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Fuzzy Soft Games

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Abstract. In this work, after introduce the definition of fuzzy soft sets and their basic operations we define two person fuzzy soft games which can apply to problems contain vagueness and uncertainty. We then give four different types solution methods of the games which are fuzzy soft saddle points, fuzzy soft lower and upper values, fuzzy soft dominated strategies and fuzzy soft Nash equilibrium. We also give a probabilistic equilibrium solution method for the two persons fuzzy soft games ($tpfs$-games) which show that every finite $tpfs$-game with probabilistic strategies has one solution. Finally, we give an application which shows that the methods can be successfully applied to a financial problem and extended the two person fuzzy soft games to n-person fuzzy soft games.

1. Introduction

Soft set theory, was first introduced by Molodtsov [39] in 1999 for dealing with uncertain, not clearly defined objects. In [39], Molodtsov gave applications of soft sets on some fields, such as Riemann-integration, probability, game theory, operations research and so on. Recently, works on soft set theory has been progressing rapidly and is finding applications in a wide variety of fields, for examples; theory of soft sets (e.g: [2, 11, 18, 35, 43]), soft decision making (e.g: [12, 13, 22, 23, 27, 36]), algebraic structures of soft sets (e.g: [1, 3, 26]), soft topologies (e.g: [15, 37, 42, 44, 49]), fuzzy soft sets (e.g: [16, 17]), intuitionistic fuzzy soft sets (e.g: [24, 25, 33, 34, 38]), neutrosophic parameterized soft set theory (e.g: [6, 20]) and interval valued neutrosophic soft sets (e.g: [5, 7]).

Game theory was introduced by von Neumann and Morgenstern [40] in 1944 and then they started modern game theory. Game theory has successfully used in logic, decision making process, economics, political science, computer science and so on. In recent years, many interesting applications of game theory have been expanded by embedding the ideas of fuzzy sets (e.g. [9, 10, 28, 30, 31, 47]). The game theory have also been expanded by using the ideas of interval data (e.g. [21, 29, 32]). The linear programming problems with fuzzy parameters is introduced (e.g. [4, 8]).

The notion of soft games is given by Çağman and Deli in [14]. In this work, we define a fuzzy soft game for dealing with uncertainties that is based on both soft sets and fuzzy sets. Therefore, payoff functions of the fuzzy soft game are set valued function and solution of the soft games obtained by using the operations of soft sets and fuzzy sets that make this game very convenient and easily applicable in practice.

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This paper is organized as follows. In the next section, the fundamental definition and most of operations of fuzzy soft sets are presented. In Section 3, we construct two person fuzzy soft games and then give four different solution methods for the games which are soft saddle points, soft lover and soft upper value, soft dominated strategies and soft Nash equilibrium. In section 4, we give a probabilistic equilibrium solution method for the two persons fuzzy soft games ($tpfs$-games). In section 5, we give an application for two person fuzzy soft games. In section 6, we give n-person fuzzy soft games that is extension of the two person fuzzy soft games. In final Section, we concluded the work.

The present expository paper is a condensation of part of the dissertation [19].

2. Preliminary

In this section, we introduce the basic definitions of soft sets [11, 39], fuzzy sets [48] and fuzzy soft sets [17] which are useful for subsequent discussions. More detailed explanations related to the soft sets, fuzzy sets and fuzzy soft sets can be found in [2, 11, 18, 35], [48] and [16, 17]), respectively.

Notion of the soft set theory is first given by Molodtsov [39]. Then the definition of soft set is modified by Çağman and Enginoğlu [11] as follows.

Definition 2.1. *[11] Let U be a universe, E be a set of parameters that are describe the elements of U, and $A \subseteq E$. Then, a soft set F_A over U is a set defined by a set valued function f_A representing a mapping*

$$f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \in E - A$$

where f_A is called approximate function of the soft set F_A. Generally, f_A, g_B, h_C, ... will be used as an approximate functions of F_A, G_B, H_C, ..., respectively. The value of approximate function $f(x)$ may be arbitrary, some of them may be empty, some may have nonempty intersection.

It is noting that the soft set is a parametrized family of subsets of the set U, and therefore it can be written a set of ordered pairs

$$F_A = \{(x, f_A(x)) : x \in E\}$$

The subscript A in the f_A indicates that f_A is the approximate function of F_A.

Note that if $f_A(x) = \emptyset$, then the element $(x, f_A(x))$ is not appeared in F_A.

Definition 2.2. *[48] Let U be the universe. A fuzzy set X over U is a set defined by a membership function μ_X representing a mapping*

$$\mu_X : U \to [0, 1].$$

The value $\mu_X(x)$ for the fuzzy set X is called the membership value or the grade of membership of $x \in U$. The membership value represents the degree of x belonging to the fuzzy set X. Then a fuzzy set X on U can be represented as follows,

$$X = \{(\mu_X(x)/x) : x \in U, \mu_X(x) \in [0, 1]\}.$$

Definition 2.3. *[17] Let U be an initial universe, $F(U)$ be all fuzzy sets over U. E be the set of all parameters and $A \subseteq E$. An fuzzy soft set Γ_A on the universe U is defined by the set of ordered pairs as follows,*

$$\Gamma_A = \{(x, \gamma_A(x)) : x \in E, \gamma_A(x) \in F(U)\}$$

where $\gamma_A : E \to F(U)$ such that $\gamma_A(x) = \emptyset$ if $x \notin A$, and for all $x \in E$

$$\gamma_{A(x)} = \{\mu_{\gamma_{A(x)}}(u)/u : u \in U, \mu_{\gamma_{A(x)}}(u) \in [0, 1]\}$$

is a fuzzy set over U.

The subscript A in the γ_A indicates that γ_A is the approximate function of Γ_A.

Note that if $\gamma_A(x) = \emptyset$, then the element $(x, \gamma_A(x))$ is not appeared in Γ_A.

Example 2.4. *Suppose that $U = \{u_1, u_2, u_3, u_4\}$ is the universe contains four house under consideration in an auto agent and $E = \{x_1, x_2, x_3, x_4\}$ is the set of parameters, where x_i $(i = 1, 2, 3, 4)$ stand for 'garden", "cheap", "modern" and "large", respectively.*

A customer to select a house from the real estate agent, can construct a fuzzy soft set Γ_A that describes the characteristic of houses according to own requests. Assume that $A = \{x_1, x_2, x_3, x_4\} \subseteq E$ and $\gamma_A(x_1) = \{0.4/u_1, 0.3/u_4\}$, $\gamma_A(x_2) = \{0.5/u_2\}$, $\gamma_A(x_3) = \emptyset$, $\gamma_A(x_4) = \{0.2/u_1, 0.8/u_2\}$ then the fuzzy soft-set Γ_A is written by

$$\Gamma_A = \{(x_1, \{0.4/u_1, 0.3/u_4\}), (x_2, \{0.5/u_2\}), (x_4, \{0.2/u_1, 0.8/u_2\})\}$$

By using same parameter set A, another customer to select a house from the same real estate agent, can construct a fuzzy soft set $\acute{\Gamma}_A$ according to own requests. Here $\acute{\Gamma}_A$ may be different then Γ_A. Assume that $\acute{\gamma}_A(x_1) = \emptyset$, $\acute{\gamma}_A(x_2) = \{0.3/u_2, 0.5/u_3, 0.1/u_4\})$, $\acute{\gamma}_A(x_3) = \{0.4/u_1, 0.4/u_2, 0.9/u_3\}$, $\acute{\gamma}_A(x_4) = \{0.7/u_4\}$, then the fuzzy soft set $\acute{\Gamma}_A$ is written by

$$\acute{\Gamma}_A = \{(x_2, \{0.3/u_2, 0.5/u_3, 0.1/u_4\}), (x_3, \{0.4/u_1, 0.4/u_2, 0.9/u_3\}), (x_4, \{0.7/u_4\})\}$$

Definition 2.5. *[17] Let Γ_A and Γ_B be two fuzzy soft sets. Then,*

a) *Complement of Γ_A is denoted by $\Gamma_A^{\tilde{c}}$. Its approximate function $\gamma_{A^{\tilde{c}}}$ is defined by*

$$\gamma_{A^{\tilde{c}}}(x) = \gamma_A^c(x), \text{ for all } x \in E,$$

b) *Union of Γ_A and Γ_B is denoted by $\Gamma_A \tilde{\cup} \Gamma_B$. Its fuzzy approximate function $\gamma_{A\tilde{\cup}B}$ is defined by*

$$\gamma_{A\tilde{\cup}B}(x) = \gamma_A(x) \cup \gamma_B(x) \quad \text{for all } x \in E.$$

c) *Intersection of Γ_A and Γ_B is denoted by $\Gamma_A \tilde{\cap} \Gamma_B$. Its fuzzy approximate function $\gamma_{A\tilde{\cap}B}(x)$ is defined by*

$$\gamma_{A\tilde{\cap}B}(x) = \gamma_A(x) \cap \gamma_B(x) \quad \text{for all } x \in E.$$

d) *Γ_A is an fuzzy soft subset of Γ_B, denoted by $\Gamma_A \tilde{\subseteq} \Gamma_B$, if $\gamma_A(x) \subseteq \gamma_B(x)$ for all $x \in E$.*

3. Two Person Fuzzy Soft Games

In this section, we construct two person fuzzy soft games with fuzzy soft payoffs. We then give four solution methods for the fuzzy soft games. For basic definitions and preliminaries of the soft set, game and soft game theory we refer to [14, 18, 39–41, 45].

In the soft game [14], the strategy sets and the soft payoffs are crisp. But in fuzzy soft game, while the strategy sets are crisp, the fuzzy soft payoffs are fuzzy subsets of U. To avoid the confusion we will use Γ_A^k, Γ_B^k, Γ_C^k,..., etc. for two person fuzzy soft game and γ_A^k, γ_B^k, γ_C^k,..., etc. for their fuzzy soft payoffs, respectively.

In the following, some definition and results on game theory defined in [40, 41, 45], we extend this definition to fuzzy soft game by using fuzzy soft set.

Definition 3.1. *Let E be a set of strategy and $X, Y \subseteq E$. A choice of behaviour in a fuzzy soft game is called an action. The elements of $X \times Y$ are called action pairs. That is, $X \times Y$ is the set of available actions.*

Definition 3.2. *Let U be a set of alternatives, $F(U)$ be all fuzzy sets over U, E be a set of strategies, $X, Y \subseteq E$. Then, a set valued function*

$$\gamma_{X\times Y} : X \times Y \to F(U)$$

is called a fuzzy soft payoff function. For each $(x, y) \in X \times Y$, the value $\gamma_{X\times Y}(x, y)$ is called a fuzzy soft payoff.

Definition 3.3. *Let $X \times Y$ be a set of action pairs. Then, an action $(x^*, y^*) \in X \times Y$ is called an optimal action if*

$$\gamma_{X\times Y}(x^*, y^*) \supseteq \gamma_{X\times Y}(x, y) \text{ for all } (x, y) \in X \times Y.$$

Definition 3.4. *Let $X \times Y$ be a set of action pairs and $(x_i, y_j), (x_r, y_s) \in X \times Y$. Then,*

a) *if $\gamma_{X\times Y}(x_i, y_j) \supset \gamma_{X\times Y}(x_r, y_s)$, we says that a player strictly prefers action pair (x_i, y_j) over action (x_r, y_s),*
b) *if $\gamma_{X\times Y}(x_i, y_j) = \gamma_{X\times Y}(x_r, y_s)$, we says that a player is indifferent between the two actions,*
c) *if $\gamma_{X\times Y}(x_i, y_j) \supseteq \gamma_{X\times Y}(x_r, y_s)$, we says that a player either prefers (x_i, y_j) to (x_r, y_s) or is indifferent between the two actions.*

Definition 3.5. *Let $\gamma^k_{X\times Y}$ be a fuzzy soft payoff for Player k, $(k = 1, 2)$, and (x_i, y_j), $(x_r, y_s) \in X \times Y$. Then, Player k is called rational, if the player's fuzzy soft payoff satisfies the following conditions:*

a) *Either $\gamma^k_{X\times Y}(x_i, y_j) \supseteq \gamma^k_{X\times Y}(x_r, y_s)$ or $\gamma^k_{X\times Y}(x_r, y_s) \supseteq \gamma^k_{X\times Y}(x_i, y_j)$*
b) *If $\gamma^k_{X\times Y}(x_i, y_j) \supseteq \gamma^k_{X\times Y}(x_r, y_s)$ and $\gamma^k_{X\times Y}(x_r, y_s) \supseteq \gamma^k_{X\times Y}(x_i, y_j)$, then $\gamma^k_{X\times Y}(x_i, y_j) = \gamma^k_{X\times Y}(x_r, y_s)$.*

Definition 3.6. *Let X and Y be a set of strategies of Player 1 and 2, respectively, U be a set of alternatives and $\gamma^k_{X\times Y} : X \times Y \to F(U)$ be a fuzzy soft payoff function for player k, $(k = 1, 2)$. Then, for each Player k, a two person fuzzy soft game (tpfs-game) is defined by a fuzzy soft set over U as*

$$\Gamma^k_{X\times Y} = \{((x, y), \gamma^k_{X\times Y}(x, y)) : (x, y) \in X \times Y\}$$

The tpfs-game is played as follows: at a certain time Player 1 chooses a strategy $x_i \in X$, simultaneously Player 2 chooses a strategy $y_j \in Y$ and once this is done each player k (k=1,2) receives the fuzzy soft payoff $\gamma^k_{X\times Y}(x_i, y_j)$.

If $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$, then the fuzzy soft payoffs of $\Gamma^k_{X\times Y}$ can be arranged in the form of the $m \times n$ matrix shown in Table 1.

$\Gamma^k_{X\times Y}$	y_1	y_2	...	y_n
x_1	$\gamma^k_{X\times Y}(x_1, y_1)$	$\gamma^k_{X\times Y}(x_1, y_2)$	...	$\gamma^k_{X\times Y}(x_1, y_n)$
x_2	$\gamma^k_{X\times Y}(x_2, y_1)$	$\gamma^k_{X\times Y}(x_2, y_2)$	...	$\gamma^k_{X\times Y}(x_2, y_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_m	$\gamma^k_{X\times Y}(x_m, y_1)$	$\gamma^k_{X\times Y}(x_m, y_2)$	...	$\gamma^k_{X\times Y}(x_m, y_n)$

Table 1: The two person fuzzy soft game

Now, we can give an example for *tpfs*-game.

Example 3.7. *Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$be a set of alternatives, $F(U)$ be all fuzzy sets over U, $E = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of strategies and $X = \{x_1, x_2, x_4\}$ and $Y = \{x_1, x_2\}$ be a set of the strategies Player 1 and 2, respectively.*

If Player 1 constructs a tpfs-games as follows,

$$\Gamma^1_{X\times Y} = \Big\{((x_1, x_1), \{0.7/u_1, 0.6/u_2, 0.4/u_5\}), (x_1, x_2), \{0.2/u_1, 0.3/u_2, 0.8/u_3, 0.1/u_4, 0.9/u_5\}), ((x_3, x_1), \{0.8/u_1, 0.1/u_3\}), (x_3, x_2), \{0.5/u_1, 0.3/u_2, 0.8/u_3, 0.7/u_5\}), ((x_5, x_1), \{0.5/u_3, 0.7/u_5, 0.3/u_4\}), (x_5, x_2), \{0.5/u_1, 0.6/u_2, 0.5/u_3, 0.7/u_4, 0.3/u_5\})\Big\}$$

then the fuzzy soft payoffs of the game can be arranged as in Table 2,

$\Gamma^1_{X\times Y}$	x_1	x_2
x_1	$\{0.7/u_1, 0.6/u_2, 0.4/u_5\}$	$\{0.2/u_1, 0.3/u_2, 0.8/u_3, 0.1/u_4, 0.9/u_5\}$
x_3	$\{0.8/u_1, 0.1/u_3\}$	$\{0.5/u_1, 0.3/u_2, 0.8/u_3, 0.7/u_5\}$
x_5	$\{0.5/u_3, 0.7/u_5, 0.3/u_4\}$	$\{0.5/u_1, 0.6/u_2, 0.5/u_3, 0.7/u_4, 0.3/u_5\}$

Table 2

Let us explain some element of this game; if Player 1 select x_3 and Player 2 select x_2, then the value of game will be a fuzzy soft payoff $\gamma^1_{X\times Y}(x_3, x_2) = \{0.5/u_1, 0.3/u_2, 0.8/u_3, 0.7/u_5\}$. In this case, Player 1 wins the set of alternatives $\{0.5/u_1, 0.3/u_2, 0.8/u_3, 0.7/u_5\}$ and Player 2 lost the same set of alternatives.

Similarly, if Player 2 constructs a tpfs-game as follows,

$$\Gamma^2_{X\times Y} = \Big\{((x_1, x_1), \{0.7/u_3, 0.6/u_4, 0.4/u_6\}), (x_1, x_2), \{0.2/u_6\}), ((x_3, x_1), \{0.8/u_2, 0.1/u_4, 0.3/u_5, 0.8/u_6\}), (x_3, x_2), \{0.5/u_4, 0.3/u_6\}), ((x_5, x_1), \{0.5/u_1, 0.7/u_2, 0.3/u_6\}), (x_5, x_2), \{0.5/u_6\})\Big\}$$

then the fuzzy soft payoffs of the game can be arranged as in Table 3,

$\Gamma^2_{X\times Y}$	x_1	x_2
x_1	$\{0.7/u_3, 0.6/u_4, 0.4/u_6\}$	$\{0.2/u_6\}$
x_3	$\{0.8/u_2, 0.1/u_4, 0.3/u_5, 0.8/u_6\}$	$\{0.5/u_4, 0.3/u_6\}$
x_5	$\{0.5/u_1, 0.7/u_2, 0.3/u_6\}$	$\{0.5/u_6\}$

Table 3

Let us explain some element of this tpfs-game; if Player 1 select x_3 and Player 2 select x_2, then the value of game will be fuzzy soft payoff $\gamma^2_{X\times Y}(x_3, x_2) = \{0.5/u_4, 0.3/u_6\}$. In this case, Player 1 wins the set of alternatives $\{0.5/u_4, 0.3/u_6\}$ and Player 2 lost $\{0.5/u_4, 0.3/u_6\}$.

Now the two person zero sum game on the classical game theory will be a two person empty intersection game on the fuzzy soft game theory. It is given in following definition.

Definition 3.8. *A tpfs-game is called a two person empty intersection fuzzy soft game if intersection of the fuzzy soft payoff of players is empty set for each action pairs.*

For instance, Example 3.7 is a two person empty intersection fuzzy soft game.

Definition 3.9. *Let $\gamma^k_{X\times Y}$ be a fuzzy soft payoff function of a tpfs-game $\Gamma^k_{X\times Y}$. If the following properties hold*

a) $\bigcup_{i=1}^{m} \gamma^k_{X\times Y}(x_i, y_j) = \gamma^k_{X\times Y}(x, y)$

b) $\bigcap_{j=1}^{n} \gamma^k_{X\times Y}(x_i, y_j) = \gamma^k_{X\times Y}(x, y)$

then $\gamma^k_{X\times Y}(x, y)$ is called a fuzzy soft saddle point value and (x, y) is called a fuzzy soft saddle point of Player k's in the tpfs-game.

Note that if (x, y) is a fuzzy soft saddle point of a *tpfs*-game $\Gamma^1_{X\times Y}$, then Player 1 can then win at least by choosing the strategy $x \in X$, and Player 2 can keep her/his loss to at most $\gamma^1_{X\times Y}(x, y)$ by choosing the strategy $y \in Y$. Hence the fuzzy soft saddle point is a value of the *tpfs*-game.

Example 3.10. *Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$ be a set of alternatives, $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be the strategies for Player 1 and 2, respectively. Then, tpfs-game of Player 1 is given as in Table 4,*

$\Gamma^1_{X\times Y}$	y_1	y_2
x_1	$\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$	$\{0.9/u_1, 0.6/u_2, 0.6/u_3, 0.9/u_4, 0.5/u_5\}$
x_2	$\{0.7/u_1, 0.1/u_2\}$	$\{0.5/u_1, 0.8/u_3, 0.8/u_5\}$
x_3	$\{0.2/u_1, 0.1/u_4\}$	$\{0.5/u_1, 0.5/u_3, 0.7/u_4\}$

Table 4

Clearly,

$$\bigcup_{i=1}^{3} \gamma^1_{X\times Y}(x_i, y_1) = \{0.8/u_1, 0.4/u_2, 0.6/u_4\},$$
$$\bigcup_{i=1}^{3} \gamma^1_{X\times Y}(x_i, y_2) = \{0.9/u_1, 0.6/u_2, 0.8/u_3, 0.9/u_4, 0.8/u_5\}$$

and

$$\bigcap_{j=1}^{2} \gamma^1_{X\times Y}(x_1, y_j) = \{0.8/u_1, 0.4/u_2, 0.6/u_4\},$$
$$\bigcap_{j=1}^{2} \gamma^1_{X\times Y}(x_2, y_j) = \{0.5/u_1\},$$
$$\bigcap_{j=1}^{2} \gamma^1_{X\times Y}(x_3, y_j) = \{0.2/u_1, 0.1/u_4\}.$$

Therefore, $\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$ *is a fuzzy soft saddle point value of the* $tpfs$*-game, since the intersection of the forth row is equal to the union of the third column. So, the value of the* $tpfs$*-game is* $\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$.

Note that every $tpfs$-game has not a fuzzy soft saddle point. (For instance, in the above example, if $\{0.8/u_1, 0.4/u_2\}$ is replaced with $\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$ in fuzzy soft payoff $\gamma^1_{X\times Y}(x_1, y_1)$, then a fuzzy soft saddle point of the game can not be found.) Saddle point can not be used for a $tpfs$-game, fuzzy soft upper and fuzzy soft lower values of the $tpfs$-game may be used is given in the following definition.

Definition 3.11. *Let* $\Gamma_{X\times Y}$ *be a* $tpfs$*-game with its fuzzy soft payoff function* $\gamma_{X\times Y}$*. Then,*

i. *Fuzzy soft upper value of the* $tpfs$*-game, denoted* $\overline{v}$*, is defined by*

$$\overline{v} = \cap_{y\in Y}(\cup_{x\in X}(\gamma_{X\times Y}(x, y)))$$

ii. *Fuzzy soft lower value of the* $tpfs$*-game, denoted* $\underline{v}$*, is defined by*

$$\underline{v} = \cup_{x\in X}(\cap_{y\in Y}(\gamma_{X\times Y}(x, y)))$$

iii. *If fuzzy soft upper and fuzzy soft lower value of a* $tpfs$*-game are equal, they are called value of the* $tpfs$*-game, noted by* v*. That is* $v = \underline{v} = \overline{v}$*.*

Example 3.12. *Let us consider Table 4 in Example 3. It is clear that fuzzy soft upper value* $\overline{v} = \{0.8/u_1, 0.4/u_2, 0.6/u_4\}$ *and fuzzy soft lower value* $\underline{v} = \{0.8/u_1, 0.4/u_2, 0.6/u_4\}$*, hence* $\underline{v} = \overline{v}$*. It means that value of the* $tpfs$*-game is* $\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$.

Theorem 3.13. $\underline{v}$ *and* $\overline{v}$ *be a fuzzy soft lower and fuzzy soft upper value of a* $tpfs$*-game, respectively. Then, the fuzzy soft lower value is subset or equal to the fuzzy soft upper value, that is,*

$$\underline{v} \subseteq \overline{v}$$

Proof: Assume that $\underline{v}$ be a fuzzy soft lower value, $\overline{v}$ be a fuzzy soft upper value of a $tpfs$-game and $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ are sets of the strategies for Player 1 and 2, respectively.

We choose $x^*_i \in X$ and $y^*_j \in Y$. Then,

$$\begin{aligned}
\underline{v} &= \cup_{x\in X}(\cap_{y\in Y}(\gamma_{X\times Y}(x, y))) \\
&\subseteq \cap_{y\in Y}(\gamma_{X\times Y}(x^*, y)) \\
&\subseteq \gamma_{X\times Y}(x^*, y^*) \\
&\subseteq \cup_{x\in X}(\gamma_{X\times Y}(x, y^*)) \\
&\subseteq \cap_{y\in Y}(\cup_{x\in X}(\gamma_{X\times Y}(x, y)))
\end{aligned}$$

i.e.:

$$\underline{v} = \cup_{x\in X}(\cap_{y\in Y}(\gamma_{X\times Y}(x, y))) \subseteq \overline{v} = \cap_{y\in Y}(\cup_{x\in X}(\gamma_{X\times Y}(x, y)))$$

proof is valid.

Example 3.14. *Let us consider fuzzy soft upper value* $\overline{v}$ *and fuzzy soft lower value* $\underline{v}$ *in Example 3.12. It is clear that* $\overline{v} = \{0.8/u_1, 0.4/u_2, 0.6/u_4\} \subseteq \underline{v} = \{0.8/u_1, 0.4/u_2, 0.6/u_4\}$*, hence* $\underline{v} \subseteq \overline{v}$*.*

Theorem 3.15. *Let $\gamma_{X\times Y}(x,y)$ be a fuzzy soft saddle point value, $\underline{v}$ be a fuzzy soft lower value and $\overline{v}$ be a fuzzy soft upper value of a tpfs-game. Then,*

$$\underline{v} \subseteq \gamma_{X\times Y}(x,y) \subseteq \overline{v}$$

Proof: Assume that $\gamma_{S_k}(x^*,y^*)$ be a fuzzy soft saddle point value, $\underline{v}$ be a fuzzy soft lower value, $\overline{v}$ be a fuzzy soft upper value of a $tpfs$-game and $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ are sets of the strategies for Player 1 and 2, respectively.

We choose $x_i^* \in X$ and $y_j^* \in Y$. Then,

Since $\gamma_{S_k}(x^*,y^*)$ is a fuzzy soft saddle point value, we have

$$\bigcup_{i=1}^{m} \gamma_{S_k}(x_i,y_j) = \bigcap_{j=1}^{n} \gamma_{S_k}(x_i,y_j) = \gamma_{S_k}(x^*,y^*)$$

Clearly,

$$eV_L = \cup_{x\in X}(\cap_{y\in Y}(\gamma_{X\times Y}(x,y))) \subseteq \bigcup_{i=1}^{m} \gamma_{S_k}(x_i,y_j) = \gamma_{S_k}(x^*,y^*) \tag{1}$$

and

$$\gamma_{S_k}(x^*,y^*) = \bigcap_{j=1}^{n} \gamma_{S_k}(x_i,y_j) \subseteq eV_U = \cap_{y\in Y}(\cup_{x\in X}(\gamma_{X\times Y}(x,y)) \tag{2}$$

Then, from (1) and (2)

$$eV_L \subseteq \gamma_{X\times Y}(x,y) \subseteq eV_U$$

proof is valid.

Corollary 3.16. *Let (x,y) be a fuzzy soft saddle point, $\underline{v}$ be a fuzzy soft lower value and $\overline{v}$ be a fuzzy soft upper value of a tpfs-game. If $v = \underline{v} = \overline{v}$, then $\gamma_{X\times Y}(x,y)$ is exactly v.*

Example 3.17. *Let us consider Table 4 in Example 3 and fuzzy soft upper value $\overline{v}$ and fuzzy soft lower value $\underline{v}$ in Example 3.12. It is clear that fuzzy soft saddle point value $\gamma_{X\times Y}(x,y)$ is exactly $v = \underline{v} = \overline{v} = \{0.8/u_1, 0.4/u_2, 0.6/u_4\}$.*

Note that in every $tpfs$-game, the fuzzy soft lower value $\underline{v}$ can not be equals to the fuzzy soft upper value $\overline{v}$. (For instance, in the above example, if fuzzy soft payoff $\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$ is replaced with $\{0.1/u_1, 0.4/u_2, 0.6/u_4\}$ in fuzzy soft payoff $\gamma^1_{X\times Y}(x_1,y_1)$, then the fuzzy soft lower value $\underline{v}$ can not be equals to the fuzzy soft upper value $\overline{v}$.) If in a $tpfs$-game $\underline{v} \neq \overline{v}$, then to get the solution of the game fuzzy soft dominated strategy may be used. We define fuzzy soft dominated strategy for $tpfs$-game as follows.

Definition 3.18. *Let $\Gamma^1_{X\times Y}$ be a tpfs-game with its fuzzy soft payoff function $\gamma^1_{X\times Y}$. Then,*

a) *a strategy $x_i \in X$ is called a fuzzy soft dominated to another strategy $x_r \in X$, if $\gamma^1_{X\times Y}(x_i,y) \supseteq \gamma^1_{X\times Y}(x_r,y)$ for all $y \in Y$,*
b) *a strategy $y_j \in Y$ is called a fuzzy soft dominated to another strategy $y_s \in Y$, if $\gamma^1_{X\times Y}(x,y_j) \subseteq \gamma^1_{X\times Y}(x,y_s)$ for all $x \in X$*

By using fuzzy soft dominated strategy, $tpfs$-games may be reduced by deleting rows and columns that are obviously bad for the player who uses them. This process of eliminating fuzzy soft dominated strategies sometimes leads us to a solution of a $tpfs$-game. Such a method of solving $tpfs$-game is called a fuzzy soft elimination method.

The following $tpfs$-game can be solved by using the method.

Example 3.19. *Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$ be a set of alternatives, $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be the strategies for Player 1 and 2, respectively. Then, tpfs-game of Player 1 is given as in Table 5,*

$\Gamma^1_{X\times Y}$	y_1	y_2
x_1	$\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$	$\{0.9/u_1, 0.6/u_2, 0.6/u_3, 0.9/u_4, 0.5/u_5\}$
x_2	$\{0.7/u_1, 0.1/u_2\}$	$\{0.5/u_1, 0.8/u_2, 0.8/u_5\}$
x_3	$\{0.2/u_1, 0.1/u_4\}$	$\{0.5/u_1, 0.5/u_3, 0.7/u_4\}$

Table 5

The last column is dominated by the first column. Deleting the last column we can obtain Table 6 as:

$\Gamma^1_{X\times Y}$	y_1
x_1	$\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$
x_2	$\{0.7/u_1, 0.1/u_2\}$
x_3	$\{0.2/u_1, 0.1/u_4\}$

Table 6

Now, in Table 6, the bottom and middle row is dominated by the top row. (Note that this is not the case in Table 5). Deleting the bottom and middle row we obtain Table 7 as:

$\Gamma^1_{X\times Y}$	y_1
x_1	$\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$

Table 7

The solution using the method is (x_1, y_1), that is, value of the tpfs-game is $\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$.

Note that the fuzzy soft elimination method cannot be used for some $tpfs$-games which do not have a fuzzy soft dominated strategies. In this case, we can use fuzzy soft Nash equilibrium that is defined as follows.

Definition 3.20. *Let $\Gamma^k_{X\times Y}$ be a tpfs-game with its fuzzy soft payoff function $\gamma^k_{X\times Y}$ for $k = 1, 2$. If the following properties hold*

a) $\gamma^1_{X\times Y}(x^*, y^*) \supseteq \gamma^1_{X\times Y}(x, y^*)$ *for each* $x \in X$
b) $\gamma^2_{X\times Y}(x^*, y^*) \supseteq \gamma^2_{X\times Y}(x^*, y)$ *for each* $y \in Y$

then, $(x^*, y^*) \in X \times Y$ is called a fuzzy soft Nash equilibrium of a tpfs-game.

Note that if $(x^*, y^*) \in X \times Y$ is a fuzzy soft Nash equilibrium of a $tpfs$-game, then Player 1 can then win at least $\gamma^1_{X\times Y}(x^*, y^*)$ by choosing strategy $x^* \in X$, and Player 2 can win at least $\gamma^2_{X\times Y}(x^*, y^*)$ by choosing strategy $y^* \in Y$. Hence the fuzzy soft Nash equilibrium is an optimal action for $tpfs$-game, therefore, $\gamma^k_{X\times Y}(x^*, y^*)$ is the solution of the $tpfs$-game for Player k, $k = 1, 2$.

Following game, given in Example 3.21, can be solved by fuzzy soft Nash equilibrium, but it is very difficult to solve by using the others methods.

Example 3.21. *Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$ be a set of alternatives, $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be the strategies Player 1 and 2, respectively. Then, tpfs-game of Player 1 is given as in Table 8,*

$\Gamma^1_{X\times Y}$	y_1	y_2
x_1	$\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$	$\{0.6/u_1, 0.7/u_2, 0.6/u_3, 0.9/u_4, 0.5/u_5\}$
x_2	$\{0.7/u_1, 0.1/u_2\}$	$\{0.5/u_1, 0.8/u_2, 0.8/u_5\}$
x_3	$\{0.2/u_1, 0.1/u_4\}$	$\{0.5/u_1, 0.5/u_3, 0.7/u_4\}$

Table 8

and $tpfs$-game of Player 2 is given as in Table 9,

$\Gamma^1_{X\times Y}$	y_1	y_2
x_1	$\{0.9/u_1, 0.6/u_2, 0.6/u_3, 0.9/u_4, 0.5/u_5\}$	$\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$
x_2	$\{0.5/u_1, 0.8/u_2, 0.8/u_5\}$	$\{0.7/u_1, 0.1/u_2\}$
x_3	$\{0.2/u_1, 0.1/u_4\}$	$\{0.5/u_1, 0.5/u_3, 0.7/u_4\}$

Table 9

From the tables, we have

a) $\gamma^1_{X\times Y}(x_1, y_1) \supseteq \gamma^1_{X\times Y}(x, y_1)$ for each $x \in X$, and
b) $\gamma^2_{X\times Y}(x_1, y_1) \supseteq \gamma^2_{X\times Y}(x_1, y)$ for each $y \in Y$

then, $(x_1, y_1) \in X \times Y$ is a fuzzy soft Nash equilibrium. Therefore, $\gamma^1_{X\times Y}(x_1, y_1) = \{0.8/u_1, 0.4/u_2, 0.6/u_4\}$ and $\gamma^2_{X\times Y}(x_1, y_1) = \{0.9/u_1, 0.6/u_2, 0.6/u_3, 0.9/u_4, 0.5/u_5\}$ are the solution of the $tpfs$-game for Player 1 and Player 2, respectively.

Definition 3.22. *[40] Let X be a set of strategies. A probabilistic choice function ρ_X over X is defined by*

$$\rho_X : X \to [0,1] \text{ such that } \sum_{x\in X} \rho_X(x) = 1$$

Here, $\rho_X(x)$ is probability value for each $x \in X$. A set of probabilistic strategies $\widetilde{\rho}_X$ over X can be represented as follows

$$\widetilde{\rho}_X = \{(\rho_X(x)/x) : x \in X\}$$

Here, $\rho_X(x)/x$ is called probabilistic strategy for each $x \in X$.

Note that the subscript X in the ρ_X indicates that ρ_X is defined over X.

4. A solution method of $tpfs$-games

In this section, we give a probabilistic equilibrium solution method for the $tpfs$-games.

In the following, some definition and results on game theory defined in [40, 41, 45], we extend this definition to fuzzy soft game by using fuzzy soft set.

Definition 4.1. *Let $\Gamma^k_{X\times Y}$ be a tpfs-game with its fuzzy soft payoff function $\gamma^k_{X\times Y}$. Then a* relation form *of $\Gamma^k_{X\times Y}$ is defined by*

$$R_{\Gamma^k} = \{(\mu_{R_{\Gamma^k}}((x,y),u)/((x,y),u)) : (x,y) \in X \times Y, u \in U\}$$

$$\mu_{R_{\Gamma^k}} : (X \times Y) \times U \to [0,1] \quad \text{and} \quad \mu_{R_{\Gamma^k}}((x,y),u) = \mu_{\gamma^k(x)}(u)$$

If $U = \{u_1, u_2, ..., u_k\}$, $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$, then the R_{Γ^k} can be presented by a table as in the following form

R_{Γ^k}	u_1	u_2	...	u_k
(x_1, y_1)	$\mu_{R_{\Gamma^k}}((x_1, y_1), u_1)$	$\mu_{R_{\Gamma^k}}((x_1, y_1), u_2)$	...	$\mu_{R_{\Gamma^k}}((x_1, y_1), u_k)$
(x_1, y_2)	$\mu_{R_{\Gamma^k}}((x_1, y_2), u_1)$	$\mu_{R_{\Gamma^k}}((x_1, y_2), u_2)$	...	$\mu_{R_{\Gamma^k}}((x_1, y_2), u_k)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
(x_2, y_1)	$\mu_{R_{\Gamma^k}}((x_2, y_1), u_1)$	$\mu_{R_{\Gamma^k}}((x_2, y_1), u_2)$	...	$\mu_{R_{\Gamma^k}}((x_2, y_1), u_k)$
(x_2, y_2)	$\mu_{R_{\Gamma^k}}((x_2, y_2), u_1)$	$\mu_{R_{\Gamma^k}}((x_2, y_2), u_2)$	...	$\mu_{R_{\Gamma^k}}((x_2, y_2), u_k)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
(x_m, y_n)	$\mu_{R_{\Gamma^k}}((x_m, y_n), u_1)$	$\mu_{R_{\Gamma^k}}((x_m, y_n), u_2)$	...	$\mu_{R_{\Gamma^k}}((x_m, y_n), u_k)$

If $\widetilde{a_{ij}}^t = \mu_{R_{\Gamma^k}}((x_i, y_j), u_t)$, $t \in \{1, 2, ..., k\}$ *then*

$$[\widetilde{a_{ij}}^t]_{mn\times k} = \begin{bmatrix} \widetilde{a_{11}}^1 & \widetilde{a_{11}}^2 & \cdots & \widetilde{a_{11}}^k \\ \widetilde{a_{12}}^1 & \widetilde{a_{12}}^2 & \cdots & \widetilde{a_{1m}}^k \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a_{21}}^1 & \widetilde{a_{21}}^2 & \cdots & \widetilde{a_{21}}^k \\ \widetilde{a_{22}}^1 & \widetilde{a_{22}}^2 & \cdots & \widetilde{a_{2m}}^k \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a_{mn}}^1 & \widetilde{a_{mn}}^2 & \cdots & \widetilde{a_{mn}}^k \end{bmatrix}$$

is called an $m \cdot n \times k$ *fuzzy soft payoff matrix of* $\Gamma^k_{X\times Y}$ *over* U.

Definition 4.2. *Let* $\Gamma^k_{X\times Y}$ *be a tpfs-game with its* $m \cdot n \times k$ *fuzzy soft payoff matrix* $[a^t_{ij}]_{m\cdot n\times k}$. *Then,*

$$[\widetilde{a_{ij}}^t]^e_{m\cdot n\times k} = \left[\begin{array}{c|cccc} (x_1, y_1) & \widetilde{a_{11}}^1 & \widetilde{a_{11}}^2 & \cdots & \widetilde{a_{11}}^k \\ (x_1, y_2) & \widetilde{a_{12}}^1 & \widetilde{a_{12}}^2 & \cdots & \widetilde{a_{1m}}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (x_2, y_1) & \widetilde{a_{21}}^1 & \widetilde{a_{21}}^2 & \cdots & \widetilde{a_{21}}^k \\ (x_2, y_1) & \widetilde{a_{22}}^1 & \widetilde{a_{22}}^2 & \cdots & \widetilde{a_{2m}}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (x_m, y_n) & \widetilde{a_{mn}}^1 & \widetilde{a_{mn}}^2 & \cdots & \widetilde{a_{mn}}^k \end{array}\right]$$

is called an $m \cdot n \times k$ *extended fuzzy soft payoff matrix of* $\Gamma^k_{X\times Y}$ *over* U.

Definition 4.3. *Let* $\widetilde{\Gamma^k}_{X\times Y}$ *be a* $tpfs$*-game with its extended fuzzy soft payoff matrix* $[\widetilde{a_{ij}}^t]^e_{m\cdot n\times k}$. *Then,*

1. *pure fuzzy soft upper impact value of the* $tpfs$*-game, denoted* eI_U, *is defined by*

$$eI_U = min_{y_j\in Y} max_{x_i\in X} \sum_{t=1}^{k} \widetilde{a_{ij}}^t$$

2. *pure fuzzy soft lower impact value of the* $tpfs$*-game, denoted* eI_L, *is defined by*

$$eI_L = max_{x_i\in X} min_{y_j\in Y} \sum_{t=1}^{k} \widetilde{a_{ij}}^t$$

3. *If pure fuzzy soft upper impact value and pure fuzzy soft lower impact value of a* $tpfs$*-game are equal, then the solution of the tpfs-game is this values.*

Theorem 4.4. *Suppose that* $\widetilde{\Gamma^k}_{X\times Y}$ *be a* $tpfs$*-game with its extended fuzzy soft payoff matrix* $[\widetilde{a_{ij}}^t]^e_{m\cdot n\times k}$. *Then,*

$$eI_L \leq eI_U$$

is valid.

Proof: Assume that $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ be a set of strategies of Player 1 and 2, respectively. Choose $x^*_i \in X$ and $y^*_j \in Y$; then, we have

$$min_{y_j\in Y} \sum_{t=1}^{k} \widetilde{a_{i^*j}}^t \leq \sum_{t=1}^{k} \widetilde{a_{i^*j^*}}$$

Hence for every $x_i^* \in X$

$$eI_L = \quad max_{x_i \in X} min_{y_j \in Y} \sum_{t=1}^{k} \widetilde{a_{ij}}^t \leq max_{x_i \in X} \sum_{t=1}^{k} \widetilde{a_{ij^*}}^t$$

Similarly, for every $y_i^* \in Y$

$$eI_L = \quad max_{x_i \in X} min_{y_j \in Y} \sum_{t=1}^{k} \widetilde{a_{ij}}^t \leq eI_U = min_{y_j \in Y} max_{x_i \in X} \sum_{t=1}^{k} \widetilde{a_{ij}}^t$$

Consequently, theorem is valid.

Definition 4.5. *Let U be a set of alternatives, X and Y be two sets of strategies, $\widetilde{\rho}_X = (\rho_X(x_1)/x_1, \rho_X(x_2)/x_2, ..., \rho_X(x_m)/x_m)$ and $\widetilde{\rho}_Y = (\rho_Y(y_1)/y_1, \rho_Y(y_2)/y_2, ..., \rho_Y(y_n)/y_n)$ be a set of probabilistic strategies of Player 1 and 2, respectively, and $\widetilde{\gamma^k}_{X\times Y} : X \times Y \to P(U)$ be a fuzzy soft payoff function for player k, $(k = 1, 2)$. Then, for each Player k, a $tpfs$-game with probabilistic strategies is defined by*

$$\widetilde{\Gamma^k}_{X\times Y} = \{((\rho_X(x), \rho_Y(y))/(x, y), \gamma^k_{X\times Y}(x, y)) : (x, y) \in X \times Y\}$$

where

$$\rho_X : X \to [0.1], \quad \rho_Y : Y \to [0.1], \sum_{x\in X} \rho_X(x) = 1 \quad and \quad \sum_{y\in Y} \rho_Y(y) = 1.$$

Example 4.6. *Suppose that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ is a set of alternatives, $\widetilde{\rho}_X = \{\rho_X(x_1)/x_1, \rho_X(x_2)/x_2\}$ and $\widetilde{\rho}_Y = \{\rho_Y(y_1)/y_1, \rho_Y(y_2)/y_2, \rho_Y(y_3)/y_3\}$ are sets of the probabilistic strategies for Player 1 and 2, respectively. If Player 1 constructs a $tpfs$-games as follows,*

$$\begin{aligned}\widetilde{\Gamma^1}_{X\times Y} = \Big\{ &((\rho_X(x_1), \rho_Y(y_1))/(x_1, y_1), \{0.5/u_1, 0.7/u_2, 0.1/u_3, 0.9/u_5, 1.0/u_6\}),\\ &((\rho_X(x_1), \rho_Y(y_2))/(x_1, y_2), \{0.4/u_1, 0.7/u_2, 0.2/u_3, 0.9/u_4, 0.8/u_6\}),\\ &((\rho_X(x_1), \rho_Y(y_3))/(x_1, y_3), \{0.5/u_1, 0.7/u_3, 1.0/u_4, 0.9/u_5, 0.2/u_6\}),\\ &((\rho_X(x_2), \rho_Y(y_1))/(x_2, y_1), \{0.7/u_2, 0.1/u_3, 0.5/u_4, 0.9/u_5, 1.0/u_6\}),\\ &((\rho_X(x_2), \rho_Y(y_2))/(x_2, y_2), \{0.3/u_1, 0.8/u_2, 0.1/u_3, 0.5/u_5, 1.0/u_6\}),\\ &((\rho_X(x_2), \rho_Y(y_3))/(x_2, y_3), \{0.2/u_1, 0.7/u_2, 0.9/u_3, 0.4/u_4, 0.5/u_5\})\Big\}\end{aligned}$$

then the extended soft payoff matrix of $\widetilde{\Gamma^1}_{X\times Y}$ over U for Player 1 as;

$$[\widetilde{a_{ij}}^t]^e_{2\cdot 3\times 6} = \left[\begin{array}{c|cccccc} (\rho_X(x_1), \rho_Y(y_1))/(x_1, y_1) & 0.5 & 0.7 & 0.1 & 0 & 0.9 & 1\\ (\rho_X(x_1), \rho_Y(y_2))/(x_1, y_2) & 0.4 & 0.7 & 0.2 & 0.9 & 0 & 0.8\\ (\rho_X(x_1), \rho_Y(y_3))/(x_1, y_3) & 0.5 & 0 & 0.7 & 1.0 & 0.9 & 0.2\\ (\rho_X(x_2), \rho_Y(y_1))/(x_2, y_1) & 0 & 0.7 & 0.1 & 0.5 & 0.9 & 1\\ (\rho_X(x_2), \rho_Y(y_2))/(x_2, y_2) & 0.3 & 0.8 & 0.1 & 0 & 0.5 & 1\\ (\rho_X(x_2), \rho_Y(y_3))/(x_2, y_3) & 0.2 & 0.7 & 0.9 & 0.4 & 0.5 & 0 \end{array}\right]$$

Definition 4.7. *Let $\widetilde{\Gamma^1}_{X\times Y}$ and $\widetilde{\Gamma^2}_{X\times Y}$ be a $tpfs$-game with probabilistic strategies with their fuzzy soft payoff matrix $[\widetilde{a_{ij}}^t]_{m\cdot n\times k}$ and $[\widetilde{b_{ij}}^t]_{m\cdot n\times k}$, respectively. If Player 1 and 2 choose the strategy $x_i \in X$ and $y_j \in Y$, respectively. Then $[\rho_X(x_i).\widetilde{a_{ij}}^t.\rho_Y(y_j)]_{m\cdot n\times k}$ and $[\rho_X(x_i).\widetilde{b_{ij}}^t.\rho_Y(y_j)]_{m\cdot n\times k}$ are called expected fuzzy soft payoff matrix for Player 1 and 2, respectively.*

Definition 4.8. *Let $\widetilde{\Gamma^k}_{X\times Y}$ for $k = 1, 2$ be a $tpfs$-game with probabilistic strategies with their fuzzy soft payoff matrix $[\widetilde{a_{ij}}^t]_{m\cdot n\times k}$ and $[\widetilde{b_{ij}}^t]_{m\cdot n\times k}$, respectively. If $[\rho_X(x_i).\widetilde{a_{ij}}^t.\rho_Y(y_j)]_{m\cdot n\times k}$ and $[\rho_X(x_i).\widetilde{b_{ij}}^t.\rho_Y(y_j)]_{m\cdot n\times k}$ are given, then we shall say that the $tpfs$-game with probabilistic strategies is in its bimatrix form.*

Definition 4.9. *Let $\widetilde{\Gamma^k}_{X\times Y}$, (k=1,2), be a tpfs-game with probabilistic strategies with its expected fuzzy soft payoff matrix $[\rho_X(x_i).\widetilde{a^t_{ij}}.\rho_Y(y_j)]_{m\cdot n\times k}$ and $[\rho_X(x_i).\widetilde{b^t_{ij}}.\rho_Y(y_j)]_{m\cdot n\times k}$ for Player 1 and 2, respectively. Then, impact values of the tpfs-game are defined by*

$$\widetilde{v_1} = \sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}\rho_X(x_i).\widetilde{a_{ij}}^t.\rho_Y(y_j)$$

and

$$\widetilde{v_2} = \sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}\rho_X(x_i).\widetilde{b^t_{ij}}.\rho_Y(y_j)$$

for Player 1 and 2, respectively.

Definition 4.10. *A tpfs-game with probabilistic strategies is said to be a finite tpfs-game with probabilistic strategies if both strategy sets X and Y, and alternatives set U are finite sets.*

Definition 4.11. *Let $\widetilde{\Gamma^k}_{X\times Y}$ be a tpfs-game with probabilistic strategies with its expected fuzzy soft payoff matrix $[\rho_X(x_i).\widetilde{a^t_{ij}}.\rho_Y(y_j)]_{m\cdot n\times k}$. Then,*

1. *fuzzy soft upper impact value of the tpfs-game, denoted eI^M_U, is defined by*

$$eI^M_U = min_{y_j\in Y}(max_{x_i\in X}(\sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho_X(x_i).\widetilde{a_{ij}}^t.\rho_Y(y_j))$$

2. *fuzzy soft lower impact value of the tpfs-game, denoted eI^M_L, is defined by*

$$eI^M_L = max_{x_i\in X}(min_{y_j\in Y}(\sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho_X(x_i).\widetilde{a_{ij}}^t.\rho_Y(y_j))$$

3. *If fuzzy soft upper impact value and fuzzy soft lower impact value of a tpfs-game are equal for $\widetilde{\rho_X} = (\rho_X(x_1)/x_1, \rho_X(x_2)/x_2, ..., \rho_X(x_m)/x_m)$ and $\widetilde{\rho_Y} = (\rho_Y(y_1)/y_1, \rho_Y(y_2)/y_2, ..., \rho_Y(y_n)/y_n)$, then the solution of the tpfs-game is $\widetilde{\rho_X}$ and $\widetilde{\rho_Y}$.*

Theorem 4.12. *Suppose that $\widetilde{\Gamma^k}_{X\times Y}$ be a tpfs-game, eI^M_L and eI^M_U be a fuzzy soft lower impact value and fuzzy soft upper impact value of the game, respectively. Then*

$$eI^M_L \leq eI^M_U$$

is valid.

Proof: Suppose that $\widetilde{\Gamma^k}_{X\times Y}$ be a *tpfs*-game with its extended fuzzy soft payoff matrix $[\widetilde{a_{ij}}^t]^e_{m\cdot n\times k}$, $\widetilde{\rho_X} = (\rho_X(x_1)/x_1, \rho_X(x_2)/x_2, ..., \rho_X(x_m)/x_m)$ and $\widetilde{\rho_Y} = (\rho_Y(y_1)/y_1, \rho_Y(y_2)/y_2, ..., \rho_Y(y_n)/y_n)$ be a set of probabilistic strategies of Player 1 and 2, respectively.

If

$$\sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho_X(x_i).\widetilde{a_{ij}}^t.\rho_Y(y_j)$$

then, we handle the $x^*_i \in X$ and $y^*_j \in Y$ strategies of Player 1 and 2, respectively.

$$\begin{aligned} min_{y_j\in Y} \quad & (\textstyle\sum_{t=1}^{k}\sum_{j=1}^{n}(\rho_X(x^*_i).\widetilde{a_{i^*j}}^t.\rho_Y(y_j) \\ & \leq \textstyle\sum_{t=1}^{k}(\rho_X(x^*_i).\widetilde{a_{i^*j^*}}^t.\rho_Y(y^*_j) \end{aligned}$$

Hence for every $x_i^* \in X$

$$\begin{aligned} eI_L^M &= max_{x_i \in X}(min_{y_j \in Y}(\sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho_X(x_i).\widetilde{a_{ij}}^t.\rho_Y(y_j)) \\ &\leq max_{x_i \in X}(\sum_{t=1}^{k}\sum_{i=1}^{m}(\rho_X(x_i).\widetilde{a_{ij^*}}^t.\rho_Y(y_j^*)) \end{aligned}$$

Similarly, for every $y_i^* \in Y$

$$\begin{aligned} eI_L^M &= max_{x_i \in X}(min_{y_j \in Y}(\sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho_X(x_i).\widetilde{a_{ij}}^t.\rho_Y(y_j)) \\ &\leq eI_U^M = min_{y_j \in Y}(max_{x_i \in X}(\sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho_X(x_i).\widetilde{a_{ij}}^t.\rho_Y(y_j)) \end{aligned}$$

Consequently, theorem is valid.

Theorem 4.13. *Let $\widetilde{\Gamma}^k{}_{X\times Y}$ be a tpfs-game with probabilistic strategies with its expected fuzzy soft payoff matrix $[\rho_X(x_i).\widetilde{a_{ij}^t}.\rho_Y(y_j)]_{m\cdot n\times k}$. Then every finite tpfs-game with probabilistic strategies has one solution, that is, if expected impact values of the tpfs-game are defined by*

$$v(x_i, y_i) = \sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho_X(x_i).\widetilde{a_{ij}}^t.\rho_Y(y_j)$$

then,

$$eI_L^M = eI_U^M.$$

Proof: Assume that U be a set of alternatives, $\widetilde{\rho}_X$ and $\widetilde{\rho}_Y$ are sets of the probabilistic strategies for Player 1 and 2, respectively. Then, we have eI_U^M since $v(x_i, y_j)$ function is a continuous function for $\widetilde{\rho}_X$ and $\widetilde{\rho}_Y$.

Similarly, we have eI_L^M.

Now, consider the matrix for $I \in R$, $[a_{ij} - I]m \times n$ and assume that expected impact values of the matrix as

$$v_I(x_i, y_j) = \sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho_X(x_i).(a_{ij}^t - I).\rho_Y(y_j) \tag{3}$$

consider the following inequations not implemented for (3)

$$maxminv_I(x_i, y_j) < 0 < minmaxv_I(x_i, y_j) \tag{4}$$

from (3) we have

$$v_I(x_i, y_j) = \sum_{t=1}^{k}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho_X(x_i).a_{ij}^t.\rho_Y(y_j) - I \tag{5}$$

from (4) and (5)

$$maxminv(x_i, y_j) - I < 0 < minmaxv(x_i, y_j)) - I \tag{6}$$

and the last inequality added to each side I we have

$$maxminv(x_i, y_j) < I < minmaxv(x_i, y_j) \tag{7}$$

and then (7) is not true for every I. Therefore,

$$maxminv(x_i, y_j) < minmaxv(x_i, y_j) \tag{8}$$

inequality is false. Here

$$maxminv(x_i, y_j) \geq minmaxv(x_i, y_j) \tag{9}$$

is seen.

Finally, from Theorem 4.12 and (9)

$$maxminv(x_i, y_j) = minmaxv(x_i, y_j) \tag{10}$$

proof is provided

5. An Application

In this section, we give a financial problem that are solved by using both fuzzy soft dominated strategy and fuzzy soft saddle point methods. Now, we modified the application, given in [14], by using fuzzy soft set as follows;

There are two companies, say Player 1 and Player 2, who competitively want to increase sale of produces in the country. Therefore, they give advertisements. Assume that two companies have a set of different products $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ where for $i = 1, 2, ..., 8$, the product u_i stand for "oil", "salt",, "honey", " jam", " cheese", "sugar", "cooker", and "jar", respectively. The products can be characterized by a set of strategy $E = \{x_i : i = 1, 2, 3\}$ which contains styles of advertisement where for $j = 1, 2, 3$, the strategies x_j stand for "TV", "radio" and " newspaper", respectively.

Suppose that $X = \{x_1, x_2, x_3\}$ and $Y = \{x_1, x_2\}$ are strategies of Player 1 and 2, respectively. Then, a $tpfs$-game of Player 1 is given as in Table 10.

$\Gamma^1_{X\times Y}$	x_1	x_2
x_1	$\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$	$\{0.9/u_1, 0.7/u_2, 0.6/u_3, 0.9/u_4, 0.5/u_5\}$
x_2	$\{0.7/u_1, 0.1/u_2\}$	$\{0.7/u_1, 0.8/u_4, 0.8/u_5\}$
x_3	$\{0.2/u_1, 0.1/u_4\}$	$\{0.5/u_1, 0.5/u_3, 0.7/u_4\}$

Table 10

In Table 10, let us explain action pair (x_1, x_1); if Player 1 select $x_1 = "TV"$ and Player 2 select $x_1 = "TV"$, then the fuzzy soft payoff of Player 1 is a set $\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$, that is, $\gamma^1_{X\times Y}(x_1, x_1) = \{0.8/u_1, 0.4/u_2, 0.6/u_4\}$. In this case, Player 1 increase sale of $\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$ and Player 2 decrease sale of $\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$.

We can now solve the game. It is seen in Table 10,

$$\begin{array}{rcl} \{0.8/u_1, 0.4/u_2, 0.6/u_4\} & \subseteq & \{0.2/u_1, 0.1/u_4\} \\ \{0.9/u_1, 0.7/u_2, 0.6/u_3, 0.9/u_4, 0.5/u_5\} & \subseteq & \{0.5/u_1, 0.5/u_3, 0.7/u_4\} \end{array}$$

the bottom row is dominated by the top row. We then deleting the bottom row we obtain Table 11.

$\Gamma^1_{X\times Y}$	x_1	x_2
x_1	$\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$	$\{0.9/u_1, 0.7/u_2, 0.6/u_3, 0.9/u_4, 0.5/u_5\}$
x_2	$\{0.7/u_1, 0.1/u_2\}$	$\{0.7/u_1, 0.8/u_4, 0.8/u_5\}$

Table 11

In Table 11, there is no another fuzzy soft dominated strategy, we can use fuzzy soft saddle point method.

$$\begin{array}{rcl} \bigcup_{i=1}^{2} \gamma^1_{X\times Y}(x_i, x_1) & = & \{0.8/u_1, 0.4/u_2, 0.6/u_4\} \\ \bigcup_{i=1}^{2} \gamma^1_{X\times Y}(x_i, x_2) & = & \{0.7/u_1, 0.7/u_2, 0.6/u_3, 0.9/u_4, 0.8/u_5\} \\ \bigcap_{j=1}^{2} \gamma^1_{X\times Y}(x_1, y_j) & = & \{0.8/u_1, 0.4/u_2, 0.6/u_4\} \\ \bigcap_{j=1}^{2} \gamma^1_{X\times Y}(x_2, y_j) & = & \{0.7/u_1\} \end{array}$$

Here, optimal strategy of the game is (x_1, y_1) since

$$\bigcup_{i=1}^{2} \gamma^1_{X\times Y}(x_i, y_1) = \bigcap_{j=1}^{2} \gamma^1_{X\times Y}(x_1, y_j)$$

Therefore, value of the $tpfs$-game is $\{0.8/u_1, 0.4/u_2, 0.6/u_4\}$.

6. *n*-Person Soft Games

In many applications the fuzzy soft games can be often played between more than two players. Therefore, $tpfs$-games can be extended to n-person fuzzy soft games.

From now on, $X_n^\times$ will be used for $X_1 \times X_2 \times ... \times X_n$.

Definition 6.1. *Let U be a set of alternatives, $F(U)$ be all fuzzy sets over U, E be a set of strategies, $X_1, X_2, ..., X_n \subseteq E$, and X_k is the set of strategies of Player k, $(k = 1, 2, ..., n)$. Then, for each Player k, an n-person fuzzy soft game (nps-game) is defined by a fuzzy soft set over U as*

$$\Gamma^k_{X_n^\times} = \{((x_1, x_2, ..., x_n), \gamma^k_{X_n^\times}(x_1, x_2, ..., x_n)) : (x_1, x_2, ..., x_n) \in X_n^\times\}$$

where $\gamma^k_{X_n^\times}$ is a fuzzy soft payoff function of Player k.

The nps-game is played as follows: at a certain time Player 1 chooses a strategy $x_1 \in X_1$ and simultaneously each Player k $(k = 2, ..., n)$ chooses a strategy $x_k \in X_k$ and once this is done each player k receives the fuzzy soft payoff $\gamma^k_{X_n^\times}(x_1, x_2, ..., x_n)$.

Definition 6.2. *Let $\Gamma^k_{X_n^\times}$ be an nps-game with its fuzzy soft payoff function $\gamma^k_{X_n^\times}$ for $k = 1, 2, ..., n$. Then, a strategy $x_k \in X_k$ is called a fuzzy soft dominated to another strategy $x \in X_k$, if*

$$\gamma^k_{X_n^\times}(x_1, ..., x_{k-1}, x_k, x_{k+1}, ..., x_n) \supseteq \gamma^k_{X_n^\times}(x_1, ..., x_{k-1}, x, x_{k+1}, ..., x_n)$$

for each strategy $x_i \in X_i$ of player i $(i = 1, 2, ...k - 1, k + 1, ..., n)$, respectively.

Definition 6.3. *Let $\gamma^k_{X_n^\times}$ be a fuzzy soft payoff function of a nps-game $\Gamma^k_{X_n^\times}$. If for each player k (k=1,2,...,n) the following properties hold*

$$\gamma^k_{X_n^\times}(x_1^*, ..., x_{k-1}^*, x_k^*, x_{k+1}^*, ..., x_n^*) \supseteq \gamma^k_{X_n^\times}(x_1^*, ..., x_{k-1}^*, x, x_{k+1}^*, ..., x_n^*)$$

for each $x \in X_k$, then $(x_1^*, x_2^*, ..., x_n^*) \in X_n^\times$ is called a fuzzy soft Nash equilibrium of an nps-game.

7. Conclusion

In this paper, we first present the basic definitions and results of fuzzy soft set theory. We then construct $tpfs$-games with fuzzy soft payoffs which is set value and the solution operations based on the set operations. We also give four solution methods for the $tpfs$-games with examples. To applied the game to the real world problem we give an application which shows the methods can be successfully applied to a financial problem. Finally, we extended the two person fuzzy soft games to n-person fuzzy soft games. The fuzzy soft games may be applied to many fields and more comprehensive in the future to solve the related problems, such as; computer science, decision making, and so on.

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