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FUZZY P-VALUE OF HYPOTHESES TESTS WITH CRISP DATA USING NON-ASYMPTOTIC FUZZY ESTIMATORS

NIKOS MYLONAS AND BASIL K. PAPADOPOULOS*

ABSTRACT. In a fuzzy hypothesis test the null hypothesis is rejected or accepted with a degree of rejection or acceptance. A way to carry out such a test is to use fuzzy significance level and fuzzy p -value, which is a fuzzy number constructed by a fuzzy estimator. This approach is particularly useful in critical situations, where subtle comparisons between almost equal statistical quantities have to be made. In such cases the fuzzy hypotheses tests give better results than the crisp tests, since they give us the possibility of partial rejection or acceptance of H_0 using a degree of rejection or acceptance obtained by ordering the fuzzy numbers of p -value and significance level.

1. Introduction

In classical crisp statistics the problem of testing a hypothesis for a parameter θ of the distribution of a random variable X is to decide whether to reject or accept at a significance level γ the null hypothesis $H_0 : \theta = \theta_0$ against the alternative hypothesis H_1 from a random sample of observations of X using a test statistic U which is evaluated for the sample observations, resulting in a value u . The space of possible values of U is decomposed into a rejection region and its complement, the acceptance region. Depending on the alternative hypothesis H_1 , the rejection region has one of the following forms [6]:

a) $U \leq u_{c1}$, if the alternative hypothesis is $H_1 : \theta < \theta_0$ (one sided test from the left), where

$$P(U \leq u_{c1}) = \gamma$$

b) $U \geq u_{c2}$, if the alternative hypothesis is $H_1 : \theta > \theta_0$ (one sided test from the right), where

$$P(U \geq u_{c2}) = \gamma$$

c) $U \leq u_{c3}$ or $U \geq u_{c4}$, if the alternative hypothesis is $H_1 : \theta \neq \theta_0$ (two sided test), where

$$P(U \leq u_{c3}) = \frac{\gamma}{2} \quad \text{and} \quad P(U \geq u_{c4}) = \frac{\gamma}{2}$$

where u_{ci} the critical values of the test, which are determined by the distribution of U . So, H_0 is rejected, if the value u of the test statistic is in the rejection region and is not rejected, if u is in the acceptance region.

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An equivalent way to test the null hypothesis H_0 against the alternative hypothesis H_1 is to calculate the p-value, which is defined for the above cases (a), (b), and (c) as (u is the sample value of the test statistic) [6]:

$$\text{a) } p = P(U \leq u), \quad \text{b) } p = P(U \geq u), \quad \text{c) } p = 2 \min\{P(U \leq u), P(U \geq u)\} \quad (1.1)$$

If the p-value is less or equal to γ , then H_0 is rejected at the significance level γ , otherwise H_0 is not rejected.

Many approaches have been developed for fuzzy hypotheses testing. Among them [13] and [5] deal with the problem of hypotheses testing when the hypotheses are fuzzy from crisp or fuzzy data respectively using the concept of fuzzy p -value.

In [2] a method for testing crisp hypotheses has been proposed, which uses fuzzy test statistics obtained by fuzzy estimators produced by a set of confidence intervals and fuzzy critical values. In [12], from the various types of existing fuzzy estimators we use the well known non-asymptotic fuzzy estimators of [14] for the construction of the fuzzy test statistics. These are more convenient, because they are functions, whereas the fuzzy estimators of [2] are not functions, since they contain two small vertical segments.

Continuing this work, we present in this article a method for testing crisp hypotheses using fuzzy p -value \bar{P} obtained by fuzzy test statistics, which are constructed by non-asymptotic fuzzy estimators. Fuzzifying the significance level γ to \bar{S} and using fuzzy ordering between \bar{P} and \bar{S} , we decide for the rejection or acceptance of H_0 at the significance level γ with a degree of confidence d , as explained in Sections 1.1 and 2.

If the p -value of the test is close to the significance level, then the crisp hypothesis test is unstable, since two similar samples may lead to contradicted result (one gives rejection and the other acceptance of H_0), as in Examples 3 and 6. This problem does not exist in the respective fuzzy test in which we use a fuzzy test statistic \bar{U} produced by a fuzzy estimator $\bar{\theta}$ of θ (instead of a crisp estimator) and a degree of rejection or acceptance of H_0 [12]. In such case a fuzzy hypothesis test gives similar results for the two similar samples: very low degree of rejection or acceptance, which means “no decision”.

1.1. Ordering fuzzy numbers. The fuzzy hypotheses tests are based on ordering fuzzy numbers, for which we will use one of the several procedures used ([2],[3],[8],[9],[10]), according to which the degree of the inequality $A \leq B$ that counts the degree to which the fuzzy number A is less or equal than the fuzzy number B is defined as [2]

$$v(A \leq B) = \max \{ \min(A(x), B(y)) \mid x \leq y \} \quad (1.2)$$

If $v(A \leq B) = 1$, then the degree of confidence (truth-value) of $A < B$ is defined as

$$v(A < B) = d \Leftrightarrow v(A \leq B) = 1 \quad \text{and} \quad v(B \leq A) = 1 - d \quad (1.3)$$

If $v(B \leq A) = 1$, then the truth-value of $B < A$ is defined as

$$v(B < A) = d \Leftrightarrow v(B \leq A) = 1 \quad \text{and} \quad v(A \leq B) = 1 - d \quad (1.4)$$

The truth-value η of $A \approx B$ is defined as

$$v(A \approx B) = \eta \Leftrightarrow v(A \leq B) = 1 \quad \text{and} \quad v(B \leq A) = \eta$$

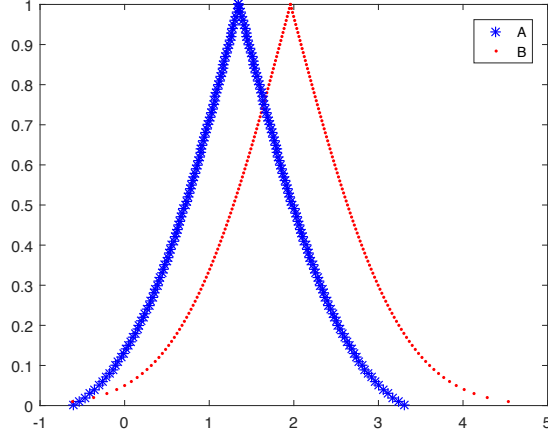


FIGURE 1. Ordering triangular shaped fuzzy numbers

$$\text{or } v(B \leq A) = 1 \text{ and } v(A \leq B) = \eta \quad (1.5)$$

In the case of the triangular shaped fuzzy numbers of Figure 1, which appears often in fuzzy hypotheses tests, we can see that:

$v(A \leq B) = 1$ according to (1.2), since the core of A is at the left of the core of B , $v(B \leq A) = y_0$ according to (1.2), where y_0 the truth level of the point of intersection of the right part of A and the left part of B .

Therefore according to (1.3) and (1.5),

$$v(A < B) = 1 - y_0 \text{ and } v(A \approx B) = y_0$$

2. Fuzzy p-value, Fuzzy Significance Level and Degree of Rejection or Acceptance of H_0

Using Zadeh's extension principle [15], a fuzzy p-value is defined as a fuzzy number which is found by its α -cuts.

In a fuzzy test of the null hypothesis $H_0 : \theta = \theta_0$ for the parameter θ of a distribution from a random sample of observations we use a fuzzy test statistic \bar{U} produced by a fuzzy estimator $\bar{\theta}$ of θ . So, the α -cuts of the fuzzy p-value of the test as described in [5] are:

a) for **one sided test from the left** (alternative hypothesis $H_1 : \theta < \theta_0$):

$$\bar{P}[\alpha] = [\Pr(U \leq \bar{U}_l[\alpha]), \Pr(U \leq \bar{U}_r[\alpha])] \quad (2.1)$$

where U the test statistic and $\bar{U}_l[\alpha]$ and $\bar{U}_r[\alpha]$ the α -cuts of the fuzzy test statistic \bar{U} for the given sample,

b) for **one sided test from the right** (alternative hypothesis $H_1 : \theta > \theta_0$):

$$\bar{P}[\alpha] = [\Pr(U \geq \bar{U}_r[\alpha]), \Pr(U \geq \bar{U}_l[\alpha])] \quad (2.2)$$

c) for **two sided test** (alternative hypothesis $H_1 : \theta \neq \theta_0$):

$$\bar{P}[\alpha] = \begin{cases} [2(\Pr(U \leq \bar{U}_l[\alpha])), \\ \quad \min\{1, 2(\Pr(U \leq \bar{U}_r[\alpha]))\}], & u_0 < m_U \\ [2\Pr(U \geq \bar{U}_r[\alpha]), \\ \quad \min\{1, 2\Pr(U \geq \bar{U}_l[\alpha])\}], & u_0 \geq m_U \end{cases} \quad (2.3)$$

where m_U the median of the distribution of U under the null hypothesis H_0 and u_0 the core of the fuzzy test statistic \bar{U} (where $\bar{U}(u_0) = 1$).

Since the p-value is fuzzy, the significance level has to be a fuzzy set \bar{S} , too.

A simple way to fuzzify a significance level γ is to use a triangular fuzzy numbers of one of the forms [7]

$$(a) \bar{S} = T(0, \gamma, 2\gamma) \quad \text{or} \quad (b) \bar{S} = T(\gamma - c, \gamma, 2\gamma + c), \quad c \text{ constant} \quad (2.4)$$

which expresses the notion “the significance level is approximately γ ”.

So, using fuzzy ordering between the fuzzy numbers \bar{P} and \bar{S} (Section 1.1):

H_0 is rejected at the significance level γ with degree of rejection d , if the truth value of $\bar{P} < \bar{S}$ is d ,

H_0 is accepted (not rejected) at the significance level γ with degree of acceptance d if the truth value of $\bar{P} > \bar{S}$ is d ,

if the truth value of $\bar{P} \approx \bar{S}$ is d_1 , then no decision can be made on the rejection or the acceptance of H_0 at the significance level γ with any degree of confidence $d > 1 - d_1$.

3. Fuzzy p-value of the Mean of a Normal Distribution or of Any Distribution From Large Sample

In the following subsections we present the development of the fuzzy p -value of hypotheses tests for the mean of a normal random variable with known variance, of a random variable that follows any distribution from a large sample and for the equality of the means of two normal random variables with known variances, using fuzzy statistics constructed by the non-asymptotic fuzzy estimator of the mean developed in [2] and [14].

3.1. Fuzzy p-value of the hypotheses tests for the mean of a normal random variable with known variance. We test at significance level γ the null hypothesis that the mean μ of a random variable X that follows normal distribution with known variance σ^2 is equal to μ_0 ,

$$H_0 : \mu = \mu_0$$

using a random sample of observations of X of size n with mean \bar{x} .

3.1.1. Two sided test. In the two sided test of H_0 the alternative is

$$H_1 : \mu \neq \mu_0$$

In the crisp case we use the statistic [6]

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \quad (3.1)$$

where \bar{X} is the statistic of the sample mean. It is known [6] that under the null hypothesis ($\mu = \mu_0$) Z follows the standard normal distribution $N(0, 1)$, so from a random sample of observations H_0 is rejected if $p \leq \gamma$, where according to (1.1)

$$p = \begin{cases} 2\Phi(z_0), & z_0 \leq 0 \\ 2(1 - \Phi(z_0)), & z_0 > 0 \end{cases} \quad (3.2)$$

the crisp p -value of the test, $\Phi(z)$ the distribution function of the standard normal distribution and

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \quad (3.3)$$

the sample value of the statistic (3.1). If $p > \gamma$, then H_0 is not rejected.

In the fuzzy case, the test of H_0 is based on the fuzzy statistic

$$\bar{Z} = \frac{\bar{\mu} - \mu_0}{\sigma/\sqrt{n}}$$

which is generated by substituting \bar{X} in (3.1) with a fuzzy estimator $\bar{\mu}$ of the mean [2].

In our approach we use the non-asymptotic fuzzy estimator $\bar{\mu}$ the α -cuts of which for the given sample are [14]

$$\bar{\mu}[\alpha] = \left[\bar{x} - z_{h(\alpha)} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{h(\alpha)} \frac{\sigma}{\sqrt{n}} \right], \quad \alpha \in [0, 1] \quad (3.4)$$

where \bar{x} the sample mean,

$$h : [0, 1] \rightarrow \left[\frac{\gamma_0}{2}, \frac{1}{2} \right] \quad h(\alpha) = \left(\frac{1}{2} - \frac{\gamma_0}{2} \right) \alpha + \frac{\gamma_0}{2}, \quad \gamma_0 = 0.01 \quad (3.5)$$

and

$$z_{h(\alpha)} = \Phi^{-1}(1 - h(\alpha)) \quad (3.6)$$

where Φ^{-1} the inverse distribution function of the standard normal distribution. From (3.1)-(3.6) and interval arithmetics follows that the α -cuts of the fuzzy statistic \bar{Z} for the given sample are [2]

$$\bar{Z}_0[\alpha] = [z_0 - z_{h(\alpha)}, z_0 + z_{h(\alpha)}], \quad \alpha \in [0, 1]$$

The median of the standard normal distribution is equal to its mean, which is zero, and the core of \bar{Z} is obtained for $\alpha = 1$ to be $z_0 - z_{h(1)} = z_0 + z_{h(1)} = z_0$, since according to (3.5) $h(1) = 0.5$, so $z_{h(1)} = \Phi^{-1}(1 - 0.5) = 0$. Hence, since the statistic Z follows the standard normal distribution, according to (2.3) the α -cuts of the p -value of this test are:

$$\begin{aligned} \bar{P}[\alpha] &= \begin{cases} [2\Pr(Z \leq z_0 - z_{h(\alpha)}), \\ \quad \min\{1, 2\Pr(Z \leq z_0 + z_{h(\alpha)})\}], & z_0 \leq 0 \\ [2(\Pr(Z \geq z_0 + z_{h(\alpha)})), \\ \quad \min\{1, 2\Pr(Z \geq z_0 - z_{h(\alpha)})\}], & z_0 > 0 \end{cases} \\ &= \begin{cases} [2\Phi(z_0 - z_{h(\alpha)}), \\ \quad \min\{1, 2\Phi(z_0 + z_{h(\alpha)})\}], & z_0 \leq 0 \\ [2(1 - \Phi(z_0 + z_{h(\alpha)})), \\ \quad \min\{1, 2(1 - \Phi(z_0 - z_{h(\alpha)}))\}], & z_0 > 0 \end{cases} \end{aligned} \quad (3.7)$$

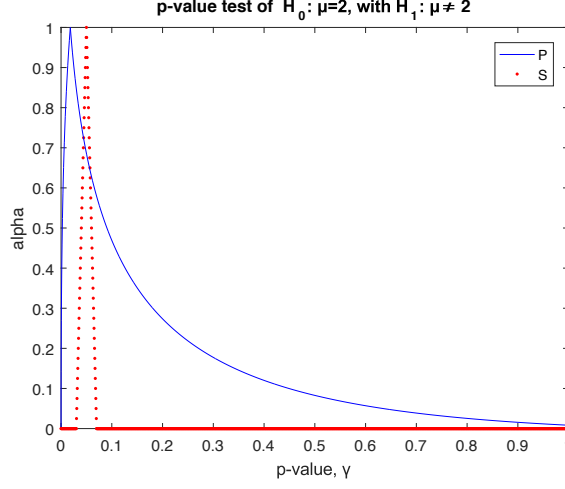


FIGURE 2. Fuzzy p-value \bar{P} and significance level \bar{S} for the two sided test of $H_0 : \mu = 2$ of Example 1

where $\Phi(z)$ the distribution function of the standard normal distribution and z_0 and $z_{h(\alpha)}$ are given by (3.3) and (3.6).

Using the fuzzy significance level \bar{S} given in (2.4), H_0 is rejected at significance level γ if $\bar{P} < \bar{S}$ and not rejected if $\bar{P} > \bar{S}$ with a degree of rejection or acceptance as described in Section 2.

3.1.2. One sided test. If the alternative is (one sided test from the left)

$$H_1 : \mu < \mu_0,$$

then according to (1.1) the crisp p -value is

$$p = \Phi(z_0)$$

According to (2.1) the α -cuts of the p -value of this test are

$$\begin{aligned} \bar{P}[\alpha] &= [\Pr(Z \leq z_0 - z_{h(\alpha)}), \Pr(Z \leq z_0 + z_{h(\alpha)})] \\ &= [\Phi(z_0 - z_{h(\alpha)}), \Phi(z_0 + z_{h(\alpha)})] \end{aligned} \quad (3.8)$$

If the alternative is (one sided test from the right)

$$H_1 : \mu > \mu_0,$$

then according to (1.1) the crisp p -value is

$$p = 1 - \Phi(z_0)$$

According to (2.2) the α -cuts of the p -value of this test are

$$\begin{aligned} \bar{P}[\alpha] &= [\Pr(Z \geq z_0 + z_{h(\alpha)}), \Pr(Z \geq z_0 - z_{h(\alpha)})] \\ &= [1 - \Phi(z_0 + z_{h(\alpha)}), 1 - \Phi(z_0 - z_{h(\alpha)})] \end{aligned} \quad (3.9)$$

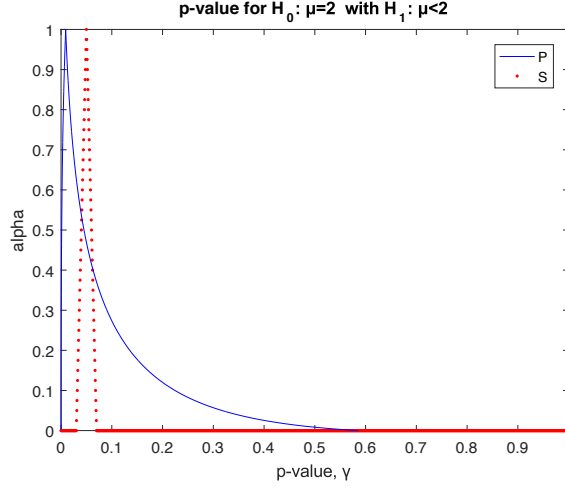


FIGURE 3. Fuzzy p -value \bar{P} and significance level \bar{S} for the one sided test of $H_0 : \mu = 2$ with $H_1 : \mu < 2$ of Example 1

Example 1. a) We test the null hypothesis $H_0 : \mu = 2$ at significance level $\gamma = 0.05$ with alternative $H_1 : \mu \neq 2$ (two sided test) for the mean μ of a random variable X that follows normal distribution with variance $\sigma^2 = 4$ using a random sample of $n = 25$ observations with sample mean $\bar{x} = 1,056$.
b) We test H_0 at significance level $\gamma = 0.05$ with alternative ($H_1 : \mu < 2$ one sided test from left) using the same sample.

a) In the crisp test the sample value of the statistic (3.1) is given by (3.3),

$$z_0 = \frac{1,056 - 2}{2/\sqrt{25}} = -2.36$$

so the crisp p -value of the two sided test is given by (3.2),

$$p = 2\Phi(-2.36) = 0.018 < \gamma = 0.05$$

Therefore H_0 is rejected by this crisp test.

For $\gamma = 0.05$, (3.5) and (3.6) give

$$z_{h(\alpha)} = \Phi^{-1} \left(1 - \left[\left(\frac{1}{2} - \frac{0.05}{2} \right) \alpha + \frac{0.05}{2} \right] \right) = 0.98 - 0.48a$$

so according to (3.7), the α -cuts of the fuzzy p -value are ($z_0 = -2.36$)

$$\begin{aligned} \bar{P}[\alpha] &= [2\Phi(-2.36 - (0.98 - 0.48a)), \min\{1, 2\Phi(-2.36 + (0.98 - 0.48a))\}] \\ &= [2(\Phi(-3.34 + 0.48a)), \min\{1, 2\Phi(-1.38 - 0.48a)\}]. \end{aligned} \quad (3.10)$$

According to (2.4b) the significance level $\gamma = 0.05$ can be fuzzified to a triangular fuzzy number as

$$\bar{S}(a) = T(\gamma - 0.02, \gamma, \gamma + 0.02) = T(0.03, 0.05, 0.07) \quad (3.11)$$

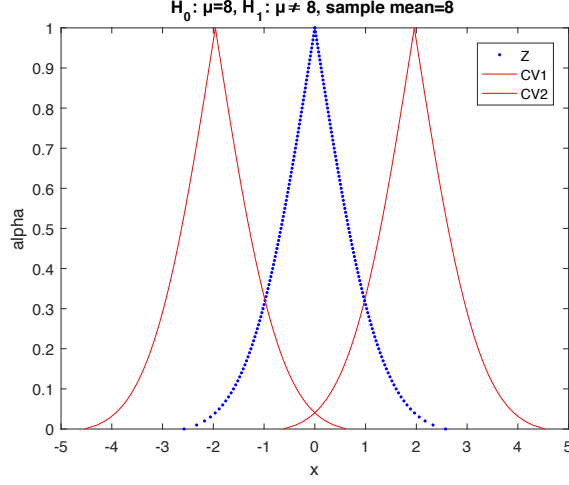


FIGURE 4. Fuzzy statistic \bar{Z} and critical values \overline{CV}_i for the two sided test of Example 2

So implementing (3.10) and (3.11), we get the results of Figure 2, where the core of the fuzzy number \bar{P} is at the left of the core of \bar{S} and their point of intersection has $y_0 = 0.72$, so the truth value of $\bar{P} < \bar{S}$ is $1 - 0.72 = 0.28$. Therefore, H_0 is rejected the two sided test with degree of rejection 0.28.

b) The crisp p -value of the one sided test from the left is

$$p = \Phi(-2.36) = 0.009 < \gamma = 0.05,$$

so H_0 is rejected by this test.

Implementing (3.8) and (3.11), we get the results of Figure 3, where the core of the fuzzy number \bar{P} is at the left of \bar{S} and their point of intersection has $y_0 = 0.51$, so the truth value of $\bar{P} < \bar{S}$ is $1 - 0.51 = 0.49$. Therefore, H_0 is rejected by the one sided test from the left with degree of rejection 0.49.

Example 2. We test at significance level 0.05 the null hypothesis $H_0 : \mu = 8$ with alternative $H_1 : \mu \neq 8$ (two sided test) for the mean μ of a random variable X that follows normal distribution with variance $\sigma^2 = 4$ from a sample of $n = 25$ observations with sample mean $\bar{x} = 8$ with two sided fuzzy test which uses:

- fuzzy critical values, developed in [2] and [12],
- fuzzy p -value.

The value of the test statistic Z is

$$z_0 = \frac{8 - 8}{2/\sqrt{25}} = 0$$

Since

$$z_0 = 0 < z_{0.05/2} = 1.96$$

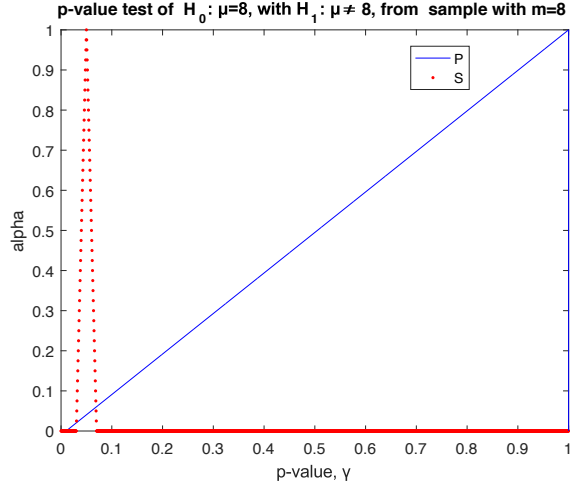


FIGURE 5. Fuzzy p -value \bar{P} and significance level \bar{S} for the two sided test of Example 2

H_0 is not rejected by this crisp test.

Also, the crisp p -value of the test is given by (3.2),

$$p = 2\Phi(0) = 2 \cdot 0.5 = 1 > \gamma = 0.05$$

In this case the crisp test gives no rejection of H_0 with the largest possible difference between the test statistic and the critical values since the value of the test statistic is exactly in the middle of the no rejection region (the crisp p -value has its maximum possible value, one), so it is the best case for acceptance of H_0 .

a) Applying the two sided fuzzy test of [12], which uses the fuzzy critical values of [2], we obtain the results of Figure 4, where the point of intersection of the fuzzy numbers \bar{Z} and \overline{CV}_1 is $y_1 = 0.31$ and of \bar{Z} and \overline{CV}_2 is $y_2 = 0.32$, so

$$\min(v(\bar{Z} > \overline{CV}_1), v(\bar{Z} < \overline{CV}_2)) = \min(1 - y_1, 1 - y_2) = 0.68$$

Therefore, H_0 is accepted by this test with degree of acceptance $d = 0.68$.

b) Implementing (3.7) and (3.11) for the two sided test of H_0 , we get the results of Figure 5, where the core of the fuzzy number \bar{P} is at the right of \bar{S} and their point of intersection has $y_0 = 0.05$, so the truth value of $\bar{P} > \bar{S}$ is $1 - 0.05 = 0.95$. Therefore, H_0 is accepted by this test with degree of acceptance 0.95.

3.2. Fuzzy p -value of hypotheses tests for the mean of a random variable from large sample. We test at significance level γ the null hypothesis that the mean μ of a random variable X is equal to μ_0

$$H_0 : \mu = \mu_0$$

using a random sample of observations of X of large size n ($n > 30$).

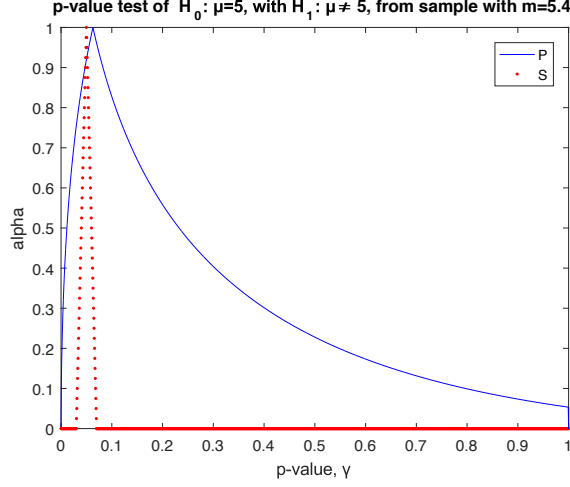


FIGURE 6. Fuzzy p-value \bar{P} and significance level \bar{S} for the two sided test of $H_0 : \mu = 5$ of Example 3 from a sample with $\bar{x} = 5.4$

In the crisp case we test H_0 using the statistic

$$Z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \quad (3.12)$$

where \bar{X} the statistic of the sample mean and s the sample standard deviation. It is known [6] that under the null hypothesis ($\mu = \mu_0$) Z follows the standard normal distribution $N(0,1)$ according to the central limit theorem, so H_0 is rejected if $p \leq \gamma$, where the crisp p -value of the test is found as in Section 3.1 for the sample value of the statistic (3.12),

$$z_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \quad (3.13)$$

If $p > \gamma$, then H_0 is not rejected.

In the fuzzy case, for the test of H_0 we use the fuzzy statistic

$$\bar{Z} = \frac{\bar{\mu} - \mu_0}{s/\sqrt{n}} \quad (3.14)$$

which is generated by substituting \bar{X} in (3.12) with the non-asymptotic fuzzy estimator of the mean of [14], the α -cuts of which for the given sample are

$$\bar{\mu}[\alpha] = \left[\bar{x} - z_{h(\alpha)} \frac{s}{\sqrt{n}}, \bar{x} + z_{h(\alpha)} \frac{s}{\sqrt{n}} \right], \quad \alpha \in [0, 1]$$

So, the α -cuts of the fuzzy p -value are given by (3.7) for the two sided test and by (3.8) or (3.9) for the one sided tests, for the value z_0 of (3.13).

Example 3. We test the null hypothesis $H_0 : \mu = 5$ at significance level $\gamma = 0.05$ with alternative the $H_1 : \mu \neq 5$ (two sided test) for the mean μ of a random variable X for two random large samples of $n = 100$ observations with sample

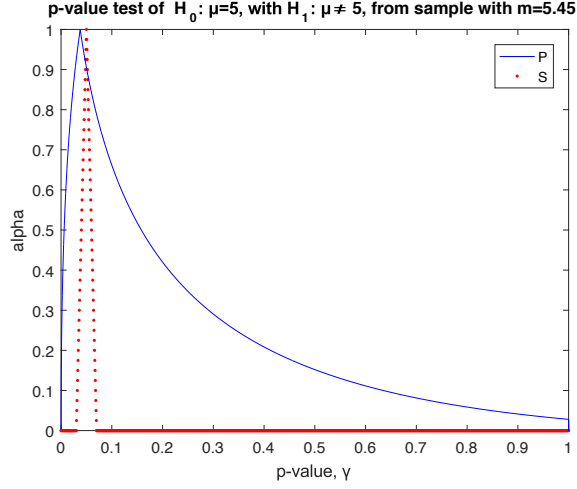


FIGURE 7. Fuzzy p-value \bar{P} and significance level \bar{S} for the two sided test of $H_0 : \mu = 5.45$ of Example 3 from a sample with $\bar{x} = 5.45$

means and variances:

$$\text{a) } \bar{x} = 5.40 \text{ and } s^2 = 4.62, \quad \text{b) } \bar{x}_2 = 5.45 \text{ and } s^2 = 4.69$$

We evaluate the value of the statistic (3.12) for the first sample from (3.13)

$$z_0 = \frac{5.4 - 5}{\sqrt{4.62}/\sqrt{100}} = 1.86,$$

so the crisp p -value of the test is found by (3.2) to be

$$p = 2(1 - \Phi(1.86)) = 0.063 > \gamma = 0.05$$

Therefore H_0 is not rejected by this test.

For the second sample, which has sample mean and variance $\bar{x}_2 = 5.45$ and $s^2 = 4.69$ (very close to these of the first sample), (3.13) gives

$$z_0 = \frac{5.45 - 5}{\sqrt{4.69}/\sqrt{100}} = 2.08,$$

so the crisp p -value of the test is found by (3.2) to be

$$p = 2(1 - \Phi(2.08)) = 0.038 < \gamma = 0.05$$

Therefore H_0 is rejected by this test.

a) Implementing (3.7) for the first sample fuzzifying the significance level to a triangular fuzzy number as in (3.11), we get the results of Figure 6, where the point of intersection of the fuzzy numbers \bar{P} and \bar{S} is $y_0 = 0.93$. This according to (1.5) means that $v(\bar{P} \approx \bar{S}) = 0.93$, so according to Section 2, we cannot make a decision on rejecting or not H_0 by this test with any degree of confidence $d > 0.07$.

b) For the second sample ($\bar{x} = 5.45$), we plot the fuzzy numbers \bar{P} and \bar{S} in Figure 7, where their point of intersection has $y_0 = 0.94$, so $v(\bar{P} \approx \bar{S}) = 0.91$.

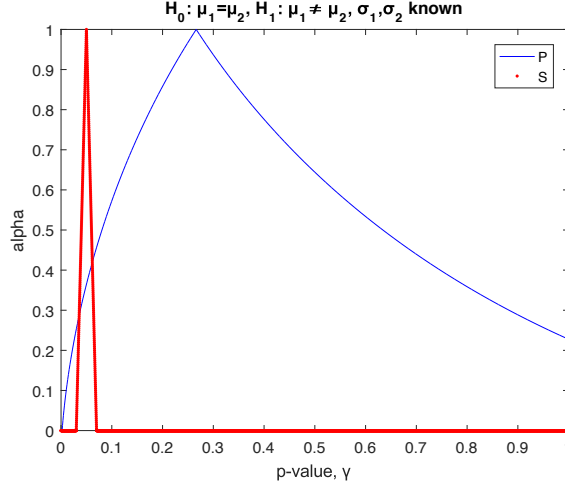


FIGURE 8. Fuzzy p-value \bar{P} and significance level \bar{S} of the two sided test of $H_0 : \mu_1 = \mu_2$ of Example 4

Therefore, we cannot make a decision on rejecting or not H_0 from this test with any degree of confidence $d > 0.09$.

3.3. Fuzzy p-value of the tests of $H_0 : \mu_1 = \mu_2$, for the means μ_1, μ_2 of two normal random variables with known variances. We test at significance level γ the null hypothesis that the means μ_1 and μ_2 of two random variables X_1 and X_2 that follow normal distributions with variances σ_1^2 and σ_2^2 are equal,

$$H_0 : \mu_1 = \mu_2$$

using two independent random samples of observations of X_1 and X_2 of sizes n_1 and n_2 with sample means and variances \bar{x}_1, \bar{x}_2 and s_1^2, s_2^2 .

It is known [6] that under the null hypothesis (equal means) the test statistic

$$Z = \frac{\bar{X}_1 - \bar{X}_2}{\sigma_0} \quad \text{where} \quad \sigma_0 = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \quad (3.15)$$

(\bar{X}_1, \bar{X}_2 the statistics of the sample means) follows standard normal distribution $N(0, 1)$, so H_0 is rejected from a random sample of observations if $p \leq \gamma$, where the crisp p -value of the test is found as in Section 3.1 for the value of the statistic (3.15) for the given sample,

$$z_0 = \frac{\bar{x}_1 - \bar{x}_2}{\sigma_0} \quad (3.16)$$

If $p > \gamma$, then H_0 is not rejected.

In the fuzzy test of H_0 , we use the fuzzy statistic

$$\bar{Z} = \frac{\bar{\mu}_{12}}{\sigma_0} \quad (3.17)$$

that is generated by substituting $\bar{X}_1 - \bar{X}_2$ in (3.15) with the non-asymptotic fuzzy estimator $\bar{\mu}_{12}$ of the difference of the sample means [2].

The α -cuts of $\bar{\mu}_{12}$ for the given sample are found as in Section 3.1 to be

$$\bar{Z}[\alpha] = [z_0 - z_{h(\alpha)}, z_0 + z_{h(\alpha)}], \quad \alpha \in (0, 1]$$

So, the α -cuts of the fuzzy p-value are given by (3.7) for the two sided test and by (3.8) or (3.9) for the one sided tests for the value z_0 of (3.16).

Example 4. We test the null hypothesis $H_0 : \mu_1 = \mu_2$ at significance level $\gamma = 0.05$ with alternative the $H_1 : \mu_1 \neq \mu_2$ (two sided test) for the means μ_1 and μ_2 of the variables X_1 and X_2 that follow normal distributions with variances $\sigma_1^2 = 5.2$ and $\sigma_2^2 = 3.7$ from two independent random samples of observations of sizes $n_1 = 15$ and $n_2 = 8$ and sample means $\bar{x}_1 = 50.2$ and $\bar{x}_2 = 51.1$ with two sided fuzzy test which uses:

- a) fuzzy p -value,
- b) fuzzy critical values, developed in [2] and [12].

According to (3.16), the value of the test statistic (3.15) is

$$z_0 = \frac{50.2 - 51.1}{\sqrt{\frac{5.2}{15} + \frac{3.7}{8}}} = -1$$

so the crisp p-value of the test is given by (3.2)

$$p = 2\Phi(-1) = 0.387 > \gamma = 0.05$$

Therefore H_0 is not rejected by the crisp test.

a) Implementing (3.7) for $z_0 = -1$ and fuzzifying the significance level to a triangular fuzzy number as in (3.11), we get the results of Figure 8, where the core of \bar{P} is at the right of the core of \bar{S} and their point of intersection has $y_0 = 0.43$, so $v(\bar{P} > \bar{S}) = 1 - 0.43 = 0.57$. Therefore H_0 is accepted by this test with degree of acceptance 0.57.

b) Applying the fuzzy test which uses fuzzy critical values developed in [2] and [12], we get the results of Figure 9, where the core of \bar{Z}_0 is between the cores of \bar{CV}_1 and \bar{CV}_2 and the point of intersection of \bar{Z}_0 and \bar{CV}_1 has truth value $y_1 = 0.63$ and of \bar{Z}_0 and \bar{CV}_2 , $y_2 = 0.12$, so that

$$\min(v(\bar{Z}_0 > \bar{CV}_1), v(\bar{Z}_0 < \bar{CV}_2)) = \min(1 - y_1, 1 - y_2) = 0.37$$

Therefore, H_0 is accepted by this test with degree of acceptance $d = 0.37$.

4. p-value of the Test for the Variance of a Normal Random Variable

In this section we present the hypotheses testing of the variance of a normal random variable and in the next section of the variances of two normal random variables using fuzzy p -value produced by fuzzy statistics [2] constructed by the non-asymptotic fuzzy estimator of the variance given in [14].

We test at significance level γ the null hypothesis that the variance σ^2 of a normal random variable X is equal to σ_0^2 ,

$$H_0 : \sigma^2 = \sigma_0^2$$

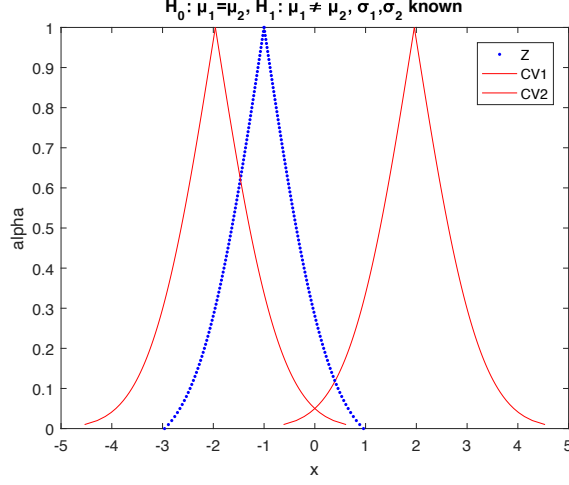


FIGURE 9. The fuzzy test statistic Z and critical values for the test of Example 4

using a random sample of n observations of X with sample variance s^2 . In the crisp case we test H_0 using the test statistic [6]

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$$

where S^2 is the statistic of the sample variance.

It is known that under the null hypothesis ($\sigma^2 = \sigma_0$) the statistic χ^2 follows χ_{n-1}^2 distribution with $n-1$ degrees of freedom [6], so from a sample of observations H_0 is rejected if $p \leq \gamma$, where the crisp p -value of the test according to (1.1) is: for the one sided test from the left,

$$p = F_{n-1}(\chi_0^2)$$

for the one sided test from the right,

$$p = 1 - F_{n-1}(\chi_0^2)$$

for the two sided test,

$$p = \min \{2F_{n-1}(\chi_0^2), 2(1 - F_{n-1}(\chi_0^2))\} \quad (4.1)$$

where F_{n-1} the distribution function of the χ_{n-1}^2 distribution and

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} \quad (4.2)$$

the value of the statistic χ^2 for the given sample. If $p > \gamma$, then H_0 is not rejected.

The fuzzy statistic of this test is derived by substituting S^2 in χ^2 with a fuzzy estimator $\overline{\sigma^2}$ of the variance,

$$\overline{\chi^2} = \frac{(n-1)\overline{\sigma^2}}{\sigma_0^2} \quad (4.3)$$

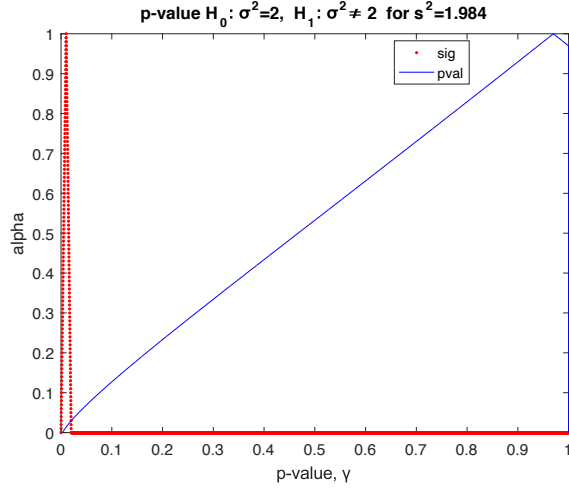


FIGURE 10. Fuzzy p-value \bar{P} and significance level \bar{S} of the two sided test of Example 5

In our approach we use the non-asymptotic fuzzy estimator, the α -cuts of which for the given sample are found in [14] to be

$$\bar{\sigma}^2[\alpha] = \left[\frac{(n-1)s^2}{\chi_{R,h(\alpha);n-1}^2}, \frac{(n-1)s^2}{\chi_{L,h(\alpha);n-1}^2} \right] \quad (4.4)$$

where

$$\chi_{L,h(\alpha);n-1}^2 = F_{n-1}^{-1}(h(\alpha)) \quad \text{and} \quad \chi_{R,h(\alpha);n-1}^2 = F_{n-1}^{-1}(1-h(\alpha)) \quad (4.5)$$

From (4.2)-(4.4) and interval arithmetics we deduce that the α -cuts of $\bar{\chi}^2$ for the given sample are

$$\bar{\chi}^2[\alpha] = \left[\frac{(n-1)\chi_0^2}{\chi_{R,h(\alpha);n-1}^2}, \frac{(n-1)\chi_0^2}{\chi_{L,h(\alpha);n-1}^2} \right] \quad (4.6)$$

The core of $\bar{\chi}^2$ is obtained by (4.6) for $\alpha = 1$ to be $(n-1)\chi_0^2/m$ (from (3.5) $h(1) = 0.5$, so $\chi_{R,h(1);n-1}^2 = \chi_{L,h(1);n-1}^2 = F_{n-1}^{-1}(0.5) = m$), where m the median of the χ_{n-1}^2 distribution, so

$$\frac{(n-1)\chi_0^2}{m} < m \Leftrightarrow \chi_0^2 < \frac{m^2}{n-1}$$

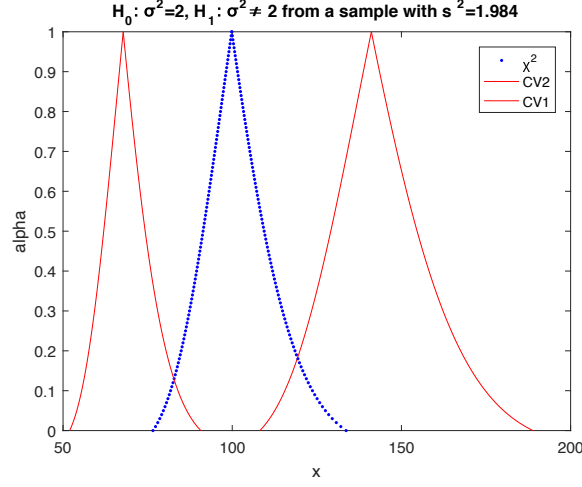


FIGURE 11. Fuzzy statistic \bar{Z} and critical values \overline{CV}_1 , \overline{CV}_2 of the two sided test of Example 5

Therefore, according to (2.3) and (4.6) the α -cuts of the fuzzy p -value of the two sided test of H_0 are

$$\bar{P}[\alpha] = \begin{cases} \left[\begin{array}{l} 2\Pr \left(\chi^2 < \frac{(n-1)\chi_0^2}{\chi_{R;h(\alpha);n-1}^2} \right), \\ \min \left\{ 1, 2\Pr \left(\chi^2 < \frac{(n-1)\chi_0^2}{\chi_{L;h(\alpha);n-1}^2} \right) \right\} \end{array} \right], & \chi_0^2 \leq \frac{m^2}{n-1} \\ \left[\begin{array}{l} 2\Pr \left(\chi^2 > \frac{(n-1)\chi_0^2}{\chi_{L;h(\alpha);n-1}^2} \right), \\ \min \left\{ 1, 2\Pr \left(\chi^2 > \frac{(n-1)\chi_0^2}{\chi_{R;h(\alpha);n-1}^2} \right) \right\} \end{array} \right], & \chi_0^2 > \frac{m^2}{n-1} \end{cases} \quad (4.7)$$

So since χ^2 follows χ_{n-1}^2 distribution with $n-1$ degrees of freedom, (4.7) gives:

$$\text{if } \chi_0^2 \leq \frac{m^2}{n-1},$$

$$\bar{P}[\alpha] = \left[2F \left(\frac{(n-1)\chi_0^2}{\chi_{R;h(\alpha);n-1}^2} \right), \min \left\{ 1, 2F \left(\frac{(n-1)\chi_0^2}{\chi_{L;h(\alpha);n-1}^2} \right) \right\} \right], \quad (4.8)$$

$$\text{if } \chi_0^2 > \frac{m^2}{n-1},$$

$$\bar{P}[\alpha] = \left[2 \left(1 - F \left(\frac{(n-1)\chi_0^2}{\chi_{L;h(\alpha);n-1}^2} \right) \right), \min \left\{ 1, 2 \left(1 - F \left(\frac{(n-1)\chi_0^2}{\chi_{R;h(\alpha);n-1}^2} \right) \right) \right\} \right] \quad (4.9)$$

where $F(x)$ the distribution function of the χ_{n-1}^2 distribution.

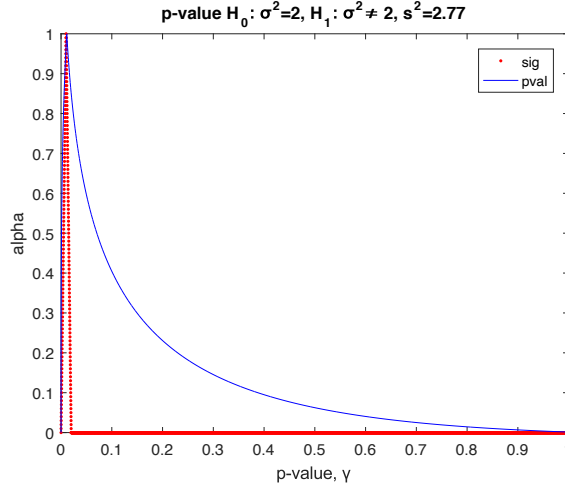


FIGURE 12. Fuzzy p-value \bar{P} and significance level \bar{S} of the two sided test of Example 6 for the first sample

According to (2.1) the α -cuts of the fuzzy p -value of the one sided test from the left are

$$\bar{P}[\alpha] = \left[F \left(\frac{(n-1)\chi_0^2}{\chi_{R;h(\alpha);n-1}^2} \right), F \left(\frac{(n-1)\chi_0^2}{\chi_{L;h(\alpha);n-1}^2} \right) \right]$$

and according to (2.2), of the one sided test from the right

$$\bar{P}[\alpha] = \left[1 - F \left(\frac{(n-1)\chi_0^2}{\chi_{L;h(\alpha);n-1}^2} \right), 1 - F \left(\frac{(n-1)\chi_0^2}{\chi_{R;h(\alpha);n-1}^2} \right) \right]$$

Using the fuzzy significance level \bar{S} given in (2.4), H_0 is rejected at significance level γ if $\bar{P} < \bar{S}$ and not rejected if $\bar{P} > \bar{S}$ as described in Section 2.

Example 5. We test the null hypothesis $H_0 : \sigma^2 = 2$ at significance level $\gamma = 0.01$ with alternative $H_1 : \sigma^2 \neq 2$ from a random sample of 101 observations with sample variance $s^2 = 1.984$ with two sided fuzzy test which uses:

- fuzzy p -value,
- fuzzy critical values, developed in [2] and [12].

The crisp value of the test statistic χ^2 is found by (4.2) to be

$$\chi_0^2 = \frac{(101-1) \cdot 1.984}{2} = 99.2$$

So, according to (4.1) the p-value of the crisp test is

$$\begin{aligned} p &= \min \{ 2F_{100}(99.2), 2(1 - F_{100}(99.2)) \} \\ &= 0.9924 > \gamma = 0.01 \end{aligned}$$

Therefore H_0 is accepted by this test.

- Implementing (4.8)-(4.9) and fuzzifying the significance level to a triangular

fuzzy number as in (2.4a) we get the results of Figure 10, from which we see that the core of the fuzzy number \bar{P} is at the right of the core of \bar{S} and their point of intersection has $y_0 = 0.02$, so $\bar{P} > \bar{S}$ with degree of confidence $1 - 0.02 = 0.98$. Hence, H_0 is accepted by this test with degree of acceptance 0.98.

b) Applying the fuzzy hypothesis test developed in [2] and [12], which uses fuzzy critical values, we get the results of Figure 11, where the core of \bar{Z}_0 is between the cores of \bar{CV}_1 and \bar{CV}_2 and the point of intersection of \bar{Z}_0 and \bar{CV}_1 has $y_1 = 0.16$ and of \bar{Z}_0 and \bar{CV}_2 , $y_2 = 0.19$. So

$$\min(v(\bar{Z}_0 > \bar{CV}_1), v(\bar{Z}_0 < \bar{CV}_2)) = \min(1 - y_1, 1 - y_2) = 0.81$$

Therefore, H_0 is accepted by this test with degree of acceptance $d = 0.81$.

Example 6. We test the null hypothesis $H_0 : \sigma^2 = 2$ at significance level $\gamma = 0.01$ with alternative the $H_1 : \sigma^2 \neq 2$ (two sided test) using a random sample of 101 observations with sample variance: a) $s^2 = 2.77$, b) $s^2 = 2.83$

For the first sample (variance $s^2 = 2.77$) the sample value of the test statistic χ^2 is found by (4.2) to be

$$\chi_0^2 = \frac{(101 - 1) \cdot 2.77}{2} = 138.5,$$

so according to (4.1), the p -value of the crisp test is

$$\begin{aligned} p &= \min\{2F_{100}(138.5), 2(1 - F_{100}(138.5))\} \\ &= \min\{1.987, 0.013\} = 0.013 > \gamma = 0.01 \end{aligned}$$

Therefore, H_0 is accepted by this crisp test.

For the second sample (variance $s^2 = 2.83$) (4.2) gives

$$\chi_0^2 = \frac{(101 - 1) \cdot 2.83}{2} = 141.5,$$

so according to (4.1), the p -value of the crisp test is

$$\begin{aligned} p &= \min\{2F_{100}(141.5), 2(1 - F_{100}(141.5))\} \\ &= \min\{1.992, 0.008\} = 0.008 < \gamma = 0.01 \end{aligned}$$

Therefore, H_0 is rejected by this crisp test.

a) Implementing (4.8)-(4.9) and fuzzifying the significance level to a triangular fuzzy number as in (2.4a), we get the results of Figure 12, where the point of intersection of \bar{P} and \bar{S} has $y_0 = 0.99$, so $v(\bar{P} \approx \bar{S}) = 0.99$. Hence according to Section 2, we cannot make a decision on rejecting or not H_0 from this test with any degree of confidence $d \geq 0.01$.

b) For the second sample ($\bar{x} = 5.45$) we plot the fuzzy numbers of \bar{P} and \bar{S} in Figure 13, where their point of intersection has $y_0 = 0.96$, so $v(\bar{P} \approx \bar{S}) = 0.96$. Therefore, we cannot make a decision on rejecting H_0 or not from this test with any degree of confidence $d \geq 0.04$.

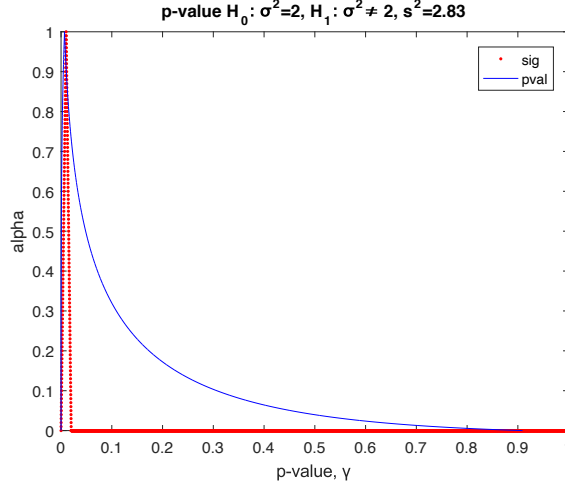


FIGURE 13. Fuzzy p-value \bar{P} and significance level \bar{S} of the two sided test of Example 6 for the second sample

5. p-value of the Tests of $H_0 : \sigma_1^2 = \sigma_2^2$, for the Variances σ_1^2, σ_2^2 of two Normal Distributions

We test at significance level γ the null hypothesis that the variances σ_1^2 and σ_2^2 of two normal random variables X_1 and X_2 are equal,

$$H_0 : \sigma_1^2 = \sigma_2^2,$$

using two independent random samples of observations of X_1 and X_2 of sizes n_1, n_2 and variances s_1^2, s_2^2 .

It is known [6] that

$$\chi_1^2 = \frac{(n_1 - 1)S_1^2}{\sigma_1^2} \sim \chi_{n_1 - 1}^2 \quad \text{and} \quad \chi_2^2 = \frac{(n_2 - 1)S_2^2}{\sigma_2^2} \sim \chi_{n_2 - 1}^2$$

where S_1^2, S_2^2 the statistics of the sample variances. So, in the crisp case under the null hypothesis (equal variances) the test statistic

$$F = \frac{\frac{\chi_1^2}{n_1 - 1}}{\frac{\chi_2^2}{n_2 - 1}} = \frac{S_1^2}{S_2^2} \quad (5.1)$$

follows a $F_{n_1 - 1, n_2 - 1}$ distribution with $n_1 - 1$ degrees of freedom for the numerator and $n_2 - 1$ degrees of freedom for the denominator [6]. Therefore, H_0 is rejected if $p \leq \gamma$, where according to (1.1) the crisp p -value of the test is: for the one sided test from the left,

$$p = F_{n_1 - 1, n_2 - 1} \left(\frac{s_1^2}{s_2^2} \right)$$

for the one sided test from the right,

$$p = 1 - F_{n_1-1, n_2-1} \left(\frac{s_1^2}{s_2^2} \right)$$

for the two sided test,

$$p = \min \left\{ 2F_{n_1-1, n_2-1} \left(\frac{s_1^2}{s_2^2} \right), 2 \left(1 - F_{n_1-1, n_2-1} \left(\frac{s_1^2}{s_2^2} \right) \right) \right\}$$

where $F_{n_1-1, n_2-1}(x)$ the distribution function of the F_{n_1-1, n_2-1} distribution. If $p > \gamma$, then H_0 is not rejected.

In the fuzzy test of H_0 we use the statistic

$$\bar{F} = \frac{\bar{\sigma}_1^2}{\sigma_2^2}$$

created by the fuzzification of F . Since \bar{F} is the ratio of the fuzzy estimators $\bar{\sigma}_1^2$ and $\bar{\sigma}_2^2$, its α -cuts are found using interval arithmetics for their α -cuts, which in our approach are given by (4.4). So, the α -cuts of the fuzzy statistic \bar{F} for the given sample are found to be

$$\begin{aligned} \bar{F}[\alpha] &= \left[\frac{(n_1-1)s_1^2}{\chi_{R;h(a);n_1-1}^2}, \frac{(n_1-1)s_1^2}{\chi_{L;h(a);n_1-1}^2} \right] \\ &= \left[\frac{(n_1-1)s_1^2}{\chi_{L;h(a);n_2-1}^2}, \frac{(n_1-1)s_1^2}{\chi_{R;h(a);n_2-1}^2} \right] \end{aligned} \quad (5.2)$$

where $\chi_{L;h(a);n_1-1}^2$ and $\chi_{R;h(a);n_2-1}^2$ are defined by (4.5).

The core of \bar{F} is obtained by (5.2) for $\alpha = 1$ to be

$$\frac{(n_1-1)s_1^2}{(n_2-1)s_2^2} \frac{\chi_{L;h(1);n_2-1}^2}{\chi_{R;h(1);n_1-1}^2} = \frac{(n_1-1)s_1^2 m_{n_2-1}}{(n_2-1)s_2^2 m_{n_1-1}}$$

since from (3.5) $h(1) = 0.5$, so

$$\frac{\chi_{L;h(1);n_2-1}^2}{\chi_{R;h(1);n_1-1}^2} = \frac{\chi_{L;0.5;n_2-1}^2}{\chi_{R;0.5;n_1-1}^2} = \frac{F_{n_2-1}^{-1}(0.5)}{F_{n_1-1}^{-1}(0.5)} = \frac{m_{n_2-1}}{m_{n_1-1}}$$

where m_{n_1-1}, m_{n_2-1} the medians of the $\chi_{n_1-1}^2$ and $\chi_{n_2-1}^2$ distributions and $F_{n_1-1}^{-1}, F_{n_2-1}^{-1}$ their inverse distribution function. Hence, if $m = F_{n_1-1, n_2-1}^{-1}(0.5)$ the median of the F_{n_1-1, n_2-1} distribution (F_{n_1-1, n_2-1}^{-1} the inverse distribution function of the F_{n_1-1, n_2-1} distribution)

$$\frac{(n_1-1)s_1^2 m_{n_2-1}}{(n_2-1)s_2^2 m_{n_1-1}} < m \Leftrightarrow \frac{s_1^2}{s_2^2} < \frac{(n_2-1)m_{n_1-1}}{(n_1-1)m_{n_2-1}} m$$

Therefore according to (2.3), the α -cuts of the p-value of the two sided test of H_0 are:

$$\begin{aligned}
& \text{if } \frac{s_1^2}{s_2^2} \leq \frac{(n_2 - 1)m_{n_1-1}}{(n_1 - 1)m_{n_2-1}} m, \\
& \bar{P}[\alpha] = \left[2\Pr \left(F \leq \frac{(n_1 - 1)s_1^2 \chi_{L;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{R;h(a);n_1-1}^2} \right), \right. \\
& \quad \left. \min \left\{ 1, 2\Pr \left(F \leq \frac{(n_1 - 1)s_1^2 \chi_{R;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{L;h(a);n_1-1}^2} \right) \right\} \right], \tag{5.3}
\end{aligned}$$

$$\begin{aligned}
& \text{if } \frac{s_1^2}{s_2^2} > \frac{(n_2 - 1)m_{n_1-1}}{(n_1 - 1)m_{n_2-1}} m, \\
& \bar{P}[\alpha] = \left[2\Pr \left(F \geq \frac{(n_1 - 1)s_1^2 \chi_{R;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{L;h(a);n_1-1}^2} \right), \right. \\
& \quad \left. \min \left\{ 1, 2\Pr \left(F \geq \frac{(n_1 - 1)s_1^2 \chi_{L;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{R;h(a);n_1-1}^2} \right) \right\} \right] \tag{5.4}
\end{aligned}$$

Since F follows the F_{n_1-1, n_2-1} distribution, (5.3) and (5.4) give:

$$\begin{aligned}
& \text{if } \frac{s_1^2}{s_2^2} \leq \frac{(n_2 - 1)m_{n_1-1}}{(n_1 - 1)m_{n_2-1}} m, \\
& \bar{P}[\alpha] = \left[2F_{n_1-1, n_2-1} \left(\frac{(n_1 - 1)s_1^2 \chi_{L;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{R;h(a);n_1-1}^2} \right), \right. \\
& \quad \left. \min \left\{ 1, 2F_{n_1-1, n_2-1} \left(\frac{(n_1 - 1)s_1^2 \chi_{R;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{L;h(a);n_1-1}^2} \right) \right\} \right], \tag{5.5}
\end{aligned}$$

$$\begin{aligned}
& \text{if } \frac{s_1^2}{s_2^2} > \frac{(n_2 - 1)m_{n_1-1}}{(n_1 - 1)m_{n_2-1}} m, \\
& \bar{P}[\alpha] = \left[2 \left(1 - F_{n_1-1, n_2-1} \left(\frac{(n_1 - 1)s_1^2 \chi_{R;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{L;h(a);n_1-1}^2} \right) \right), \right. \\
& \quad \left. \min \left\{ 1, 2 \left(1 - F_{n_1-1, n_2-1} \left(\frac{(n_1 - 1)s_1^2 \chi_{L;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{R;h(a);n_1-1}^2} \right) \right) \right\} \right] \tag{5.6}
\end{aligned}$$

According to (2.1), the α -cuts of the fuzzy p -value of the one sided test from the left are

$$\begin{aligned}
& \bar{P}[\alpha] = \left[F_{n_1-1, n_2-1} \left(\frac{(n_1 - 1)s_1^2 \chi_{L;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{R;h(a);n_1-1}^2} \right), \right. \\
& \quad \left. F_{n_1-1, n_2-1} \left(\frac{(n_1 - 1)s_1^2 \chi_{R;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{L;h(a);n_1-1}^2} \right) \right]
\end{aligned}$$

and according to (2.2), of the one sided test from the right

$$\bar{P}[\alpha] = \left[1 - F_{n_1-1, n_2-1} \left(\frac{(n_1 - 1)s_1^2 \chi_{R;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{L;h(a);n_1-1}^2} \right), \right.$$

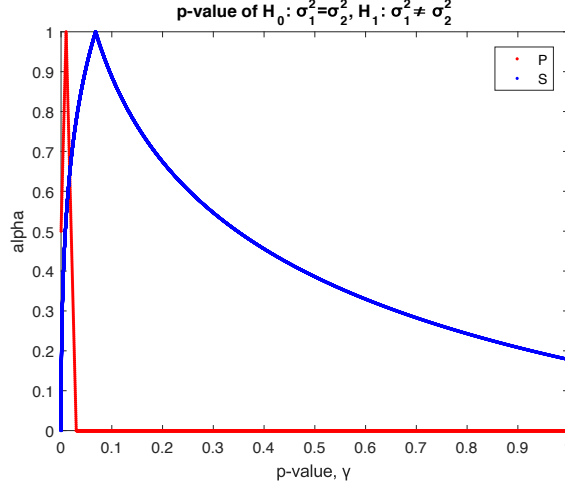


FIGURE 14. The fuzzy p-value \bar{P} and significance level \bar{S} of the two sided test of Example 7

$$1 - F_{n_1-1, n_2-1} \left(\frac{(n_1 - 1)s_1^2 \chi_{L;h(a);n_2-1}^2}{(n_2 - 1)s_2^2 \chi_{R;h(a);n_1-1}^2} \right)$$

Example 7. We test at significance level 0.01 the null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ with alternative $H_1 : \sigma_1^2 \neq \sigma_2^2$ (two sided test) using two random samples of $n_1 = 11$ and $n_2 = 13$ observations with sample variances $s_1^2 = 0.24$ and $s_2^2 = 0.8$ with two sided fuzzy test which uses:

- i) fuzzy p -value,
- ii) fuzzy critical values, developed in [2] and [12].

a) The sample value of the test statistic (5.1) is

$$F = \frac{s_1^2}{s_2^2} = \frac{0.24}{0.8} = 0.3$$

So, the p -value of the two sided crisp test is $(F_{10,13}(x)$ the distribution function of the $F_{10,13}$ distribution that follows the statistic F).

$$\begin{aligned} p &= \min \{2F_{10,13}(0.3), 2(1 - F_{10,13}(0.3))\} \\ &= \min \{0.064, 1.872\} = 0.064 > \gamma = 0.05 \end{aligned}$$

so H_0 is not rejected by this test.

Implementing (5.5)-(5.6) and fuzzifying the significance level to a triangular fuzzy number as in (2.4a), we get the results of Figure 14 where the core of \bar{P} is at the right of \bar{S} and their point of intersection has $y_0 = 0.6$. This according to (1.5) means that $v(\bar{P} > \bar{S}) = 1 - 0.6 = 0.4$. So H_0 is accepted by this test with degree of acceptance $d = 0.4$.

b) Applying the fuzzy test developed in [2] and [12], which uses fuzzy critical values, we get the results of Figure 15, where the core of \bar{Z}_0 is between the cores

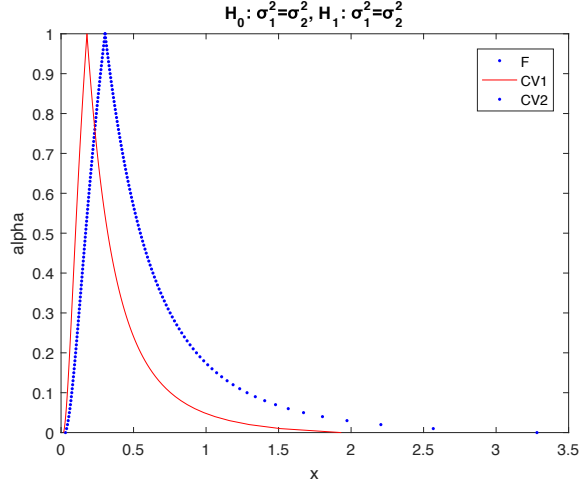


FIGURE 15. The fuzzy test statistic \bar{Z} and critical values of the two sided test of Example 7

of \overline{CV}_1 and \overline{CV}_2 and the point of intersection of \bar{Z}_0 and \overline{CV}_1 has $y_1 = 0.78$ (the point of intersection of \bar{Z}_0 and \overline{CV}_2 , not shown, has $y_2 = 0.08$), so that

$$\min(v(\bar{Z}_0 > \overline{CV}_1), v(\bar{Z}_0 < \overline{CV}_2)) = \min(1 - y_1, 1 - y_2) = 0.22$$

Therefore as described in Section 2, H_0 is accepted by this test with degree of acceptance $d = 0.22$.

6. Hypotheses Tests for the Mean of a Normal Variable With Unknown Variance

We test at significance level γ the null hypothesis that the mean μ of a random variable X which follows normal distribution with unknown variance is equal to μ_0

$$H_0 : \mu = \mu_0$$

using a random sample of n observations of X with sample mean and variance \bar{x} and s^2 .

In the crisp case we test H_0 using the statistic

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \quad (6.1)$$

where \bar{X} and S are the statistics of the sample mean and standard deviation. It is known [6] that under the null hypothesis ($\mu = \mu_0$) T follows t_{n-1} distribution with $n - 1$ degrees of freedom. Therefore, H_0 is rejected if $p \leq \gamma$, where according to (1.1) the crisp p -value of the test is:
for the one sided test from the left,

$$p = F_{n-1}(t_0)$$

for the one sided test from the right,

$$p = 1 - F_{n-1}(t_0)$$

for the two sided test,

$$p = \begin{cases} 2F_{n-1}(t_0), & t_0 \leq 0 \\ 2(1 - F_{n-1}(t_0)), & t_0 > 0 \end{cases} \quad (6.2)$$

where $F_{n-1}(x)$ the distribution function of the t_{n-1} distribution and

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \quad (6.3)$$

While if $p > \gamma$, then H_0 is not rejected.

In the fuzzy case, for the test of H_0 we use the fuzzy statistic

$$\bar{T} = \frac{\bar{\mu} - \mu_0}{\bar{\sigma}/\sqrt{n}} \quad (6.4)$$

which is generated by by substituting \bar{X} and S in (6.1) with the non-asymptotic fuzzy estimators $\bar{\mu}$ and $\bar{\sigma}$ of the mean and standard deviation. As described in [2] the α -cuts of $\bar{\mu}$ for the given sample in our approach are [14]

$$\bar{\mu}[\alpha] = \left[\bar{x} - t_{h(\alpha);n-1} \frac{s}{\sqrt{n}}, \bar{x} + t_{h(\alpha);n-1} \frac{s}{\sqrt{n}} \right], \quad \alpha \in [0, 1] \quad (6.5)$$

where $h(a)$ is given by (3.6) and

$$t_{h(\alpha);n-1} = F_{n-1}^{-1}(1 - h(\alpha)) \quad (6.6)$$

According to (4.4) the α -cuts of the non-asymptotic fuzzy estimator of the standard deviation for the given sample are

$$\bar{\sigma}[\alpha] = \left[s \sqrt{\frac{(n-1)}{\chi_{R;h(\alpha);n-1}^2}}, s \sqrt{\frac{(n-1)}{\chi_{L;1-h(\alpha);n-1}^2}} \right], \quad \alpha \in [0, 1] \quad (6.7)$$

where $\chi_{R;h(\alpha);n-1}^2$ and $\chi_{L;h(\alpha);n-1}^2$ are defined in (4.5).

Using fuzzy number arithmetics from (6.4)-(6.7) follows that the α -cuts of the fuzzy statistic \bar{T} for the given sample are

$$\begin{aligned} \bar{T}[\alpha] &= \left[\frac{\bar{x} - \mu_0 - t_{h(\alpha);n-1} \frac{s}{\sqrt{n}}}{\frac{1}{\sqrt{n}} s \sqrt{\frac{(n-1)}{\chi_{L;h(\alpha);n-1}^2}}}, \frac{\bar{x} - \mu_0 + t_{h(\alpha);n-1} \frac{s}{\sqrt{n}}}{\frac{1}{\sqrt{n}} s \sqrt{\frac{(n-1)}{\chi_{R;h(\alpha);n-1}^2}}} \right] \\ &= \left[\sqrt{\frac{\chi_{L;h(\alpha);n-1}^2}{n-1}} (t_0 - t_{h(\alpha)}), \sqrt{\frac{\chi_{R;h(\alpha);n-1}^2}{n-1}} (t_0 + t_{h(\alpha)}) \right] \end{aligned}$$

where $t_{h(\alpha);n-1}$ is given by (6.6).

The median of the t distribution is equal to its mean, which is zero and the core of the $\bar{T}[\alpha]$ is obtained for $\alpha = 1$ to be

$$\sqrt{\frac{\chi_{L;0.5;n-1}^2}{n-1}} (t_0 - t_{h(1);n-1}) = \sqrt{\frac{m}{n-1}} t_0$$

where m the median of the χ_{n-1}^2 distribution, since according to (3.5) $h(1) = 0.5$, so $t_{h(1);n-1} = t_{0.5;n-1} = F_{1-0.5;n-1}^{-1} = 0$. Therefore, since the statistic T follows t_{n-1} distribution, according to (2.3) the α -cuts of the p-value of the two sided test of H_0 are:

if $\sqrt{\frac{m}{n-1}}t_0 \leq 0 \Leftrightarrow t_0 \leq 0$,

$$\begin{aligned} \bar{P}[\alpha] = & \left[2\Pr \left(T \leq \sqrt{\frac{\chi_{L;h(\alpha);n-1}^2}{n-1}}(t_0 - t_{h(\alpha);n-1}) \right), \right. \\ & \left. \min \left\{ 1, 2\Pr \left(T \leq \sqrt{\frac{\chi_{R;h(\alpha);n-1}^2}{n-1}}(t_0 + t_{h(\alpha);n-1}) \right) \right\} \right], \end{aligned} \quad (6.8)$$

if $t_0 > 0$,

$$\begin{aligned} \bar{P}[\alpha] = & \left[2\Pr \left(T \geq \sqrt{\frac{\chi_{R;h(\alpha);n-1}^2}{n-1}}(t_0 + t_{h(\alpha);n-1}) \right), \right. \\ & \left. \min \left\{ 1, 2\Pr \left(T \geq \sqrt{\frac{\chi_{L;h(\alpha);n-1}^2}{n-1}}(t_0 - t_{h(\alpha);n-1}) \right) \right\} \right] \end{aligned} \quad (6.9)$$

Since T follows t_{n-1} distribution, (6.8) and (6.9) give:

if $t_0 \leq 0$,

$$\begin{aligned} \bar{P}[\alpha] = & \left[2F_{n-1} \left(\sqrt{\frac{\chi_{L;h(\alpha);n-1}^2}{n-1}}(t_0 - t_{h(\alpha);n-1}) \right), \right. \\ & \left. \min \left\{ 1, 2F_{n-1} \left(\sqrt{\frac{\chi_{R;h(\alpha);n-1}^2}{n-1}}(t_0 + t_{h(\alpha);n-1}) \right) \right\} \right] \end{aligned} \quad (6.10)$$

if $t_0 > 0$,

$$\begin{aligned} \bar{P}[\alpha] = & \left[2 \left(1 - F_{n-1} \left(\sqrt{\frac{\chi_{R;h(\alpha);n-1}^2}{n-1}}(t_0 + t_{h(\alpha);n-1}) \right) \right), \right. \\ & \left. \min \left\{ 1, 2 \left(1 - F_{n-1} \left(\sqrt{\frac{\chi_{L;h(\alpha);n-1}^2}{n-1}}(t_0 - t_{h(\alpha);n-1}) \right) \right) \right\} \right] \end{aligned} \quad (6.11)$$

where $F_{n-1}(t)$ the distribution function of the t_{n-1} distribution.

According to (2.1) the α -cuts of the fuzzy p-value of the one sided test from the left are

$$\bar{P}[\alpha] = \left[F_{n-1} \left(\sqrt{\frac{\chi_{L;h(\alpha);n-1}^2}{n-1}}(t_0 - t_{h(\alpha);n-1}) \right), \right.$$

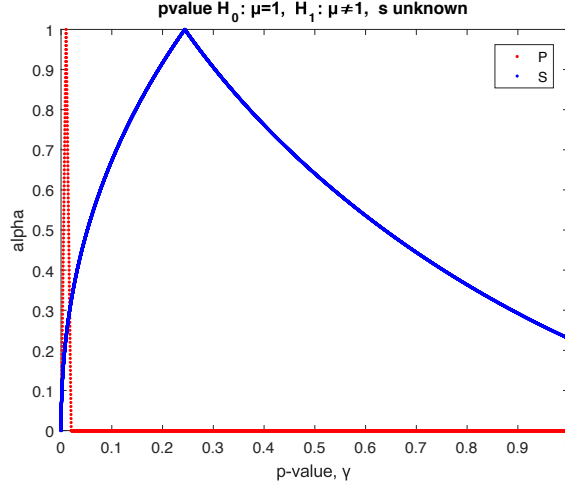


FIGURE 16. Fuzzy p-value \bar{P} and significance level \bar{S} for the two sided test of $H_0 : \mu = 1$ of Example 8

$$F_{n-1} \left(\sqrt{\frac{\chi_{R;h(\alpha);n-1}^2}{n-1}} (t_0 + t_{h(\alpha);n-1}) \right) \Bigg]$$

and according to (2.2) of the one sided test from the right

$$\bar{P}[\alpha] = \left[1 - F_{n-1} \left(\sqrt{\frac{\chi_{R;h(\alpha);n-1}^2}{n-1}} (t_0 + t_{h(\alpha);n-1}) \right), \right. \\ \left. 1 - F_{n-1} \left(\sqrt{\frac{\chi_{L;h(\alpha);n-1}^2}{n-1}} (t_0 - t_{h(\alpha);n-1}) \right) \right]$$

Example 8. We test the null hypothesis $H_0 : \mu = 1$ at significance level $\gamma = 0.01$ with alternative the $H_1 : \mu \neq 1$ (two sided test) for the mean μ of a normal random variable X using a sample of $n = 25$ observations with sample mean and variance $\bar{x} = 1.32$ and $s^2 = 3.04$.

In the crisp test the sample value of the statistic (6.3) is

$$t_0 = \frac{1.32 - 1}{\sqrt{3.04}/\sqrt{25}} = 0.918,$$

so according to (6.2) the p -value of the crisp test is

$$p = 2(1 - F_{25-1}(0.918)) = 0.3678 > \gamma = 0.01$$

Therefore H_0 is not rejected by the crisp test.

Implementing (6.10)-(6.11) and fuzzifying the significance level to a triangular fuzzy number as in (2.4a), we get Figure 16, where the core of \bar{P} is at the right of

\bar{S} and the point of intersection of \bar{P} and \bar{S} has $y_0 = 0.31$. So, H_0 is accepted by this test with degree of acceptance $d = 1 - 0.31 = 0.69$.

7. Hypotheses Tests for the Probability of Success p of a Binomial Distribution

We test at significance level γ the null hypothesis that the probability of success p of a binomial distribution is equal to p_0 ,

$$H_0 : p = p_0$$

using a random sample of observations of size n in which happened x successes.

In the crisp case, we test H_0 using the statistic [6]

$$Z = \frac{P - p_0}{\sqrt{p_0 q_0 / n}} \quad (7.1)$$

where $q_0 = 1 - p_0$ and $P = X/n$ the statistic of the sample proportion of successes. It is known that if n is sufficiently large, then we can use the normal approximation to the binomial, so Z under the null hypothesis follows the standard normal distribution $N(0, 1)$. Therefore, H_0 is rejected if p -value $\leq \gamma$, where the crisp p -value of the test is found as in Section 3.1 for the sample value of the statistic Z ,

$$z_0 = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}} \quad (7.2)$$

and $\hat{p} = \frac{x}{n}$ is the proportion of successes for the given sample. If p -value $> \gamma$, then H_0 is not rejected.

In the fuzzy case, the test of H_0 is based on the fuzzy statistic

$$\bar{Z} = \frac{\bar{p} - p_0}{\sqrt{p_0 q_0 / n}} \quad (7.3)$$

which is generated by substituting P in (7.1) with a fuzzy estimator \bar{p} [2].

In our approach we use the non-asymptotic fuzzy estimator \bar{p} , the α -cuts of which for the given sample are

$$\bar{p}[\alpha] = \left[\hat{p} - z_{h(\alpha)} \sqrt{\frac{\hat{p}\hat{q}}{n}}, \hat{p} + z_{h(\alpha)} \sqrt{\frac{\hat{p}\hat{q}}{n}} \right], \quad \alpha \in [0, 1] \quad (7.4)$$

where $\hat{q} = 1 - \hat{p}$ and $h(a)$ and $z_{h(\alpha)}$ are given in (3.5) and (3.6). From (7.1)-(7.4) and interval arithmetics follows that the α -cuts of the fuzzy test statistic \bar{Z} for the given sample are

$$\bar{Z}[\alpha] = \left[z_0 - z_{h(\alpha)} \sqrt{\frac{\hat{p}\hat{q}}{p_0 q_0}}, z_0 + z_{h(\alpha)} \sqrt{\frac{\hat{p}\hat{q}}{p_0 q_0}} \right], \quad \alpha \in [0, 1] \quad (7.5)$$

Since Z follows the standard normal distribution its median is equal to its mean, which is zero. So according to (2.3), the α -cuts of the p -value of the two sided

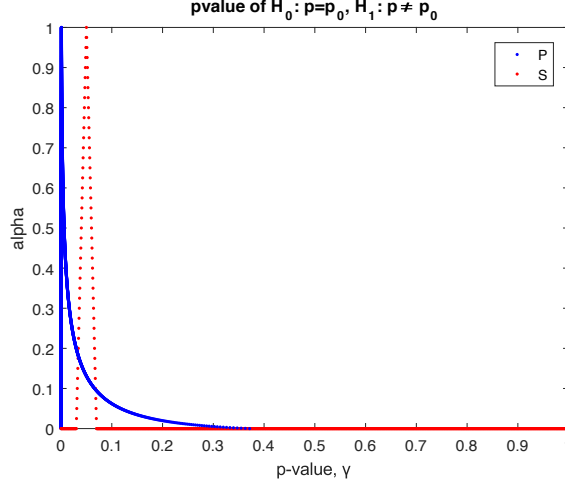


FIGURE 17. Fuzzy p-value \bar{P} and significance level \bar{S} for the two sided test of $H_0 : p = p_0$ with $H_1 : p \neq p_0$ of Example 9

test of H_0 are:
if $z_0 \leq 0$,

$$\bar{P}[\alpha] = \left[2\Pr \left(Z \leq z_0 - z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0q_0}} \right), \right. \\ \left. \min \left\{ 1, 2\Pr \left(Z \leq z_0 + z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0q_0}} \right) \right\} \right] \quad (7.6)$$

if $z_0 > 0$,

$$\bar{P}[\alpha] = \left[2 \left(\Pr(Z \geq z_0 + z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0q_0}}) \right), \right. \\ \left. \min \left\{ 1, 2 \left(\Pr \left(Z \geq z_0 - z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0q_0}} \right) \right) \right\} \right] \quad (7.7)$$

Since Z follows the standard normal distribution, (7.6) and (7.7) give:
if $z_0 \leq 0$,

$$\bar{P}[\alpha] = \left[2\Phi \left(z_0 - z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0q_0}} \right), \right. \\ \left. \min \left\{ 1, 2\Phi \left(z_0 + z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0q_0}} \right) \right\} \right], \quad (7.8)$$

if $z_0 > 0$,

$$\bar{P}[\alpha] = \left[2 \left(1 - \Phi \left(z_0 + z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0q_0}} \right) \right), \right.$$

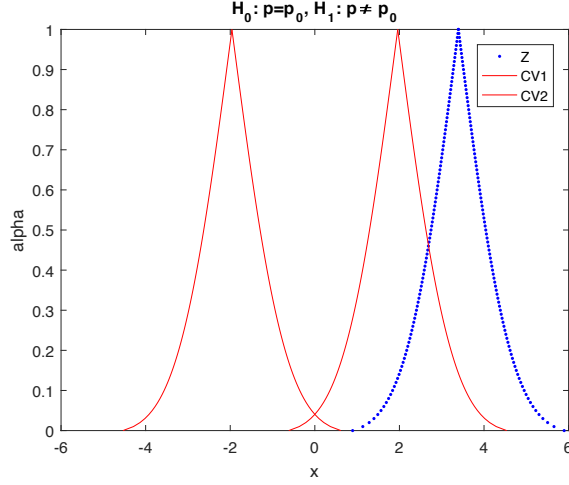


FIGURE 18. The fuzzy test statistic Z and critical values for the two sided test of Example 9

$$\min \left\{ 1, 2 \left(1 - \Phi \left(z_0 - z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0 q_0}} \right) \right) \right\} \quad (7.9)$$

where $\Phi(z)$ the distribution function of the standard normal distribution and z_0 and $z_{h(\alpha)}$ are given by (7.2) and (3.6).

According to (2.1), the α -cuts of the fuzzy p -value of the one sided test from the left are

$$\bar{P}[\alpha] = \left[\Phi \left(z_0 - z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0 q_0}} \right), \Phi \left(z_0 + z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0 q_0}} \right) \right] \quad (7.10)$$

and according to (2.2) of the one sided test from the right,

$$\bar{P}[\alpha] = \left[1 - \Phi \left(z_0 + z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0 q_0}} \right), 1 - \Phi \left(z_0 - z_{h(\alpha)} \sqrt{\frac{\widehat{pq}}{p_0 q_0}} \right) \right] \quad (7.11)$$

Example 9. a) We test at significance level $\gamma = 0.05$ the null hypothesis that the probability of success p of a binomial distribution is equal to $p_0 = 0.5$

$$H_0 : p = 0.5$$

with alternative $H_1 : p \neq 0.5$ (two sided test) using a random sample of $n = 200$ observations in which happened 124 successes with two sided fuzzy test which uses:

i) fuzzy p -value,

ii) fuzzy critical values, developed in [2] and [12].

b) We test H_0 at significance level $\gamma = 0.05$ with alternative $H_1 : p > 0.5$ using the same sample.

a) In the crisp case we test H_0 evaluating the sample value (7.2) of the statistic Z for $\hat{p} = \frac{124}{200} = 0.62$

$$z_0 = \frac{0.62 - 0.5}{\sqrt{0.62(1 - 0.62)/200}} = 3.5$$

So, the crisp p -value of the test is found as in Section 3.1 to be

$$p\text{-value} = 2(1 - \Phi(3.5)) = 0.0005 < \gamma = 0.05,$$

Therefore H_0 is rejected by the crisp test.

i) Implementing (7.8)-(7.9) and fuzzifying the significance level to a triangular fuzzy number as in (3.11), we get the result of Figure 17, where the core of \bar{P} is at the left of the core \bar{S} and their point of intersection has $y_0 = 0.18$. So according to Section 2, H_0 is rejected by this test with degree of rejection $d = 1 - 0.18 = 0.82$.

ii) Applying the fuzzy test developed in [2] and [12], which uses fuzzy critical values, we get the results of Figure 18, where the core of \bar{Z}_0 is at the right of the core of \bar{CV}_2 and their point of intersection has $y_1 = 0.45$. So according to Section 2, H_0 is rejected by this test with degree of rejection $d = 1 - 0.45 = 0.55$.

b) For the one sided test from the right, as in Section 3.1 the crisp p -value of the test is

$$p\text{-value} = 1 - \Phi(3.5) = 0.00025 < \gamma = 0.05,$$

so H_0 is rejected by the crisp test.

Implementing (7.11) and (3.11) for z_0 of (7.2) we get the results of Figure 19, where the core of the fuzzy number \bar{P} is at the left of the core of \bar{S} and the point of intersection of \bar{P} and \bar{S} has $y_0 = 0.11$, so the truth value of $\bar{P} < \bar{S}$ is $1 - 0.11 = 0.89$. Therefore, H_0 is rejected by the one sided test with degree of rejection 0.89.

8. Hypotheses Tests of $H_0 : p_1 = p_2$ for the Probabilities of Success p_1, p_2 of Two Binomial Distributions

We test at significance level γ the null hypothesis that the probabilities of success p_1 and p_2 of two binomial distributions are equal,

$$H_0 : p_1 = p_2$$

using two random samples of observations of sizes n_1 and n_2 in which happened x_1 and x_2 successes respectively .

In the crisp case we test H_0 using the statistic [6]

$$Z = \frac{P_1 - P_2}{s_p} \tag{8.1}$$

where $P_1 = \frac{X_1}{n_1}$, $P_2 = \frac{X_2}{n_2}$ the statistics of the sample proportions of successes,

$$s_p = \sqrt{\hat{p}\hat{q} \left(\frac{1}{n_1} + \frac{1}{n_2} \right)} \tag{8.2}$$

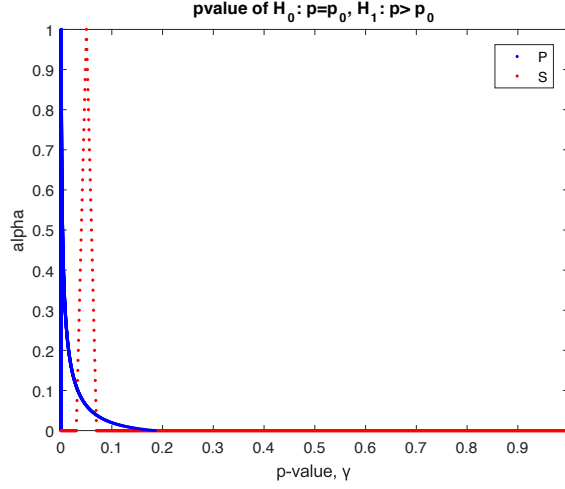


FIGURE 19. Fuzzy p-value \bar{P} and significance level \bar{S} for the one sided test of $H_0 : p = p_0$ with $H_1 : p > p_0$ of Example 9

and

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} \quad (8.3)$$

a “pooled” estimator of p under H_0 and $\hat{q} = 1 - \hat{p}$.

It is known that if n_1 and n_2 are sufficiently large, then we can use the normal approximation to the binomial [6]. Then under the null hypothesis ($p_1 = p_2$), Z follows the standard normal distribution $N(0, 1)$, so H_0 is rejected if p -value $\leq \gamma$, where the crisp p -value of the test is found as in Section 3.1 for the sample value of the test statistic (8.1)

$$z_0 = \frac{\hat{p}_1 - \hat{p}_2}{s_p} \quad (8.4)$$

where $\hat{p}_1 = \frac{x_1}{n_1}$, $\hat{p}_2 = \frac{x_2}{n_2}$ the proportions of successes in the given samples.

In the fuzzy case the test of H_0 is based on the fuzzy statistic

$$\bar{Z} = \frac{\bar{p}_{12}}{s_p} \quad (8.5)$$

which is generated by substituting the difference $P_1 - P_2$ in (8.1) with a fuzzy estimator \bar{p}_{12} [2].

In our approach we use the non-asymptotic fuzzy estimator \bar{p}_{12} which is constructed in [2] and generalized in [12], the α -cuts of which for the given samples are

$$\bar{p}_{12}[\alpha] = [\hat{p}_1 - \hat{p}_2 - z_{h(\alpha)}s_0, \hat{p}_1 - \hat{p}_2 + z_{h(\alpha)}s_0] \quad (8.6)$$

where $\widehat{q}_1 = 1 - \widehat{p}_1$, $\widehat{q}_2 = 1 - \widehat{p}_2$ and

$$s_0 = \sqrt{\frac{\widehat{p}_1 \widehat{q}_1}{n_1} + \frac{\widehat{p}_2 \widehat{q}_2}{n_2}} \quad (8.7)$$

From (8.1)-(8.7) and interval arithmetics follows that the α -cuts of the fuzzy statistic \overline{Z} for the given samples are

$$\overline{Z}_0[\alpha] = \left[z_0 - z_{h(\alpha)} \frac{s_0}{s_p}, z_0 + z_{h(\alpha)} \frac{s_0}{s_p} \right], \quad \alpha \in [0, 1] \quad (8.8)$$

Since Z follows the standard normal distribution, its median is equal to its mean, which is zero. So according to (2.3), the α -cuts of the fuzzy p -value of two sided test are:

if $z_0 \leq 0$,

$$\overline{P}[\alpha] = \left[2\Phi \left(z_0 - z_{h(\alpha)} \frac{s_0}{s_p} \right), \min \left\{ 1, 2\Phi \left(z_0 + z_{h(\alpha)} \frac{s_0}{s_p} \right) \right\} \right], \quad (8.9)$$

if $z_0 > 0$,

$$\begin{aligned} \overline{P}[\alpha] = & \left[2 \left(1 - \Phi \left(z_0 + z_{h(\alpha)} \frac{s_0}{s_p} \right) \right), \right. \\ & \left. \min \left\{ 1, 2 \left(1 - \Phi \left(z_0 - z_{h(\alpha)} \frac{s_0}{s_p} \right) \right) \right\} \right] \end{aligned} \quad (8.10)$$

where $\Phi(z)$ the distribution function of the standard normal distribution and z_0, s_p, s_0 and $z_{h(\alpha)}$ are given by (8.4), (8.2), (8.7) and (3.6).

According to (2.1) the α -cuts of the fuzzy p -value of the one sided test from the left are

$$\overline{P}[\alpha] = \left[\Phi \left(z_0 - z_{h(\alpha)} \frac{s_0}{s_p} \right), \Phi \left(z_0 + z_{h(\alpha)} \frac{s_0}{s_p} \right) \right] \quad (8.11)$$

and according to (2.2) of the one sided test from the right,

$$\overline{P}[\alpha] = \left[\left(1 - \Phi \left(z_0 + z_{h(\alpha)} \frac{s_0}{s_p} \right) \right), 1 - \left(1 - \Phi \left(z_0 - z_{h(\alpha)} \frac{s_0}{s_p} \right) \right) \right] \quad (8.12)$$

Example 10. We test at significance level $\gamma = 0.05$ the null hypothesis that the probabilities of success p_1 and p_2 of two binomial distributions are equal $H_0 : p_1 = p_2$ with alternative $H_1 : p_1 \neq p_2$ (two sided test) using two random samples of observations of sizes $n_1 = 250$ and $n_2 = 400$ in which occurred $x_1 = 40$ and $x_2 = 55$ successes respectively.

In the crisp case we test H_0 evaluating the value of the statistic (8.1) for the given sample, where

$$\widehat{p}_1 = \frac{40}{250} = 0.16, \quad \widehat{p}_2 = \frac{55}{400} = 0.138$$

and

$$\widehat{p} = \frac{40 + 55}{250 + 400} = 0.146$$

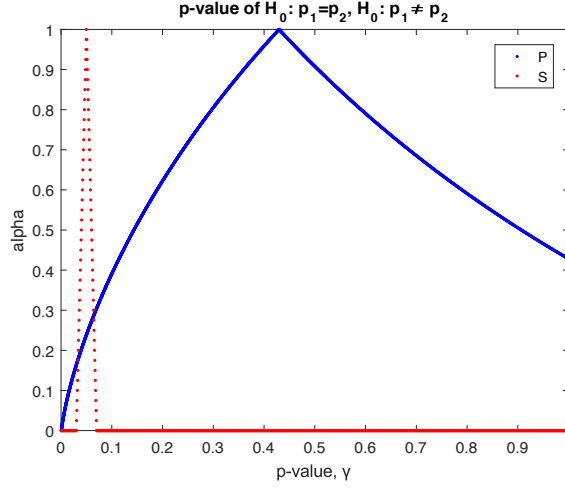


FIGURE 20. Fuzzy p-value \bar{P} and significance level \bar{S} for the two sided test of $H_0 : p_1 = p_2$ of Example 10

So from (8.2) we get

$$s_p = \sqrt{0.146(1 - 0.146) \left(\frac{1}{250} + \frac{1}{400} \right)} = 0.029$$

and from (8.4)

$$z_0 = \frac{0.16 - 0.138}{0.029} = 0.759$$

Thus, the crisp p -value of the test is

$$p\text{-value} = 2(1 - \Phi(0.759)) = 0.448 > \gamma = 0.05$$

so H_0 is not rejected by the crisp test.

Implementing (8.9)-(8.10) and fuzzifying the significance level to a triangular fuzzy number as in (3.11), we get the result of Figure 20, where the core of \bar{P} is at the right of the core \bar{S} and their point of intersection has $y_0 = 0.28$. So according to Section 2, H_0 is accepted by this test with degree of acceptance $d = 1 - 0.28 = 0.72$.

9. Conclusions

If the value of the test statistic of a hypothesis test is close to the critical values of the test, then the crisp test is unstable, since a very small change in the sample may lead from rejection to no rejection of H_0 or vice-versa, as in Examples 3 and 6. Whereas, our approach, which uses non-asymptotic fuzzy estimators for the construction of the fuzzy p -value and a degree of rejection or acceptance of the null hypothesis H_0 , gives better results than the crisp test, since it gives us the possibility to make a decision on a partial rejection or acceptance of H_0 , as shown

in Examples 3 and 6 where we get a very low degree of rejection or acceptance of the null hypothesis, which means “no decision”.

The advantage of our approach compared to the approach of [13] is that it tests crisp hypotheses using fuzziness, which leads to a degree of rejection or acceptance of the null hypothesis, whereas the tests of [13] give crisp results when applied to crisp hypotheses, as explained in its examples.

In the crisp statistics the hypotheses tests which use critical values are equivalent to the tests which use p -value. This does not happen in the fuzzy hypotheses tests, since in general the tests which use fuzzy critical values give different degrees of rejection or acceptance of H_0 than the tests which use fuzzy p -value. The two tests give similar results only in cases in which the p -value of the test is close to the significance level (or equivalently if the value of the test statistic is close to a critical value), but in the other cases the former gives significantly lower degree of rejection or acceptance than the latter (Examples 2,4,5,7 and 9). Characteristic is the case of the hypotheses tests of the Examples 2 and 5, in which the crisp p -value is one (the value of the test statistic is in the center of the acceptance region), which is the best case of acceptance. As shown in these examples, the tests that use fuzzy critical values give significantly lower degree of acceptance of H_0 than the expected, which is a value close to one, ($d = 0.68$ in Example 2 and $d = 0.81$ in Example 5), whereas the tests that use fuzzy p -value give degree of acceptance of H_0 near to one ($d = 0.95$ in Example 2 and $d = 0.98$ in Example 5). So, since the tests which use fuzzy p -value are in much better agreement with the crisp hypotheses tests than the tests which use fuzzy critical values in [2] and [12], we can conclude that the former give more reliable results than the latter.

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