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FUZZY MODELS OF ROUGH MATHEMATICS

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Показано, що введений відомий формальний опис неточних множин може бути інтерпретований у термінах нечітких множин. Це дозволяє для розв'язання багатьох задач неточної математики використати розвинений апарат нечіткої математики. Наведено приклад розв'язання задачі лінійного програмування, параметри якої визначені неточно. Для опису неточних параметрів задачі використані функції (L-R)-типу. Для розв'язання задачі введено складений критерій. Чисельне значення критерію враховує міру близькості отримуваного результату до модального рішення і рівень компактності функції приналежності значення цільової функції

Ключові слова: неточна математика, нечіткі моделі неточних чисел, рішення задач неточної математики, неточне лінійне програмування

Показано, что введённое известное формальное описание неточных множеств может быть интерпретировано в терминах нечётких множеств. Это позволяет для решения многих задач неточной математики использовать развитый аппарат нечёткой математики. Приведён пример решения задачи линейного программирования, параметры которой определены неточно. Для описания неточных параметров задачи использованы функции (L-R)-типа. При решении задачи введён составной критерий. Численное значение критерия учитывает меру близости получаемого результата к модальному решению и уровень компактности функции принадлежности получаемого значения целевой функции

Ключевые слова: неточная математика, нечёткие модели неточных чисел, решение задач неточной математики, неточное линейное программирование

1. Introduction

The practical problems of analyzing and synthesizing complex systems are solved under conditions of uncertainty. The degree of uncertainty is determined by the level of knowledge on the state and behavior of a system under study and the environment in which the system operates. It is essential to take this uncertainty into account while solving problems of assessing and predicting the states of systems in engineering [1, 2], military affairs [3, 4], medicine [5], economy [6, 7], as well as problems of structural and parametric optimization [8–10].

Over the past few decades, the emergence and rapid development of the fuzzy set theory [11–21] have significantly expanded the range of tasks for which it has become possible to use a strict formal mathematical apparatus. The presence of a rod element in this theory means a fuzzy value membership function as a natural analogue of the distribution density of random variables, which creates some necessary preconditions for an axiomatic construction of fuzzy mathematics.

However, in practical situations of "information hunger", a priori assignment of the membership function is not well-founded. Its adequate recovery from experimental data can not be implemented. The impossibility of using the fuzzy set theory in such cases correctly has been an impetus for developing a theory of rough sets, introduced in [22, 23]. The basic theoretical premise of this theory is the possibility of approximating an ill-defined rough set with a couple of precise sets, which are called a lower approximation and an upper approximation. The lower approximation is a subset containing objects that definitely belong to a set under consideration, and the upper approximation is a subset that includes objects that may belong to the considered set.

2. Literature review and problem statement

Let us consider the basic provisions of the rough set theory that are presented in [22, 23] and specified in [24–27].

May U be a universal set and may X be a subset within U. We shall introduce $\underline{X}$ to denote the lower approximation and $\overline{X}$ to denote the upper approximation of X. Then, according to [25], $\underline{X} \subset X \subset \overline{X}$. Let us consider an example of a constructive setting of a rough number. Let Λ be a non-empty set, may A be a set of all the subsets of Λ, and let Δ be an element of A. We shall introduce a four-element set in which

$$c \le a \le b \le d,\ \ \Lambda = \{x | c \le x \le d\} \text{ and } \Delta = \{x | a \le x \le b\}.$$

Then the value of $\xi \in \Delta$, definitely belonging to the interval [a, b] and possibly belonging to the interval [c, d], is a rough value given by the intervals

$$(\Delta, \Lambda) = ([a,b],[c,d]).$$

A special case of a rough value is the interval [a, b] number, which can be regarded as some rough value of the type

$$([a,b],[a,b]).$$

The possibilities of the direct use of the rough set theory to solve practical problems are determined by the rules of processing rough values [24, 27].

May

$$\xi = ([a_1,a_2],[a_3,a_4]) \text{ and } \eta = ([b_1,b_2],[b_3,b_4]),$$

be two rough values. Then

$$\xi + \eta = ([a_1+b_1, a_2+b_2],[a_3+b_3, a_4+b_4]);$$

$$\xi - \eta = ([a_1-b_2, a_2-b_1],[a_3-b_4, a_4-b_3]);$$

$$k\xi = \begin{cases} [ka_1,ka_2],[ka_3,ka_4],\ k \ge 0, \\ [ka_2,ka_1],[ka_4,ka_3],\ k < 0; \end{cases} \qquad (1)$$

$$\xi \times \eta = ([a_1b_1, a_2b_2],[a_3b_3, a_4b_4]),\ a_3 \ge 0,\ b_3 \ge 0;$$

and

$$\frac{\xi}{\eta} = \left(\left[\frac{a_1}{b_2}, \frac{a_2}{b_1}\right],\left[\frac{a_3}{b_4}, \frac{a_4}{b_3}\right]\right),\ a_3 \ge 0,\ b_3 > 0.$$

In a special case of interval numbers, we obtain the following interval arithmetic [28-31]:

$$[a_1,a_2]+[b_1,b_2]=[a_1+b_1, a_2+b_2];$$

$$[a_1,a_2]-[b_1,b_2]=[a_1-b_2, a_2-b_1];$$

$$k[a_1,a_2] = \begin{cases} k[a_1,a_2],\ k \ge 0, \\ k[a_2,a_1],\ k < 0; \end{cases} \qquad (2)$$

$$[a_1,a_2]\times[b_1,b_2]=[a_1b_1, a_2b_2],\ a_1 \ge 0,\ b_1 \ge 0;$$

and

$$\frac{[a_1,a_2]}{[b_1,b_2]} = \left[\frac{a_1}{b_2}, \frac{a_2}{b_1}\right],\ a_1 \ge 0,\ b_1 > 0.$$

There are undoubted theoretical benefits of the formalism that is introduced in [22]. However, some weaknesses are also obvious. It is clear that the quality, utility and efficiency of using any models are primarily assessed in terms of their adequacy. A reasonable claim against the adequacy level of describing a rough value with the model ([a, b], [c, d]) is based on the lack of smoothness and derivative continuity of all transitions between the subsets containing objects that *certainly* belong, *possibly* belong, and *certainly do not* belong to a variety under study. Besides, all elements of the subset ([c, d]/[a, b]) are equally, although it is not clear how, similar to each other in respect of a possible belonging to the considered set. These elements are *equally* distinct from the objects that definitely belong to this set, which does not fully correspond to the true understanding of the nature of things. Furthermore, the summation of two rough numbers produces a rough number, but its type of uncertainty differs from that of the summands. The subset $([a_3+b_3, a_4+b_4] \setminus [a_1+b_1, a_2+b_2])$ of the elements of the resulting rough value is not as homogeneous as the original subsets $([a_3,a_4] \setminus [a_1,a_2])$ and $([b_3,b_4] \setminus [b_1,b_2])$. It is because the elements of the resulting set, for which the uncertainty remains, are produced by combining, in particular with the elements that undoubtedly belong to the summand sets for which there is no uncertainty. Thus, the resulting rough number can not be further adequately described by the model of the type $([c_1,c_2],[c_3,c_4])$; therefore, rule (1) must be adjusted. The natural direction of the adjustment is fuzzy mathematics [11–16]. The use of advanced tools of fuzzy mathematics to solve problems with an imprecise description of the source data significantly expands the toolbox of mathematical methods of research under conditions of uncertainty. It determines the urgency of the problem.

The definition of a rough set, first described in [22], through its lower and upper approximations allows interpreting it in terms of the fuzzy set theory. In this case, the rough number $\xi = ([a,b],[c,d])$ can be described, for example, by a trapezoidal fuzzy number x with a membership function

$$\mu(x) = \begin{cases} 0, & x < c, \\ \dfrac{x-c}{a-c}, & c \le x < a, \\ 1, & a \le x < b, \\ \dfrac{d-x}{d-b}, & b \le x \le d, \\ 0, & x > d. \end{cases}$$

The fuzzy number x *definitely* belongs to the interval [a, b] and *possibly* belongs to the interval [c, d], which is quite consistent with the canonical definition of the rough value of $([a,b],[c,d])$, suggested in [22–24]. Moreover, this definition of a rough value is equally consistent with any unimodal fuzzy number with the normal membership function (i. e. if $\max_x \mu(x)=1$). In terms of rough mathematics, such a fuzzy number has a subset Λ, coinciding with the support of the fuzzy number, whereas the subset Δ contains one element that is the kernel of the corresponding fuzzy set. This makes it possible to extend some of the known results of fuzzy mathematics to rough mathematics.

3. The purpose and objectives of the study

The theoretical and practical aims of this study consist in developing a method and structural techniques of describing the elements of the rough set theory as well as the rules of processing rough numbers in terms of fuzzy mathematics.

Therefore, our objective is to use terms, models, methods and tools of fuzzy mathematics to solve practical problems of rough mathematics. To achieve the aim, it is necessary to solve the following tasks:

– to develop a method of describing models of rough sets by means of fuzzy sets;

– to select the type of membership functions of fuzzy numbers that are approximations of rough numbers;

– to calculate the parameters of the membership functions of fuzzy numbers used to describe rough numbers;

– to adapt the fuzzy mathematics algebra to solving problems with roughly set initial data.

4. The main results. The development of a technology for applying the mathematical apparatus of fuzzy mathematics to solve problems of rough mathematics

In accordance with [32], let us consider an arbitrary binary operation (addition, subtraction, multiplication, and division) that assigns the result B to the elements of the composition numbers A_1 and A_2. At the same time, we introduce a "reverse" operation ⊗ that can help use the composition result B and one of the components (e. g., A_1) to determine the second component.

Let

$$B=A_1*A_2=A_1+A_2,\quad B=A_1*A_2=A_1-A_2,$$

and

$$B=A_1*A_2=A_1A_2,\quad B=A_1*A_2=A_1/A_2.$$

Then the use of the "reverse" operation will respectively produce

$$A_2=B\otimes A_1=B-A_1,\quad A_2=B\otimes A_1=A_1-B,$$

$$A_2=B\otimes A_1=\frac{B}{A_1},\ \text{and}\ A_2=B\otimes A_1=\frac{A_1}{B}.$$

The membership function of the binary composition $B=A_1*A_2$ is defined by the correlation

$$\mu_B(z)=\int_{-\infty}^{\infty}\mu_1(t)\mu_2(z\otimes t)dt.$$

We normalize the obtained membership function by rationing its maximum value. In particular, if * is a summation operation, the membership function corresponding to the result will have the form

$$\mu_B(z)=\int_{-\infty}^{\infty}\mu_1(t)\mu_2(z-t)dt.$$

In this case, the normalization produces

$$\tilde{\mu}_B(z)=\left[\max_z\{\mu_B(z)\}\right]^{-1}\int_{-\infty}^{\infty}\mu_1(t)\mu_2(z-t)dt.$$

These rules can be applied when using fuzzy models for rough values.

It is noteworthy that more varied possibilities of a practical use of fuzzy models of rough values appear if the corresponding membership functions are reduced to the form

$$\hat{\mu}(x)=\frac{\mu(x)}{\int_c^d \mu(x)dx}.$$

Then

$$\int_c^d \hat{\mu}(x)dx=1,\quad \hat{\mu}(x)>0,\quad x\in[c,d],$$

and the normalized membership function acquires the properties of the distribution density. In this case, in particular, it is possible to calculate the expected value for a rough value with the membership function $\hat{\mu}(x)$, and it will be a natural analogue of a mathematical expectation of a random variable.

Let us calculate the expected value of a rough value

$$\zeta=([a,b],[c,d])$$

with the membership function

$$\hat{\mu}(x)=\frac{1}{S}\begin{cases}0, & x<c,\\ \frac{x-c}{a-c}, & c\le x<a,\\ 1, & a\le x<b,\\ \frac{d-x}{d-b}, & b\le x\le d,\\ 0, & x>d,\end{cases} \tag{3}$$

where

$$S=\int_c^d \mu(x)dx=\frac{(d-c)+(b-a)}{2}.$$

Consequently,

$$m=E[\zeta]=\int_c^d x\hat{\mu}(x)dx=$$

$$=\frac{1}{S}\left[\int_c^a x\frac{x-c}{a-c}dx+\int_a^b xdx+\int_b^d \frac{d-x}{d-b}dx\right]=J_1+J_2+J_3.$$

Then

$$J_1=\frac{1}{S(a-c)}\int_c^a x(x-c)dx=$$
$$=\frac{1}{S(a-c)}\left[\frac{x^3}{3}\Big|_c^a-c\frac{x^2}{2}\Big|_c^a\right]=$$
$$=\frac{1}{S(a-c)}\left(\frac{a^3-c^3}{3}-c\frac{a^2-c^2}{2}\right)=$$
$$=\frac{1}{S}\left(\frac{a^2+ac+c^2}{3}-\frac{ac+c^2}{2}\right)=$$
$$=\frac{1}{S\cdot 6}\left(2a^2+2ac+2c^2-3ac-3c^2\right)=$$
$$=\frac{2a^2-ac-c^2}{6S};$$

$$J_2=\frac{1}{S}\frac{x^2}{2}\Big|_a^b=\frac{b^2-a^2}{2S};$$

$$J_3=\frac{1}{S(d-b)}\int_b^d x(d-x)dx=\frac{1}{S(d-b)}\left[d\frac{x^2}{2}\Big|_b^d-\frac{x^3}{3}\Big|_b^d\right]=$$
$$=\frac{1}{S(d-b)}\left[\frac{d(d^2-b^2}{2}-\frac{d^3-b^3}{3}\right]=$$
$$=\frac{1}{S}\left[\frac{d^2+db}{2}-\frac{d^2+db+b^2}{3}\right]=$$
$$=\frac{1}{6S}\left(3d^2+3db-2d^2-2db-2b^2\right)=\frac{d^2+db-2b^2}{6S}.$$

$$J_1+J_2+J_3=\frac{1}{6S}\left(2a^2-ac-c^2+3b^2-3a^2+d^2+db-2b^2\right)=$$
$$=\frac{1}{6S}\left(b^2+d^2+db-a^2-c^2-ac\right)=$$
$$=\frac{1}{6S}\left[\frac{1}{2}\left(b^2+2db+d^2\right)-\frac{1}{2}\left(a^2+2ac+c^2\right)+\right.$$
$$\left.+\frac{1}{2}b^2+\frac{1}{2}d^2-\frac{1}{2}a^2-\frac{1}{2}c^2\right]=$$
$$=\frac{1}{6S}\left[\frac{1}{2}\left[\left(d+b\right)^2-\left(a+c\right)^2\right]+\right.$$
$$+\frac{1}{4}\left[\left(b^2+d^2+2db+b^2+d^2-2db\right)-\right.$$
$$\left.\left.-\left(a^2+c^2+2ac+a^2+c^2-2ac\right)\right]\right]=$$
$$=\frac{1}{6S}\left[\frac{1}{2}\left[\left(d+b\right)^2-\left(a+c\right)^2\right]+\right.$$
$$\left.+\frac{1}{4}\left[\left(b+d\right)^2+\left(d-b\right)^2-\left(a+c\right)^2-\left(a-c\right)^2\right]\right]=$$
$$=\frac{1}{6S}\left[\frac{3}{4}\left[\left(d+b\right)^2-\left(a+c\right)^2\right]+\right.$$
$$\left.+\frac{1}{4}\left[\left(d-b\right)^2-\left(a-c\right)^2\right]\right].$$

Since

$$\left(d+b\right)^2-\left(a+c\right)^2=\left(a+b+c+d\right)\left(d-c+b-a\right),$$

$$S=\frac{d-c+b-a}{2},$$

then

$$m=\frac{1}{4}\left(a+b+c+d\right)+\frac{1}{12}\frac{\left(d-b\right)^2-\left(a-c\right)^2}{d-c+b-a}. \qquad (4)$$

The obtained correlation can be de presented in another form, which in some cases is more convenient than the initial expression

$$m=\frac{3\left(a+b+c+d\right)\left(d-c+b-a\right)+\left(d-b\right)^2-\left(a-c\right)^2}{12\left(d-c+b-a\right)}=$$
$$=\frac{G}{12\left(d-c+b-a\right)}.$$

The numerator of this ratio is equal to

$$G=3ad-3ac+3ab-3a^2+3bd-$$
$$-3bc+3b^2-3ab+3cd-3c^2+3cb-$$
$$-3ac+3d^2-3cd+3bd-3ad+$$
$$+d^2-2bd+b^2-a^2+2ac-c^2=$$
$$=4\left(b^2+d^2+bd-a^2-ac-c^2\right).$$

In this case,

$$m=\frac{b^2+d^2+bd-a^2-ac-c^2}{3\left(d-c+b-a\right)}.$$

In the most realistic particular case, if the interval [a, b] within the interval [c, d] is symmetric, then d–b=a–c, that is, correlation (4) is simplified in the form

$$m=\frac{a+b}{2}. \qquad (5)$$

In another particular case, if a rough number is of an interval type, then a=c, b=d and correlation (4) can be again reduced to formula (5).

Finally, let us consider another special case in which the lower approximation of a rough number is reduced to a point, that is, a=b. Then

$$m=\frac{1}{4}\left(d+c+2a\right)+\frac{1}{12}\frac{\left(d-a+a-c\right)\left(d-a-a+c\right)}{d-c}=$$
$$=\frac{1}{4}\left(d+c+2a\right)+\frac{1}{12}\left(d+c-2a\right)=$$
$$=\frac{1}{12}\left(3d+3c+ba+d+c-2a\right)=\frac{1}{3}\left(d+a+c\right). \qquad (6)$$

If in this case d–a=a–c, then $a=\frac{c+d}{2}$, and we re-obtain formula (5).

The uncertainty level of a rough number that is defined by formula (3) is determined by the extent by which this number deviates from its expected value. This extent can be naturally measured by

$$D=E\left[\left(\zeta-m\right)^2\right]=\int_c^d (x-m)^2\hat{\mu}(x)dx=$$
$$=\frac{1}{S}\left[\int_c^d x^2\mu(x)dx-2m\int_c^d x\mu(x)dx+m^2\int_c^d \mu(x)dx\right]=\left(I_1+I_2+I_3\right).$$

$$I_1=\frac{1}{S}\int_c^d x^2\mu(x)dx=\frac{1}{S}\left[\int_c^a x^2\frac{x-c}{a-c}dx+\int_a^b x^2dx+\int_b^d x^2\frac{d-x}{d-b}dx\right]=$$

$$=\frac{1}{S}\left[\frac{1}{(a-c)}\int_c^a x^3dx-\frac{c}{a-c}\int_c^a x^2dx+\int_a^b x^2dx+\frac{d}{d-b}\int_b^d x^2dx-\frac{1}{d-b}\int_b^d x^3dx=\right.$$

$$=\frac{1}{S}\left[\frac{1}{(a-c)}\left(\frac{a^4}{4}-\frac{c^4}{4}\right)-\int_c^a x^2dx+\int_a^b x^2dx+\frac{d}{d-b}\int_b^d x^2dx-\frac{1}{d-b}\int_b^d x^3dx=\right.$$

$$=\frac{1}{S}\left[\frac{1}{(a-c)}\left(\frac{a^4}{4}-\frac{c^4}{4}\right)-\frac{c}{a-c}\left(\frac{a^3}{3}-\frac{c^3}{3}\right)+\left(\frac{b^3}{3}-\frac{a^3}{3}\right)+\frac{d}{d-b}\left(\frac{d^3}{3}-\frac{b^3}{3}\right)-\right.$$

$$\left.-\frac{1}{d-b}\left(\frac{d^4}{4}-\frac{b^4}{4}\right)\right]=$$

$$=\frac{1}{12S}\left(3a^3+3ac^2+3ca^2+3c^3-4a^2c-4ac^2-4c^3+4b^3-4a^3+\right.$$

$$\left.+4d^3+4d^2b+4db^2-3d^3-3db^2-3bd^2-3b^3\right)=$$

$$=\frac{1}{12S}\left(-a^3-ac^2-a^2c-c^3+b^3+d^3+d^2b+db^2\right)=$$

$$=\frac{1}{12\cdot S}\left[b^2(b+d)+d^2(b+d)-a^2(a+c)-c^2(a+c)\right]=$$

$$=\frac{1}{12S}\left[(b+d)(b^2+d^2)-(a+c)(a^2+c^2)\right]=$$

$$=\frac{(b+d)(b^2+d^2)-(a+c)(a^2+c^2)}{6(d-c+b-a)};$$

$$I_2=-2m\int_c^d x\hat{\mu}dx=-2m^2;\ \ I_2+I_3=-m^2.$$

Then

$$I_1+I_2+I_3=$$

$$=\frac{1}{6}\frac{b^3+d^3+d^2b+db^2-a^3-c^3-a^2c-ac^2}{d-c+b-a}-$$

$$-\frac{1}{9}\frac{\left(b^2+d^2+bd-a^2-ac-c^2\right)^2}{\left(d-c+b-a\right)^2}.$$

In the previously considered particular cases, the resulting expression is, of course, simplified.

If the rough number is of an interval type, then a=c and b=d. Then

$$D=\frac{4b^3-4a^3}{6\cdot 2(b-a)}-\frac{(a+b)^2}{4}=\frac{b^2+ab+a^2}{3}-\frac{a^2+2ab+b^2}{4}=$$

$$=\frac{4b^2+4ab+4a^2-3a^2-6ab+3b^2}{12}=\frac{b^2-2ab+a^2}{12}=\frac{(b-a)^2}{12}.$$

If the interval [a, b] degenerates into a point, then a=b and

$$D=\frac{d^3+d^2a+da^2+a^3-a^3-c^3-a^2c-ac^2}{6(d-c)}-\frac{(d+c+a)^2}{9}=$$

$$=\frac{d^3-c^3+a^2(d-c)+a(d^2-c^2)}{6(d-c)}-\frac{(d+c+a)^2}{9}=$$

$$=\frac{d^2+a^2+c^2+ad+cd+ac}{6}-\frac{(d+c+a)^2}{9}=$$

$$=\frac{3d^2+3a^2+3c^2+3ad+3cd+3ac-2d^2-2c^2-2a^2-4dc-4ac-4ad}{18}=$$

$$=\frac{a^2+c^2+d^2-ad-cd-ac}{18}.$$

Thus, if the points c and d are symmetrically arranged relative to a, that is, a–c=d–a, then $a=\frac{c+d}{2}$ and

$$D=\frac{1}{18}\left[\frac{(c+d)^2}{4}+c^2+d^2-\frac{c+d}{2}(c+d)-cd\right]=$$

$$=\frac{1}{18}\left[-\frac{(c+d)^2}{2}+c^2+d^2-cd\right]=$$

$$=\frac{2c^2+2d^2-2cd+c^2+d^2+2cd}{36}=\frac{c^2+d^2}{12}.$$

Finally, if the points c and d are arranged symmetrically with respect to [a, b], that is, a–c=d–b, then d–a=b–c and d–c+b–a=2(d–a). Consequently,

$$D=\frac{b^3+d^3+b^2d+d^2b-a^3-c^3-a^2c-ac^2}{12(d-a)}-\frac{(a+b)^2}{4}.$$

These correlations make it possible to formulate and solve more complex problems in which rough source data can be converted into fuzzy data, for example, it concerns the problem of rough mathematical programming [33] and the problem of regression analysis [34].

We introduce the rough vector

$$C=\left\{\left([c_{11},c_{12}],[c_{13},c_{14}]\right),\left([c_{21},c_{22}],[c_{23},c_{24}]\right),...,\left([c_{n1},c_{n2}],[c_{n3},c_{n4}]\right)\right\},$$

the rough vector

$$B=\left\{\left([b_{11},b_{12}],[b_{13},b_{14}]\right),\left([b_{21},b_{22}],[b_{23},b_{24}]\right),...,\left([b_{m1},b_{m2}],[b_{m3},b_{m4}]\right)\right\},$$

the rough matrix

$$A=\left\{[a_{ij1},a_{ij2}],[a_{ij3},a_{ij4}]\right\},\ \ i=1,2,...,n,\ \ j=1,2,...,m,$$

and the vector $X=(x_1,x_2,...,x_n)$. Then, for example, the problem of rough linear programming is formulated as follows: to find a vector X, which maximizes

$$L(C,X)=E\left[\sum_{j=1}^{n}\left([c_{j1},c_{j2}],[c_{j3},c_{j4}]\right)x_j\right] \tag{7}$$

and satisfies the constraints

$$G_i(A,B,X)=E\left[\sum_{j=1}^{n}\left([a_{ij1},a_{ij2}],[a_{ij3},a_{ij4}]\right)x_j\right]=$$

$$=E\left[\left([b_{i1},b_{i2}],[b_{i3},b_{i4}]\right)\right],$$

$$i=1,2,...,m,\ \ x_j\geq 0,\ \ j=1,2,...,n. \tag{8}$$

Here $E[\ ]$ is the operator for calculating the expected value of a rough value. Relations (7) and (8) are converted into the form

$$L(C,X)=\sum_{j=1}^{n}\bar{c}_j x_j, \tag{9}$$

$$G_i(A,B,X)=\sum_{j=1}^{n}\bar{a}_{ij}x_j=\bar{b}_i,\ \ j=1,2,...,m. \tag{10}$$

Here $\bar{c}_j, \bar{a}_{ij}, \bar{b}_i$ $i=1,2,...,m$, and $j=1,2,...,n$, are the expected values of the relevant rough numbers, calculated according to formulae (4)–(6), depending on specific characteristics of their descriptions.

The resulting problem is a regular problem of linear programming.

Let us consider a simple example: to find a set X=(x_1, x_2), maximizing

$$L(C,X)=([c_{11},c_{12}],[c_{13},c_{14}])x_1+([c_{21},c_{22}],[c_{23},c_{24}])x_2$$

and satisfying the constraints

$$x_1+2x_2=5,\ \ x_1\ge 0,\ \ x_2\ge 0.$$

$$c_{11}=3,\ \ c_{12}=4,\ \ c_{13}=1,\ \ c_{14}=6,$$

$$c_{21}=4,\ \ c_{22}=6,\ \ c_{23}=2,\ \ c_{24}=8.$$

Using (5), we get $\bar{c}_1=3{,}5,\ \ \bar{c}_2=5$, and the problem is converted into the form: to find a set (x_1, x_2), which maximizes

$$L(X)=3{,}5x_1+5x_2,$$

and satisfies the constraints

$$x_1+2x_2=5,\ \ x_1\ge 0,\ \ x_2\ge 0.$$

Let us assume that $y_{1^*}=x_1,\ \ y_{2^*}=2x_2$. Then the objective function is reduced to the following:

$$L(Y)=3{,}5y_1+2{,}5y_2,$$

and the restrictions are as follows:

$$y_1+y_2=5,\ \ y_1\ge 0,\ \ y_2\ge 0.$$

The problem solution is obvious: $y_1=5,\ \ y_2=0$. At the same time, returning to the original variables, we find the answer:

$$x_1^*=5,\ \ x_2^*=0.$$

An even simpler example is to find (x_1, x_2), maximizing

$$L(C,X)=([c_{11},c_{11}],[c_{13},c_{14}])x_1+([c_{21},c_{21}],[c_{23},c_{24}])x_2,$$

and satisfying the constraints

$$x_1+2x_2=5,\ \ x_1\ge 0,\ \ x_2\ge 0,$$

$$c_{11}=c_{21}=1,\ \ c_{13}=c_{23}=0,\ \ c_{14}=c_{24}=2.$$

Then $\bar{c}_1=\bar{c}_2=1$, and the problem is converted into the form: to find the set $X=(x_1,x_2)$, maximizing $L(X)=x_1+x_2$ and satisfying the constraints $x_1+2x_2=5$, $x_1\ge 0$, and $x_2\ge 0$. It is obvious that the problem in this case will be solved by the set

$$x_1^*=5,\ \ x_2^*=0.$$

The situation changes if the orthodox concept of fuzzy sets is used to assume that the rough model of the number $\xi=([a,a],[c,d])$ is compliant with a fuzzy number of an L-R type [13] with the membership function

$$M(x)=\begin{cases} L\left(\dfrac{m-x}{\alpha}\right), & x\le m,\\ R\left(\dfrac{x-m}{\beta}\right), & x>m,\end{cases} \qquad (11)$$

where m is the mode of the fuzzy number and (α, β) are the left and right factors of the fuzziness.

Let us consider the technology of solving one of the standard rough problems of mathematical programming by using model (11).

We shall introduce the rough numbers $\xi_1=([1,1],[0,2])$, and $\xi_2=([2,2],[0,4])$, as well as formulate the problem: to find the set $X=(x_1,x_2)$, maximizing

$$L(C,X)=\xi_1x_1+\xi_2x_2 \qquad (12)$$

and satisfying the constraints

$$x_1+x_2=2,\ \ x_1\ge 0,\ \ x_2\ge 0. \qquad (13)$$

To describe the rough numbers ξ_1 and ξ_2, we apply the model of the (L-R) type.

Since in this particular problem for both numbers ξ_1 and ξ_2 the interval [c, d] is located symmetrically with respect to the modal value of a, it is convenient to represent the (L-R) function by a Gaussian function, setting that

$$\mu_1(u)=\exp\left\{-\frac{(u-1)^2}{2\left(\frac{1}{3}\right)^2}\right\},\ \ \mu_2(u)=\exp\left\{-\frac{(u-2)^2}{2\left(\frac{2}{3}\right)^2}\right\}.$$

The parameters of these membership functions – the modal value and the factors of fuzziness – are calculated by the formulae

$$m=\frac{a+b}{2},\ \ \alpha=\frac{m-c}{3},\ \text{and}\ \ \beta=\frac{d-m}{3}.$$

Now, in accordance with the rules of processing fuzzy numbers [32], let us construct the membership function of a fuzzy value of the objective function (12). We have

$$\mu(L(X))=\exp\left\{-\frac{(L-m_\Sigma(X))^2}{2\sigma_\Sigma^2(X)}\right\}, \qquad (14)$$

$$m_\Sigma(X)=m_1x_1+m_2x_2=x_1+2x_2 \qquad (15)$$

and

$$\sigma_\Sigma^2(X)=\frac{1}{9}x_1^2+\frac{4}{9}x_2^2.$$

To solve the problem, we use the following two-stage procedure. In the first stage, we obtain a modal solution, assuming the following: $\xi_1^{(0)}=m_1,\ \xi_2^{(0)}=m_2$. In this case, the task of (12)-(15) is converted into the form: to find the set (x_1, x_2), maximizing $L(X)=x_1+2x_2$ and satisfying the constraints of (13). This trivial problem is solved by the set of $X^{(0)}=(x_1^{(0)},x_2^{(0)})=(0,2)$.

Actually, the problem is solved in the second stage, with the optimization of the complex criterion the numerical value of which takes into account, on the one hand, the extent of the proximity between the desired solution to the problem

X and the modal solution $X^{(0)}$ and, on the other hand, describes the compactness of the membership function of the fuzzy value of L(x) of the objective function of the problem within the set X.

The extent of closeness of the set X to the modal set $X^{(0)}$ in the Euclidean metric is determined by the formula

$$J_1 = (X - X^{(0)})^T (X - X^{(0)}).$$

The extent of the membership function compactness $\mu(L(X))$ of the objective function can be assumed as the square of the area under the objective function $\mu(L(X))$. This measure, taking into account (14), is calculated as follows:

$$J_2 = S^2(\mu(L(X))) = \left[\int_{-\infty}^{\infty} \mu(L(X))dL\right]^2 =$$

$$= \left[\int_{-\infty}^{\infty} \exp\frac{(L - m_\Sigma(X))^2}{2\sigma_\Sigma^2(X)} dL\right]^2 =$$

$$= \left[\sqrt{2\pi}\sigma_\Sigma \frac{1}{\sqrt{2\pi}\sigma_\Sigma}\int_{-\infty}^{\infty} \exp\left\{-\frac{(L - m_\Sigma)^2}{2\sigma_\Sigma^2}\right\} dL\right]^2 = 2\pi\sigma_\Sigma^2(X).$$

In forming a complex criterion, it is expedient that the introduced measures J_1 and J_2 be normalized. Then the expression for the criterion will have the form

$$J(x) = \lambda \frac{\left(X - X^{(0)}\right)^T \left(X - X^{(0)}\right)}{\left(X^{(0)}\right)^T X^{(0)}} + (1 - \lambda)\frac{\sigma_\Sigma^2(X)}{\sigma_\Sigma^2(X^{(0)})}. \quad (16)$$

Here λ is a regularizing parameter $(\lambda \in [0;1])$, defining the acceptable level of compromise between the individual criteria J_1 and J_2.

Let us express criterion (16), taking into account the parameters of the problem being solved.

$$J(X) = \lambda \frac{x_1^2 + (x_2 - 2)^2}{4} + (1 - \lambda)\frac{\frac{1}{9}x_1^2 + \frac{4}{9}x_2^2}{\frac{16}{9}} =$$

$$= \frac{1}{4}\left[\lambda\left(x_1^2 + (x_2 - 2)^2\right) + (1 - \lambda)\left(\frac{x_1^2}{4} + x_2^2\right)\right]. \quad (17)$$

The minimization of (17) with the restrictions of (13) is equivalent to the minimization of the following function of the variable x_1:

$$f(x_1) = \lambda 2x_1^2 + (1 - \lambda)\left(\frac{x_1^2}{4} + (2 - x_1)^2\right).$$

The minimum f(x) is achieved at

$$x_1 = \frac{8}{8v + 5}, \quad v = \frac{\lambda}{1 - \lambda}.$$

The problem solution for various values of the regularization parameter λ is reduced to the table below. Here we also specify the relevant normalization of the values of the specific criteria J_1 and J_2 and the complex criterion J.

The analysis of the table data produces the following conclusions. With an increase in the numerical value of the regularization parameter λ, the sequence of the problem solutions converges into a modal solution. Meanwhile, there is an increase in the area under the curve that describes the membership function of the fuzzy value of the objective function. If the height of the membership function is fixed, it means some sprawling and stretching of the body of the membership function, i.e. an increase of uncertainty as to the result of solving the problem. Moreover, the minimum value of the complex criterion is achieved at λ=0.5.

Table 1

The results of solving the problem

λ	0	0.09	0.33	0.5	0.9
(x_1^*, x_2^*)	(1.6;0.4)	(1.38;0.62)	(0.89;1.11)	(0.62;1.38)	(0.17;1.83)
$\hat{J}_2$	0.2	0.22	0.36	0.5	0.84
$\hat{J}_1$	1.28	0.95	0.4	0.19	0.01
J	1.48	1.17	0.76	0.69	0.85

5. Discussion of the results of developing fuzzy models of rough mathematics

Thus, the study has determined that the formal definition of rough sets [22] allows making their consistent interpretation in terms of fuzzy sets. It has been shown that the developed mathematical apparatus of the fuzzy set theory can be effectively used to solve many practical problems in which the initial data are rough values. The development of appropriate technologies for empowering mathematical analysis, linear algebra, methods of solving linear and nonlinear equations and their systems, as well as optimization methods in terms of rough input information are directions for further research.

6. Conclusions

1. The study suggests a method for describing models of rough sets in terms and by means of using the mathematical apparatus of fuzzy sets.

2. Rules have been developed for performing arithmetic processing of rough values. The introduced rules have the structure and the technology of their implementing that comply with the same rules of processing fuzzy numbers.

3. The suggested mathematical methods for calculating the numerical characteristics of rough values are analogues to the mathematical expectation and the variance of random variables. These techniques are based on using fuzzy models of functions that analytically describe rough numbers.

4. Examples are given on the simplest tasks of mathematical programming when the parameters are given roughly.

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