{w}^{(e)} & \ldots & \widetilde{\mathrm{cim}}_{mm}^{(e)} \otimes \tilde{w}^{(e)} \end{bmatrix}_{m\times m} \end{array}$$

$$= \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \ldots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\ \begin{bmatrix} \widetilde{\mathrm{eim}}_{11}^{(e)} & \widetilde{\mathrm{eim}}_{12}^{(e)} & \widetilde{\mathrm{eim}}_{13}^{(e)} & \widetilde{\mathrm{eim}}_{14}^{(e)} & \ldots & \widetilde{\mathrm{eim}}_{1m}^{(e)} \\ \widetilde{\mathrm{eim}}_{21}^{(e)} & \widetilde{\mathrm{eim}}_{22}^{(e)} & \widetilde{\mathrm{eim}}_{23}^{(e)} & \widetilde{\mathrm{eim}}_{24}^{(e)} & \ldots & \widetilde{\mathrm{eim}}_{2m}^{(e)} \\ \widetilde{\mathrm{eim}}_{31}^{(e)} & \widetilde{\mathrm{eim}}_{32}^{(e)} & \widetilde{\mathrm{eim}}_{33}^{(e)} & \widetilde{\mathrm{eim}}_{34}^{(e)} & \ldots & \widetilde{\mathrm{eim}}_{3m}^{(e)} \ldots \\ \widetilde{\mathrm{eim}}_{41}^{(e)} & \widetilde{\mathrm{eim}}_{42}^{(e)} & \widetilde{\mathrm{eim}}_{43}^{(e)} & \widetilde{\mathrm{eim}}_{44}^{(e)} & \ldots & \widetilde{\mathrm{eim}}_{4m}^{(e)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \widetilde{\mathrm{eim}}_{m1}^{(e)} & \widetilde{\mathrm{eim}}_{m4}^{(e)} & \widetilde{\mathrm{eim}}_{m3}^{(e)} & \widetilde{\mathrm{eim}}_{m4}^{(e)} & \ldots & \widetilde{\mathrm{eim}}_{mm}^{(e)} \end{bmatrix}_{m\times m} \end{array}$$

$$\text{where} \quad e = 1, 2, \ldots, n$$

Step 6. Integration of all decision-making matrices for the decision-making experts

The decision-making matrices for all experts are then integrated into a single matrix. The **final (F)**-integrated decision-making matrix is represented as follows:

$$\tilde{F} = \frac{1}{n}\sum_{e=1}^{n} \widetilde{\mathrm{EIM}}^{(e)}$$

$$= \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \ldots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & \ldots & A_m \end{array} \\ \begin{bmatrix} \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{11}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{12}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{13}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{14}^{(e)} & \ldots & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{1m}^{(e)} \\ \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{21}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{22}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{23}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{24}^{(e)} & \ldots & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{2m}^{(e)} \\ \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{31}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{32}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{33}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{34}^{(e)} & \ldots & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{3m}^{(e)} \ldots \\ \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{41}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{42}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{43}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{44}^{(e)} & \ldots & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{4m}^{(e)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{m1}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{m4}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{m3}^{(e)} & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{m4}^{(e)} & \ldots & \sum_{e=1}^{n} \widetilde{\mathrm{eim}}_{mm}^{(e)} \end{bmatrix}_{m\times m} \end{array}$$

Step 7. Defuzzification of the integrated decision-making matrix

Table 4 Linguistic terms for weights corresponding to fuzzy numbers

Linguistic terms	Triangular fuzzy numbers
Extremely non-important (ENI)	(0.0, 1.5, 1.5)
Not important (NIP)	(0.5, 1.5, 2.5)
Slightly not important (SNI)	(1.5, 2.5, 3.5)
Neutral (N)	(2.5, 3.5, 4.5)
Slightly important (SIP)	(3.5, 4.5, 5.5)
Important (IP)	(4.5, 5.5, 6.5)
Extremely important (EIP)	(5.5, 7.0, 7.0)

Table 5 Weights for decision-making experts corresponding to fuzzy numbers

Expert nos	Linguistic terms	Fuzzy number weights
E1	EIP	(5.5, 7.0, 7.0)
E2	NIP	(0.5, 1.5, 2.5)
E3	NIP	(0.5, 1.5, 2.6)
E4	NIP	(0.5, 1.5, 2.7)
E5	SNI	(1.5, 2.5, 3.5)
E6	EIP	(5.5, 7.0, 7.0)
E7	IP	(4.5, 5.5, 6.5)
E8	SIP	(3.5, 4.5, 5.5)
E9	EIP	(5.5, 7.0, 7.0)
E10	EIP	(5.5, 7.0, 7.0)
E11	IP	(4.5, 5.5, 6.5)

This study applied maximizing and minimizing set methods (Chen 1985) to defuzzify the fuzzy numbers. Equations are represented as follows:

Maximizing Set $R = \{(x, f_R(x)) | x \in R|\}$.

$$\text{and} \quad f_R(x) = \begin{cases} (x - x_1)/(x_2 - x_1), & x_1 \le x \le x_2 \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

Minimizing Set $L = \{(x, f_L(x)) | x \in R|\}$.

$$\text{and} \quad f_R(x) = \begin{cases} (x - x_2)/(x_1 - x_2), & x_1 \le x \le x_2 \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

$$U_M(i) = \sup_x (f_M(x) \wedge f_{A_i}(x)), \quad i = 1, 2, \ldots, n \tag{16}$$

$$U_G(i) = \sup_x (f_G(x) \wedge f_{A_i}(x)), \quad i = 1, 2, \ldots, n \tag{17}$$

$$U_T(i) = [U_M(i) + 1 - U_G(i)]/2, \quad i = 1, 2, \ldots, n \tag{18}$$

Step 8. Ranking the crisp values of all alternatives

Table 6 Weights of criteria based on the importance of the experts

Criteria	Fuzzy numbers
Sincerity	(0.0843, 0.1877, 0.4124)
Excitement	(0.0945, 0.2056, 0.4458)
Competence	(0.0997, 0.2182, 0.4755)
Sophistication	(0.0868, 0.1909, 0.4340)
Ruggedness	(0.0896, 0.1976, 0.4306)

4 Empirical example

In this section, Fuzzy InLinPreRa was employed to demonstrate a quantitative basis for analytically determining from a managerial viewpoint the ranking of popular brand personalities in the sports shoe industry (Khazaei Pool et al. 2018). The research procedure was conducted according to the following steps:

Step 1 Determine the evaluation criteria, alternatives, and decision-making experts.

The construction of brand personality has received considerable attention in consumer behavior research. Brand personality, a five-dimensional framework developed by Aaker (1997), is comprised of the integers of sincerity, excitement, competence, sophistication, and ruggedness. It refers to a series of human characteristics associated with given brands. This characterization plays an important role in promoting interaction between consumers and brands, thus helping to create, develop, and maintain strong brands (Seimiene and Kamarauskaite 2014). Based on the self-concept and self-congruity theories, consumer behavior research has suggested that consumers prefer brands that they believe to be similar in various respects to themselves. Brand self-congruity refers to a match between a brand's image and an individual's self-concept, strongly influencing brand success, such as through positive consumer brand recognition, customer satisfaction, and customer loyalty (Matzler et al. 2016).

We considered ten of the current leading brands in the sports shoe industry: Adidas, Asics, Champion, Converse, New Balance, Nike, Puma, Reebok, Skechers, and Under Armour (Khazaei Pool et al. 2018). There were 11 decision-making experts who worked in the sports shoe industry had extensive experience, and held different positions (Table 3).

In this case, the decision-makers were denoted as E_e, where $e = 1, 2, \ldots, 11$; the evaluating criteria as C_r, where $r = 1, 2, \ldots, 5$; and the alternatives as A_i, where $i = 1, 2, \ldots, 10$.

Step 2 Data collection.

This study employed a questionnaire divided into three parts: evaluation of alternatives, weight of criteria, and personal information. The survey was distributed to 11 decision-making experts who had worked as sales and marketing executives in the sports shoe industry between 2 and 15 years. The linguistic terms for the weights corresponded to the fuzzy numbers represented (as in the table). The weights of the criteria and experts corresponding to the fuzzy numbers are shown in Tables 4 and 5, respectively.

The weights of the criteria were calculated as referents to the importance of the experts, according to Eq. (5). The results are shown in Table 6.

Example

The left fuzzy number of criterion 1 (^1w) was calculated as follows:

$$\frac{(5.5 \times 5.5 + 0.5 \times 0.5 + \ldots + 4.5 \times 1.5)}{[(5.5 \times 5.5 + 0.5 \times 0.5 + \ldots + 4.5 \times 1.5) + \ldots + (5.5 \times 4.5 + 0.5 \times 0.5 + \ldots + 4.5 \times 2.5)]} = 0.0843$$

Step 3 Construction of the decision-making matrix

This study utilized the algorithm of horizontal pairwise comparison to construct all of the experts' evaluations under each criterion. The original matrix evaluated by expert E_1 for criteria C_1 was represented as:

C_1	A_1	A_2	A_3	…	A_{10}
A_1	(0.4, 0.5, 0.6)	(0.4, 0.5, 0.6)	(0.6, 0.7, 0.8)	…	(0.4, 0.5, 0.6)
A_2	×	(0.4, 0.5, 0.6)	×	…	×
A_3	×	×	(0.4, 0.5, 0.6)	…	×
…	…	…	…	…	…
A_{10}	×	×	×	…	(0.4, 0.5, 0.6)

The fuzzy numbers in the first column were produced according to the rule of mapping, (i.e., Eq. 11) and the matrix was as follows:

C_1	A_1	A_2	A_3	…	A_{10}
A_1	(0.4, 0.5, 0.6)	(0.4, 0.5, 0.6)	(0.6, 0.7, 0.8)	…	(0.4, 0.5, 0.6)
A_2	(0.4, 0.5, 0.6)	(0.4, 0.5, 0.6)	×	…	×
A_3	(0.2, 0.3, 0.4)	×	(0.4, 0.5, 0.6)	…	×
…	…	…	…	…	…
A_{10}	(0.4, 0.5, 0.6)	×	×	…	(0.4, 0.5, 0.6)

All unknown variables × the upper half of the triangle were calculated according to Eq. (10).

C_1	A_1	A_2	A_3	…	A_{10}
A_1	(0.4, 0.5, 0.6)	(0.4, 0.5, 0.6)	(0.6, 0.7, 0.8)	…	(0.4, 0.5, 0.6)
A_2	(0.4, 0.5, 0.6)	(0.4, 0.5, 0.6)	**(1.0, 1.2, 1.4)**	…	(0.8, 1.0, 1.2)
A_3	(0.2, 0.3, 0.4)	×	(0.4, 0.5, 0.6)	…	(0.6, 0.8, 1.0)
…	…	…	…	…	…
A_{10}	(0.4, 0.5, 0.6)	×	(0.2, 0.2, 0.2)	…	(0.4, 0.5, 0.6)

Step 4 Conversion of the fuzzy numbers into the boundary $\tilde{S_4} = (0.8, 0.9, 1.0)$.

The maximum fuzzy numbers that exceeded the boundary, such as (1.0, 1.2, 1.4), were sorted. Then, all of the fuzzy numbers were converted according to Eq. (13).

Table 7 Overall ranking of the alternatives

Brand	New Balance	Asics	Nike	Adidas	Under Armour	Skechers	Reebok	Champion	Converse	Puma
Avg	0.3264	0.3075	0.3578	0.4260	0.3093	0.3385	0.3499	0.2682	0.3478	0.4180
Rank	7	9	3	1	8	6	4	10	5	2

The unknown variables × in the lower half of the triangle were derived from Eq. (28). The complete fuzzy preference decision-making matrix was represented as:

C_1	A_1	A_2	A_3	…	A_{10}
A_1	(0.4, 0.5, 0.6)	(0.4, 0.5, 0.6)	(0.6, 0.7, 0.8)	…	(0.4, 0.5, 0.6)
A_2	(0.4, 0.5, 0.6)	(0.4, 0.5, 0.6)	(0.73, 0.83, 0.93)	…	(0.58, 0.69, 0.8)
A_3	(0.2, 0.3, 0.4)	(0.07, 0.17, 0.27)	(0.4, 0.5, 0.6)	…	(0.44, 0.55, 0.67)
…	…	…	…	…	…
A_{10}	(0.4, 0.5, 0.6)	(0.22, 0.31, 0.4)	(0.36, 0.45, 0.53)	…	(0.4, 0.5, 0.6)

Step 5 Multiplication by the fuzzy number for each criterion's weight

C_1	A_1	A_2	…	A_{10}
A_1	(0.4, 0.5, 0.6)⊗ (0.0843, 0.1877, 0.4124)	(0.4, 0.5, 0.6)⊗ (0.0843, 0.1877, 0.4124)	…	(0.4, 0.5, 0.6)⊗ (0.0843, 0.1877, 0.4124)
A_2	(0.4, 0.5, 0.6)⊗ (0.0843, 0.1877, 0.4124)	(0.4, 0.5, 0.6)⊗ (0.0843, 0.1877, 0.4124)	…	(0.58, 0.69, 0.8)⊗ (0.0843, 0.1877, 0.4124)
A_3	(0.2, 0.3, 0.4)⊗ (0.0843, 0.1877, 0.4124)	(0.07, 0.17, 0.27)⊗(0.0843, 0.1877, 0.4124)	…	(0.44, 0.55, 0.67)⊗ (0.0843, 0.1877, 0.4124)
…	…	…	…	…
A_{10}	(0.4, 0.5, 0.6)⊗ (0.0843, 0.1877, 0.4124)	(0.22, 0.31, 0.4)⊗ (0.0843, 0.1877, 0.4124)	…	(0.4, 0.5, 0.6)⊗ (0.0843, 0.1877, 0.4124)

Step 6 Integration of the five-criteria matrix for each expert's evaluation of the alternatives, then dividing by 5

E_1	A_1	A_2	…	A_{10}
A_1	[(0.0337, 0.0939, 0.2474)⊕ (0.0378, 0.1028, 0.2675)⊕… ⊕(0.0358, 0.0988, 0.2584)]/5	[(0.0337, 0.0939, 0.2474)⊕ (0.0283, 0.0823, 0.2229)⊕… ⊕(0.0448, 0.1185, 0.3014)]/5	…	[(0.0337, 0.0939, 0.2474)⊕ (0.0472, 0.1234, 0.3121)⊕… ⊕(0.0269, 0.0790, 0.2153)]/5
A_2	[(0.0337, 0.0939, 0.2475)⊕ (0.0472, 0.1234, 0.3121)⊕… ⊕(0.0269, 0.0790, 0.2153)]/5	[(0.0337, 0.0939, 0.2474)⊕ (0.0378, 0.1028, 0.2675)⊕… ⊕(0.0358, 0.0988, 0.2584)]/5	…	[(0.0491, 0.1300, 0.3299)⊕ (0.0630, 0.1586, 0.3901)⊕… ⊕(0.0358, 0.1016, 0.2691)]/5
⋮	⋮	⋮	…	⋮
A_{10}	[(0.0337, 0.0939, 0.2475)⊕ (0.0283, 0.0823, 0.2229)⊕… ⊕(0.0448, 0.1185, 0.3014)]/5	[(0.0184, 0.0578, 0.1649)⊕ (0.0126, 0.0470, 0.0814)⊕… ⊕(0.0358.0.0960, 0.2476)]/5	…	[(0.0337, 0.0939, 0.2474)⊕ (0.0378, 0.1028, 0.2675)⊕… ⊕(0.0358, 0.0988, 0.2584)]/5

Example Integration of the five-criteria preference evaluations for A_1 and A_{10} by expert E_1, calculated as follows:

$$\begin{aligned}&[(0.0337,\ 0.0939,\ 0.2474)\oplus\\&\quad((0.0472,\ 0.1234,\ 0.3121)\oplus(0.0399,\ 0.1091,\ 0.2853)\\&\quad\oplus(0.0521,\ 0.1336,\ 0.3472)\oplus(0.0269,\ 0.0790,\ 0.2153)]/5\\&=(0.0400,\ 0.1078,0.2815)\end{aligned}$$

Step 7 Multiplication by the fuzzy number for each expert's weight

E_1	A_1	A_2	…	A_{10}
A_1	(0.0364,0.1000,0.2638) ⊗(5.5,7,7)	(0.0346,0.0960,0.2548) ⊗(5.5,7,7)	…	(0.0100,0.1078,0.2815) ⊗(5.5,7,7)
A_2	(0.0382,0.1040,0.2728) ⊗(5.5,7,7)	0.0364,0.1000,0.2638) ⊗(5.5,7,7)	…	0.0511,0.1340,0.3415) ⊗(5.5,7,7)
⋮	⋮	⋮	⋮	⋮
A_{10}	(0.0328,0.0932,0.2461) ⊗(5.5,7,7)	(0.0216,0.0660,0.1860) ⊗(5.5,7,7)	…	(0.0364,0.1000,0.2638) ⊗(5.5,7,7)

Step 8 Integration of the matrix of the 11 experts' evaluations of alternatives, then dividing by 11

E	A_1	A_1	…	A_1
A_1	[(0.2003, 0.7000, 1.8465)⊕ (0.0182, 0.1500, 0.6595)⊕… ⊕(0.1638, 0.4500, 1.1871)]/11	[(0.1901, 0.6721, 1.7836)⊕ (0.0173, 0.1440, 0.6370)⊕… ⊕ 0.1967, 0.5226, 1.347)]/11	…	[(0.2198, 0.7546, 1.9702)⊕ (0.0200, 0.1617, 0.7036)⊕… ⊕(0.2638, 0.6693, 1.6692)]/11
A_1	[(0.2102, 0.7279, 1.9094)⊕ (0.0191, 0.1560, 0.6153)⊕… ⊕(0.1309, 0.3774, 1.0267)]/11	[(0.2002, 0.7000, 1.8465)⊕ (0.0182, 0.1500, 0.6595)⊕… ⊕(0.1638, 0.4500, 1.1871)]/11	…	[(0.2813, 0.9380, 2.3907)⊕ (0.0310, 0.2360, 0.9790)⊕… ⊕(0.2809, 0.7118, 1.7705)]/11
⋮	⋮	⋮	⋮	⋮
A_1	[(0.1805, 0.6454, 1.7229)⊕ (0.0164, 0.1383, 0.6153)⊕… ⊕(0.0637, 0.2307, 0.7049)]/11	[(0.1190, 0.4620, 1.3023)⊕ (0.0054, 0.0640, 0.3399)⊕… ⊕(0.0467, 0.1882, 0.6036)]/11	…	[(0.2002, 0.7000, 1.8465)⊕ (0.0182, 0.1500, 0.6595)⊕… ⊕(0.1638, 0.4500, 1.1871)]/11

Example: Integration of the matrices for the 11 experts' preference evaluations for A_1 and A_{10}, calculated as follows:

[(0.2198, 0.7546, 1.9702) ⊕ (0.0200, 0.1617, 0.7036)⊕(0.0293, 0.2233, 0.9292) ⊕ (0.0293, 0.2233, 0.9293) ⊕ (0.0904, 0.2285, 0.5685)⊕(0.1895, 0.5270, 1.3994) ⊕ (0.2199, 0.5741, 1.4637) ⊕ (0.2288, 0.5729, 1.4130)⊕(0.1588, 0.4596, 1.2524) ⊕ (0.1588, 0.4596, 1.2524) ⊕ (0.2305, 0.6167, 1.5983)⊕(0.2638, 0.6693, 1.6692)]/11 = (0.1527, 0.4556, 1.2633).

Step 9 Defuzzifying the integrated decision-making matrices

E_1-E_{11}	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	0.3389	0.3542	0.3510	0.3874	0.3464	0.3716	0.3510	0.3619	0.2101	0.3874
A_2	0.3084	0.4067	0.4070	0.4247	0.3999	0.4101	0.4070	0.4098	0.2673	0.4247
A_3	0.3180	0.4115	0.4118	0.4293	0.4047	0.4149	0.4118	0.2834	0.4046	0.4293
A_4	0.3231	0.3389	0.4148	0.4323	0.4078	0.4179	0.4148	0.2864	0.4076	0.4323
A_5	0.4037	0.3256	0.5003	0.5185	0.3389	0.4245	0.4214	0.2936	0.4143	0.4387
A_6	0.3040	0.2480	0.4114	0.4289	0.2395	0.3389	0.4114	0.2825	0.4042	0.4289
A_7	0.3265	0.2520	0.3389	0.4320	0.2435	0.2563	0.3389	0.2862	0.4073	0.4320
A_8	0.3150	0.2484	0.2488	0.4292	0.2398	0.2527	0.2488	0.2111	0.4045	0.4292
A_9	0.3399	0.2610	0.2614	0.4389	0.2526	0.2653	0.2614	0.1476	0.3389	0.4389
A_{10}	0.2860	0.2285	0.2326	0.3389	0.2197	0.2329	0.2326	0.1199	0.2195	0.3389
Avg	0.3264	0.3075	0.3578	0.4260	0.3093	0.3385	0.3499	0.2682	0.3478	0.4180

Step 10 Ranking the alternatives.

The results showed that Adidas, Puma, Nike, Reebok, and Converse were, respectively, ranked from first to fifth in terms of preference (Table 7).

5 Conclusions

MCDM techniques can help decision-makers choose viable alternatives for real world decision-making problems involving multiple conflicting criteria. Multi-criteria analysis problems require decision-makers to make qualitative evaluations concerning the performance of alternatives with regard to the relative importance of each independent criterion, and each independent criterion with regard to the overall goal of the problem set. Due to the relative complexities and uncertainties of decision-making problems and inherent subjectivity of human judgment, accurate conclusions are often unrealistic or unfeasible. Decision-makers often find that assigning linguistic variables to judgments feels more natural and is easier than to fixed value judgments (Chen et al. 2011). The use of fuzzy sets is more compatible with the vague interpretations of human language (Khazaei Pool et al. 2018). Therefore, it is better to use fuzzy instead of crisp numbers to indicate the data (Chen et al. 2011; Yang and Wang 2013).

This study presented Fuzzy InLinPreRa as a means of addressing increasingly complex decision-making problems resulting from rapid economic development and profound social change (Chen et al. 2022a; Peng et al. 2022). Triangular fuzzy numbers were used here to quantify linguistic variables in Fuzzy InLinPreRa because their simplicity and ease of use has made them the most commonly employed to represent linguistic information in practical applications (Tavana et al. 2021). The theoretical contributions of this study can be summarized as follows. Fuzzy InLinPreRa is an alternative additive transitivity property-based estimation of the use of the fuzzy set method. It considers more objective weights of criteria and weights of decision-makers, allowing decision-making in imprecise and vague environments and solving inconsistent problems. When decision-makers process pairwise comparisons for criteria with the least number of judgments (i.e., $n-1$ judgments), comparisons can be carried out more efficiently and do not generate inconsistent problems; this makes the decision-making process more efficient and accurate. Each decision-maker can unrestrictedly choose the explicit index for pairwise comparisons, named horizontal, vertical, and oblique comparisons. The rest of the unknown variables can be obtained through adjoining additions and their corresponding opposite relationship algorithms, and then quickly produce a complete matrix (Hsu and Wang 2011). This study also presented a formula for considering the weights of decision-makers according to their positions and work experiences, in order to obtaining a more reasonable ranking of alternatives.

5.1 Managerial implications

This analytical framework was used to evaluate and rank the personalities of selected brands of sports shoes and verify the feasibility of the proposed approach. The results showed that Fuzzy InLinPreRa is capable of providing invaluable insights for use in strategic marketing decisions. The evaluation and ranking of brands is useful for both academic research and practice. Researchers can measure the competences of each brand by evaluating them, and industrialists can extract the competitive advantages of the brands selected (Khazaei Pool et al. 2018). In addition, this method assures consistency and flexibility for a number of alternatives, attributions, criteria, and hierarchical levels related to decision-making issues. The method can be used as a powerful tool in solving decision-making problems in academic research and practice.

6 Limitations and future research directions

Future directions for this research will focus on consumers' perceptions and preferences, exploring new insights and further considering consumers' heterogeneity (Chen et al. 2022b). The investigation of consumer numbers should go beyond that of expert opinions. It is recommended that software be developed to facilitate analyses of larger decision-making groups.

Appendix 1

Two preference relations for the fuzzy preference relations (Herrera-Viedma et al. 2004):

- Multiplicative preference relation

The multiplicative preference relation A on a set of alternatives X is denoted by a matrix $A \subset X \times X$, $A = (a_{ij})$, a_{ij} is expressed as the ratio of the preference degree of alternative x_i over x_j, and A is assumed to be a multiplicative reciprocal:

$$a_{ij} \cdot a_{ji} = 1 \quad \forall i, j \in \{1, \ldots, n\} \tag{19}$$

- Additive fuzzy preference relation

The fuzzy preference relation supposes that P on a set of alternatives X is denoted by $P = (p_{ij})$, $p_{ij} = \mu_p(x_i, x_j)$ and p_{ij} is regarded as a different preference degree of alternative x_i over x_j. If $p_{ij} = 1/2$ denotes no difference between x_i and $x_j (x_i \sim x_j)$, $p_{ij} = 1$ denotes that x_i is absolutely preferred over x_j, $p_{ij} = 0$ denotes that x_j is absolutely preferred over x_i, $p_{ij} > 1/2$ indicates that x_i is preferred over $x_j (x_i \succ x_j) p$, and the preference matrix is assumed to be an additive reciprocal:

$$p_{ij} + p_{ji} = 1 \quad \forall i, j \in \{1, \ldots, n\} \tag{20}$$

Proposition *Assume a set of alternatives $X = (x_1, \ldots, x_n)$ and a reciprocal multiplicative preference relation $A = (a_{ij})$ with $a_{ij} \in [1/9, 9]$ being associated with it; then, the corresponding reciprocal additive fuzzy preference relation $P = (p_{ij})$ with $p_{ij} \in [0, 1]$ to $A = (a_{ij})$ is given as follows*:

$$P_{ij} = g(a_{ij}) = \frac{1}{2} \cdot \left(1 + \log_9 a_{ij}\right) \tag{21}$$

Using this transformation function g, we can relate the research issues obtained for the two preference relations. In order to make a consistent choice, when a fuzzy preference relationship is assumed, the consistent properties are proposed to satisfy this relation. One of the most important properties concerning preference is transitivity, which represents the preference value obtained by directly comparing two alternatives being equal to or greater than the preference value between those two alternatives obtained from an indirect chain of alternatives. The properties are given below (Herrera-Viedma et al. 2004; Hsu and Wang 2011):

- Additive transitivity consistency fuzzy preference relation

  A reciprocal additive fuzzy preference relation is consistent if:

$$p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \quad \forall i, j, k \tag{22}$$

- Construction of a consistent fuzzy preference relation A

  A set of alternatives $X = \{x_1, x_2, \ldots, x_n, n \geq 2\}$ as a consistent fuzzy preference relation P from $n - 1$ preference values $\{p_{12}, p_{23}, \ldots, p_{n-1n}\}$ can be constructed as follows. The set of preference values B is calculated as:

$$B = \left\{p_{ij}, i < j \wedge p_{ij} \notin \{p_{12}, p_{23}, \ldots, p_{n-1n}\}\right\}$$
$$p_{ji} = \frac{j - i + 1}{2} - p_{ij+1} - p_{i+1,i+2,\ldots} - p_{ij} \tag{23}$$

$$a = |\min\{B \cup \{p_{12}, p_{23}, \ldots, p_{n-1n}\}\}| \tag{24}$$

$$P = \{p_{12}, p_{23}, \ldots, p_{n-1n}\} \cup B \cup \{1 - \tilde{p}_{12}, 1 - \tilde{p}_{23}, \ldots, 1 - \tilde{p}_{n-1n}\} \cup \neg B \tag{25}$$

  The consistent fuzzy preference relation P is obtained as $P = f(P)$:

$$f : [-a, 1+a] \rightarrow [0, 1], \quad f(x) = \frac{x + a}{1 + 2a} \tag{26}$$

Appendix 2

The relevant definitions of incomplete linguistic preference relations are as follows (Hsu and Wang 2011; Shih and Hsu 2016; Xu 2006):

Definition 1 (*Incomplete linguistic preference additive relation*) Let $A = (a_{ij})_{n \times n}$ be a linguistic preference relation. Assume A is an incomplete linguistic preference relation that decision-makers can use to carry out pairwise comparison to satisfy Eq. (9).

$$a_{ij} \in S, \quad a_{ij} \oplus a_{ji} = S_0, \quad a_{ii} = S_0, \quad \text{for all} \quad i, j \tag{27}$$

Definition 2 (*Incomplete linguistic consistent additive preference relation*) Let $A = (a_{ij})_{n \times n}$ be a complete consistent additive preference relation, a type of additive transitivity represented as Eq. (10), which interprets that the a_{ik} value represents the intensity of the preference for alternative $x_i(A_i)$ over $x_k(A_k)$ and the a_{kj} value represents the intensity of the preference for alternative x_k over $x_j(A_j)$. Then, it can reasonably be assumed that the intensity of the preference for alternative x_i over x_j should be equal to the sum of the preference intensities regarding alternative x_k as an intermediate.

$$a_{ik} \oplus a_{kj} = a_{ij}, \quad \text{for all} \quad i, j, k \tag{28}$$

If $a_{ij} = S_0$, $a_{ij} = 0$ represents x_i and x_j indifference, both can satisfy.

$a_{ik} = a_{kj} = a_{ij} = S_0$.

Definition 3 (*Incomplete linguistic preference adjoining preference relation*) Let $A = (a_{ij})_{n \times n}$ be a linguistic preference relation. Assume A is an incomplete linguistic preference relation. If $(i, j) \cap (k, l) \neq \emptyset$, the elements a_{ij} and a_{kl} are named in the adjoining relation.

Definition 4 *(Incomplete linguistic preference indirect relation)* Let $A = (a_{ij})_{n \times n}$ be a linguistic preference relation. Assume A is an incomplete linguistic preference relation and $a_{i_0 j_0}$ is the unknown value in preference matrix A. The element $a_{i_0 j_0}$ is indirectly named available, as derived from the two known adjoining elements $a_{i_0 k}$ and $a_{k j_0}$.

Definition 5 (*Acceptable alternative for incomplete linguistic preference*) Let $A = (a_{ij})_{n \times n}$ be an incomplete linguistic preference relation. Assume A is an incomplete linguistic preference relation. If each unknown element can be obtained through its adjoining known elements, then it is called an acceptable alternative. The acceptable alternative for incomplete linguistic preference A could be the known value in a column or row, having $n-1$ contrasting values by pairs.

Employing the incomplete linguistic preference relations proposed by Xu (2006) to construct the decision-making matrix, as described through the following steps (Chen et al. 2011):

Step 1 Let $D=\{d_1, d_2, \ldots, d_m\}$ be the set of decision-makers and the weight vector of the decision-makers be $W=(w_1, w_2, \ldots, w_m)^{\mathrm{T}}$, $w_k \geq 0$, where $k=1,2,\ldots,m$, $\sum_{k=1}^{m} w_k = 1$. The decision-maker $d_k \in D$ uses linguistic variables to compare all n alternative numbers, where an acceptable incomplete linguistic preference relation matrix $A_k=(a_{ij}^{(k)})_{n\times n}$ will be constructed through $n-1$ times pairwise comparisons, among which $a_{ij}^{(k)}$ denotes that the kth expert counters the preference relation values of the pairwise comparison of alternatives i, j.
Step 2 Apply the known variables in A_k $(k=1,2,\ldots,m)$ and determine all the unknown variables according to Eq. (10), $a_{ij}=a_{ik}\oplus a_{kj}$ in A_k,$(k=1,2,\ldots,m)$. Then, the corresponding consistent complete linguistic preference relations are obtained:

$$\overline{A}=(\overline{a}_{ij}^{(k)})_{n\times n}\,(k=1,2,\ldots,m)$$

Step 3 Each expert's decision preference matrix is multiplied by the weight vector of the decision-maker to integrate a complete decision-making matrix, shown as follows:

$$\overline{a}_{ij}=w_1\overline{a}_{ij}^{(1)}\oplus w_2\overline{a}_{ij}^{(2)}\oplus\cdots\oplus w_m\overline{a}_{ij}^{(m)},\quad \text{for all } i,j \tag{29}$$

Step 4 To calculate the average of all the preference degrees $\overline{a}_{ij}$, $(j=1,2,\ldots,m)$ is in the ith row of $\overline{A}$. Then, the final decision-making preference matrix is obtained.

$$\overline{a}=\frac{1}{n}\overline{a}_{i1}\oplus\frac{1}{n}\overline{a}_{i2}\oplus\cdots\oplus\frac{1}{n}\overline{a}_{in},\quad \text{for all } i \tag{30}$$

Step 5 Rank all alternatives $x_i(i=1,2,\ldots,n)$ and choose the optimal one(s) according to the value of $\overline{a}_i(i=1,2,\ldots,n)$.

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Declarations

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