Fuzzy Interpolative Reasoning for Sparse Fuzzy-Rule-Based Systems Based on the Areas of Fuzzy Sets

Yu-Chuan Chang, Shyi-Ming Chen, *Senior Member, IEEE*, and Churn-Jung Liau

***Abstract*—Fuzzyinterpolative reasoning is an inference technique for dealing with the sparse rules problem in sparse fuzzy-rule-based systems. In this paper, we present a new fuzzy interpolative reasoning method for sparse fuzzy-rule-based systems based on the areas of fuzzy sets. The proposed method uses the weighted average method to infer the fuzzy interpolative reasoning results and has the following advantages: 1) it holds the normality and the convexity of the fuzzy interpolative reasoning result, 2) it can deal with fuzzy interpolative reasoning with complicated membership functions, 3) it can deal with fuzzy interpolative reasoning when the fuzzy sets of the antecedents and the consequents of the fuzzy rules have different kinds of membership functions, 4) it can handle fuzzy interpolative reasoning with multiple antecedent variables, 5) it can handle fuzzy interpolative reasoning with multiple fuzzy rules, and 6) it can handle fuzzy interpolative reasoning with logically consistent properties with respect to the ratios of fuzziness. We use some examples to compare the fuzzy interpolative reasoning results of the proposed method with those of the existing fuzzy interpolative reasoning methods. In terms of the six evaluation indices, the experimental results show that the proposed method performs more reasonably than the existing methods. The proposed method provides us a useful way to deal with fuzzy interpolative reasoning in sparse fuzzy-rule-based systems.**

***Index Terms*—Fuzzy interpolative reasoning, fuzzy rules, multiple antecedent variables, multiple fuzzy rules interpolation, polygonal fuzzy sets, ratios of fuzziness, sparse fuzzy-rule-based systems.**

I. INTRODUCTION

FUZZY interpolative reasoning is an important inference technique for sparse fuzzy-rule-based systems, where the fuzzy sets appearing in the antecedents of the fuzzy rules do not cover the whole input universe of discourse. In other words, there is an empty space between two adjacent membership functions of the fuzzy sets appearing in the antecedents of the fuzzy rules. If an observation occurs in the empty space, there is no rule fired and no consequence will be derived. In recent years, some fuzzy interpolative reasoning methods have been presented for sparse fuzzy-rule-based systems [1]–[16], [18]–[23]. Baranyi *et al.* [1] presented a fuzzy interpolation method that can infer the fuzzy interpolative reasoning result based on the fuzzy relation and the semantic relation of fuzzy sets. Baranyi *et al.* [2] presented a fuzzy interpolative reasoning method that modifies the α-cut-based rule interpolation method to avoid abnormal conclusions. Bouchon *et al.* [3] presented a fuzzy interpolative reasoning method based on the concept of graduality, which infers a conclusion by means of the transformations of location and shape gradually. Hsiao *et al.* [5] presented an interpolative reasoning method based on the slopes of triangular fuzzy sets. Huang and Shen [7] presented a fuzzy interpolative reasoning method based on the representative values of fuzzy sets and presented their scale and move transformation operators to deal with fuzzy interpolative reasoning. Huang [8] improved the method [7] to handle multiple fuzzy rules interpolation and fuzzy rule extrapolation. Jenei [9] presented an approach of interpolation and extrapolation based on compact fuzzy quantities. Jenei [10] presented a method for dealing with multidimensional fuzzy interpolative reasoning. Koczy and Hirota [11]–[13] presented a linear fuzzy interpolative reasoning method, which uses the proportions of fuzzy distance between the observation and rule antecedents to infer the fuzzy interpolative reasoning result. Li *et al.* [14] presented a weighted fuzzy interpolative reasoning method based on the like-gravity-center of trapezoidal fuzzy sets. Marsala *et al.* [15] presented a fuzzy interpolative reasoning method with multiple variable rules. Qiao *et al.* [16] presented a similarity transfer reasoning model to improve Koczy-and-Hirota's fuzzy interpolative reasoning method in sparse fuzzy-rule-based systems. Shi *et al.* [17] pointed out that the Koczy-and-Hirota's fuzzy interpolative reasoning method [11] does not always lead to convex conclusions. Tikk and Baranyi [18] compared the modified α-cuts-based fuzzy interpolation method [21] and Koczy-and-Hirota's fuzzy interpolative reasoning method [11] and made a comprehensive analysis. Wang *et al.* [19] presented a new fuzzy interpolative reasoning method to infer a conclusion based on the similarities of fuzzy sets. Wong *et al.* [20] presented an improved fuzzy rule interpolation technique to handle multidimensional input spaces. Yam *et al.* [22] presented a fuzzy interpolative reasoning method with function space representation of membership functions. Yam and Koczy [23] presented a method for representing membership functions as points in high-dimensional spaces for fuzzy interpolation and fuzzy extrapolation.

Manuscript received December 4, 2006; revised May 3, 2007 and September 9, 2007; accepted November 28, 2007. First published April 30, 2008; current version published October 8, 2008. This work was supported in part by the National Science Council, under Grant NSC 95-2221-E-011-116-MY2.

Y.-C. Chang is with the Department of Computer Science and Information Engineering, National Taiwan University of Science and Technology, Taipei 106, Taiwan (e-mail: D9315003@mail.ntust.edu.tw).

S.-M. Chen is with the Department of Computer Science and Information Engineering, National Taiwan University of Science and Technology, Taipei 106, Taiwan. He is also with the Department of Computer Science and Information Engineering, Jinwen University of Science and Technology, Taipei County, Taiwan (e-mail: smchen@mail.ntust.edu.tw).

C.-J. Liau is with the Institute of Information Science, Academia Sinica, Taipei 115, Taiwan (e-mail: liaucj@iis.sinica.edu.tw).

Digital Object Identifier 10.1109/TFUZZ.2008.924340

Although many methods have been proposed to deal with fuzzy interpolative reasoning, there have been some drawbacks in these methods. Some methods cannot preserve the convexity of the fuzzy interpolative reasoning result [6], [11]; some methods can only deal with triangular membership functions and trapezoidal membership functions [3], [5], [6], [14]; some methods cannot deal with the interpolation when the antecedents and the consequences of the given fuzzy rules are different kinds of membership functions [5], [6], [14]; some methods do not deal with the fuzzy interpolative reasoning with multiple antecedent variables [3], [5], [6], [9], [11], [13], [15], [16], [19]; and some methods do not deal with the fuzzy interpolative reasoning of multiple fuzzy rules [3], [5], [6], [9]–[16], [19], [20]–[23]. For simplicity and efficiency, most interpolation methods handle fuzzy interpolative reasoning based on the two nearest fuzzy rules. However, the interpolation based on the two selected fuzzy rules is not flexible enough to deal with the fuzzy interpolative reasoning in sparse fuzzy-rule-based systems. Multiple fuzzy rule interpolation is desirable for getting more relationships between the fuzzy rules in sparse fuzzy rule bases. If we can perform fuzzy interpolative reasoning involving more rules, then there is room for more flexibility.

In this paper, we present a new fuzzy interpolative reasoning method based on the areas of fuzzy sets. It can overcome the drawbacks mentioned earlier. The proposed method has the following advantages: 1) it preserves the normality and the convexity of the fuzzy interpolative reasoning result, 2) it can deal with fuzzy interpolative reasoning with complicated membership functions (e.g., polygonal membership functions and Gaussian membership functions), 3) it can deal with fuzzy interpolative reasoning when the fuzzy sets of the antecedents and the consequences of the fuzzy rules have different kinds of membership functions, 4) it can handle fuzzy interpolative reasoning with multiple antecedent variables, 5) it can handle fuzzy interpolative reasoning with multiple fuzzy rules, and 6) it can handle fuzzy interpolative reasoning with logically consistent properties with respect to the ratios of fuzziness. We use some examples to compare the proposed method with the existing methods and the comparison shows that the fuzzy interpolative reasoning results of the proposed method are more reasonable than those of the KH method [11], the HCL method [5], the HTY method [6], and the HS method [7].

The rest of this paper is organized as follows. In Section II, we present a new fuzzy interpolative reasoning method for sparse fuzzy-rule-based systems based on the areas of fuzzy sets and present the logically consistent properties with respect to the ratios of fuzziness. In Section III, we use some examples [7], [8] to compare the proposed method with the existing methods in terms of the six evaluation indices. The conclusions are given in Section IV.

II. New Fuzzy Interpolative Reasoning Method

In this section, we present a new fuzzy interpolative reasoning method for sparse fuzzy-rule-based systems based on the areas of membership functions of fuzzy sets. In the following, we describe the proposed method by means of different kinds of membership functions.

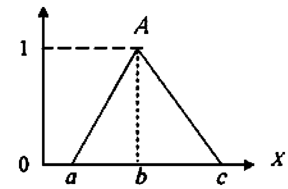

Fig. 1. Triangular fuzzy set.

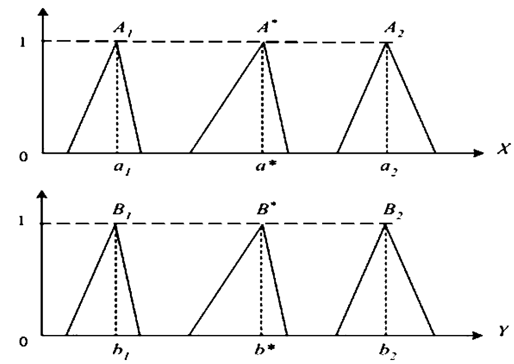

Fig. 2. Fuzzy interpolative reasoning using triangular membership functions.

A. Fuzzy Interpolative Reasoning With Triangular Fuzzy Sets

Triangular membership functions are widely used in fuzzy-rule-based systems. A triangular fuzzy set is typically denoted by (a, b, c), as shown in Fig. 1, where a, b, and c are called the "left extreme point," the "normal point," and the "right extreme point," respectively. An example of fuzzy interpolative reasoning using triangular fuzzy sets is shown in Fig. 2.

Assume that there are two adjacent and disjoint fuzzy rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, where the triangular fuzzy sets A_1 and A_2 are the antecedent fuzzy sets of the fuzzy rules and the triangular fuzzy sets B_1 and B_2 are the consequences of the fuzzy rules. Assume that the observation fuzzy set A^* occurs between the fuzzy sets A_1 and A_2, and fuzzy set B^* denotes the fuzzy interpolative reasoning result. In Fig. 2, a_1, a^*, a_2, b_1, b^*, and b_2 are the normal points of the triangular fuzzy sets A_1, A^*, A_2, B_1, B^*, and B_2, respectively. The proposed fuzzy interpolative reasoning method with triangular fuzzy sets is now presented as follows.

Step 1: Calculate the normal point b^* of the triangular fuzzy set B^* by the linear interpolation, which is defined as follows:

$$\frac{b_2 - b_1}{a_2 - a_1} = \frac{b^* - b_1}{a^* - a_1}. \tag{1}$$

From (1), the normal point b^* is calculated as follows:

$$b^* = b_1 + \frac{(a^* - a_1) \times (b_2 - b_1)}{(a_2 - a_1)}. \tag{2}$$

Step 2: Determine the left and the right extreme points of the fuzzy set B^*. We divide a triangular fuzzy set A into the left area S_L and the right area S_R, as shown in Fig. 3, where d_l and d_r denote the "left bottom length" and the "right bottom length"

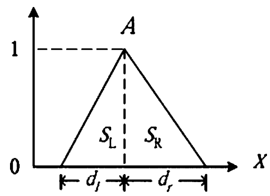

Fig. 3. Left area S_L and the right area S_R of the triangular fuzzy set.

of A, respectively. We use (3) to calculate the left area $S_L(B^*)$ and the right area $S_R(B^*)$ of B^*, respectively, where

$$S_K(B^*) = \begin{cases} S_K(A^*) \times \sum\limits_{\substack{i=1,\\ S_K(A_i)>0}}^{2} W_i \times \dfrac{S_K(B_i)}{S_K(A_i)}, & \text{if } \exists i S_K(A_i) > 0 \\ S_K(A^*), & \text{if } \forall i S_K(A_i) = 0 \end{cases} \quad (3)$$

$K \in \{L,\ R\}, S_L(B^*)$ denotes the left area of $B^*, S_R(B^*)$ denotes the right area of $B^*, W_1 = 1 - (a^* - a_1)/(a_2 - a_1), W_2 = 1 - (a_2 - a^*)/(a_2 - a_1)$, and $W_1 + W_2 = 1$.

Step 3: From Fig. 3, we can see that $S_L(B^*)$ is also equal to $(d_l(B^*) \times h)/2$ and $S_R(B^*)$ is also equal to $(d_r(B^*) \times h)/2$, where $d_l(B^*)$ denotes the left bottom length of B^* and $d_r(B^*)$ denotes the right bottom length of B^*, and h denotes the height of B^*. Because the value of h is equal to 1 when the fuzzy set B^* is normal, the values of $d_l(B^*)$ and $d_r(B^*)$ are equal to $2S_L(B^*)$ and $2S_R(B^*)$, respectively. Then, we can obtain the left extreme point $b^* - 2S_L(B^*)$ and the right extreme point $b^* + 2S_R(B^*)$ of the fuzzy set B^*, respectively. Finally, the fuzzy interpolative reasoning result B^* is derived, where $B^* = (b^* - 2S_L(B^*),\ b^*,\ b^* + 2S_R(B^*))$.

The top equation of (3) is used to infer the areas of the interpolated fuzzy set B^* if there exists a fuzzy rule whose area of the antecedent part is larger than zero. Otherwise, the bottom equation of (3) is used when the areas of the antecedent part of the given fuzzy rules are zero (e.g., both the antecedent fuzzy sets have vertical slopes at their left-hand sides). Generally speaking, the larger the area of the membership function of a fuzzy set is, the more fuzziness the fuzzy set has. The fuzzy interpolative reasoning result inferred by (3) satisfies the logically consistent properties with respect to the ratios of fuzziness based on the two-fuzzy-rules interpolative reasoning technique, where the ratio of fuzziness $RF_K(A,B)$ of the consequence fuzzy set B with respect to the antecedent fuzzy set A is calculated as follows:

$$RF_K(A,B) = \frac{S_K(B)}{S_K(A)} \quad (4)$$

where $S_K(A) > 0, K \in \{L,\ R\}$, the fuzzy sets A and B are the antecedent fuzzy set and the consequence fuzzy set of a fuzzy rule, respectively, $RF_L(A,B)$ denotes the ratio of fuzziness of the left area of B with respect to the left area of $A, RF_R(A,B)$ denotes the ratio of fuzziness of the right area of B with respect to the right area of $A, S_L(A)$ and $S_L(B)$ denote the left areas of A and B, respectively, and $S_R(A)$ and $S_R(B)$ denote the right areas of A and B, respectively. For example, from Fig. 4,

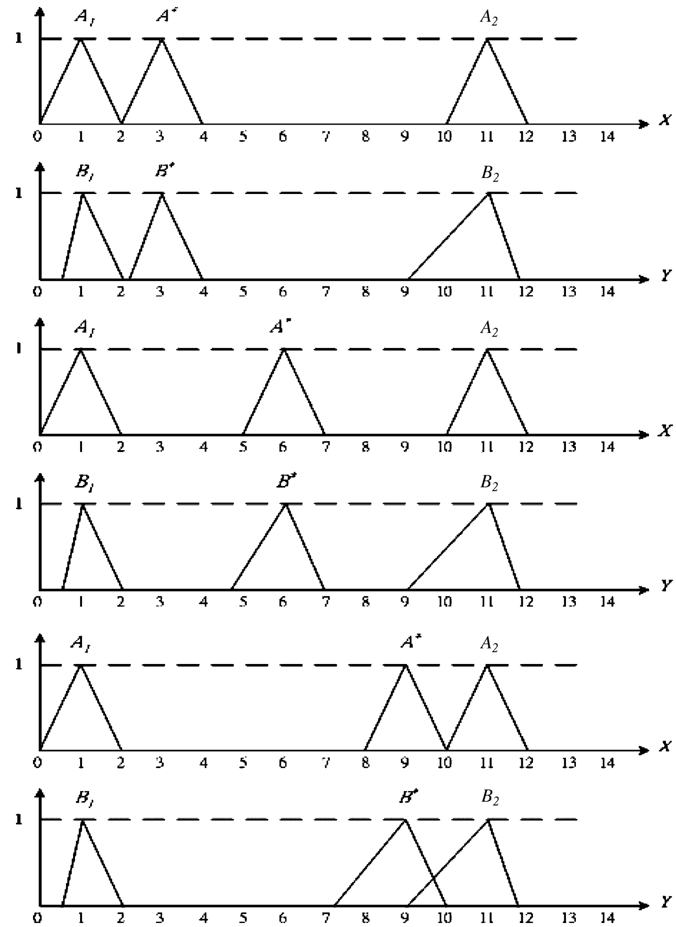

Fig. 4. Fuzzy interpolative reasoning results for the gradual observations.

we can see that $RF_L(A_1, B_1) = 1/2, RF_L(A_2, B_2) = 2$, and $RF_R(A_1, B_1) = RF_R(A_2, B_2) = 1$. For the ratio of fuzziness $RF_K(A,B)$ shown in (4), it does not consider the situation that the antecedent fuzzy sets have vertical slopes at their left-hand side or right-hand side (i.e., $S_L(A) = 0$ or $S_R(A) = 0$).

Assume that there are two adjacent fuzzy rules $A_1 => B_1,\ A_2 => B_2$ and one observation A^*, where A^* occurs between A_1 and A_2. The fuzzy interpolative reasoning result B^* obtained by (3) satisfies the following two properties.

Property 1: $\text{Min}(RF_K(A_1,\ B_1),\ RF_K(A_2,\ B_2)) \le RF_K(A^*,\ B^*) \le \text{Max}(RF_K(A_1,\ B_1),\ RF_K(A_2,\ B_2))$, where $K \in \{L, R\}$.

Proof: Based on (3) and (4), the value of $RF_K(A^*, B^*)$ is calculated as follows:

$$RF_K(A^*, B^*) = \frac{W_1 \times RF_K(A_1, B_1) + W_2 \times RF_K(A_2, B_2)}{W_1 + W_2} \quad (5)$$

where $0 \le W_1 \le 1, 0 \le W_2 \le 1, W_1 + W_2 = 1$, and $K \in \{L, R\}$. Let $W_1 = 1 - W_2$. Then, (5) becomes

$$RF_K(A^*, B^*) = (1 - W_2) \times RF_K(A_1, B_1) + W_2 \times RF_K(A_2, B_2) \quad (6)$$

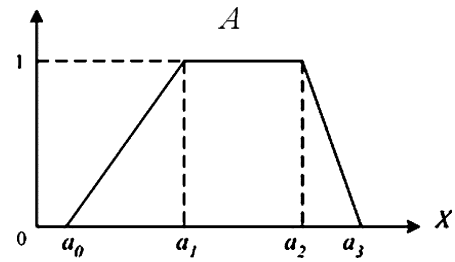

Fig. 5. Trapezoidal fuzzy set.

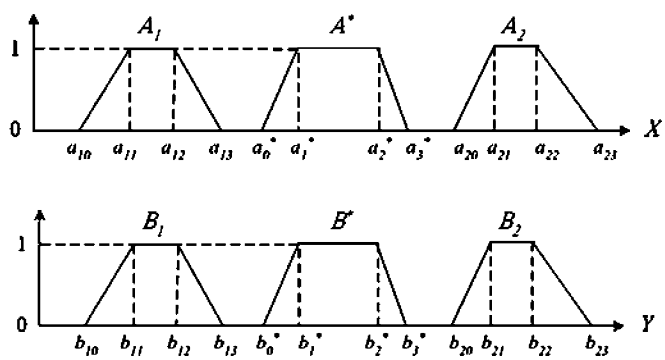

Fig. 6. Fuzzy interpolative reasoning using trapezoidal membership functions.

where $0 \leq W_2 \leq 1$ and $K \in \{L, R\}$. From (6), we can see that $\mathrm{Min}(RF_K(A_1,\ B_1),\ RF_K(A_2,\ B_2)) \leq RF_K(A^*,\ B^*) \leq \mathrm{Max}(RF_K(A_1,\ B_1),\ RF_K(A_2,\ B_2))$, where $K \in \{L, R\}$. ■

Property 2: If $RF_K(A_1, B_1) = RF_K(A_2, B_2) = C$, then $RF_K(A^*, B^*) = C$, where $C \geq 0$ and $K \in \{L, R\}$.

Proof: Based on *Property 1*, if $RF_K(A_1, B_1) = RF_K(A_2, B_2) = C$, then we can see that $\mathrm{Min}(RF_K(A_1,\ B_1), RF_K(A_2,\ B_2)) = \mathrm{Max}(RF_K(A_1, B_1), RF_K(A_2,\ B_2)) = C$ and $RF_K(A^*, B^*) = C$, where $C \geq 0$ and $K \in \{L, R\}$. ■

It is obvious that *Property 1* and *Property 2* are logically consistent with respect to the ratios of fuzziness based on the two-fuzzy-rules interpolative reasoning technique. Based on (3), the weight W_i of $RF_K(A_i, B_i)$ contributing to $RF_K(A^*, B^*)$ is determined by the distance of the normal points between A_i and A^*. That is, the closer the normal point of A^* to the normal point of A_i, the larger the weight of $RF_K(A_i, B_i)$, where $i = 1, 2$ and $K \in \{L, R\}$. From Fig. 4, we can see that $RF_L(A^*, B^*)$ is closer to $RF_L(A_1, B_1)$ when the normal point of A^* is closer to the normal point of A_1, and $RF_L(A^*, B^*)$ is closer to $RF_L(A_2, B_2)$ when the normal point of A^* is closer to the normal point of A_2.

B. Fuzzy Interpolative Reasoning With Trapezoidal Fuzzy Sets

Assume thatthe fuzzy sets of the given fuzzy rules and the observation are trapezoidal fuzzy sets. A trapezoidal fuzzy set A is typically denoted by (a_0, a_1, a_2, a_3), as shown in Fig. 5, where a_1 and a_2 are called the "left normal point" and the "right normal point," respectively, and a_0 and a_3 are called the "left extreme point" and the "right extreme point," respectively. An example of fuzzy interpolative reasoning using trapezoidal fuzzy sets is shown in Fig. 6.

The proposed fuzzy interpolative reasoning method with trapezoidal fuzzy sets is now presented as follows.

Step 1: Use (7) to calculate the composite normal points $a_{1c}, a_{2c}, a_c^*, b_{1c}$, and b_{2c} of trapezoidal fuzzy sets A_1, A_2, A^*, B_1, and B_2, respectively, where

$$\text{The composite normal point} = \frac{\text{the left normal point} + \text{ the right normal point}}{2} \tag{7}$$

(e.g., $a_{1c} = (a_{11} + a_{12})/2$), and then, calculate the composite normal point b_c^* of the interpolated fuzzy set B^* based on (2).

Step 2: Calculate the left normal point b_1^* and the right normal point b_2^* of the fuzzy set B^* using (8)–(10), where

$$b_1^* = b_c^* - \frac{\overline{b_1^* b_2^*}}{2} \tag{8}$$

$$b_2^* = b_c^* + \frac{\overline{b_1^* b_2^*}}{2} \tag{9}$$

the distance $\overline{b_1^* b_2^*}$ between the left normal point b_1^* and the right normal point b_2^* is calculated as follows:

$$\overline{b_1^* b_2^*} = \begin{cases} \overline{a_1^* a_2^*} \times \sum\limits_{\substack{i=1, \\ \overline{a_{i1} a_{i2}} > 0}}^{2} W_i \times \frac{\overline{b_{i1} b_{i2}}}{\overline{a_{i1} a_{i2}}}, & \text{if } \exists \overline{a_{i1} a_{i2}} > 0 \\ \overline{a_1^* a_2^*}, & \text{if } \forall \overline{a_{i1} a_{i2}} = 0 \end{cases} \tag{10}$$

where $W_1 = 1 - (a_c^* - a_{1c})/(a_{2c} - a_{1c}), W_2 = 1 - (a_{2c} - a_c^*)/(a_{2c} - a_{1c})$, and $W_1 + W_2 = 1$.

Step 3: Calculate the left triangular area $S_L(B^*)$ of B^* between b_0^* and b_1^* and the right triangular area $S_R(B^*)$ of B^* between b_2^* and b_3^* by (3), respectively. From Fig. 5, we can also see that $S_L(B^*)$ is equal to $(b_1^* - b_0^*)/2$ and $S_R(B^*)$ is equal to $(b_3^* - b_2^*)/2$. Therefore, the left extreme point b_0^* and the right extreme point b_3^* of the fuzzy interpolative reasoning result B^* are $b_1^* - 2S_L(B^*)$ and $b_2^* + 2S_R(B^*)$, respectively. Finally, the fuzzy interpolative reasoning result B^* is derived, where $B^* = (b_1^* - 2S_L(B^*),\ b_1^*,\ b_2^*,\ b_2^* + 2S_R(B^*))$.

The top equation of (10) is used if there exists an antecedent fuzzy set of the given fuzzy rules having a top support larger than zero. Otherwise, the bottom equation of (10) is used if all the top supports of the antecedent fuzzy sets of the given fuzzy rules are zero.

C. Fuzzy Interpolative Reasoning With Hexagonal Fuzzy Sets

Assume that the fuzzy sets of the given fuzzy rules and the observation are hexagonal fuzzy sets. Let us consider a hexagonal fuzzy set A denoted by $(a_0, a_1, a_2, a_3, a_4, a_5)$, as shown in Fig. 7, where a_2 and a_3 are called the "left normal point" and the "right normal point," respectively, a_0 and a_5 are called the "left extreme point" and the "right extreme point," respectively, and a_1 and a_4 are called the "left intermediate point" and the "right intermediate point," respectively. In Fig. 7, $S_L(A)$ denotes the triangular area between a_0 and a_1 of the hexagonal fuzzy set $A, S_{L_1}(A)$ denotes the trapezoidal area between a_1 and a_2 of the hexagonal fuzzy set $A, S_{R_1}(A)$ denotes the trapezoidal area between a_3 and a_4 of the hexagonal fuzzy set A, and $S_R(A)$

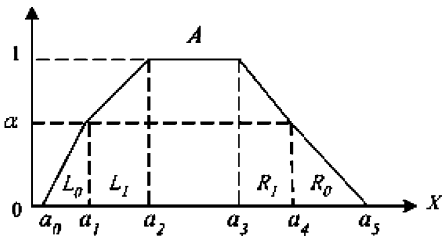

Fig. 7. Hexagonal fuzzy set.

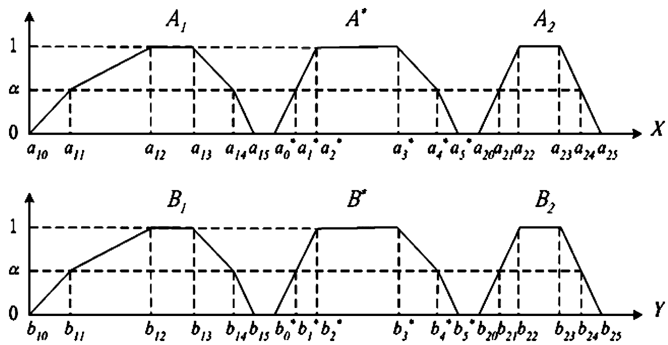

Fig. 8. Fuzzy interpolative reasoning using hexagonal membership functions.

denotes the triangular area between a_4 and a_5 of the hexagonal fuzzy set A.

An example of fuzzy interpolative reasoning using hexagonal fuzzy sets is shown in Fig. 8.

The proposed fuzzy interpolative reasoning method with hexagonal fuzzy sets is now presented as follows.

Step 1: Calculate the composite normal points $a_{1c}, a_{2c}, a_c^*, b_{1c}$, and b_{2c} of the hexagonal fuzzy sets A_1, A_2, A^*, B_1, and B_2 by (7), respectively. Then, calculate the composite normal point b_c^* of the interpolated fuzzy set B^* based on (2).

Step 2: Calculate the left normal point b_2^* and the right normal point b_3^* of fuzzy set B^* using (8)–(10).

Step 3: Calculate the trapezoidal area $S_{L_1}(B^*)$, the trapezoidal area $S_{R_1}(B^*)$, the triangular area $S_{L_0}(B^*)$, and the triangular area $S_{R_0}(B^*)$ of B^* by (3), respectively. From Fig. 7, we can also see that $S_{L_1}(B^*)$ is equal to $[(\alpha+1)\,|b_2^* - b_1^*|]/2$ and $S_{R_1}(B^*)$ is equal to $[(\alpha+1)\,|b_4^* - b_3^*|]/2$, the left intermediate point b_1^* and the right intermediate point b_4^* are $b_2^* - [2S_{L1}(B^*)/(\alpha+1)]$ and $b_3^* + [2S_{R1}(B^*)/(\alpha+1)]$, respectively. From Fig. 7, we can also see that $S_{L_0}(B^*)$ is equal to $\alpha\,|b_1^* - b_0^*|\,/2$ and $S_{R_0}(B^*)$ is equal to $\alpha\,|b_5^* - b_4^*|\,/2$, the left extreme point b_0^* and the right extreme point b_5^* are $b_1^* - [2S_{L_0}(B^*)/\alpha]$ and $b_4^* + [2S_{R_0}(B^*)/\alpha]$, respectively. Finally, the fuzzy interpolative reasoning result B^* denoted by $(b_0, b_1, b_2, b_3, b_4, b_5)$ is derived, where $b_0^* = b_2^* - [2S_{L_1}(B^*)/(\alpha+1)] - [2S_{L_0}(B^*)/\alpha]$, $b_1^* = b_2 - [2S_{L_1}(B^*)/(\alpha+1)]$, $b_4^* = b_3^* + [2S_{R_1}(B^*)/(\alpha+1)]$, and $b_5^* = b_3^* + [2S_{R_1}(B^*)/(\alpha+1)] + [2S_{R_0}(B^*)/\alpha]$.

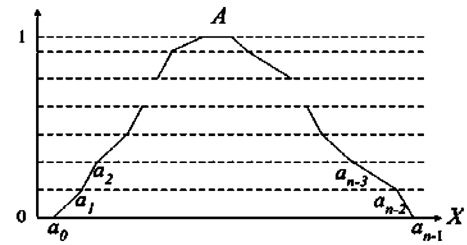

Fig. 9. Polygonal fuzzy set.

D. Fuzzy Interpolative Reasoning With Polygonal Fuzzy Sets

Assume that the given fuzzy rules and the observation are polygonal fuzzy sets. A polygonal fuzzy set A denoted by n characteristic points $(a_0, \ldots, a_{n-2}, a_{n-1})$ is shown in Fig. 9, where $a_{\lfloor (n-1)/2 \rfloor}$ and $a_{\lceil (n-1)/2 \rceil}$ are called the "left normal point" and the "right normal point," respectively, a_0 and a_{n-1} are called the "left extreme point" and the "right extreme point," respectively, and the others are called the "intermediate points." There are $\lfloor (n-1)/2 \rfloor + 1$ membership levels including the bottom level and the top level in the fuzzy set, where their membership degrees are $\alpha_0, \alpha_1, \ldots$, and $\alpha_{\lfloor (n-1)/2 \rfloor}$, respectively (i.e., $\alpha_0 = 0$ and $\alpha_{\lfloor (n-1)/2 \rfloor} = 1$).

From the previous discussion of the fuzzy interpolation of triangular, trapezoidal, and hexagonal fuzzy sets, we can see that the proposed method can preserve the normality and convexity of the fuzzy interpolative reasoning result when the fuzzy sets of the given fuzzy rules are normal and convex fuzzy sets of the same shape. In the following, we will see that when the fuzzy sets of the fuzzy rules are normal, convex, and arbitrary polygonal fuzzy sets, the fuzzy interpolative reasoning result of the proposed method is still a normal and convex fuzzy set.

Assume that the fuzzy sets of the given fuzzy rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$ are polygonal fuzzy sets, where the polygonal fuzzy sets $A_1 = (a_{10}, a_{11}, \ldots, a_{1,n-1})$ and $A_2 = (a_{20}, a_{21}, \ldots, a_{2,n-1})$ are the antecedents and the polygonal fuzzy sets $B_1 = (b_{10}, b_{11}, \ldots, b_{1,n-1})$ and $B_2 = (b_{20}, b_{21}, \ldots, b_{2,n-1})$ are the consequences. The observation $A^* = (a_0^*,\ a_1^*, \ldots,\ a_{n-1}^*)$ occurs between the fuzzy sets A_1 and A_2, and $B^* = (b_0^*,\ b_1^*, \ldots,\ b_{n-1}^*)$ denotes the fuzzy interpolative reasoning result. The proposed fuzzy interpolative reasoning method with polygonal fuzzy sets is presented as follows.

Step 1: Calculate the composite normal points $a_{1c}, a_{2c}, a_c^*, b_{1c}$, and b_{2c} of polygonal fuzzy sets A_1, A_2, A^*, B_1, and B_2 by (7), respectively. Then, calculate the composite normal point b_c^* of the interpolated fuzzy set B^* based on (2).

Step 2: Calculate the left normal point $b_{\lfloor (n-1)/2 \rfloor}^*$ and the right normal point $b_{\lceil (n-1)/2 \rceil}^*$ of the fuzzy set B^* using (8)–(10), where $b_{\lfloor (n-1)/2 \rfloor}^*$ and $b_{\lceil (n-1)/2 \rceil}^*$ are the same when n is odd. From (8)–(10), we can see that the value of the left normal point is smaller than or equal to the value of the right normal point. Therefore, the fuzzy interpolative reasoning result cannot become a twisted nonconvex fuzzy set.

Step 3: Calculate the area $S_K(B^*)$ of area K of the fuzzy set B^* by (3), where $K = L_{\lfloor (n-1)/2 \rfloor - 1}$,

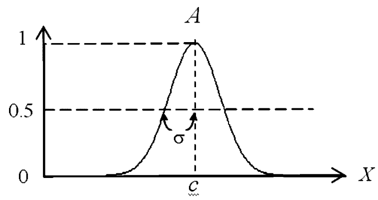

Fig. 10. Gaussian fuzzy set.

$R_{\lfloor(n-1)/2\rfloor-1}$, $L_{\lfloor(n-1)/2\rfloor-2}$, $R_{\lfloor(n-1)/2\rfloor-2}$, $\ldots$, L_1, R_1, L_0, R_0. We also can see that $S_{Li}(B^*)$ is equal to $(\alpha_i + \alpha_{i+1}) \left|b_i^* - b_{i+1}^*\right|/2$ and $S_{Ri}(B^*)$ is equal to $(\alpha_{n-i} + \alpha_{n-i-1}) \left|b_{n-i}^* - b_{n-i-1}^*\right|/2$. Finally, the fuzzy interpolative reasoning result B^* denoted by $(b_0^*,\ b_1^*, \ldots,\ b_{n-1}^*)$ is derived, as in (11), shown at the bottom of this page.

Because $b_0^* \le b_1^* \le \cdots \le b_{n-1}^*$, we can see that the proposed method can preserve the convexity of the fuzzy interpolative reasoning result with arbitrary polygonal fuzzy sets.

E. Fuzzy Interpolative Reasoning With Gaussian Membership Functions

Let us consider bell-shaped membership functions, such as the Gaussian membership function $A = e^{-(x-c)^2/2\sigma^2}$, as shown in Fig. 10, where c and σ denote the mean and the standard deviation, respectively. The normal point of the Gaussian fuzzy set A is the mean of the Gaussian membership function. Assume that the fuzzy sets of the given fuzzy rules and the observation are Gaussian fuzzy sets. An example of fuzzy interpolative reasoning using Gaussian fuzzy sets is shown in Fig. 11, where a_1, a_2, a^*, b_1, b_2, and b^* are the normal points of fuzzy sets A_1, A_2, A^*, B_1, B_2, and B^*, respectively, and $\sigma_{A_1}, \sigma_{A_2}, \sigma_{A^*}, \sigma_{B_1}, \sigma_{B_2}$, and σ_{B^*} are the standard deviations of the fuzzy sets A_1, A_2, A^*, B_1, B_2, and B^*, respectively.

The proposed fuzzy interpolative reasoning method with Gaussian fuzzy sets is now presented as follows:

Step 1: Calculate the normal point b^* of the Gaussian fuzzy set B^* by (2).

Step 2: Calculate the standard deviation σ_{B^*} of B^* as follows:

$$\sigma_{B^*} = \begin{cases} \sigma_{A^*} \times \sum\limits_{i=1,\sigma_{A_i}>0}^{2} \left(W_i \times \frac{\sigma_{B_i}}{\sigma_{A_i}}\right), & \text{if } \exists\sigma_{A_i} > 0 \\ \sigma_{A^*}, & \text{if } \forall\sigma_{A_i} = 0 \end{cases} \tag{12}$$

where σ_{B^*} denotes the standard deviation of the fuzzy set $B^*, W_1 = 1 - [(a^* - a_1)/(a_2 - a_1)], W_2 = 1 - [(a_2 - a^*)/(a_2 - a_1)]$, and $W_1 + W_2 = 1$. Then, based on the normal point b^* and the value of σ_{B^*}, the Gaussian membership function of the consequence fuzzy set $B^* = e^{-(x-b^*)^2/2\sigma_{B^*}^2}$ is derived.

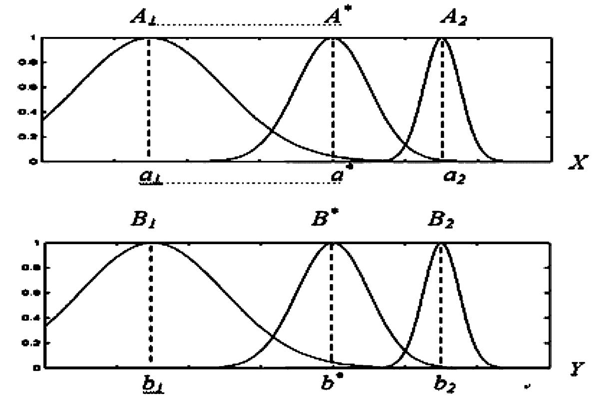

Fig. 11. Fuzzy interpolative reasoning using Gaussian membership functions.

F. Fuzzy Interpolative Reasoning With Multiple Antecedent Variables

Consider the situation that the interpolation between two adjacent fuzzy rules has multiple antecedent variables. The multiple antecedent variables fuzzy interpolative reasoning scheme is shown as follows.

Rule 1: If X_1 is A_{11} and X_2 is A_{12} and $\cdots$ and X_m is A_{1m} then Y is B_1.
Rule 2: If X_1 is A_{21} and X_2 is A_{22} and $\cdots$ and X_m is A_{2m} then Y is B_2.
Observation: X_1 is A_{11}^* and X_2 is A_{12}^* and $\cdots$ and X_m is A_m^*.

Conclusion: Y is B^*.

An example of fuzzy interpolative reasoning with two antecedent variables is shown in Fig. 12, where the antecedent fuzzy set A_{ij} is denoted by $(a_{ij0}, a_{ij_1}, a_{ij_2}, a_{ij_3})$, where $i = 1$, 2 and $j = 1$, 2, the observation fuzzy set A_j^* is denoted by $(a_{j0}^*,\ a_{j1}^*,\ a_{j2}^*,\ a_{j3}^*)$,where $j = 1$, 2, the consequent fuzzy set B_i is denoted by $(b_{i0}, b_{i1}, b_{i2}, b_{i3})$, where $i = 1$, 2, and the fuzzy interpolative reasoning result B^* is denoted by $(b_0^*,\ b_1^*,\ b_2^*,\ b_3^*)$.

The proposed multiple antecedent variables fuzzy interpolative reasoning method is now presented as follows.

Step 1: Calculate the composite normal points $a_{11c}, a_{21c}, a_{12c}, a_{22c}, a_{1c}^*, a_{2c}^*, b_{1c}$, and b_{2c} of the trapezoidal fuzzy sets $A_{11}, A_{21}, A_{12}, A_{22}, A_1^*, A_2^*, B_1$, and B_2 by (7), respectively.

$$b_i^* = \begin{cases} b^*_{\lfloor\frac{n-1}{2}\rfloor} - \frac{2S_{L_{\lfloor\frac{n-1}{2}\rfloor-1}}(B^*)}{\alpha_{\lfloor\frac{n-1}{2}\rfloor-1}+\alpha_{\lfloor\frac{n-1}{2}\rfloor}} - \frac{2S_{L_{\lfloor\frac{n-1}{2}\rfloor-2}}(B^*)}{\alpha_{\lfloor\frac{n-1}{2}\rfloor-2}+\alpha_{\lfloor\frac{n-1}{2}\rfloor-1}} - \cdots - \frac{2S_{L_i}(B^*)}{\alpha_i+\alpha_{i+1}} & \text{if } i = 0, 1, \ldots, \lfloor\frac{n-1}{2}\rfloor - 1 \\ b^*_{\lceil\frac{n-1}{2}\rceil} + \frac{2S_{R_{i-\lceil\frac{n-1}{2}\rceil-1}}(B^*)}{\alpha_{i-\lceil\frac{n-1}{2}\rceil-1}+\alpha_{i-\lceil\frac{n-1}{2}\rceil}} + \frac{2S_{R_{i-\lceil\frac{n-1}{2}\rceil-2}}(B^*)}{\alpha_{i-\lceil\frac{n-1}{2}\rceil-2}+\alpha_{i-\lceil\frac{n-1}{2}\rceil-1}} & \\ \quad + \cdots + \frac{2S_{R_{i-\lceil\frac{n-1}{2}\rceil-\lfloor\frac{n-1}{2}\rfloor}}(B^*)}{\alpha_{i-\lceil\frac{n-1}{2}\rceil-\lfloor\frac{n-1}{2}\rfloor}+\alpha_{i-\lceil\frac{n-1}{2}\rceil-(\lfloor\frac{n-1}{2}\rfloor-1)}} & \text{if } i = \lceil\frac{n-1}{2}\rceil + 1,\ \ \lceil\frac{n-1}{2}\rceil + 2, \ldots, n-1. \end{cases} \tag{11}$$

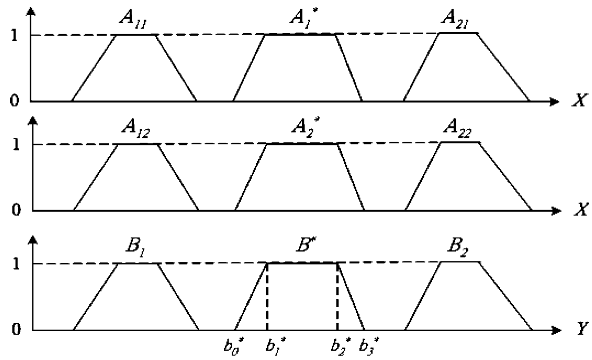

Fig. 12. Multiple antecedent variables fuzzy interpolative reasoning using trapezoidal membership functions.

Calculate the composite normal point b_c^* of the interpolated fuzzy set B^* as follows:

$$w_{ij} = 1 - \left| \frac{a_{ijc} - a_{jc}^*}{a_{2jc} - a_{1jc}} \right| \quad (13)$$

$$W_i = \frac{\sum_{j=1}^{m} w_{ij}}{m} \quad (14)$$

$$b_c^* = W_1 b_{1c} + W_2 b_{2c} \quad (15)$$

where $0 \leq w_{ij} \leq 1, w_{1j} + w_{2j} = 1, W_1 + W_2 = 1, i = 1, 2$, $j = 1, 2, \ldots, m$, and m denotes the number of antecedent variables of the given fuzzy rules.

Step 2: Calculate the left normal point b_1^* and the right normal point b_2^* of the fuzzy set B^*, as in (16), shown at the bottom of this page.

$$b_1^* = b_c^* - \frac{\overline{b_1^* b_2^*}}{2}$$

$$b_2^* = b_c^* + \frac{\overline{b_1^* b_2^*}}{2}$$

where $W_i = \sum_{j=1}^{m} w_{ij}/m, w_{ij} = 1 - |(a_{jc}^* - a_{ijc})/(a_{2jc} - a_{1jc})|$, $i = 1, 2, j = 1, 2, \ldots, m, W_1 + W_2 = 1$, and m denotes the number of antecedent variables of the given fuzzy rules.

Step 3: Calculate the left area $S_L(B^*)$ and the right area $S_R(B^*)$ of the fuzzy set B^*, as in (17), shown at the bottom of this page, where $K \in \{L, R\}$, $S_L(B^*)$ denotes the left area of $B^*, S_R(B^*)$ denotes the right area of $B^*, W_i = \sum_{j=1}^{m} w_{ij}/m, w_{ij} = 1 - \left|(a_{jc}^* - a_{ijc})/(a_{2jc} - a_{1jc})\right|, i = 1$, 2, $j = 1, 2, \ldots, m, W_1 + W_2 = 1$, and m denotes the number of antecedent variables of the given fuzzy rules. From Fig. 12, we can also see that $S_L(B^*)$ is equal to $(b_1^* - b_0^*)/2$ and $S_R(B^*)$ is equal to $(b_3^* - b_2^*)/2$. Therefore, the left extreme point b_0^* and the right extreme point b_3^* of the fuzzy interpolative reasoning result B^* are $b_1^* - 2S_L(B^*)$ and $b_2^* + 2S_R(B^*)$, respectively. Finally, the fuzzy interpolative reasoning result B^* is derived, where $B^* = (b_1^* - 2S_L(B^*),\ b_1^*,\ b_2^*,\ b_2^* + 2S_R(B^*))$.

Equation (17) is used to handle multiple antecedent variables fuzzy interpolative reasoning based on two selected fuzzy rules. The top equation of (17) is used to infer the areas of the interpolated fuzzy set B^* if there exists a fuzzy rule that at least one of the areas of its antecedent parts is larger than zero. Otherwise, the bottom equation of (17) is used when all the areas of the antecedent parts of the given fuzzy rules are zero. The fuzzy interpolative reasoning result inferred by (17) satisfies the logically consistent properties with respect to the ratios of fuzziness based on the multiple antecedent variables fuzzy interpolative reasoning, where the ratio of fuzziness $RF_K((A_1, A_2, \ldots, A_m), B)$ of the consequence fuzzy set B with respect to the antecedent fuzzy sets $A_1, A_2, \ldots, A_m$ is calculated as follows:

$$RF_K((A_1, A_2, \ldots, A_m), B) = \frac{S_K(B)}{1/m(\sum_{j=1}^{m} S_K(A_j))} \quad (18)$$

where $K \in \{L, R\}$, the fuzzy sets $A_1, A_2, \ldots, A_m$ are the antecedent fuzzy sets of a fuzzy rule and B is the consequence fuzzy set of a fuzzy rule, $RF_L((A_1, A_2, \ldots, A_m), B)$ denotes the ratio of fuzziness of the left area of B with respect to the left areas of $A_1, A_2, \ldots, A_m, RF_R((A_1, A_2, \ldots, A_m), B)$ denotes the ratio of fuzziness of the right area of B with respect to the right areas of $A_1, A_2, \ldots, A_m, S_L(A_j)$ denotes the left areas of A_j, respectively, where $j = 1, 2, \ldots, m, S_R(A)$ denotes the right areas of A_j, respectively, where $j = 1, 2, \ldots, m$, and $S_L(B)$ and $S_R(B)$ denote the left area and the right area of B, respectively.

$$\overline{b_1^* b_2^*} = \begin{cases} \left(\sum_{j=1}^{m} \overline{a_{j1}^* a_{j2}^*} \right) \times \left(\sum_{\substack{i=1, \\ \exists j \overline{a_{ij1} a_{ij2}} > 0}}^{2} W_i \times \frac{\overline{b_{i1} b_{i2}}}{\sum_{j=1}^{m} \overline{a_{ij1} a_{ij2}}} \right), & \text{if } \exists ij\ \overline{a_{ij1} a_{ij2}} > 0 \\ \frac{\sum_{j=1}^{m} \overline{a_{j1}^* a_{j2}^*}}{m}, & \text{if } \forall ij\ \overline{a_{ij1} a_{ij2}} = 0 \end{cases} \quad (16)$$

$$S_K(B^*) = \begin{cases} \left(\sum_{j=1}^{m} S_K(A_j^*) \right) \times \left(\sum_{\substack{i=1 \\ \exists j S_K(A_{ij}) > 0}}^{2} W_i \times \frac{S_K(B_i)}{\sum_{j=1}^{m} S_K(A_{ij})} \right), & \text{if } \exists ij\ S_K(A_{ij}) > 0 \\ \frac{\sum_{j=1}^{m} S_K(A_j^*)}{m}, & \text{if } \forall ij\ S_K(A_{ij}) = 0 \end{cases} \quad (17)$$

Assume that there are two fuzzy rules $A_{11} \wedge A_{12} \wedge \cdots \wedge A_{1m} => B_1$, $A_{21} \wedge A_{22} \wedge \cdots \wedge A_{2m} => B_2$ and m observations $A_1^*, A_2^*, \ldots, A_m^*$, where A_j^* occurs between A_{1j} and A_{mj} and $j = 1, 2, \ldots, M$. Let $\text{MIN}RF_K = \text{Min}(RF_K((A_{11}, A_{12}, \ldots, A_{1m}), B_1), RF_K((A_{21}, A_{22}, \ldots, A_{2m})B_2))$, and let $\text{MAX}RF_K = \text{Max}(RF_K((A_{11}, A_{12}, \ldots, A_{1m}), B_1), RF_K((A_{21}, A_{22}, \ldots, A_{2m}), B_2))$, where $K \in \{L, R\}$. The fuzzy interpolative reasoning result B^* obtained by (17) satisfies the following two properties.

Property 3: $\text{MIN}\, RF_K \leq RF_K((A_1^*, A_2^*, \ldots, A_m^*), B^*) \leq \text{MAX}\, RF_K$, where $K \in \{L, R\}$.

Proof: Based on (17) and (18), the value of $RF_K((A_1^*, A_2^*, \ldots, A_m^*), B^*)$ is calculated as follows:

$$RF_K((A_1^*, A_2^*, \ldots, A_m^*), B^*)$$

$$= \frac{\left(\sum_{j=1}^{m} S_K(A_j^*)\right) \times \left(\sum_{\substack{i=1 \\ \exists j S_K(A_{ij}) > 0}}^{2} W_i \times \left(S_K(B_i) / \sum_{j=1}^{m} S_K(A_{ij})\right)\right)}{1/m\left(\sum_{j=1}^{m} S_K(A_j^*)\right)}$$

$$= m \times \sum_{i=1}^{2} W_i \times \frac{S_K(B_i)}{\sum_{j=1}^{m} S_K(A_{ij})}$$

$$= \sum_{i=1}^{2} W_i \times \frac{S_K(B_i)}{1/m\left(\sum_{j=1}^{m} S_K(A_{ij})\right)}$$

$$= \sum_{i=1}^{2} W_i \times RF_K((A_{i1}, A_{i2}, \ldots, A_{im}), B_i) \tag{19}$$

where $0 \leq W_1 \leq 1, 0 \leq W_2 \leq 1, W_1 + W_2 = 1$, and $K \in \{L, R\}$. From (19), we can see that

$$\sum_{i=1}^{2} W_i \times RF_K((A_{i1}, A_{i2}, \ldots, A_{im}), B_i) \geq \sum_{i=1}^{2} W_i \times \text{MIN}RF_K = \text{MIN}RF_K \tag{20}$$

$$\sum_{i=1}^{2} W_i \times RF_K((A_{i1}, A_{i2}, \ldots, A_{im}), B_i) \leq \sum_{i=1}^{2} W_i \times \text{MAX}RF_K = \text{MAX}RF_K \tag{21}$$

where $0 \leq W_1 \leq 1, 0 \leq W_2 \leq 1, W_1 + W_2 = 1$, and $K \in \{L, R\}$. Therefore, we can see that $\text{MIN}RF_K \leq RF_K((A_1^*, A_2^*, \ldots, A_m^*), B^*) \leq \text{MAX}RF_K$, where $K \in \{L, R\}$. ∎

Property 4: If $RF_K((A_{11}, A_{12}, \ldots, A_{1m}), B_1) = RF_K((A_{21}, A_{22}, \ldots, A_{2m}), B_2) = C$ then $RF_K((A_1^*, A_2^*, \ldots, A_m^*), B^*) = C$, where $C > 0$ and $K \in \{L, R\}$.

Proof: Based on *Property 3*, if $RF_K((A_{11}, A_{12}, \ldots, A_{1m}), B_1) = RF_K((A_{21}, A_{22}, \ldots, A_{2m})B_2) = C$, then we can see that $\text{MIN}RF_K = \text{MAX}RF_K = C$ and $RF_K((A_1^*, A_2^*, \ldots, A_m^*), B^*) = C$, where $C > 0$ and $K \in \{L, R\}$. ∎

It is obvious that *Property 3* and *Property 4* are logically consistent with respect to the ratios of fuzziness based on the two fuzzy rules interpolative reasoning technique with multiple antecedent variables. From (19), we can see that the weight W_i of $RF_K((A_{i_1}, A_{i_2}, \ldots, A_{im}), B_i)$ contributing to $RF_K((A_1^*, A_2^*, \ldots, A_m^*), B^*)$ is determined by the distance of the composite normal points between A_{i1} and A_1^*, the distance of the composite normal points between A_{i_2} and A_2^*, ..., and the distance of the composite normal points between A_{im} and A_m^*, respectively, where $i = 1, 2$ and $K \in \{L, R\}$. That is, the closer the composite normal points of A_j^* to the composite normal points of A_{ij}, where $j = 1, 2, \ldots, m$, the larger the weight of $RF_K((A_{i_1}, A_{i_2}, \ldots, A_{im}), B_i)$, where $i = 1, 2$ and $K \in \{L, R\}$. Moreover, from (19), we can see that the proposed method uses the weighted average method to calculate $RF_K((A_1^*, A_2^*, \ldots, A_m^*), B^*)$ based on the weights of $RF_K((A_{i_1}, A_{i2}, \ldots, A_{im}), B_i)$, where $i = 1, 2$ and $K \in \{L, R\}$.

G. Fuzzy Interpolative Reasoning With Multiple Fuzzy Rules and Multiple Antecedent Variables

Let us consider the interpolation with multiple fuzzy rules and multiple antecedent variables. The multiple antecedent variables fuzzy interpolative reasoning scheme with multiple fuzzy rules is shown as follows:

Rule 1: If X_1 is A_{11} and X_2 is A_{12} and $\cdots$ and X_m is A_{1m} then Y is B_1.
Rule 2: If X_1 is A_{21} and X_2 is A_{22} and $\cdots$ and X_m is A_{2m} then Y is B_2.
⋮
Rule n: If X_1 is A_{n1} and X_2 is A_{n2} and $\cdots$ and X_m is A_{nm} then Y is B_n.
Observation: X_1 is A_{11}^* and X_2 is A_{12}^* and $\cdots$ and X_m is A_m^*.

Conclusion: Y is B^*.

An example of four fuzzy rules interpolative reasoning with two antecedent variables is shown in Fig. 13, where four adjacent fuzzy rules $A_{11} \wedge A_{12} \Rightarrow B_1$, $A_{21} \wedge A_{22} \Rightarrow B_2$, $A_{31} \wedge A_{32} \Rightarrow B_3$ and $A_{41} \wedge A_{42} \Rightarrow B_4$ are given; the antecedent fuzzy set A_{ij} is denoted by $(a_{ij0}, a_{ij_1}, a_{ij_2}, a_{ij_3})$, where $i = 1, 2, 3, 4$ and $j = 1, 2$; the observation fuzzy sets A_1^* and A_2^* are denoted by $(a_{10}^*, a_{11}^*, a_{12}^*, a_{13}^*)$ and $(a_{20}^*, a_{21}^*, a_{22}^*, a_{23}^*)$, respectively; the consequent fuzzy set B_i is denoted by $(b_{i0}, b_{i1}, b_{i2}, b_{i3})$, where $i = 1, 2, 3, 4$, and the fuzzy interpolative reasoning result B^* is denoted by $(b_0^*, b_1^*, b_2^*, b_3^*)$.

The proposed multiple antecedent variables and multiple fuzzy rules interpolative reasoning method is now presented as follows:

Step 1: Calculate the composite normal points a_{11c}, a_{21c}, a_{31c}, a_{41c}, a_{1c}^* a_{12c}, a_{22c}, a_{32c}, a_{42c}, a_{2c}^*, b_{1c}, b_{2c} b_{3c} and b_{4c} of the trapezoidal fuzzy sets $A_{11}, A_{21}, A_{31}, A_{41}, A_1^*, A_{12}, A_{22}, A_{32}, A_{42}, A_2^*, B_1, B_2, B_3$, and B_4 by (7), respectively. Calculate the composite normal point b_c^* of the

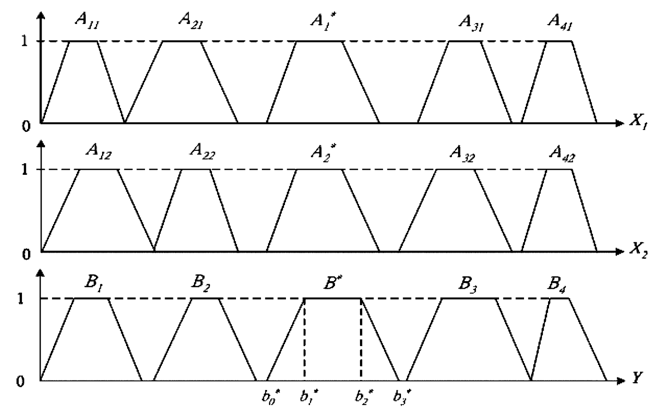

Fig. 13. Four fuzzy rules interpolative reasoning using trapezoidal membership functions.

interpolated fuzzy set B^* as follows:

$$w_{ij} = 1 - \left| \frac{a_{ijc} - a_{jc}^*}{a_{njc} - a_{1jc}} \right| \tag{22}$$

$$W_i = \frac{\sum_{j=1}^{m} w_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}} \tag{23}$$

$$b_c^* = \sum_{i=1}^{n} W_i b_{ic} \tag{24}$$

where $0 \le w_{ij} \le 1, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, m$ denotes the number of antecedent variables of the given fuzzy rules, and n denotes the number of given fuzzy rules (i.e., $m = 2$ and $n = 4$).

Step 2: Calculate the left normal point b_1^* and the right normal point b_2^* of fuzzy set B^* as follows:

$$\overline{b_1^* b_2^*} = \begin{cases} \left(\sum_{j=1}^{m} \overline{a_{j1}^* a_{j2}^*}\right) \times \left(\sum_{\substack{i=1,\\ \exists j\, \overline{a_{ij1} a_{ij2}} > 0}}^{n} W_i \times \frac{\overline{b_{i1} b_{i2}}}{\sum_{j=1}^{m} \overline{a_{ij1} a_{ij2}}} \right), & \text{if } \exists ij\, \overline{a_{ij1} a_{ij2}} > 0 \\ \frac{\sum_{j=1}^{m} \overline{a_{j1}^* a_{j2}^*}}{m}, & \text{if } \forall ij\, \overline{a_{ij1} a_{ij2}} = 0 \end{cases} \tag{25}$$

$$b_1^* = b_c^* - \frac{\overline{b_1^* b_2^*}}{2}$$

$$b_2^* = b_c^* + \frac{\overline{b_1^* b_2^*}}{2}$$

where $W_i = \sum_{j=1}^{m} w_{ij} / \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}, w_{ij} = 1 - |(a_{jc}^* - a_{ijc}) / (a_{njc} - a_{1jc})|, 0 \le w_{ij} \le 1, \sum_{i=1}^{n} W_i = 1, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, m$ denotes the number of antecedent variables of the given fuzzy rules, and n denotes the number of given fuzzy rules (i.e., $m = 2$ and $n = 4$).

Step 3: Calculate the left triangular area $S_L(B^*)$ of the fuzzy set B^* between b_0^* and b_1^* and calculate the right triangular area $S_R(B^*)$ of the fuzzy set B^* between b_2^* and b_3^*, respectively, where (26) as shown at the bottom of this page, $S_L(B^*)$ denotes the left area of $B^*, S_R(B^*)$ denotes the right area of $B^*, W_i = \sum_{j=1}^{m} w_{ij} / \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}, w_{ij} = 1 - \left|(a_{jc}^* - a_{ijc}) / (a_{njc} - a_{1jc})\right|, 0 \le w_{ij} \le 1, \sum_{i=1}^{n} W_i = 1, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, m$ denotes the number of antecedent variables of the given fuzzy rules, and n denotes the number of given fuzzy rules (i.e., $m = 2$ and $n = 4$). From Fig. 12, we also can see that $S_L(B^*)$ is equal to $(b_1^* - b_0^*)/2$ and $S_R(B^*)$ is equal to $(b_3^* - b_2^*)/2$. Therefore, the left extreme point b_0^* and the right extreme point b_3^* of the fuzzy interpolative reasoning result B^* are $b_1^* - 2S_L(B^*)$ and $b_2^* + 2S_R(B^*)$, respectively. Finally, the fuzzy interpolative reasoning result B^* is derived, where $B^* = (b_1^* - 2S_L(B^*),\ b_1^*,\ b_2^*,\ b_2^* + 2S_R(B^*))$.

Equation (26) is used to handle multiple antecedent variables fuzzy interpolative reasoning based on multiple fuzzy rules. The top equation of (26) is used to infer the areas of the interpolated fuzzy set B^* if there exists a fuzzy rule that at least one of the areas of its antecedent parts is larger than zero. Otherwise, the bottom equation of (26) is used when all the areas of the antecedent parts of the given fuzzy rules are zero. The fuzzy interpolative reasoning result inferred by (26) satisfies the two logically consistent properties with respect to the ratios of fuzziness based on the multiple antecedent variables and multiple fuzzy rules interpolation.

Assume that there are n fuzzy rules with m antecedent variables $A_{11} \wedge A_{12} \wedge \cdots \wedge A_{1m} => B_1,\ A_{21} \wedge A_{22} \wedge \cdots \wedge A_{2m} => B_2, \ldots,\ A_{n1} \wedge A_{n2} \wedge \cdots \wedge A_{nm} => B_n$, and assume that there are m observations $A_1^*,\ A_2^*, \ldots,$ and A_m^*, where A_1^* occurs between A_{t1} and $A_{(t+1)1}, A_2^*$ occurs between A_{t_2} and $A_{(t+1)2}, \ldots,\ A_m^*$ occurs between A_{tm} and $A_{(t+1)m}$, and $1 \le t \le n$. Let $\text{MIN}RF_K = \text{Min}(RF_K((A_{11}, A_{12}, \ldots, A_{1m}), B_1), RF_K((A_{21}, A_{22}, \ldots, A_{2m}), B_2), \ldots, RF_K((A_{n1}, A_{n2}, \ldots, A_{nm}), B_n))$, and let $\text{MAX}RF_K = \text{Max}(RF_K((A_{11}, A_{12}, \ldots, A_{1m}), B_1), RF_K((A_{21}, A_{22}, \ldots, A_{2m}), B_2), \ldots, RF_K((A_{n1}, A_{n2}, \ldots, A_{nm}), B_n))$. The fuzzy interpolative reasoning result B^* obtained by (26) satisfies the following logically consistent properties.

Property 5: $\text{MIN}RF_K \le RF_K((A_1^*,\ A_2^*, \ldots,\ A_m^*),\ B^*) \le \text{MAX}RF_K$, where $K \in \{L, R\}$.

$$S_K(B^*) = \begin{cases} \left(\sum_{j=1}^{m} S_K(A_j^*)\right) \times \left(\sum_{\substack{i=1,\\ \exists j S_K(A_{ij}) > 0}}^{n} W_i \times \frac{S_K(B_i)}{\sum_{j=1}^{m} S_K(A_{ij})} \right), & \text{if } \exists ij\, S_K(A_{ij}) > 0 \\ \frac{\sum_{j=1}^{m} S_K(A_j^*)}{m}, & \text{if } \forall ij\, S_K(A_{ij}) = 0 \end{cases} \tag{26}$$

Proof: Based on (18) and (26), the value of $RF_K((A_1^*,\ A_2^*,\dots,\ A_m^*),\ B^*)$ is calculated as follows:

$$\begin{aligned} &RF_K((A_1^*, A_2^*, \dots, A_m^*), B^*) \\ &= \frac{\left(\sum_{j=1}^{m} S_K(A_j^*)\right)\times\left(\sum_{\substack{i=1,\\ \exists j S_K(A_{ij})>0}}^{n} W_i \times \left(\frac{S_K(B_i)}{\sum_{j=1}^{m} S_K(A_{ij})}\right)\right)}{(1/m)\sum_{j=1}^{m} S_K(A_j^*)} \\ &= m \times \sum_{i=1}^{n} W_i \times \frac{S_K(B_i)}{\sum_{j=1}^{m} S_K(A_{ij})} \\ &= \sum_{i=1}^{n} W_i \times \frac{S_K(B_i)}{1/m\left(\sum_{j=1}^{m} S_K(A_{ij})\right)} \\ &= \sum_{i=1}^{n} W_i \times RF_K((A_{i1}, A_{i2}, \dots, A_{im}), B_i) \end{aligned} \tag{27}$$

where $0 \le W_i \le 1, \sum_{i=1}^{n} W_i = 1$, and $K \in \{L, R\}$. From (27), we can see that

$$\sum_{i=1}^{n} W_i \times RF_K((A_{i1}, A_{i2}, \dots, A_{im}), B_i) \ge \sum_{i=1}^{n} W_i \times \mathrm{MIN}RF_K = \mathrm{MIN}RF_K \tag{28}$$

$$\sum_{i=1}^{n} W_i \times RF_K((A_{i1}, A_{i2}, \dots, A_{im}), B_i) \le \sum_{i=1}^{n} W_i \times \mathrm{MAX}RF_K = \mathrm{MAX}RF_K \tag{29}$$

where $0 \le W_i \le 1, \sum_{i=1}^{n} W_i = 1$, and $K \in \{L, R\}$. Therefore, we can see that $\mathrm{MAX}RF_K \le RF_K((A_1^*,\ A_2^*,\dots,\ A_m^*),\ B^*) \le \mathrm{MAX}RF_K$, where $K \in \{L, R\}$. ■

Property 6: If $RF_K((A_{i_1}, A_{i_2}, \dots, A_{im}), B_i) = C$, then $RF_K((A_1^*,\ A_2^*,\dots,\ A_m^*),\ B^*) = C$, where $C > 0$, $i = 1, 2, \dots, N$ and $K \in \{L, R\}$.

Proof: Based on *Property 5*, if $RF_K((A_{i_1}, A_{i_2}, \dots, A_{im}), B_i) = C, i = 1, 2, \dots, N$, then we can see that $\mathrm{MIN}RF_K = \mathrm{MAX}RF_K = C$ and $RF_K((A_1^*,\ A_2^*,\ \dots,\ A_m^*),\ B^*) = C$, where $C > 0$ and $K \in \{L, R\}$. ■

It is obvious that *Property 5* and *Property 6* are logically consistent with respect to the ratios of fuzziness based on multiple fuzzy rules interpolative reasoning with multiple antecedent variables. From (27), we can see that the weight W_i of $RF_K((A_{i1}, A_{i2}, \dots, A_{im}), B_i)$ contributing to $RF_K((A_1^*,\ A_2^*,\dots,\ A_m^*),\ B^*)$ is determined by the distance of the composite normal points between A_{i1} and A_1^*, the distance of the composite normal points between A_{i2} and $A_2^*, \dots,$ and the distance of the composite normal points between A_{im} and A_m^* respectively, where $i = 1, 2, \dots, n$ and $K \in \{L, R\}$. That is, the closer the composite normal points of A_j^* to the composite normal points of A_{ij}, where $j = 1, 2, \dots, m$, the larger the weight of $RF_K((A_{i1}, A_{i2}, \dots, A_{im}), B_i)$, where $i = 1, 2, \dots, n$ and $K \in \{L, R\}$. Moreover, from (27), we can see that the proposed method uses the weighted average method to calculate $RF_K((A_1^*,\ A_2^*,\dots,\ A_m^*),\ B^*)$ based on the weights of $RF_K((A_{i1}, A_{i2}, \dots, A_{im}), B_i)$, where $i = 1, 2, \dots, n$ and $K \in \{L, R\}$.

III. Experimental Results

In this section, we compare the fuzzy interpolative reasoning results of the proposed method with the ones of the KH method [11], the HCL method [5], the HTY method [6], and the HS method [7] by using some examples.

Example 3.1 [7]*:* Let us consider the situation that the fuzzy sets of the given fuzzy rules and the observation are triangular fuzzy sets. All the conditions and the fuzzy interpolative reasoning results are shown in Fig. 14. From Fig. 14, we can see that the KH method [11] generated an abnormal fuzzy set, whereas the HCL method [5], the HTY method [6], the HS method [7], and the proposed method have convex results. Based on (4) and Table I, we can see that $RF_L(A_1, B_1) = 0.4$ and $RF_L(A_2, B_2) = 0.5$. In the same way, we can see that $RF_L(A^*, B^*)$ of the HCL method [5], the HTY method [6], the HS method [7], and the proposed method are 0.22, 0.66, 0.43, and 0.44, respectively. We can see that both $RF_L(A^*, B^*)$ of the proposed method and the HS method [7] satisfy *Property 1*. Based on (4) and Table I, we can also see that $RF_R(A_1, B_1) = RF_R(A_2, B_2) = 2$, and $RF_R(A^*, B^*)$ of the HCL method [5], the HTY method [6], the HS method [7], and the proposed method are 0.8, 0.88, 1.12, and 2, respectively. We can see that only $RF_R(A^*, B^*)$ of the proposed method satisfies *Property 2*. Therefore, we can see that only the fuzzy interpolative reasoning result of the proposed method is logically consistent in terms of *Property 1* and *Property 2*.

Example 3.2 [7]*:* Let us consider the situation that the antecedents of the given fuzzy rules are crisp values and the observation is a fuzzy set, where all the conditions and the fuzzy interpolative reasoning results are shown in Table II and Fig. 15. From Fig. 15, we can see that the KH method [11], the HCL method [5], the HS method [7], and the proposed method generated a convex triangular fuzzy set, whereas the HTY method [6] cannot handle the situation in this example.

Example 3.3 [7]*:* Let us consider the situation that the antecedents of the given fuzzy rules are fuzzy sets, but the observation is a crisp value. The conditions and the fuzzy interpolative reasoning results are shown in Table III and Fig. 16. From Fig. 16, we can see that the KH method [11] generated an abnormal fuzzy set and the HCL method [5] generated a rectangular fuzzy set. The HTY method [6], the HS method [7], and the proposed method generated the singleton results, which are more reasonable than the KH method [11] and the HCL method [5] in terms of the shapes of the observations.

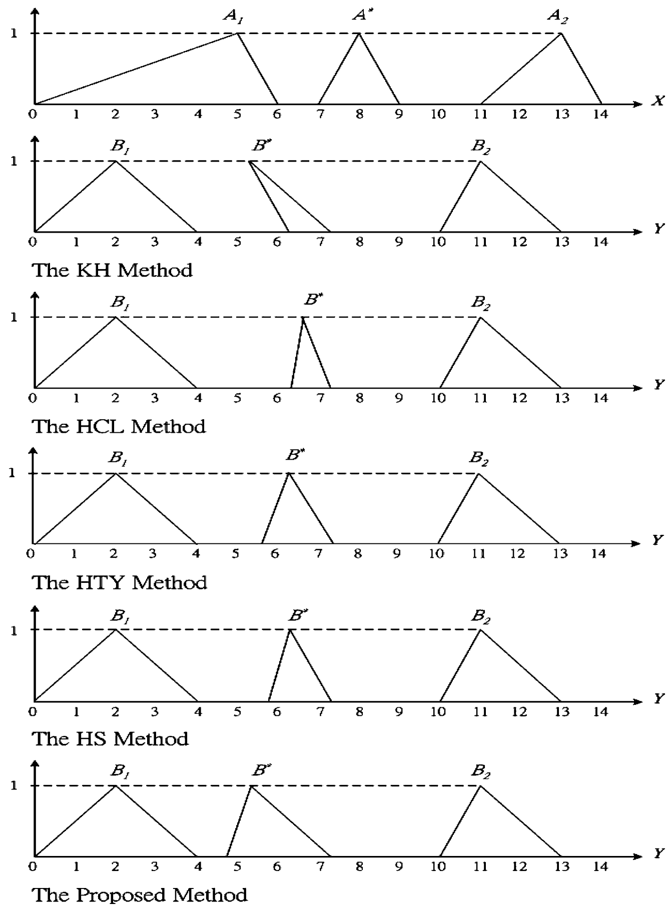

Fig. 14. A comparison of fuzzy interpolative reasoning results of Example 3.1 for different methods.

TABLE I
FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3.1

Attribute Values	Methods	Fuzzy Interpolative Reasoning Results B^*
A_1 = (0, 5, 6)	KH Method [11]	(6.36, 5.38, 7.38)
A_2 = (11, 13, 14)	HCL Method [5]	(6.36, 6.58, 7.38)
B_1 = (0, 2, 4)	HTY Method [6]	(5.76, 6.42, 7.30)
B_2 = (10, 11, 13)	HS Method [7]	(5.83, 6.26, 7.38)
A^* = (7, 8, 9)	The Proposed Method	(4.94, 5.38, 7.38)

Example 3.4 [7]*:* Let us consider the situation that the fuzzy sets of the given fuzzy rules are trapezoidal fuzzy sets and the observation is a rectangular fuzzy set. The conditions and the fuzzy interpolative reasoning results are shown in Table IV and Fig. 17. There is no obvious indication for the HCL method [5] to handle trapezoidal fuzzy sets. From Fig. 17, we can see that the KH method [11] generated an abnormal trapezoidal fuzzy set and the HTY method [6] generated a twisted abnormal fuzzy set. The HS method [7] and the proposed method generated convex trapezoidal fuzzy sets. Based on (4) and Table IV, we can see that $RF_R(A_1, B_1) = RF_R(A_2, B_2) = 1$. In the same way, we can see that $RF_R(A^*, B^*)$ of the HS method [7] and the proposed method are 0.71 and 1, respectively. Therefore, we can see that only $RF_R(A^*, B^*)$ of the proposed method is consistent in terms of *Property 2*.

TABLE II
FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3.2

Attribute Values	Methods	Fuzzy Interpolative Reasoning Results B^*
A_1 = (3, 3, 3)	KH Method [11]	(5.33, 6.33, 9.00)
A_2 = (12, 12, 12)	HCL Method [5]	(5.33, 6.55, 9.00)
B_1 = (4, 4, 4)	HTY Method [6]	–
B_2 = (10, 11, 13)	HS Method [7]	(5.71, 6.28, 8.16)
A^* = (5, 6, 8)	The Proposed Method	(5.33, 6.33, 8.33)

Note: The symbol "–" denotes there is no obvious indication for the method to handle the situation.

TABLE III
FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3.3

Attribute Values	Methods	Fuzzy Interpolative Reasoning Results B^*
A_1 = (0, 5, 6)	KH Method [11]	(6.25, 5.38, 7.27)
A_2 = (11, 13, 14)	HCL Method [5]	[7.27, 6.25]
B_1 = (0, 2, 4)	HTY Method [6]	(6.49, 6.49, 6.49)
B_2 = (10, 11, 13)	HS Method [7]	(6.49, 6.49, 6.49)
A^* = (8, 8, 8)	The Proposed Method	(5.38, 5.38, 5.38)

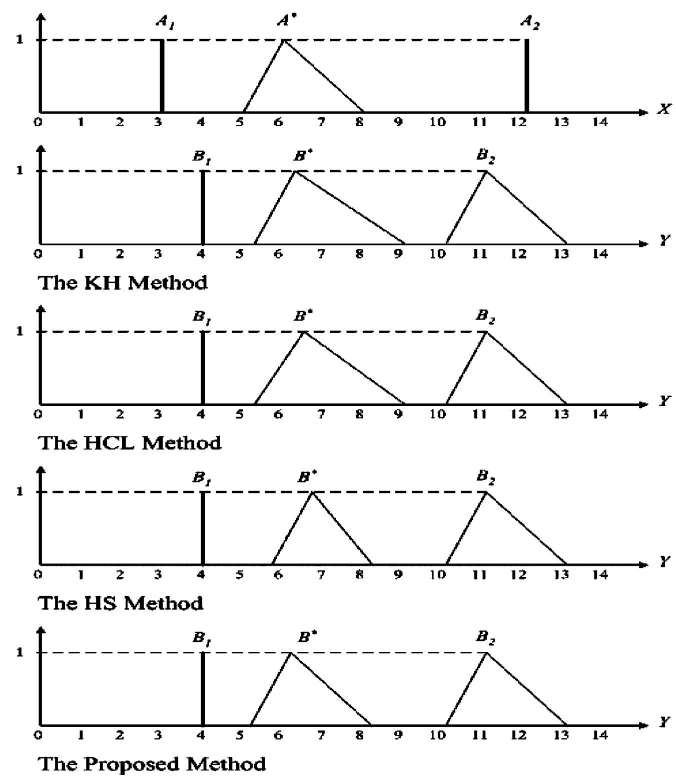

Fig. 15. A comparison of fuzzy interpolative reasoning results of Example 3.2 for different methods.

Example 3.5 [7]*:* Let us consider the situation that the fuzzy sets of the given fuzzy rules are hexagonal fuzzy sets. The conditions and the fuzzy interpolative reasoning results are shown in Table V and Fig. 18. In Table V, the HS1 method [7], the HS2 method [7], the HS3 method [7], and the HS4 method [7] use four different strategies (i.e., the average, the compatible, the weighted average, and the center of core) to calculate the representative value, respectively. There is no obvious

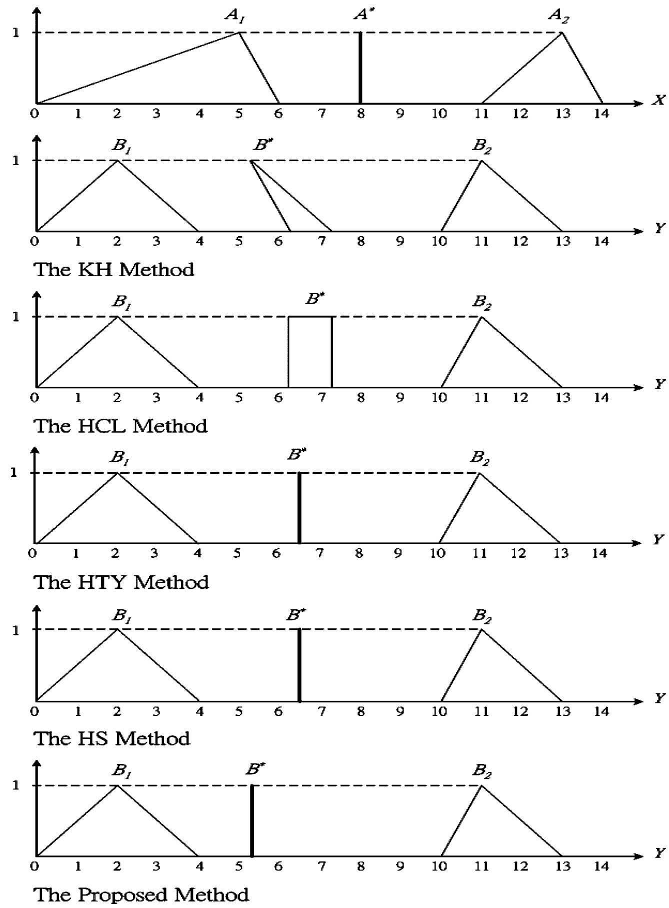

Fig. 16. A comparison of fuzzy interpolative reasoning results of Example 3.3 for different methods.

TABLE IV
FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3.4

Attribute Values	Methods	Fuzzy Interpolative Reasoning Results B^*
A_1 = (0, 4, 5, 6)	KH Method [11]	(5.45, 4.25, 7.5, 8.5)
A_2 = (11, 12, 13, 14)	HCL Method [5]	–
B_1 = (0, 2, 3, 4)	HTY Method [6]	(4.98, 7.44, 6.44, 8.06)
B_2 = (10, 11, 12, 13)	HS Method [7]	(5.23, 5.23, 7.61, 8.32)
	The Proposed Method	(4.25, 4.25, 7.5, 8.5)

Note: The symbol "–" denotes there is no obvious indication for the method to handle the situation.

indication for the HCL method [5] and the HTY method [6] to handle hexagonal fuzzy sets. From Fig. 18, we can see that the KH method [11] generated an abnormal hexagonal fuzzy set. The HS1 method [7], the HS2 method [7], the HS3 method [7], the HS4 method [7], and the proposed method all generated convex hexagonal results.

Example 3.6 [7]*:* Let us consider the situation that the fuzzy sets of the given fuzzy rules and the observation are Gaussian fuzzy sets. The conditions and the fuzzy interpolative reasoning results are shown in Table VI and Fig. 19. There is no obvious indication for the KH method [11], the HCL method [5], and the HTY method [6] to handle Gaussian fuzzy sets. From Fig. 19, we can see that both the HS method [7] and the proposed method generated the Gaussian fuzzy sets. Based on Table VI,

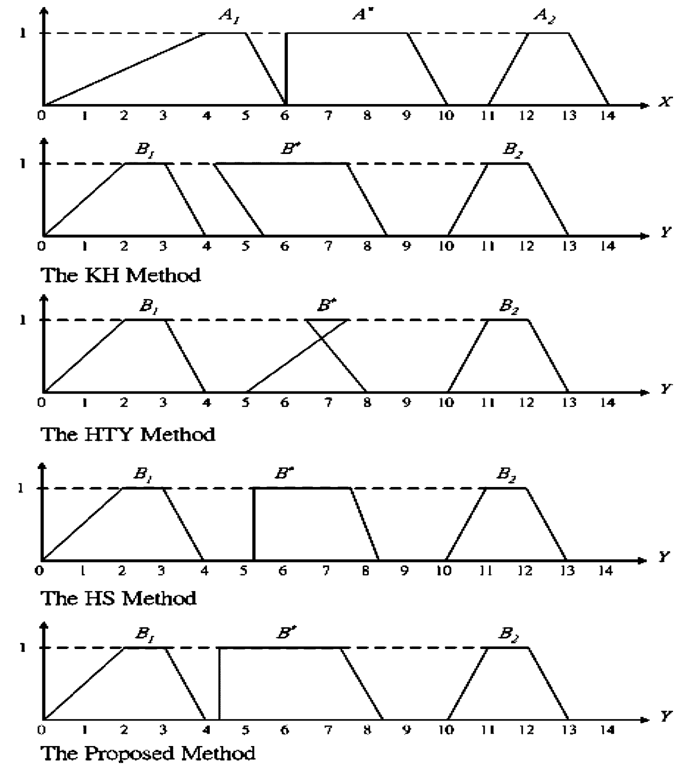

Fig. 17. A comparison of fuzzy interpolative reasoning results of Example 3.4 for different methods.

TABLE V
FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3.5

Attribute Values	Methods	Fuzzy Interpolative Reasoning Results B^*
A_1 = (0, 4, 5, 6)	KH Method [11]	(5.45, 4.25, 7.5, 8.5)
A_2 = (11, 12, 13, 14)	HCL Method [5]	–
B_1 = (0, 2, 3, 4)	HTY Method [6]	(4.98, 7.44, 6.44, 8.06)
B_2 = (10, 11, 12, 13)	HS Method [7]	(5.23, 5.23, 7.61, 8.32)
	The Proposed Method	(4.25, 4.25, 7.5, 8.5)

Note: The symbol "–" denotes there is no obvious indication for the method to handle the situation.

we can see that the ratio of fuzziness $RF_K(A_1, B_1) = 0.5$ and $RF_K(A_2, B_2) = 3$, where $K \in \{L, R\}$. In the same way, we can see that $RF_K(A^*, B^*)$ of the HS method [7] and the proposed method are 1.24 and 2.06, respectively, where $K \in \{L, R\}$. Therefore, we can see that both $RF_K(A^*, B^*)$ of the proposed method and the HS method [7] are logically consistent in terms of *Property 1*.

Example 3.7: Let us consider the situation that the antecedents of the given fuzzy rules are triangular fuzzy sets and the consequences of the given fuzzy rules are trapezoidal fuzzy sets. The conditions and the fuzzy interpolative reasoning results are shown in Table VII and Fig. 20. There is no obvious indication for the HCL method [5] to handle trapezoidal fuzzy sets. From Fig. 20, we can see that the KH method [11] generated an abnormal trapezoidal fuzzy set, whereas the proposed method, the HTY method [6], and the HS method [7] generated a convex trapezoidal fuzzy set. Based on (4) and Table VII, we can see that $RF_L(A_1, B_1) = 0.4$ and $RF_L(A_2, B_2) = 0.5$.

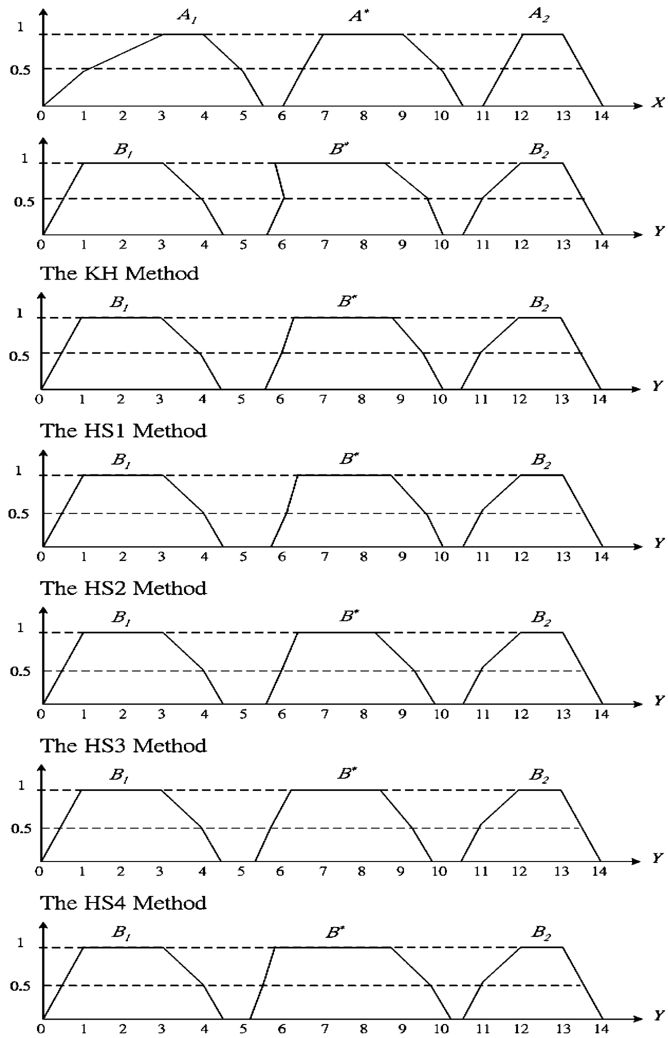

Fig. 18. A comparison of fuzzy interpolative reasoning results of Example 3.5 for different methods.

TABLE VI
FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3.6

Attribute Values	Methods	Fuzzy Interpolative Reasoning Results B^*
$A_1 = e^{\frac{-(x-3)^2}{2\times 2^2}}$	KH Method [11]	–
$A_2 = e^{\frac{-(x-11)^2}{2\times 0.5^2}}$	HCL Method [5]	–
$B_1 = e^{\frac{-(x-6)^2}{2\times 1^2}}$	HTY Method [6]	–
$B_2 = e^{\frac{-(x-13)^2}{2\times 1.5^2}}$	HS Method [7]	$B^* = e^{\frac{-(x-10.38)^2}{2\times 1.24^2}}$
$A^* = e^{\frac{-(x-8)^2}{2\times 1^2}}$	The Proposed Method	$B^* = e^{\frac{-(x-10.38)^2}{2\times 2.06^2}}$

Note: The symbol "–" denotes there is no obvious indication for the method to handle the situation.

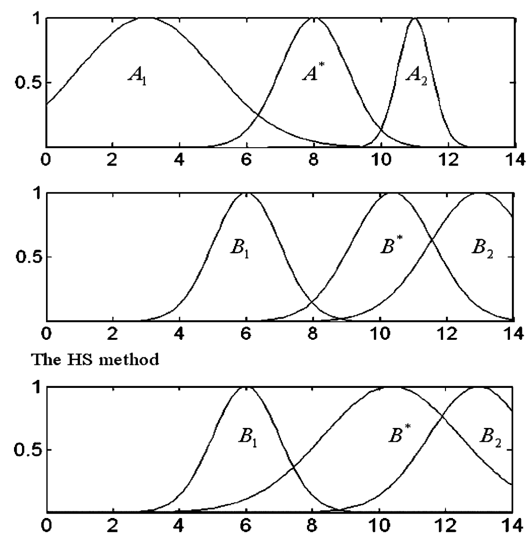

Fig. 19. A comparison of fuzzy interpolative reasoning results of Example 3.6 for different methods.

TABLE VII
FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3.7

Attribute Values	Methods	Fuzzy Interpolative Reasoning Results B^*
A_1 = (0, 5, 5, 6)	KH Method [11]	(6.36, 5.38, 6.36, 7.38)
A_2 = (11, 13, 13, 14)	HCL Method [5]	–
B_1 = (0, 2, 3, 4)	HTY Method [6]	(5.87, 6.20, 7.20, 7.41)
B_2 = (10, 11, 12, 13)	HS Method [7]	(5.93, 6.36, 6.80, 7.47)
	The Proposed Method	(4.94, 5.38, 6.38, 7.38)

Note: The symbol "–" denotes there is no obvious indication for the method to handle the situation.

In the same way, we can see that $RF_L(A^*, B^*)$ of the HTY method [6], the HS method [7], and the proposed method are 0.33, 0.43, and 0.44, respectively. Both $RF_L(A^*, B^*)$ of the proposed method and the HS method [7] satisfy *Property 1*. Based on (4) and Table VII, we can also see that the ratios of fuzziness $RF_R(A_1, B_1) = RF_R(A_2, B_2) = 1$, and $RF_R(A^*, B^*)$ of the HTY method [6], the HS method [7], and the proposed method are 0.21, 0.67, and 1, respectively. We can see that only $RF_R(A^*, B^*)$ of the proposed method satisfies *Property 2*. Therefore, we can see that only the fuzzy interpolative reasoning results of the proposed method are logically consistent in terms of *Property 1* and *Property 2*.

Example 3.8: Contrary to Example 3.7, the antecedents of the given fuzzy rules are trapezoidal fuzzy sets and the consequences of the given fuzzy rules are triangular fuzzy sets. The conditions and the fuzzy interpolative reasoning results are shown in Table VIII and Fig. 21. There is no obvious indication for the HCL method [5] to handle trapezoidal fuzzy sets. From Fig. 21, we can see that the KH method [11] generated an abnormal fuzzy set and the HTY method [6] generated a twisted abnormal fuzzy set, whereas the proposed method and the HS Method [7] generated a convex triangular fuzzy set. Based on (4) and Table VIII, we can see that $RF_R(A_1, B_1) = RF_R(A_2, B_2) = 2$, and $RF_R(A^*, B^*)$ of the HS method [7] and the proposed method are 3.11 and 2, respectively. Therefore, we can see that only $RF_R(A^*, B^*)$ of the proposed method is logically consistent in terms of *Property 2*.

Example 3.9 [7]*:* Let us consider the situation of the fuzzy interpolation with multiple antecedent variables. Assume that

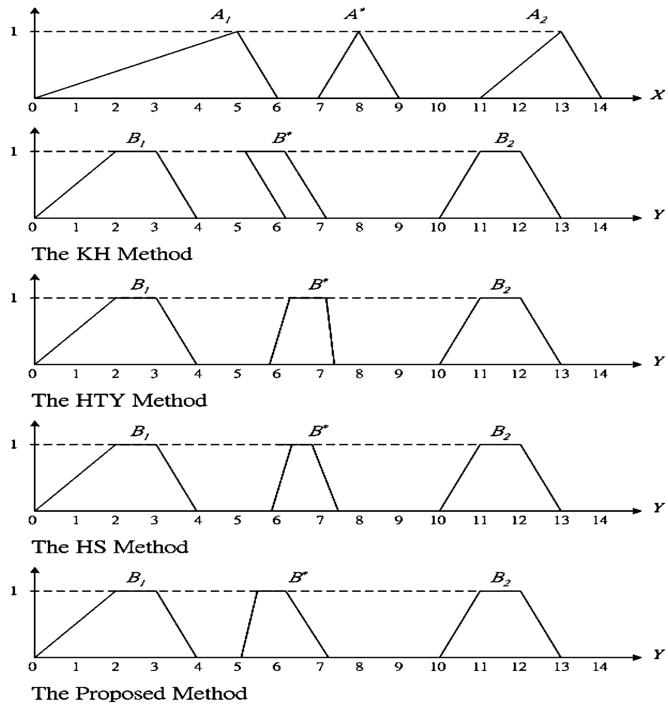

Fig. 20. A comparison of fuzzy interpolative reasoning results of Example 3.7 for different methods.

TABLE VIII
FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3.8

Attribute Values	Methods	Fuzzy Interpolative Reasoning Results B^*
$A_1 = (0, 4, 5, 6)$	KH Method [11]	(5.45, 4.25, 6.5, 8.5)
$A_2 = (11, 12, 13, 14)$	HCL Method [5]	–
$B_1 = (0, 2, 2, 4)$	HTY Method [6]	(5.07, 7.26, 5.26, 8.15)
$B_2 = (10, 11, 11, 13)$	HS Method [7]	(5.46, 5.46, 5.46, 8.55)
	The Proposed Method	(5.38, 5.38, 5.38, 7.38)

Note: The symbol "–" denotes there is no obvious indication for the method to handle the situation.

the fuzzy sets of the given fuzzy rules and the observation are trapezoidal fuzzy sets, where the given fuzzy rules $A_{11} \wedge A_{12} \Rightarrow B_1$, $A_{21} \wedge A_{22} \Rightarrow B_2$ and the observations A_1^* and A_2^* are given to determine the consequence B^*. The conditions and the fuzzy interpolative reasoning results are shown in Table IX and Fig. 22. There is no obvious indication for the HCL method [5] and the HTY method [6] to handle the fuzzy interpolation with multiple antecedent variables. From Fig. 22, we can see that the KH method [11], the HS method [7], and the proposed method all generated convex results. Based on (4) and Table IX, we can see that $\text{Min}(RF_L((A_{11}, A_{12}), B_1), RF_L((A_{21}, A_{22}), B_2)) = \text{Min}(0.8, 1) = 0.8$ and $\text{Max}(RF_L((A_{11}, A_{12}), B_1), RF_L((A_{21}, A_{22}), B_2)) = \text{Max}(0.8, 1) = 1$. In the same way, we can see that the values of $RF_L((A_1^*, A_2^*), B^*)$ of the KH method [11], the HS method [7], and the proposed method are 0.33, 0.79, and 0.9, respectively. We can see that only the fuzzy interpolative reasoning results of the proposed method satisfy *Property 3*. Based on (4) and Table IX, we can also see that $RF_R((A_{11}, A_{12}), B_1) = RF_R((A_{21}, A_{22}), B_2) = 1$, and

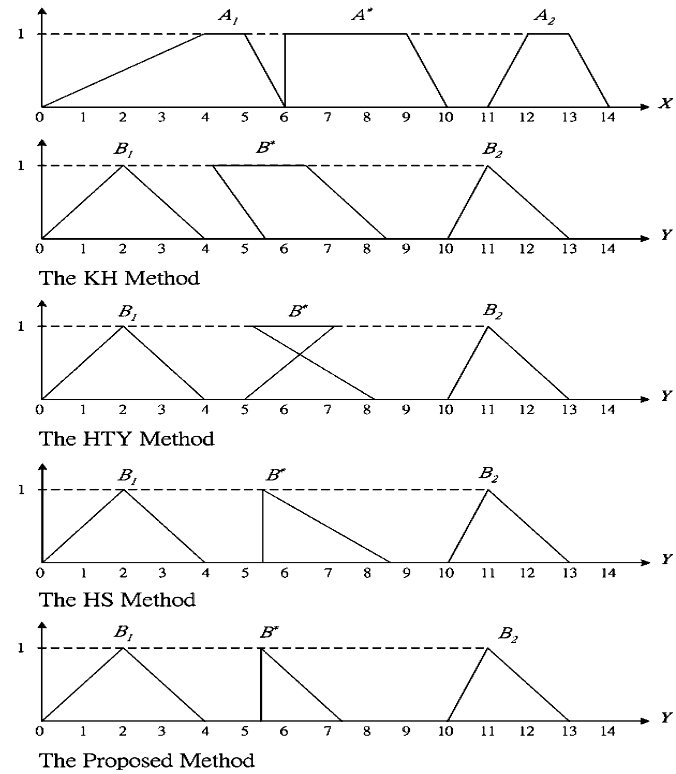

Fig. 21. A comparison of fuzzy interpolative reasoning results of Example 3.8 for different methods.

TABLE IX
FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3.9

Attribute Values	Methods	Fuzzy Interpolative Reasoning Results B^*
$A_{11} = (0, 4, 5, 6)$, $A_{21} = (11, 12, 13, 14)$	KH Method [11]	(5.45, 5.94, 7.13, 8.31)
$A_{12} = (1, 2, 3, 4)$	HCL Method [5]	–
$A_{22} = (12, 14, 15, 16)$, $B_1 = (0, 2, 3, 4)$	HTY Method [6]	–
$B_2 = (10, 11, 12, 13)$, $A_1^* = (6, 7, 9, 11)$	HS Method [7]	(4.37, 5.55, 7.48, 9.33)
$A_2^* = (6, 8, 10, 12)$	The Proposed Method	(4.82, 6.17, 7.83 9.83)

Note: The symbol "–" denotes there is no obvious indication for the method to handle the situation.

$RF_R((A_1^*, A_2^*), B^*)$ of the KH method [11], the HS method [7], and the proposed method are 0.59, 0.93, and 1, respectively. Only the fuzzy interpolative reasoning result of the proposed method satisfies *Property 4*. Therefore, we can see that only the fuzzy interpolative reasoning results of the proposed are logically consistent in terms of *Property 3* and *Property 4*.

Example 3.10 [8]*:* Let us consider the situation of the multiple fuzzy rules interpolation with multiple antecedent variables. Assume that the fuzzy sets of the given fuzzy rules and the observation are triangular fuzzy sets, where the fuzzy rules $A_{11} \wedge A_{12} \Rightarrow B_1$, $A_{21} \wedge A_{22} \Rightarrow B_2$, $A_{31} \wedge A_{32} \Rightarrow B_3$ and the observations A_1^* and A_2^* are given to determine the consequence B^*. The conditions and the fuzzy interpolative reasoning results are shown in Table X and Fig. 23. From Fig. 23, we can see that the HS method [8] and the proposed method generated a convex result, whereas there is no obvious indication for the

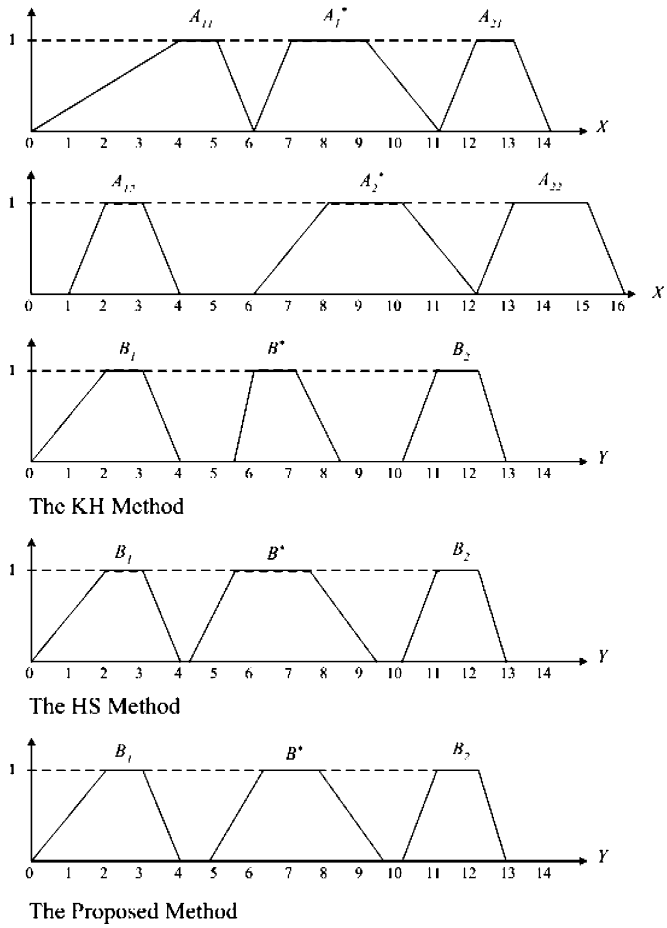

Fig. 22. A comparison of fuzzy interpolative reasoning results of Example 3.9 for different methods.

TABLE X
FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3.10

Attribute Values	Methods	Fuzzy Interpolative Reasoning Results B^*
A_{11} = (0, 1, 3) A_{21} = (8, 9, 10)	KH Method [11]	–
A_{31} = (11, 13, 14) A_{12} = (1, 2, 3) A_{22} = (7, 9, 10)	HCL Method [5]	–
A_{32} = (11, 12, 13) B_1 = (1, 2, 3)	HTY Method [6]	–
B_2 = (9, 10, 11) B_3 = (12, 13, 14)	HS Method [8]	(6.33, 7.7, 8.7)
A_1^* = (3.5, 5, 7) A_1^* = (5, 6, 7)	The Proposed Method	(6.19, 7.65, 8.96)

Note: The symbol "–" denotes there is no obvious indication for the method to handle the situation.

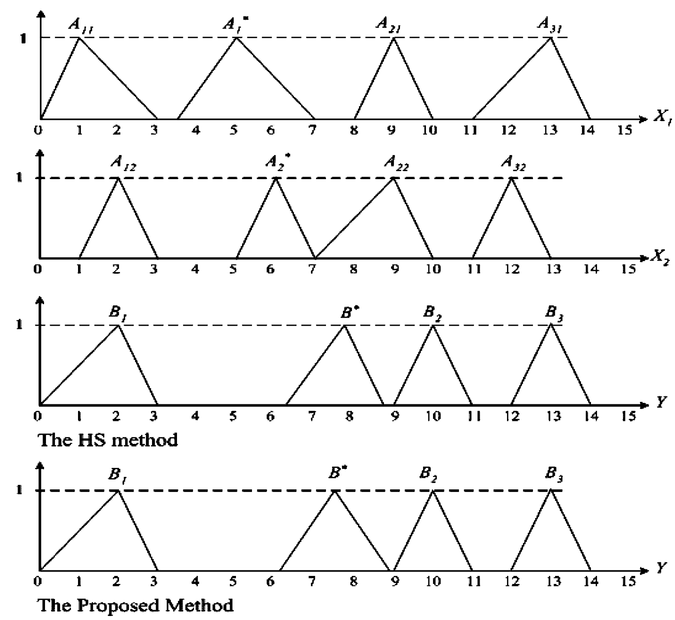

Fig. 23. A comparison of fuzzy interpolative reasoning results of Example 3.10 for different methods.

KH method [11], the HCL method [5], and the HTY method [6] to handle multiple fuzzy rules interpolation with multiple antecedent variables. Based on (4) and Table X, we can see that $\text{Min}(RF_L((A_{11}, A_{12}), B_1), RF_L((A_{21}, A_{22}), B_3), RF_L((A_{31}, A_{32}), B_3)) = \text{Min}(2, 0.66, 0.66) = 0.66$ and $\text{Max}(RF_L((A_{11}, A_{12}), B_1), RF_L((A_{21}, A_{22}), B_3), RF_L((A_{31}, A_{32}), B_3)) = \text{Max}(2, 0.66, 0.66) = 2$. In the same way, the values of $RF_L((A_1^*, A_2^*), B^*)$ of the HS method [7] and the proposed method are 1.09 and 1.17, respectively. Based on (4) and Table X, we can also see that $\text{Min}(RF_R((A_{11}, A_{12}), B_1), RF_R((A_{21}, A_{22}), B_3), RF_R((A_{31}, A_{32}), B_3)) = \text{Min}(0.66, 1, 1) = 0.66$ and $\text{Max}(RF_R((A_{11}, A_{12}), B_1), RF_R((A_{21}, A_{22}), B_3), RF_R((A_{31}, A_{32}), B_3)) = \text{Max}(0.66, 1, 1) = 1$. In the same way, we can see that the values of $RF_R((A_1^*, A_2^*), B^*)$ of the HS method [7] and the proposed method are 0.66 and 0.87, respectively. Therefore, we can see that both the fuzzy interpolative reasoning results of the proposed method and the HS method [7] are logically consistent in terms of *Property 5*.

In the following, we use six evaluation indices (i.e., "normality and convexity," "whether handle complicated membership functions," "whether the antecedent membership functions and the consequent membership functions can be different," "whether handle multiple antecedent variables," "whether handle multiple fuzzy rules interpolation," and "logically consistent with respect to the ratios of fuzziness") to compare the proposed method with the four existing methods (i.e., the KH method [11], the HCL method [5], the HTY method [6], and the HS method [7]), as shown in Table XI. From Table XI, we can see that only the proposed method satisfies these six evaluation indices.

IV. CONCLUSION

In this paper, we have presented a new fuzzy interpolative reasoning method for sparse fuzzy-rule-based systems based on the areas of fuzzy sets. The proposed method can hold the normality and the convexity of the fuzzy interpolative results. Besides triangular membership functions and trapezoidal membership functions, the proposed method can handle fuzzy interpolative reasoning with complicated membership functions, such as hexagonal membership functions, Gaussian membership functions, and polygonal membership functions. Especially, the proposed method can generate normal and convex

TABLE XI
COMPARISON OF THE EVALUATION INDICES OF THE PROPOSED METHOD WITH FOUR EXISTING METHODS

Indices \ Methods	KH Method [11]	HCL Method [5]	HTY Method [6]	HS Method [8]	The Proposed Method
Normality and Convexity	No	Yes	No	Yes	Yes
Whether Handle Complicated Membership Functions	Yes	No	No	Yes	Yes
Whether the Antecedent Membership Functions and the Consequent Membership Functions can be Different	Yes	No	Yes	Yes	Yes
Whether Handle Multiple Antecedent Variables	Yes	No	No	Yes	Yes
Whether Handle Multiple Fuzzy Rules Interpolation	No	No	No	Yes	Yes
Logically consistent with respect to the ratios of fuzziness	No	No	No	No	Yes

results when the antecedents and the consequences of the fuzzy rules have different kinds of membership functions. The proposed method can handle multiple antecedent variables fuzzy interpolative reasoning with multiple fuzzy rules. From Table XI, we can see that only the proposed method satisfies the six evaluation indices. That is, the proposed method performs more reasonably than the KH method [11], the HCL method [5], the HTY method [6], and the HS method [7]. The proposed method provides us a usefulway to deal with fuzzy interpolative reasoning in sparse fuzzy-rule-based systems.

REFERENCES

[1] P. Baranyi, L. T. Koczy, and T. D. Gedeon, "A generalized concept for fuzzy rule interpolation," *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 6, pp. 820–832, 2004.

[2] P. Baranyi, D. Tikk, Y. Yam, and L. T. Kozcy, "A new method for avoiding abnormal conclusion for α-cut based rule interpolation," in *Proc. 1999 IEEE Int. Conf. Fuzzy Syst.*, pp. 383–388.

[3] B. Bouchon-Meunier, C. Marsala, and M. Rifqi, "Interpolative reasoning based on graduality," in *Proc. 2000 IEEE Int. Conf. Fuzzy Syst.*, pp. 483–487.

[4] Y. C. Chang, S. M. Chen, and C. J. Liau, "A new fuzzy interpolative reasoning method based on the areas of fuzzy sets," in *Proc. 2007 IEEE Int. Conf. Syst., Man, Cybern.*, pp. 320–325.

[5] W. H. Hsiao, S. M. Chen, and C. H. Lee, "A new fuzzy interpolative reasoning method in sparse rule-based system," *Fuzzy Sets Syst*, vol. 93, no. 1, pp. 17–22, 1998.

[6] D. M. Huang, E. C. C. Tsang, and D. S. Yeung, "A fuzzy interpolative reasoning method," in *Proc. 2004 IEEE Int. Conf. Mach. Learning Cybern.*, vol. 3, pp. 1826–1830.

[7] Z. H. Huang and Q. Shen, "Fuzzy interpolative reasoning via scale and move transformations," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 2, pp. 340–359, Apr. 2006.

[8] Z. H. Huang, "Rule model simplification," Ph.D. dissertation, School of Informatics, Univ. Edinburgh, Edinburgh, U.K., 2006. [Online]. Available: http://hdl.handle.net/1842/904.

[9] S. Jenei, "Interpolation and extrapolation of fuzzy quantities revisited—An axiomatic approach," *Soft Comput.*, vol. 5, no. 3, pp. 179–193, 2001.

[10] S. Jenei, "Interpolation and extrapolation of fuzzy quantities—The multiple-dimensional case," *Soft Comput.*, vol. 6, no. 3/4, pp. 258–270, 2002.

[11] L. T. Koczy and K. Hirota, "Approximate reasoning by linear rule interpolation and approximation," *Int. J. Approx. Reason.*, vol. 9, no. 3, pp. 197–225, 1993.

[12] L. T. Koczy and K. Hirota, "Interpolative reasoning with insufficient evidence in sparse fuzzy rule bases," *Inf. Sci.*, vol. 71, no. 1/2, pp. 169–201, 1993.

[13] L. T. Koczy and K. Hirota, "Size reduction by interpolation in fuzzy rule bases," *IEEE Trans. Syst., Man, Cybern.*, vol. 27, no. 1, pp. 14–25, Feb. 1997.

[14] Y. M. Li, D. M. Huang, C. C. Tsang, and L. N. Zhang, "Weighted fuzzy interpolative reasoning method," in *Proc. 2005 IEEE Int. Conf. Mach. Learning Cybern.*, pp. 18–21.

[15] C. Marsala and B. Bouchon-Meuner, "Interpolative reasoning with multivariable rules," in *Proc. Joint 9th IFSA World Congress 20th NAFIPS Int. Conf.*, 2001, vol. 4, pp. 2476–2481.

[16] W. Z. Qiao, M. Mizumoto, and Y. Shi, "An improvement to Koczy and Hirota's interpolative reasoning in sparse fuzzy rule bases," *Int. J. Approx. Reason.*, vol. 15, no. 3, pp. 185–201, 1996.

[17] Y. Shi, M. Mizumoto, and W. Z. Qiao, "Reasoning conditions on Kozcy's interpolative reasoning method in sparse fuzzy rule bases," *Fuzzy Sets Syst.*, vol. 75, no. 1, pp. 63–71, 1995.

[18] D. Tikk and P. Baranyi, "Comprehensive analysis of a new fuzzy rule interpolation method," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 3, pp. 281–296, Jun. 2000.

[19] B. Wang, X. Li, W. Liu, and Y. Shi, "A new sparse rule-based fuzzy reasoning method," in *Proc. 2004 IEEE Int. Conf. Hybrid Int. Syst.*, pp. 462–467.

[20] K. W. Wong, D. Tikk, T. D. Gedeon, and L. T. Koczy, "Fuzzy rule interpolation for multidimensional input spaces with applications: A case study," *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 6, pp. 809–819, 2005.

[21] Y. Yam, P. Baranyi, D. Tikk, and L. T. Koczy, "Eliminating the abnormality problem of α-cut based interpolation," in *Proc. 8th IFSA World Congress*, 1999, vol. 2, pp. 726–766.

[22] Y. Yam, M. L. Wong, and P. Baranyi, "Interpolation with function space representation of membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 3, pp. 398–411, 2006.

[23] Y. Yam and L. T. Koczy, "Representing membership functions as points in high dimensional spaces for fuzzy interpolation and extrapolation," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 6, pp. 761–772, 2000.

Yu-Chuan Chang received the B.S. degree in 2000 from the Department of Computer Science and Information Engineering, Fu-Jen University, Taipei, Taiwan, R.O.C., and the M.S. degree in computer science and information engineering in 2004 from the National Taiwan University of Science and Technology, Taipei, where he is currently working toward the Ph.D. degree in computer science and information engineering, National Taiwan University of Science and Technology, Taipei, Taiwan.

His current research interests include fuzzy information systems, fuzzy reasoning, and artificial intelligence.

Shyi-Ming Chen (M'88–SM'96) received the Ph.D. degree in electrical engineering from the National Taiwan University, Taipei, Taiwan, R.O.C., in 1991.

From August 1987 to July 1989 and from August 1990 to July 1991, he was with the Department of Electronic Engineering, Fu-Jen University, Taipei, Taiwan. From August 1991 to July 1996, he was an Associate Professor in the Department of Computer and Information Science, National Chiao Tung University, Hsinchu, Taiwan. From August 1996 to July 1998, he was a Professor in the Department of Computer and Information Science. From August 1998 to July 2001, he was a Professor in the Department of Electronic Engineering, National Taiwan University of Science and Technology, Taipei. Since August 2001, he has been a Professor in the Department of Computer Science and Information Engineering. During 1999, he was a Visiting Scholar in the Department of Electrical Engineering and Computer Science, University of California, Berkeley. During 2003, he was a Visiting Scholar in the Institute of Information Science, Academia Sinica, Taipei. Since August 2007, he also has been the Dean of the College of Electronic Engineering and Computer Science, Jinwen University, Taipei County, Taiwan. He is an Associate Editor of the IEEE TRANSACTION ON FUZZY SYSTEMS, IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS—PART A, IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS—PART C, IEEE

COMPUTATIONAL INTELLIGENCE MAGAZINE, *International Journal of Applied Intelligence*, *Journal of Intelligent and Fuzzy Systems*, *International Journal of Artificial Intelligence Tools*, *International Journal of Intelligent Information and Database Systems*, *Journal of Uncertain Systems*, *International Journal of Innovative Computing, Information and Control*, *International Journal of Fuzzy Systems*, and *WSEAS Transactions on Computers*. He is an Editor of the *New Mathematics and Natural Computation Journal*, *Journal of Advanced Computational Intelligence and Intelligent Informatics*, *International Journal of Soft Computing*, *Asian Journal of Information Technology*, and *Asian Journal of Information Management*. He was an Editor of the *Journal of the Chinese Grey System Association* from 1998 to 2003. He is a member of the Editorial Board of the *International Journal of Information and Communication Technology*, *WSEAS Transactions on Systems*, *International Journal of Computational Intelligence and Applications*, *Advances in Fuzzy Sets and Systems Journal*, *International Journal of Intelligence Systems Technologies and Applications*, *International Journal of Computer Applications in Technology*, and *Advances in Computer Sciences and Engineering Journal*. He is listed in the *International Who's Who of Professionals*, *Marquis Who's Who in the World*, and *Marquis Who's Who in Science and Engineering.* He is the author or coauthor of more than 300 papers published in referred journals, conference proceedings, and book chapters. His current research interests include fuzzy systems, information retrieval, knowledge-based systems, artificial intelligence, neural networks, data mining, and genetic algorithms.

Prof. Chen is the recipient of several honors and awards, including the 1994 Outstanding Paper Award of the Journal of Information and Education, the 1995 Outstanding Paper Award of the Computer Society of the Republic of China, the 1997 Outstanding Youth Electrical Engineer Award of the Chinese Institute of Electrical Engineering, Republic of China, the Best Paper Award of the 1999 National Computer Symposium, Republic of China, the 1999 Outstanding Paper Award of the Computer Society of the Republic of China, the 2001 Distinguished Talented Person Award, Republic of China, for the contributions in Information Technology, the 2002 Distinguished Electrical Engineering Professor Award granted by the Chinese Institute of Electrical Engineering (CIEE), Republic of China, the 2003 Distinguished Paper Award of the Technologically and Vocational Education Society, R.O.C., the 2005 Distinguished Research Award of the National Science Council, R.O.C., the 2006 Distinguished Research Award of the National Taiwan University of Science and Technology, the 2006 Outstanding Paper Award of the 11th Conference on Artificial Intelligence and Applications, the 2007 Best Paper Award of the 20th International Conference on Industrial, Engineering and Other Applications of Applied Intelligent Systems (IEA/AIE 2007), and the 2008 Distinguished Engineering Professor Award granted by the Chinese Institute of Engineers (CIE), R.O.C. He is currently the President of the Taiwanese Association for Artificial Intelligence (TAAI). He is a member of the Association for Computing Machinery (ACM), the International Fuzzy Systems Association (IFSA), and the Phi Tau Phi Scholastic Honor Society. From 1998 to 2004, he was an administrative committee member of the Chinese Fuzzy Systems Association (CFSA). He is a Fellow of the Institute of Engineering and Technology (IET). He is a member of the Board Governors of the IEEE Systems, Man, and Cybernetics Society, a Co-Chair of the IEEE SMC Technical Committee on Intelligent Internet Systems, the Vice Chair of the IEEE Computational Intelligence, Taipei Chapter, and an Officer of the IEEE Taipei Section.

Churn-Jung Liau received the B.S., M.S. and Ph.D. degrees in information engineering and computer science from the National Taiwan University, Taipei, Taiwan, in 1985, 1987, and 1992, respectively.

He joined the Institute of Information Science, Academia Sinica, Taiwan, where he is currently a tenured Full Research Fellow. His current research interests include artificial intelligence, uncertainty reasoning, and logic.