# Fuzzy Rough Sets: The Forgotten Step

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***Abstract*—Traditional rough set theory uses equivalence relations to compute lower and upper approximations of sets. The corresponding equivalence classes either coincide or are disjoint. This behaviour is lost when moving on to a fuzzy T-equivalence relation. However, none of the existing studies on fuzzy rough set theory tries to exploit the fact that an element can belong to some degree to several "soft similarity classes" at the same time. In this paper we show that taking this truly fuzzy characteristic into account may lead to new and interesting definitions of lower and upper approximations. We explore two of them in detail and we investigate under which conditions they differ from the commonly used definitions. Finally we show the possible practical relevance of the newly introduced approximations for query refinement.**

***Index Terms*—Fuzzy rough set, lower and upper approximation, query refinement, transitivity.**

## I. Introduction

Since its introduction in the 1960s, fuzzy set theory has had a significant impact on the way we represent and compute with vague information. More recently it has become part of the larger paradigm of soft computing, a collection of techniques that are tolerant of typical characteristics of imperfect data and knowledge—such as vagueness, imprecision, uncertainty, and partial truth—and hence adhere closer to the human mind than conventional hard computing techniques. During the last decades new approaches have been developed that generalize the original fuzzy set theory (which is also called type-1 fuzzy set theory in this context). Type-2 fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets and fuzzy rough sets have in common that they can all be formally characterized by membership functions taking values in a partially ordered set $P$, which is no longer the same (but an extension of) the set of membership degrees $[0, 1]$ used in fuzzy set theory. The introduction of such new, generalizing theories is often accompanied by lengthy discussions on issues such as the choice of terminology and the added value of the generalization.

In this paper we focus on fuzzy rough set theory. Pawlak [23] launched rough set theory as a framework for the construction of approximations of concepts when only incomplete information is available. The available information consists of a set $A$ of examples (a subset of a universe $X, X$ being a nonempty set of objects we want to say something about) of a concept $C$, and a relation $R$ in $X$. $R$ models "indiscernibility" or "indistinguishability" and therefore generally is a tolerance relation (i.e., a reflexive and symmetrical relation) and in most cases even an equivalence relation (i.e., a transitive tolerance relation).

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After a public debate reflecting rivalry between rough set theory and the slightly older fuzzy set theory, many researchers started working towards a hybrid theory (e.g., [11], [18], [20], [25], [26], [28], and [30]). In doing so, the central focus moved from elements' indistinguishability (objects are indistinguishable or not) to their similarity (objects are similar to a certain degree), represented by a fuzzy relation $R$. As a result, objects are categorized into classes with "soft" boundaries based on their similarity to one another; abrupt transitions between classes are replaced by gradual ones, allowing that an element can belong (to varying degrees) to more than one class.

Soon, researchers started exploring possible applications of the new paradigm of fuzzy-rough hybridization. For a comprehensive literature review up to 1999, we refer to [15]. Among the more recent work is that of Drwal [10] and Jensen and Shen [13], [14], who studied extensions of the well-known rough set approaches to data reduction and classification.

In Section II, we recall the necessary background leading to the definition of a fuzzy rough set as presented in [26]. This definition is an elegant fuzzification of the concept of a rough set and at the same time absorbs earlier suggestions in the same direction. The most striking aspect of all the studies on fuzzy rough set theory mentioned above however is that none of them tries to exploit the fact that an element $y$ of $X$ can belong to some degree to several "soft similarity classes" at the same time. This property does not only lie at the heart of fuzzy set theory but is also crucial in the decision on how to define lower and upper approximations. For instance, in traditional rough set theory, $y$ belongs to the lower approximation of $A$ if the equivalence class to which $y$ belongs is included in $A$. But what happens if $y$ belongs to several "soft similarity classes" at the same time? Do we then require that all of them are included in $A$? Most of them? Or just one? And then, which one? In Sections III and IV, we continue this discussion touched upon for the first time in [5].

As such it becomes clear that there is still significant room for improvement and generalization of the definition of a fuzzy rough set, beyond the most "obvious" fuzzification established so far. Furthermore Section V reveals that this generalization is not just of theoretical interest but becomes crucial in a topical application such as query refinement for searching on the WWW.

## II. From Rough Sets to Fuzzy Rough Sets

Rough set analysis makes statements about the membership of some element $y$ of $X$ to the concept of which $A$ is a set of examples, based on the indistinguishability between $y$ and the elements of $A$. To arrive at such statements, $A$ is approximated in two ways. An element $y$ of $X$ belongs to the lower approximation of $A$ if the equivalence class to which $y$ belongs is included in $A$. On the other hand $y$ belongs to the upper approximation

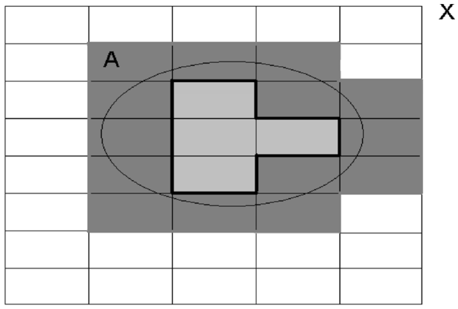



Fig. 1. Lower and upper approximation of a set $A$. The dark shaded area is the boundary region.

of $A$ if its equivalence class has a non-empty intersection with $A$.

Formally, let $X$ be a universe and $R$ an equivalence relation. The lower and the upper approximation (in the sense of Pawlak [23]) of a subset $A$ of $X$ in the approximation space $(X, R)$ are the sets $R{\downarrow}A$ and $R{\uparrow}A$ such that for all $y$ in $X$

$$y \in R{\downarrow}A \quad \text{iff } [y]_R \subseteq A \tag{1}$$

$$y \in R{\uparrow}A \quad \text{iff } [y]_R \cap A \neq \emptyset. \tag{2}$$

In other words

$$y \in R{\downarrow}A \quad \text{iff } (\forall x \in X)((x, y) \in R \Rightarrow x \in A) \tag{3}$$

$$y \in R{\uparrow}A \quad \text{iff } (\exists x \in X)((x, y) \in R \wedge x \in A) \tag{4}$$

The underlying meaning is that $R{\downarrow}A$ is the set of elements *necessarily* satisfying the concept (strong membership), while $R{\uparrow}A$ is the set of elements *possibly* belonging to the concept (weak membership); for $y$ belongs to $R{\downarrow}A$ if all elements of $X$ indistinguishable from $y$ belong to $A$ (hence, there is no doubt that $y$ also belongs to $A$), while $y$ belongs to $R{\uparrow}A$ as soon as an element of $A$ is indistinguishable from $y$. It holds that $R{\downarrow}A \subseteq R{\uparrow}A$. If $y$ belongs to the boundary region $R{\uparrow}A \backslash R{\downarrow}A$, then there is some doubt, because in this case $y$ is at the same time indistinguishable from at least one element of $A$ and at least one element of $X$ that is not in $A$. This is illustrated graphically in Fig. 1. We call $(A_1, A_2)$ a rough set (in $(X, R)$) as soon as there is a set $A$ in $X$ such that $R{\downarrow}A = A_1$ and $R{\uparrow}A = A_2$ (see e.g., [26]).

For completeness we mention that a second stream concerning rough sets in the literature was initiated by Iwinski [12] who did not use an equivalence relation or tolerance relation as an initial building block to define the rough set concept. Although his formulation provides an elegant mathematical model, the absence of the equivalence relation makes his model, as well as the fuzzy rough set theoretical models developed in the same spirit (see, e.g., [21]), hard to interpret. Therefore, we do not deal with it in this paper.

In the context of fuzzy rough set theory, $A$ is a fuzzy set in $X$, i.e., an $X \to [0,1]$ mapping, while $R$ is a fuzzy relation in $X$, i.e., a fuzzy set in $X \times X$. Recall that for all $y$ in $X$, the $R$-foreset of $y$ is the fuzzy set $Ry$ defined by

$$Ry(x) = R(x, y)$$

TABLE I
WELL-KNOWN t-NORMS; $x$ AND $y$ IN $[0, 1]$

| | | |
|---|---|---|
| $\mathcal{T}_{\mathrm{M}}(x, y)$ | $=$ | $\min(x, y)$ |
| $\mathcal{T}_{\mathrm{P}}(x, y)$ | $=$ | $xy$ |
| $\mathcal{T}_{\mathrm{W}}(x, y)$ | $=$ | $\max(x + y - 1, 0)$ |

TABLE II
WELL-KNOWN S-IMPLICATORS; $x$ AND $y$ IN $[0, 1]$

| | | |
|---|---|---|
| $\mathcal{I}_{\mathcal{T}_{\mathrm{M}},\mathcal{N}_s}(x, y)$ | $=$ | $\max(1 - x, y)$ |
| $\mathcal{I}_{\mathcal{T}_{\mathrm{P}},\mathcal{N}_s}(x, y)$ | $=$ | $1 - x + xy$ |
| $\mathcal{I}_{\mathcal{T}_{\mathrm{W}},\mathcal{N}_s}(x, y)$ | $=$ | $\min(1 - x + y, 1)$ |

TABLE III
WELL-KNOWN RESIDUAL IMPLICATORS; $x$ AND $y$ IN $[0, 1]$

| | | |
|---|---|---|
| $\mathcal{I}_{\mathcal{T}_{\mathrm{M}}}(x, y)$ | $=$ | $\begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$ |
| $\mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(x, y)$ | $=$ | $\begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise} \end{cases}$ |
| $\mathcal{I}_{\mathcal{T}_{\mathrm{W}}}(x, y)$ | $=$ | $\min(1 - x + y, 1)$ |

for all $x$ in $X$. The fuzzy logical counterparts of the connectives in (3) and (4) play an important role in this paper; we therefore recall some preliminaries in detail. Throughout this paper, let $\mathcal{T}$ and $\mathcal{I}$ denote a triangular norm and an implicator, respectively. Recall that a triangular norm (t-norm for short) $\mathcal{T}$ is any increasing, commutative and associative $[0,1]^2 \to [0,1]$ mapping satisfying $\mathcal{T}(1, x) = x$, for all $x$ in $[0,1]$. A negator $\mathcal{N}$ is a decreasing $[0,1] \to [0,1]$ mapping satisfying $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. $\mathcal{N}$ is called involutive if $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x$ in $[0,1]$. Finally, an implicator is any $[0,1]^2 \to [0,1]$-mapping $\mathcal{I}$ satisfying $\mathcal{I}(0,0) = 1, \mathcal{I}(1,x) = x$, for all $x$ in $[0,1]$. Moreover we require $\mathcal{I}$ to be decreasing in its first, and increasing in its second component. If $\mathcal{T}$ is a t-norm, the mapping $\mathcal{I}_{\mathcal{T}}$ defined by, for all $x$ and $y$ in [0, 1],

$$\mathcal{I}_{\mathcal{T}}(x, y) = \sup\{\lambda | \lambda \in [0,1] \text{ and } \mathcal{T}(x, \lambda) \leq y\} \tag{5}$$

is an implicator, usually called the residual implicator (of $\mathcal{T}$). If $\mathcal{T}$ is a t-norm and $\mathcal{N}$ is an involutive negator, then the mapping $\mathcal{I}_{\mathcal{T},\mathcal{N}}$ defined by, for all $x$ and $y$ in [0, 1],

$$\mathcal{I}_{\mathcal{T},\mathcal{N}}(x, y) = \mathcal{N}(\mathcal{T}(x, \mathcal{N}(y))) \tag{6}$$

is an implicator, usually called the S-implicator induced by $\mathcal{T}$ and $\mathcal{N}$. In Tables I–III, we mention some well known t-norms, S- and residual implicators. The S-implicators in Table II are induced by means of the standard negator $\mathcal{N}_s$ which is defined by $\mathcal{N}_s(x) = 1 - x$, for all $x$ in $[0,1]$.

Because equivalence relations are used to model equality, fuzzy $\mathcal{T}$-equivalence relations are commonly considered to represent approximate equality or similarity. Recall that a fuzzy relation $R$ in $X$ is called a fuzzy $\mathcal{T}$-equivalence relation iff for all $x, y$ and $z$ in $X$

(FE.1) $R(x, x) = 1$ (reflexivity)

(FE.2) $R(x, y) = R(y, x)$ (symmetry)

(FE.3) $\mathcal{T}(R(x, y), R(y, z)) \leq R(x, z)$ ($\mathcal{T}-$transitivity).

Paraphrasing statements (3) and (4) and absorbing earlier suggestions in the same direction, the following definition of the lower and upper approximation of a fuzzy set $A$ in $X$ was given in [26], constructed by means of an implicator $\mathcal{I}$, a t-norm $\mathcal{T}$ and a fuzzy $\mathcal{T}$-equivalence relation $R$ in $X$

$$R{\downarrow}A(y) = \inf_{x \in X} \mathcal{I}(R(x,y), A(x)) \quad (7)$$

$$R{\uparrow}A(y) = \sup_{x \in X} \mathcal{T}(R(x,y), A(x)) \quad (8)$$

for all $y$ in $X$. $(A_1, A_2)$ is called a fuzzy rough set (in $(X, R)$) as soon as there is a fuzzy set $A$ in $X$ such that $R{\downarrow}A = A_1$ and $R{\uparrow}A = A_2$. Formulas (7) and (8) for $R{\downarrow}A$ and $R{\uparrow}A$ can also be interpreted as the degree of inclusion of $Ry$ in $A$ and the degree of overlap of $Ry$ and $A$ respectively, which indicates the semantical link with (1) and (2).

## III. The Forgotten Step

### A. New Approximations

The role of the equivalence class $[y]_R$ in the crisp case [see (1) and (2)] is subsumed by the more general concept of the $R$-foreset $Ry$ in the fuzzy case [see (7) and (8)]. It is well known that in the crisp case, if we consider two equivalence classes then they either coincide or are disjoint. It is therefore not possible for $y$ to belong to two different equivalence classes at the same time. If $R$ is a fuzzy relation in $X$—in particular a fuzzy $\mathcal{T}$-equivalence relation—then it is quite normal that, because of the intermediate degrees of membership, different foresets are not necessarily disjoint, as the following examples illustrate. Recall that two (fuzzy) sets are disjoint iff their intersection is empty, and that the $\mathcal{T}$-intersection of fuzzy sets $A$ and $B$ in $X$ is defined by

$$(A \cap_{\mathcal{T}} B)(x) = \mathcal{T}(A(x), B(x))$$

for all $x$ in $X$.

*Example 1:* Let $\mathcal{T}$ be an arbitrary t-norm. One can verify that for the fuzzy $\mathcal{T}$-equivalence relation $R$ on $X = \{a, b\}$ given by

| $R$ | $a$ | $b$ |
|---|---|---|
| $a$ | 1.0 | 0.2 |
| $b$ | 0.2 | 1.0 |

it holds that

$$Ra(a) = 1.0 \quad Ra(b) = 0.2 \quad (9)$$

$$Rb(a) = 0.2 \quad Rb(b) = 1.0 \quad (10)$$

hence, for any t-norm $\mathcal{T}$

$$(Ra \cap_{\mathcal{T}} Rb)(a) = (Ra \cap_{\mathcal{T}} Rb)(b) = \mathcal{T}(1.0, 0.2) = 0.2. \quad (11)$$

The latter shows that the foresets $Ra$ and $Rb$ are not disjoint, since their intersection contains both $a$ and $b$ to degree 0.2.

*Example 2:* In applications $\mathcal{T}_{\mathrm{W}}$ is often used as a t-norm because the notion of fuzzy $\mathcal{T}_{\mathrm{W}}$-equivalence relation is dual to that of a pseudo-metric [4]. Let the fuzzy $\mathcal{T}_{\mathrm{W}}$-equivalence relation $R$ in $\mathbb{R}$ be defined by

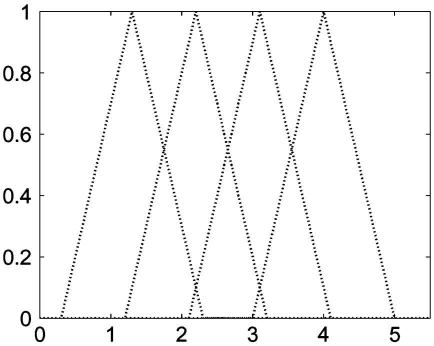


Fig. 2. Fuzzy similarity classes.

$$R(x,y) = \max(1 - |x - y|, 0)$$

for all $x$ and $y$ in $\mathbb{R}$. Fig. 2 depicts the $R$-foresets of 1.3, 2.2, 3.1, and 4. The $R$-foresets of 3.1 and 4 are clearly different. Still one can easily see that

$$R(3.1, 3.5) = 0.6$$

$$R(4.0, 3.5) = 0.5.$$

Since $\mathcal{T}_{\mathrm{W}}(0.6, 0.5) = 0.1, 3.5$ belongs to degree 0.1 to the $\mathcal{T}_{\mathrm{W}}$-intersection of the $R$-foresets of 3.1 and 4, i.e., these $R$-foresets are not disjoint.

From now on, whenever the fuzzy relation $R$ models approximate equality, we will call $Ry$ the "fuzzy similarity class" of $y$. From the previous examples, it is clear that $y$ does not only belong to $Ry$ but can also belong to other, different fuzzy similarity classes to a certain degree. Recall that, by the definition used so far, $y$ belongs to the lower approximation of $A$ to the degree to which $Ry$ is included in $A$ [see (7)]. In view of the discussion above however it makes sense to consider also the other fuzzy similarity classes to which $y$ has a non-zero membership degree, and to assess their inclusion into $A$ as well for the lower approximation, and their overlap with $A$ for the upper approximation. Informally, this immediately results in the following (inexhaustive) list of candidate definitions for the lower and the upper approximation of $A$.

1) $y$ belongs to the lower approximation of $A$ to the degree to which
   a) all fuzzy similarity classes containing $y$ are included in $A$;
   b) at least one fuzzy similarity class containing $y$ is included in $A$;
   c) $Ry$ is included in $A$.
2) $y$ belongs to the upper approximation of $A$ to the degree to which
   a) all fuzzy similarity classes containing $y$ have a nonempty intersection with $A$;
   b) at least one fuzzy similarity class containing $y$ has a nonempty intersection with $A$;
   c) $Ry$ has a nonempty intersection with $A$.

Paraphrasing these expressions, we obtain the following definitions.

*Definition 3:* Let $R$ be a fuzzy relation in $X$ and $A$ a fuzzy set in $X$.

1) The tight, loose and (usual) lower approximation of $A$ are defined as
   a) $R{\downarrow}{\downarrow}A(y) = \inf_{z\in X}\mathcal{I}(Rz(y), \inf_{x\in X}\mathcal{I}(Rz(x), A(x)))$;
   b) $R{\uparrow}{\downarrow}A(y) = \sup_{z\in X}\mathcal{T}(Rz(y), \inf_{x\in X}\mathcal{I}(Rz(x), A(x)))$;
   c) $R{\downarrow}A(y) = \inf_{x\in X}\mathcal{I}(Ry(x), A(x))$;

   for all $y$ in $X$.
2) The tight, loose and (usual) upper approximation of $A$ are defined as
   a) $R{\downarrow}{\uparrow}A(y) = \inf_{z\in X}\mathcal{I}(Rz(y), \sup_{x\in X}\mathcal{T}(Rz(x), A(x)))$;
   b) $R{\uparrow}{\uparrow}A(y) = \sup_{z\in X}\mathcal{T}(Rz(y), \sup_{x\in X}\mathcal{T}(Rz(x), A(x)))$;
   c) $R{\uparrow}A(y) = \sup_{x\in X}\mathcal{T}(Ry(x), A(x))$;

   for all $y$ in $X$.

The terminology "tight" refers to the fact that we take "all" fuzzy similarity classes into account, giving rise to a strict or tight requirement. For the "loose" approximations, we only look at "the best one" which is clearly a more flexible demand. Options (1c) and (2c) correspond to the well known definition from the literature on fuzzy rough set theory [see (7) and (8)], while (1b) and (2a) correspond to the generalized opening and closure operators defined in [1]. Furthermore in the crisp case options (1a) through (1c) coincide, as well as options (2a) through (2c) because then there is exactly one equivalence class to which $y$ belongs, namely $Ry$. As mentioned previously, however, different fuzzy similarity classes are not necessarily disjoint, hence, a further investigation on the relationships between the tight, loose and usual approximations is in order.

In the remainder of this paper we will assume that $R$ is a reflexive and symmetrical fuzzy relation in $X$, which are basic requirements if $R$ is supposed to model similarity. Some properties require additional $\mathcal{T}$-transitivity of $R$; whenever this is the case we mention it explicitly.

### B. Links Between the Approximations

We start with the following proposition which follows immediately from the definitions due to the symmetry of $R$ and proves to be very useful.

*Proposition 4:* For every fuzzy set $A$ in $X$

$$R{\downarrow}{\downarrow}A = R{\downarrow}(R{\downarrow}A) \quad (12)$$
$$R{\uparrow}{\downarrow}A = R{\uparrow}(R{\downarrow}A) \quad (13)$$
$$R{\downarrow}{\uparrow}A = R{\downarrow}(R{\uparrow}A) \quad (14)$$
$$R{\uparrow}{\uparrow}A = R{\uparrow}(R{\uparrow}A). \quad (15)$$

The following proposition supports the idea of approximating a concept from the lower and the upper side.

*Proposition 5: [26]:* For every fuzzy set $A$ in $X$

$$R{\downarrow}A \subseteq A \subseteq R{\uparrow}A. \quad (16)$$

The tight and loose approximations should exhibit a similar behaviour. To show that this is indeed the case, we recall that the lower and the upper approximation are monotonic operations due to the monotonicity of the fuzzy logical operators involved. This is reflected in the next proposition.

*Proposition 6: [26]:* For every fuzzy set $A$ and $B$ in $X$

$$A \subseteq B \text{ implies } R{\downarrow}A \subseteq R{\downarrow}B \quad (17)$$
$$A \subseteq B \text{ implies } R{\uparrow}A \subseteq R{\uparrow}B. \quad (18)$$

Combining this with Proposition 5 we conclude that the tight lower and the loose upper approximation are indeed a subset of $R{\downarrow}A$ and a superset of $R{\uparrow}A$, respectively, which justifies the terminology.

*Proposition 7:* For every fuzzy set $A$ in $X$

$$R{\downarrow}{\downarrow}A \subseteq R{\downarrow}A \subseteq A \subseteq R{\uparrow}A \subseteq R{\uparrow}{\uparrow}A. \quad (19)$$

Note that in [6] it is suggested to use $R{\downarrow}{\downarrow}A$, $R{\downarrow}A$, $R{\uparrow}A$ and $R{\uparrow}{\uparrow}A$ as representations of the modified linguistic expressions $extremely\ A$, $very\ A$, $more\ or\ less\ A$, and $roughly\ A$ respectively (for $R$ being a fuzzy relation modelling approximate equality).

From Propositions 4–6, we obtain

$$R{\downarrow}A \subseteq R{\uparrow}{\downarrow}A \subseteq R{\uparrow}A \quad (20)$$
$$R{\downarrow}A \subseteq R{\downarrow}{\uparrow}A \subseteq R{\uparrow}A \quad (21)$$

but no immediate information about a direct relationship between the loose lower and the tight upper approximation in terms of inclusion, and about how $A$ itself fits in this picture. The following proposition sheds some light on this matter.

*Proposition 8: [1]:* If $\mathcal{T}$ is a left continuous t-norm and $\mathcal{I}$ its residual implicator then for every fuzzy set $A$ in $X$

$$R{\uparrow}{\downarrow}A \subseteq A \subseteq R{\downarrow}{\uparrow}A. \quad (22)$$

Proposition 8 does not hold in general for other choices of t-norms and implicators that do not fulfill the properties

$$\mathcal{T}(x, \mathcal{I}(x, y)) \leq y \quad (23)$$
$$y \leq \mathcal{I}(x, \mathcal{T}(x, y)) \quad (24)$$

as Example 9 illustrates.

*Example 9:* Let $X$ and $R$ be defined as in Example 1 and let $A$ be the fuzzy set in $X$ defined as $A(a) = 1$ and $A(b) = 0.8$. Furthermore let $\mathcal{T} = \mathcal{T}_{\mathrm{M}}$ and $\mathcal{I} = \mathcal{I}_{\mathcal{T}_{\mathrm{M}},\mathcal{N}_s}$ be its S-implicator. Then, $R{\uparrow}A(a) = 1$ and $R{\uparrow}A(b) = 0.8$, hence

$$(R{\downarrow}{\uparrow}A)(a) = \min(\max(0,1), \max(0.8, 0.8)) = 0.8 \quad (25)$$

which makes it clear that $A \not\subseteq R{\downarrow}{\uparrow}A$.

From all of the above, we obtain

$$R{\downarrow}{\downarrow}A \subseteq R{\downarrow}A \subseteq R{\uparrow}{\downarrow}A \subseteq A \subseteq R{\downarrow}{\uparrow}A \subseteq R{\uparrow}A \subseteq R{\uparrow}{\uparrow}A$$

provided that $\mathcal{T}$ is left continuous and $\mathcal{I}$ is its residual implicator. We stress that this holds for any reflexive and symmetric fuzzy relation $R$. This justifies the name "lower" and "upper" for the new tight and loose approximations. Furthermore the following example illustrates that the new approximations indeed differ from the existing ones.

*Example 10:* Let $X = [0,1]$ and $A$ the fuzzy set in $X$ defined as $A(x) = x$, for all $x$ in $X$. Let the reflexive and symmetrical fuzzy relation $R$ in $X$ be defined as

$$R(x,y) = \begin{cases} 1, & \text{if } |x-y| < 0.1 \\ 0, & \text{otherwise} \end{cases}$$

for all $x$ and $y$ in $X$. One can verify that for $y$ in $X$:

$$\begin{aligned} R{\downarrow}A(y) &= \inf_{z\in X} \mathcal{I}(R(z,y), A(z)) \\ &= \inf\{z \mid z \in X \wedge z \in ]y-0.1, y+0.1[\} \\ &= \max(0, y-0.1). \end{aligned}$$

Hence, $R{\downarrow}A(0.95) = 0.85$. Furthermore

$$\begin{aligned} R{\uparrow}A(y) &= \sup_{z\in X} \mathcal{T}(A(z), R(z,y)) \\ &= \sup\{z \mid z \in X \wedge z \in ]y-0.1, y+0.1[\} \\ &= \min(1, y+0.1) \end{aligned}$$

hence $R{\uparrow}A(0.05) = 0.15$. We verify

$$\begin{aligned} &(R{\downarrow}{\downarrow}A)(0.95) \\ &\quad= \inf_{z\in X} \mathcal{I}(R(z,0.95), \max(0, z-0.1)) \\ &\quad= \inf\{\max(0,z-0.1) \mid z \in ]0.85, 1]\} = 0.75 \end{aligned}$$

and

$$\begin{aligned} &(R{\uparrow}{\downarrow}A)(0.95) \\ &\quad= \sup_{z\in X} \mathcal{T}(R(z,0.95), \max(0, z-0.1)) \\ &\quad= \sup\{\max(0,z-0.1) \mid z \in ]0.85, 1]\} = 0.9. \end{aligned}$$

In the same way, one can verify that $(R{\uparrow}{\uparrow}A)(0.05) = 0.25$ and $(R{\downarrow}{\uparrow}A)(0.05) = 0.1$. This illustrates that all the approximations from Definition 3 are different.

### C. Maximal Expansion and Reduction

Taking an upper approximation of $A$ in practice corresponds to expanding $A$, while a lower approximation is meant to reduce $A$. However this refining process does not go on forever. The following property says that with the loose lower and the tight upper approximation maximal reduction and expansion is achieved within one approximation phase.

*Proposition 11: [1]:* If $\mathcal{T}$ is a left continuous t-norm and $\mathcal{I}$ its residual implicator, then for every fuzzy set $A$ in $X$

$$R{\uparrow}{\downarrow}(R{\uparrow}{\downarrow}A) = R{\uparrow}{\downarrow}A \quad \text{and} \quad R{\downarrow}{\uparrow}(R{\downarrow}{\uparrow}A) = R{\downarrow}{\uparrow}A, \tag{26}$$

To investigate the behaviour of the loose upper and tight lower approximation w.r.t. expansion and reduction, we first establish links with the composition of $R$ with itself. Recall that in general the round composition of fuzzy relations $R$ and $S$ in $X$ is the fuzzy relation $R \circ S$ in $X$ defined by

$$(R \circ S)(x,z) = \sup_{y\in X} \mathcal{T}(R(x,y), S(y,z)) \tag{27}$$

for all $x$ and $z$ in $X$.

*Proposition 12:* If $\mathcal{T}$ is a left continuous t-norm then for every fuzzy set $A$ in $X$

$$R{\uparrow}{\uparrow}A = (R \circ R){\uparrow}A. \tag{28}$$

*Proof:* For all $y$ in $X$

$$\begin{aligned} &((R\circ R){\uparrow}A)(y) \\ &\quad= \sup_{x\in X} \mathcal{T}((R\circ R)(x,y), A(x)) \\ &\quad= \sup_{x\in X} \mathcal{T}(\sup_{z\in X} \mathcal{T}(R(x,z), R(z,y)), A(x)) \\ &\quad= \sup_{x\in X}\sup_{z\in X} \mathcal{T}(\mathcal{T}(R(x,z), R(z,y)), A(x)) \\ &\quad= \sup_{x\in X}\sup_{z\in X} \mathcal{T}(R(z,y), \mathcal{T}(R(x,z), A(x))) \\ &\quad= \sup_{z\in X} \mathcal{T}(R(z,y), \sup_{x\in X} \mathcal{T}(R(x,z), A(x))) \\ &\quad= \sup_{z\in X} \mathcal{T}(R(z,y), R{\uparrow}A(z)) \\ &\quad= R{\uparrow}{\uparrow}A(y). \end{aligned}$$

■

*Proposition 13:* If $\mathcal{I}$ is left continuous in its first component and right continuous in its second component, and if $\mathcal{T}$ and $\mathcal{I}$ satisfy the shunting principle

$$\mathcal{I}(\mathcal{T}(x,y), z) = \mathcal{I}(x, \mathcal{I}(y,z)) \tag{29}$$

then for every fuzzy set $A$ in $X$

$$R{\downarrow}{\downarrow}A = (R \circ R){\downarrow}A. \tag{30}$$

The proof of Proposition 13 is analogous to the proof of Proposition 12. Regarding the restrictions placed on the fuzzy logical operators involved, recall that the shunting principle is satisfied both by a left continuous t-norm and its residual implicator [22] as well as by a t-norm and an S-implicator induced by it [26].

Let us use the following notation, for $n > 1$:

$$R^1 = R \quad \text{and} \quad R^n = R \circ R^{n-1}$$

From Proposition 12 it follows that taking $n$ times successively the upper approximation of a fuzzy set under $R$ corresponds to taking the upper approximation once under the composed fuzzy relation $R^n$. Proposition 13 states a similar result for the lower approximation. Recall the following important proposition that holds for reflexive and $\mathcal{T}$-transitive fuzzy relations.

*Proposition 14: [26]:* If $R$ is a fuzzy $\mathcal{T}$-equivalence relation in $X$, then

$$R \circ R = R$$

In other words, using a $\mathcal{T}$-transitive fuzzy relation $R$, options (1a) and (1c) of Definition 3 coincide, as well as options (2b) and (2c). The following property shows that under these conditions, they also coincide with (1b), respectively, (2a).

*Proposition 15: [1], [2], [26]:* If $R$ is a fuzzy $\mathcal{T}$-equivalence relation in $X$, $\mathcal{T}$ is a left continuous t-norm and $\mathcal{I}$ its residual implicator then for every fuzzy set $A$ in $X$

$$R{\uparrow}{\downarrow}A = R{\downarrow}A \quad \text{and} \quad R{\downarrow}{\uparrow}A = R{\uparrow}A. \tag{31}$$

This means that, using a fuzzy $\mathcal{T}$-equivalence relation to model approximate equality, we will obtain maximal reduction or expansion in one phase, regardless of which of the approximations from Definition 3 is used. As Example 10 illustrates, this effect is not always exhibited by non-$\mathcal{T}$-transitive fuzzy relations.

When $R$ is not $\mathcal{T}$-transitive and the universe $X$ is finite, it is known that the $\mathcal{T}$-transitive closure of $R$ is given by $R^{|X-1|}$ (assuming $|X| \geq 2$) [19], hence

$$R \circ R^{|X-1|} = R^{|X-1|}.$$

In other words with the lower and upper approximation, maximal reduction and expansion will be reached in at most $|X-1|$ steps, while with the tight lower and the loose upper approximation it can take at most $\lceil |X-1|/2 \rceil$ steps.

## IV. Related Work and Further Comments

Although—to our knowledge—the tight and loose lower and upper approximations have never been considered in the framework of fuzzy rough set theory, their crisp counterparts have already surfaced in classical rough set theory, albeit from different angles of interpretation. The first one is due to [3]. His approach to rough sets is remarkably different from others because it does not revolve around a notion of indistinguishability or similarity, but around a dual notion of discernibility. This discernibility is represented by a so-called preclusivity relation, which is an irreflexive and symmetrical relation. It can be obtained as the set theoretical complement $co\, R$ of an equivalence relation, or more generally of that of a tolerance relation $R$. Apart from the usual set-theoretical complement $co\, A$ of a set $A$, defined by

$$y \in co\, A \text{ iff } \neg(y \in A) \tag{32}$$

for all $y$ in $X$, Cattaneo also defines the preclusive orthocomplement $R^{\#}(A)$ of $A$:

$$y \in R^{\#}(A) \text{ iff } (\forall x \in X)(x \in A \Rightarrow (x,y) \in co\, R) \tag{33}$$

$R^{\#}(A)$ is the set of elements that are discernible from all elements of $A$. Using also

$$R^{b}(A) = co(R^{\#}(co\, A)). \tag{34}$$

Cattaneo introduces the $\mathcal{P}(X) \to \mathcal{P}(X)$ mappings $\nu$, $\mathbb{I}$, $\mathbb{C}$ and $\mu$ defined by

$$\begin{aligned} \nu(A) &= R^{\#}(co\, A) \quad \text{(necessity)} \\ \mathbb{I}(A) &= R^{b}(R^{b}(A)) \quad \text{(interior)} \\ \mathbb{C}(A) &= R^{\#}(R^{\#}(A)) \quad \text{(closure)} \\ \mu(A) &= co(R^{\#}(A)) \quad \text{(possibility)} \end{aligned}$$

for all $A$ in $\mathcal{P}(X)$. Applying the law of contraposition ($p \Rightarrow q$ if and only if $\neg q \Rightarrow \neg p$) to (33) it is easy to see that

$$R^{\#}(A) = R{\downarrow}(co\, A). \tag{35}$$

Now, for every crisp relation $R$ and every crisp set $A$, $R{\uparrow}(co\, A) = co(R{\downarrow}A)$ and $R{\downarrow}(co\, A) = co(R{\uparrow}A)$ holds. This allows us to derive the following:

$$R^{b}(A) = co(R{\downarrow}A) = R{\uparrow}(co\, A) \tag{36}$$

and

$$\nu(A) = R{\downarrow}A \tag{37}$$

$$\mathbb{I}(A) = R{\uparrow}{\downarrow}A \tag{38}$$

$$\mathbb{C}(A) = R{\downarrow}{\uparrow}A \tag{39}$$

$$\mu(A) = R{\uparrow}A. \tag{40}$$

In [3] Cattaneo himself gives the full expressions of the crisp counterparts of Definition 3, parts 1(c) and 2(c) for $\nu(A)$ and $\mu(A)$, respectively.

In [15] $\mathbb{I}(A)$ and $\mathbb{C}(A)$ are linked to expressions corresponding to the crisp counterparts of Definition 3, parts 1(b) and 2(a), respectively, i.e., what we call tight lower approximation and loose upper approximation. In the crisp case, it makes sense to differentiate between $\nu(A)$ and $\mathbb{I}(A)$, and between $\mu(A)$ and $\mathbb{C}(A)$ if one is dealing with a tolerance relation $R$ which is not an equivalence relation. In [15] it is also suggested to work with "tolerance classes of some iterations of tolerance relations," most likely referring to the composition of relations.

In general in the crisp case the difference between using a tolerance relation and an equivalence relation is clear at first sight, because with the former different similarity classes need not to be disjoint, while with the latter equivalence classes are either equal or disjoint. This implies that when using an equivalence relation, it makes no sense to differentiate between "all classes containing $y$" and "at least one class containing $y$" because there is exactly one class containing $y$, namely $[y]_R$. Hence, the tight, the usual and the loose upper approximation coincide, and so do the lower approximations.

Fuzzy $\mathcal{T}$-equivalence classes are known as the fuzzy counterpart of equivalence relations, but as we illustrated at the beginning of the previous section, they do no longer satisfy the property that fuzzy similarity classes are equal or disjoint; in fact $y$ can at the same time belong to different fuzzy similarity classes to a certain degree. Hence it is not possible at first sight to rule out the usefulness of the tight and loose lower and upper approximations introduced in Definition 3 as alternatives to the existing lower and upper approximation for fuzzy rough sets.

| $R$ | mac | computer | apple | fruit | pie | recipe | store | emulator | hardware |
|---|---|---|---|---|---|---|---|---|---|
| mac | 1.00 | 0.89 | 0.89 | 0.00 | 0.01 | 0.00 | 0.75 | 0.83 | 0.66 |
| computer | | 1.00 | 0.94 | 0.44 | 0.44 | 0.56 | 0.25 | 1.00 | 0.83 |
| apple | | | 1.00 | 0.83 | 0.99 | 0.83 | 0.83 | 0.25 | 0.99 |
| fruit | | | | 1.00 | 0.44 | 0.66 | 1.00 | 0.00 | 0.03 |
| pie | | | | | 1.00 | 1.00 | 0.97 | 0.00 | 0.06 |
| recipe | | | | | | 1.00 | 1.00 | 0.00 | 0.03 |
| store | | | | | | | 1.00 | 0.34 | 0.75 |
| emulator | | | | | | | | 1.00 | 1.00 |
| hardware | | | | | | | | | 1.00 |

Fig. 3. Fuzzy thesaurus.

However careful investigation of the properties of the newly defined approximations show that interplay between suitably chosen fuzzy logical operators and the $\mathcal{T}$-transitivity of the fuzzy relation forces the new approximations to coincide with the existing ones anyway. In the next section we will illustrate that this is not always a desirable property in applications, because it does not allow for gradual expansion or reduction of a fuzzy set by iteratively taking approximations. Omitting the requirement of $\mathcal{T}$-transitivity is precisely the key that allows for a gradual expansion process.

Other undesirable effects of $\mathcal{T}$-transitivity w.r.t. approximate equality were pointed out in [7], [8]. More in particular it is observed there that fuzzy $\mathcal{T}$-equivalence relations can never satisfy the so-called Poincaré paradox. A fuzzy relation $R$ in $X$ is compatible with the Poincaré paradox iff

$$(\exists(x,y,z) \in X^3)(R(x,y) = 1 \wedge R(y,z) = 1 \wedge R(x,z) < 1)$$

This is inspired by Poincaré's [16] experimental observation that a bag of sugar of 10 grammes and a bag of 11 grammes can be perceived as indistinguishable by a human being. The same applies for a bag of 11 grammes w.r.t. a bag of 12 grammes, while the subject is perfectly capable of noting a difference between the bags of 10 and 12 grammes. Now if $R$ is a fuzzy $\mathcal{T}$-equivalence relation, then $R(x,y) = 1$ implies $Rx = Ry$ [11]. Since $Ry(z) = R(y,z) = 1$, also $Rx(z) = R(x,z) = 1$ which is in conflict with $R(x,z) < 1$. The fact that they are not compatible with the Poincaré paradox makes fuzzy $\mathcal{T}$-equivalence relations less suited to model approximate equality. The main underlying cause for this conflict is $\mathcal{T}$-transitivity.

## V. Query Refinement

One of the most common ways to retrieve information from the WWW is keyword based search: the user inputs a query consisting of one or more keywords and the search system returns a list of web documents ranked according to their relevance to the query. The same procedure is often used in e-commerce applications that attempt to relate the user's query to products from the catalogue of some company.

In the basic approach documents are not returned as search results if they do not contain (one of) the exact keywords of the query. To satisfy the user who expects search engines to come up with "what they mean and not what they say", more sophisticated techniques are needed. One option are so-called query refinement techniques that adapt the original query by adding terms that are related to the initial keywords. This requires the use of a fuzzy term-term relation, called a fuzzy thesaurus. An overview of approaches to automatically construct such fuzzy thesauri based on the co-occurrences of terms in documents is presented in [9].

Let us think of a query as a fuzzy set of keywords. If the user cares to give weights to indicate the importance of the individual keywords, these will be taken into account as the membership of the terms in the query. However we expect that in many cases the user does not want to be bothered with the need to specify his query in such detail. In these cases 1 will be used as the default membership degree of a term appearing in the query. One of the advantages of our query refinement approach is that these initial weights will be gradually adapted during the search process.

Formally, let $X$ denote the universe of terms and let the original query $A$ be a fuzzy set in $X$. Furthermore let $R$ be a fuzzy thesaurus, then query $A$ can be expanded by taking its upper approximation under $R$. In [17] a formally similar idea is promoted to expand the fuzzy set of terms associated with a document (instead of a query); there it is also suggested to use the transitive closure of $R$. In [27] the connection between query expansion and fuzzy rough sets is established.

In this section we will show that using a $\mathcal{T}$-transitive fuzzy thesaurus may result in adding too many irrelevant keywords. However even with a non $\mathcal{T}$-transitive fuzzy thesaurus we can easily run into the same problem when using the upper or the loose upper approximation. We therefore promote the use of the newly introduced tight upper approximation.

To illustrate our point, we constructed the small fuzzy thesaurus shown in Fig. 3 by taking into account the number of web pages found by a search engine for each pair of terms, as shown in Fig. 4. Let $D_{t_1}$ and $D_{t_2}$ denote the number of web pages that contain term $t_1$, respectively term $t_2$; these numbers can be found on the diagonal in Fig. 4. On the WWW there is a strong bias towards computer science related terms, hence the absolute number of web pages containing both term $t_1$ and $t_2$ cannot be used directly to express the strength of the relationship between $t_1$ and $t_2$. To level out the difference, we used the following measure:

$$\frac{|D_{t_1} \cap D_{t_2}|}{\min(|D_{t_1}|, |D_{t_2}|)}$$

as shown in Fig. 5. Finally we normalized the result using the S-function $S(.; 0.03, 0.20)$ (cfr. Fig. 6), giving rise to the fuzzy thesaurus of Fig. 3. To compute its transitive closure, depicted in Fig. 7, we used t-norm $\mathcal{T}_{\mathrm{W}}$ as we will do throughout this section. Furthermore we will keep on using $\mathcal{I}_{\mathrm{W}}$ because it is at the same

| | mac | computer | apple | fruit | pie | recipe | store | emulator | hardware |
|---|---|---|---|---|---|---|---|---|---|
| mac | 114000 | 18300 | 14900 | 1030 | 869 | 899 | 15800 | 672 | 15100 |
| computer | | 375000 | 15600 | 3760 | 2220 | 3720 | 29500 | 1170 | 26900 |
| apple | | | 93400 | 5420 | 3810 | 4590 | 14300 | 401 | 17800 |
| fruit | | | | 35400 | 2320 | 4080 | 7630 | 47 | 1630 |
| pie | | | | | 20400 | 4210 | 3740 | 30 | 1200 |
| recipe | | | | | | 31500 | 6220 | 35 | 1690 |
| store | | | | | | | 312000 | 472 | 24900 |
| emulator | | | | | | | | 4950 | 1050 |
| hardware | | | | | | | | | 178000 |

Fig. 4. Number of thousands of web pages found by Google.

| | mac | computer | apple | fruit | pie | recipe | store | emulator | hardware |
|---|---|---|---|---|---|---|---|---|---|
| mac | | 0.16 | 0.16 | 0.03 | 0.04 | 0.03 | 0.14 | 0.15 | 0.13 |
| computer | | | 0.17 | 0.11 | 0.11 | 0.12 | 0.09 | 0.25 | 0.15 |
| apple | | | | 0.15 | 0.19 | 0.15 | 0.15 | 0.09 | 0.19 |
| fruit | | | | | 0.11 | 0.13 | 0.22 | 0.01 | 0.05 |
| pie | | | | | | 0.21 | 0.18 | 0.01 | 0.06 |
| recipe | | | | | | | 0.20 | 0.01 | 0.05 |
| store | | | | | | | | 0.10 | 0.14 |
| emulator | | | | | | | | | 0.23 |
| hardware | | | | | | | | | |

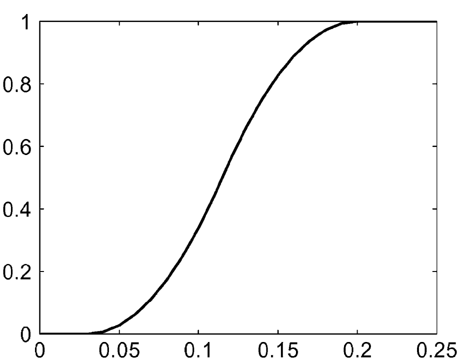
Fig. 5. Unnormalized fuzzy thesaurus.

time a residual and an S-implicator. Now, let us consider the query

$$\text{apple, pie, recipe}$$

as shown in the 2nd column in Fig. 8. The meaning of the ambiguous word "apple," which can refer both to a piece of fruit and to a computer company, is clear in this query. The disadvantage of using a $\mathcal{T}$-transitive fuzzy thesaurus becomes apparent when we compute the upper approximation $R^8{\uparrow}A$, shown in the fifth column. All the terms are added with high degrees, even though terms like "mac" and "computer" have nothing to do with the semantics of the original query. This process can be slowed down a little bit by using the non $\mathcal{T}$-transitive fuzzy thesaurus and computing $R{\uparrow}A$ which allows for some gradual refinement. However, an irrelevant term such as "emulator" shows up to a high degree in the second iteration, i.e., when computing $R{\uparrow}{\uparrow}A$.

The main problem with the query expansion process, even if it is gradual, is a fast growth of the number of less relevant or irrelevant keywords that are automatically added. This effect is caused by the use of a flexible definition of the upper approximations in which a term is added to a query as soon as it is related to one of its keywords. Therefore, we suggest the use of the tight upper approximation $R{\downarrow}{\uparrow}A$: a term $t$ will only be added to a query if all the terms that are related to $t$ are also related to at least one keyword of the query. First, the usual upper approximation of the query is computed but then it is stripped down by omitting all terms that are also related to other terms not belonging to this upper approximation. In this way terms that are sufficiently relevant, hence related to most keywords in $A$, will form a more or less closed context with few or no links outside, while a term related to only one of the keywords in $A$ in general also has many links to other terms outside $R{\uparrow}A$ and hence is omitted by taking the lower approximation. The last column of Fig. 8 shows that the tight upper approximation performs clearly better: irrelevant words such as "mac," "computer," and "hardware" are still added to the query, but to a significantly lower degree.




$$S(x;\alpha,\gamma)=\begin{cases}0, & \text{if } x\leq\alpha\\ \frac{2(x-\alpha)^2}{(\gamma-\alpha)^2}, & \text{if } \alpha\leq x\leq(\alpha+\gamma)/2\\ 1-\frac{2(x-\gamma)^2}{(\gamma-\alpha)^2}, & \text{if } (\alpha+\gamma)/2\leq x\leq\gamma\\ 1, & \text{if } \gamma\leq x\end{cases}$$

Fig. 6. S-function; $x$, $\alpha$, and $\gamma$ in $\mathbb{R}$, $\alpha < \gamma$.

## VI. Conclusion

Exploiting the truly fuzzy characteristic that an element can belong to some degree to different fuzzy similarity classes at the same time, we have defined the tight lower approximation $R{\downarrow}{\downarrow}A$, the loose lower approximation $R{\uparrow}{\downarrow}A$, the tight upper approximation $R{\downarrow}{\uparrow}A$ and the loose upper approximation $R{\uparrow}{\uparrow}A$ of a fuzzy set $A$ under a reflexive and symmetrical fuzzy relation $R$. For any left continuous t-norm $\mathcal{T}$ and its residual implicator $\mathcal{I}$ it holds that

$$R{\downarrow}{\downarrow}A\subseteq R{\downarrow}A\subseteq R{\uparrow}{\downarrow}A\subseteq A\subseteq R{\downarrow}{\uparrow}A\subseteq R{\uparrow}A\subseteq R{\uparrow}{\uparrow}A$$

| $R^8$ | mac | computer | apple | fruit | pie | recipe | store | emulator | hardware |
|---|---|---|---|---|---|---|---|---|---|
| mac | 1.00 | 0.89 | 0.89 | 0.88 | 0.88 | 0.88 | 0.88 | 0.89 | 0.89 |
| computer | | 1.00 | 0.99 | 0.99 | 0.99 | 0.99 | 0.99 | 1.00 | 1.00 |
| apple | | | 1.00 | 0.99 | 0.99 | 0.99 | 0.99 | 0.99 | 0.99 |
| fruit | | | | 1.00 | 1.00 | 1.00 | 1.00 | 0.99 | 0.99 |
| pie | | | | | 1.00 | 1.00 | 1.00 | 0.99 | 0.99 |
| recipe | | | | | | 1.00 | 1.00 | 0.99 | 0.99 |
| store | | | | | | | 1.00 | 0.99 | 0.99 |
| emulator | | | | | | | | 1.00 | 1.00 |
| hardware | | | | | | | | | 1.00 |

Fig. 7. Transitive closure of fuzzy thesaurus.

| | $A$ | $R{\uparrow}A$ | $R{\uparrow\uparrow}A$ | $R^8{\uparrow}A$ | $R{\downarrow\uparrow}A$ |
|---|---|---|---|---|---|
| mac | 0.00 | 0.89 | 0.89 | 0.89 | 0.42 |
| computer | 0.00 | 0.94 | 0.94 | 0.99 | 0.25 |
| apple | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| fruit | 0.00 | 0.83 | 1.00 | 1.00 | 0.83 |
| pie | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| recipe | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| store | 0.00 | 1.00 | 1.00 | 1.00 | 0.83 |
| emulator | 0.00 | 0.25 | 0.99 | 0.99 | 0.25 |
| hardware | 0.00 | 0.99 | 0.99 | 0.99 | 0.25 |

Fig. 8. Original and expanded queries.

which justifies the names of the new approximations. When $R$ is in addition $\mathcal{T}$-transitive, the new approximations coincide with the existing ones. However when $R$ is not $\mathcal{T}$-transitive, the approximations can be different and allow for a gradual expansion process. We have shown the practical relevance of our results for query refinement.

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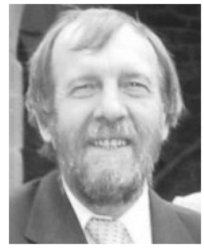

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Since 1984, he has been a Lector, and since 1991, a Full Professor at Ghent University. In 1976, he founded the Fuzziness and Uncertainty Modelling Research Unit (FUM) and since then his research has been focused on the modelling of fuzziness and uncertainty, and has resulted in a great number of contributions in fuzzy set theory and its various generalizations. The theories of fuzzy relational calculus and of fuzzy mathematical structures owe a great deal to him. Over the years he has also been a supervisor of 20 Ph.D.s on fuzzy set theory. His current research interests include fuzzy and intuitionistic fuzzy relations, fuzzy topology, and fuzzy image processing. He has authored or coauthored eleven books and more than 350 papers appeared in international refereed journals and proceedings.

Dr. Kerre is a referee for 50 international scientific journals, and also a member of the editorial board of international journals and conferences on fuzzy set theory. He was an honorary chairman at various international conferences.