

FUZZY STABILITY OF THE CAUCHY ADDITIVE AND QUADRATIC TYPE FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we investigate a fuzzy version of stability for the functional equation

$$2f(x+y) + f(x-y) + f(y-x) - 3f(x) - f(-x) - 3f(y) - f(-y) = 0$$

in the sense of M. Mirmostafae and M. S. Moslehian.

1. Introduction

A classical question in the theory of functional equations is “when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?”. Such a problem, called a *stability problem of the functional equation*, was formulated by S. M. Ulam [22] in 1940. In the next year, D. H. Hyers [6] gave a partial solution of Ulam’s problem for the case of approximate additive mappings. Subsequently, his result was generalized by T. Aoki [1] for additive mappings, and by Th. M. Rassias [20] for linear mappings, to considering the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [4], [5], [7], [9], [10], [12]-[16], [21].

In 1984, A. K. Katsaras [8] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, some mathematicians have introduced several types of fuzzy norm in different points of view. In particular, T. Bag and S. K. Samanta [2], following Cheng and Mordeson [3], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11]. In 2008, A. K. Mirmostafae and M. S. Moslehian [18] obtained a fuzzy version of stability for *the Cauchy functional equation*:

$$(1.1) \quad f(x+y) - f(x) - f(y) = 0.$$

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In the same year, they [17] proved a fuzzy version of stability for *the quadratic functional equation*:

$$(1.2) \quad f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0.$$

A solution of (1.1) is called *an additive mapping* and a solution of (1.2) is called *a quadratic mapping*. Now we consider the functional equation:

$$(1.3) \quad 2f(x+y) + f(x-y) + f(y-x) - 3f(x) - f(-x) - 3f(y) - f(-y) = 0$$

which is called *a Cauchy additive and quadratic type functional equation*. A solution of (1.3) is called *a quadratic-additive mapping*. In 2008, C.-G. Park [19] obtained a stability of the functional equation (1.3) by taking and composing an additive mapping A and a quadratic mapping Q to prove the existence of a quadratic-additive mapping F which is close to the given mapping f . In his processing, A is approximate to the odd part $\frac{f(x)-f(-x)}{2}$ of f and Q is close to the even part $\frac{f(x)+f(-x)}{2}$ of it, respectively.

In this paper, we get a general stability result of the Cauchy additive and quadratic type functional equation (1.3) in the fuzzy normed linear space in the manner of A. K. Mirmostafae and M. S. Moslehian [17]. To do it, we introduce a Cauchy sequence $\{J_n f(x)\}$ starting from a given mapping f , which converges to the desired mapping F in the fuzzy sense. As we mentioned before, in previous studies of stability problem of (1.3), he attempted to get stability theorems by handling the odd and even part of f , respectively. According to our proposal in this paper, we can take the desired approximate solution F at only one time. Therefore, this idea is a refinement with respect to the simplicity of the proof.

2. Main results

We use the definition of a fuzzy normed space given in [2] to exhibit a reasonable fuzzy version of stability for the quadratic-additive type functional equation in the fuzzy normed linear space.

Definition 2.1 ([2]). Let X be a real linear space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ (the so-called fuzzy subset) is said to be *a fuzzy norm on X* if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

- (N1) $N(x, c) = 0$ for $c \leq 0$;
- (N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
- (N3) $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$;
- (N4) $N(x+y, s+t) \geq \min\{N(x, s), N(y, t)\}$;
- (N5) $N(x, \cdot)$ is a non-decreasing function on \mathbb{R} and $\lim_{t \rightarrow \infty} N(x, t) = 1$.

The pair (X, N) is called *a fuzzy normed linear space*. Let (X, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence $\{x_n\}$* and we denote it by $N - \lim_{n \rightarrow \infty} x_n = x$. A sequence $\{x_n\}$ in X is called *Cauchy* if for each

$\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called a *fuzzy Banach space*.

Let (X, N) be a fuzzy normed space and (Y, N') a fuzzy Banach space. For a given mapping $f : X \rightarrow Y$, we use the abbreviation

$Df(x, y) := 2f(x + y) + f(x - y) + f(y - x) - 3f(x) - f(-x) - 3f(y) - f(-y)$ for all $x, y \in X$. For given $q > 0$, the mapping f is called a *fuzzy q -almost quadratic-additive mapping*, if

$$(2.1) \quad N'(Df(x, y), t + s) \geq \min\{N(x, s^q), N(y, t^q)\}$$

for all $x, y \in X$ and all $s, t \in (0, \infty)$. Now we get the general stability result in the fuzzy normed linear space.

Theorem 2.2. *Let q be a positive real number with $q \neq \frac{1}{2}, 1$. And let f be a fuzzy q -almost quadratic-additive mapping from a fuzzy normed space (X, N) into a fuzzy Banach space (Y, N') . Then there is a unique quadratic-additive mapping $F : X \rightarrow Y$ such that*

$$(2.2) \quad N'(F(x) - f(x), t) \geq \begin{cases} \sup_{t' < t} N(x, (2 - 2^p)^q t'^q) & \text{if } q > 1, \\ \sup_{t' < t} N\left(x, \left(\frac{(4-2^p)(2^p-2)}{2}\right)^q t'^q\right) & \text{if } \frac{1}{2} < q < 1 \\ \sup_{t' < t} N(x, (2^p - 4)^q t'^q) & \text{if } 0 < q < \frac{1}{2} \end{cases}$$

for each $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. It follows from (2.1) and (N4) that

$$N'(f(0), t) = N'(Df(0, 0), 4t) \geq N(0, (2t)^q) = 1$$

for all $t \in (0, \infty)$. By (N2), we have $f(0) = 0$. We will prove the theorem in three cases, $q > 1$, $\frac{1}{2} < q < 1$, and $0 < q < \frac{1}{2}$.

Case 1. Let $q > 1$ and let $J_n f : X \rightarrow Y$ be a mapping defined by

$$J_n f(x) = \frac{1}{2} (4^{-n} (f(2^n x) + f(-2^n x)) + 2^{-n} (f(2^n x) - f(-2^n x)))$$

for all $x \in X$. Notice that $J_0 f(x) = f(x)$ and

$$(2.3) \quad J_j f(x) - J_{j+1} f(x) = \frac{2^{j+1} - 1}{4^{j+2}} Df(-2^j x, -2^j x) - \frac{2^{j+1} + 1}{4^{j+2}} Df(2^j x, 2^j x)$$

for all $x \in X$ and $j \geq 0$. Together with (N3), (N4) and (2.1), this equation implies that if $n + m > m \geq 0$, then

$$(2.4) \quad N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \frac{1}{2} \left(\frac{2^p}{2}\right)^j t^p\right)$$

$$\begin{aligned}
&\geq N' \left(\sum_{j=m}^{n+m-1} (J_j f(x) - J_{j+1} f(x)), \sum_{j=m}^{n+m-1} \frac{1}{2} \left(\frac{2^p}{2} \right)^j t^p \right) \\
&\geq \min \bigcup_{j=m}^{n+m-1} \left\{ N' \left(J_j f(x) - J_{j+1} f(x), \frac{1}{2} \left(\frac{2^p}{2} \right)^j t^p \right) \right\} \\
&\geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N' \left(-\frac{(2^{j+1} + 1) Df(2^j x, 2^j x)}{4^{j+2}}, \frac{(2^{j+1} + 1) 2^{jp} t^p}{8 \cdot 4^j} \right), \right. \right. \\
&\quad \left. \left. N' \left(\frac{(2^{j+1} - 1) Df(-2^j x, -2^j x)}{4^{j+2}}, \frac{(2^{j+1} - 1) 2^{jp} t^p}{8 \cdot 4^j} \right) \right\} \right\} \\
&\geq \min \bigcup_{j=m}^{n+m-1} \{N(2^j x, 2^j t)\} = N(x, t)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Let $\varepsilon > 0$ be given. Since $\lim_{t \rightarrow \infty} N(x, t) = 1$, there is $t_0 > 0$ such that

$$N(x, t_0) \geq 1 - \varepsilon.$$

Observe that for some $\tilde{t} > t_0$, the series $\sum_{j=0}^{\infty} \frac{1}{2} \left(\frac{2^p}{2} \right)^j \tilde{t}^p$ converges for $p = \frac{1}{q} < 1$. It guarantees that, for an arbitrary given $c > 0$, there exists $n_0 \geq 0$ such that

$$\sum_{j=m}^{n+m-1} \frac{1}{2} \left(\frac{2^p}{2} \right)^j \tilde{t}^p < c$$

for each $m \geq n_0$ and $n > 0$. Together with (N5) and (2.4), this implies that

$$\begin{aligned}
&N'(J_m f(x) - J_{n+m} f(x), c) \\
&\geq N' \left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \frac{1}{2} \left(\frac{2^p}{2} \right)^j \tilde{t}^p \right) \\
&\geq N(x, \tilde{t}) \geq N(x, t_0) \geq 1 - \varepsilon
\end{aligned}$$

for all $x \in X$. Hence $\{J_n f(x)\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N') , and so we can define a mapping $F : X \rightarrow Y$ by

$$F(x) := N' - \lim_{n \rightarrow \infty} J_n f(x).$$

Moreover, if we put $m = 0$ in (2.4), we have

$$(2.5) \quad N'(f(x) - J_n f(x), t) \geq N \left(x, \frac{t^q}{\left(\sum_{j=0}^{n-1} \frac{1}{2} \left(\frac{2^p}{2} \right)^j \right)^q} \right)$$

for all $x \in X$. Next we will show that F is the desired quadratic additive function. Using (N4), we have

$$(2.6) \quad \begin{aligned} & N'(DF(x, y), t) \\ & \geq \min \left\{ N' \left(2F(x + y) - 2J_n f(x + y), \frac{t}{14} \right), N' \left(F(x - y) - J_n f(x - y), \frac{t}{14} \right), \right. \\ & \quad N' \left(F(y - x) - J_n f(y - x), \frac{t}{14} \right), N' \left(3J_n f(x) - 3F(x), \frac{t}{14} \right), \\ & \quad N' \left(J_n f(-x) - F(-x), \frac{t}{14} \right), N' \left(3J_n f(y) - 3F(y), \frac{t}{14} \right), \\ & \quad \left. N' \left(J_n f(-y) - F(-y), \frac{t}{14} \right), N' \left(DJ_n f(x, y), \frac{t}{2} \right) \right\} \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. The first seven terms on the right hand side of (2.6) tend to 1 as $n \rightarrow \infty$ by the definition of F and (N2), and the last term holds

$$\begin{aligned} & N' \left(DJ_n f(x, y), \frac{t}{2} \right) \\ & \geq \min \left\{ N' \left(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right), N' \left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right), \right. \\ & \quad \left. N' \left(\frac{Df(2^n x, 2^n y)}{2 \cdot 2^n}, \frac{t}{8} \right), N' \left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 2^n}, \frac{t}{8} \right) \right\} \end{aligned}$$

for all $x, y \in X$. By (N3) and (2.1), we obtain

$$\begin{aligned} N' \left(\frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right) &= N' \left(Df(\pm 2^n x, \pm 2^n y), \frac{4^n t}{4} \right) \\ &\geq \min \left\{ N \left(2^n x, \left(\frac{4^n t}{8} \right)^q \right), N \left(2^n y, \left(\frac{4^n t}{8} \right)^q \right) \right\} \\ &\geq \min \left\{ N \left(x, \frac{2^{(2q-1)n}}{2^{3q}} t^q \right), N \left(y, \frac{2^{(2q-1)n}}{2^{3q}} t^q \right) \right\} \end{aligned}$$

and

$$N' \left(\frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 2^n}, \frac{t}{8} \right) \geq \min \left\{ N \left(x, \frac{2^{(q-1)n}}{2^{3q}} t^q \right), N \left(y, \frac{2^{(q-1)n}}{2^{3q}} t^q \right) \right\}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Since $q > 1$, together with (N5), we can deduce that the last term of (2.6) also tends to 1 as $n \rightarrow \infty$. It follows from (2.6) that

$$N'(DF(x, y), t) = 1$$

for each $x, y \in X$ and $t > 0$. By (N2), this means that $DF(x, y) = 0$ for all $x, y \in X$.

Next we approximate the difference between f and F in a fuzzy sense. For an arbitrary fixed $x \in X$ and $t > 0$, choose $0 < \varepsilon < 1$ and $0 < t' < t$. Since F is the limit of $\{J_n f(x)\}$, there is $n \in \mathbb{N}$ such that

$$N'(F(x) - J_n f(x), t - t') \geq 1 - \varepsilon.$$

By (2.5), we have

$$\begin{aligned} N'(F(x) - f(x), t) &\geq \min \{N'(F(x) - J_n f(x), t - t'), N'(J_n f(x) - f(x), t')\} \\ &\geq \min \left\{ 1 - \varepsilon, N \left(x, \frac{t'^q}{\left(\frac{1}{2} \sum_{j=0}^{n-1} \left(\frac{2^p}{2} \right)^j \right)^q} \right) \right\} \\ &\geq \min \{1 - \varepsilon, N(x, (2 - 2^p)^q t'^q)\}. \end{aligned}$$

Because $0 < \varepsilon < 1$ is arbitrary, we get the inequality (2.2) in this case. Finally, to prove the uniqueness of F , let $F' : X \rightarrow Y$ be another quadratic-additive mapping satisfying (2.2). Then by (2.3), we get

$$(2.7) \quad \begin{cases} F(x) - J_n F(x) = \sum_{j=0}^{n-1} (J_j F(x) - J_{j+1} F(x)) = 0 \\ F'(x) - J_n F'(x) = \sum_{j=0}^{n-1} (J_j F'(x) - J_{j+1} F'(x)) = 0 \end{cases}$$

for all $x \in X$ and $n \in \mathbb{N}$. Together with (N4) and (2.2), this implies that

$$\begin{aligned} &N'(F(x) - F'(x), t) \\ &= N'(J_n F(x) - J_n F'(x), t) \\ &\geq \min \left\{ N' \left(J_n F(x) - J_n f(x), \frac{t}{2} \right), N' \left(J_n f(x) - J_n F'(x), \frac{t}{2} \right) \right\} \\ &\geq \min \left\{ N' \left(\frac{(F - f)(2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), N' \left(\frac{(f - F')(2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), \right. \\ &\quad N' \left(\frac{(F - f)(-2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), N' \left(\frac{(f - F')(-2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), \\ &\quad N' \left(\frac{(F - f)(2^n x)}{2 \cdot 2^n}, \frac{t}{8} \right), N' \left(\frac{(f - F')(2^n x)}{2 \cdot 2^n}, \frac{t}{8} \right), \\ &\quad \left. N' \left(\frac{(F - f)(-2^n x)}{2 \cdot 2^n}, \frac{t}{8} \right), N' \left(\frac{(f - F')(-2^n x)}{2 \cdot 2^n}, \frac{t}{8} \right) \right\} \\ &\geq \sup_{t' < t} N \left(x, 2^{(q-1)n-2q} (2 - 2^p)^q t'^q \right) \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Observe that, for $q = \frac{1}{p} > 1$, the last term of the above inequality tends to 1 as $n \rightarrow \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ and so we get

$$F(x) = F'(x)$$

for all $x \in X$ by (N2).

Case 2. Let $\frac{1}{2} < q < 1$ and let $J_n f : X \rightarrow Y$ be a mapping defined by

$$J_n f(x) = \frac{1}{2} \left(4^{-n} (f(2^n x) + f(-2^n x)) + 2^n \left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right) \right) \right)$$

for all $x \in X$. Then we have $J_0 f(x) = f(x)$ and

$$\begin{aligned} J_j f(x) - J_{j+1} f(x) &= -\frac{1}{4^{j+2}} Df(2^j x, 2^j x) - \frac{1}{4^{j+2}} Df(-2^j x, -2^j x) \\ &\quad + 2^{j-2} Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) - 2^{j-2} Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right) \end{aligned}$$

for all $x \in X$ and $j \geq 0$. If $n + m > m \geq 0$, then we have

$$\begin{aligned} & N' \left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{1}{4} \left(\frac{2^p}{4} \right)^j + \frac{1}{2^p} \left(\frac{2}{2^p} \right)^j \right) t^p \right) \\ & \geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N' \left(\frac{Df(2^j x, 0)}{2 \cdot 4^{j+1}}, \frac{2^{jp} t^p}{2 \cdot 4^{j+1}} \right), \right. \right. \\ & \quad N' \left(\frac{Df(-2^j x, 0)}{2 \cdot 4^{j+1}}, \frac{2^{jp} t^p}{2 \cdot 4^{j+1}} \right), N' \left(-2^{j-1} Df\left(\frac{x}{2^{j+1}}, 0\right), \frac{2^{j-1} t^p}{2^{(j+1)p}} \right), \\ & \quad \left. \left. N' \left(2^{j-1} Df\left(-\frac{x}{2^{j+1}}, 0\right), \frac{2^{j-1} t^p}{2^{(j+1)p}} \right) \right\} \right\} \\ & \geq \min \bigcup_{j=m}^{n+m-1} \left\{ N(2^j x, 2^j t), N\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right) \right\} \\ & = N(x, t) \end{aligned}$$

for all $x \in X$ and $t > 0$. In the similar argument following (2.4) of the previous case, we can define the limit $F(x) := N' - \lim_{n \rightarrow \infty} J_n f(x)$ of the Cauchy sequence $\{J_n f(x)\}$ in the Banach fuzzy space Y . Moreover, putting $m = 0$ in the above inequality, we have

$$(2.8) \quad N'(f(x) - J_n f(x), t) \geq N \left(x, \frac{t^q}{\left(\sum_{j=0}^{n-1} \left(\frac{1}{4} \left(\frac{2^p}{4} \right)^j + \frac{1}{2^p} \left(\frac{2}{2^p} \right)^j \right) \right)^q} \right)$$

for each $x \in X$ and $t > 0$. To prove that F is a quadratic additive function, we have enough to show that the last term of (2.6) in Case 1 tends to 1 as $n \rightarrow \infty$. By (N3) and (2.1), we get

$$\begin{aligned} & N' \left(DJ_n f(x, y), \frac{t}{2} \right) \\ & \geq \min \left\{ N' \left(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right), N' \left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right), \right. \\ & \quad \left. N' \left(2^{n-1} Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{t}{8} \right), N' \left(2^{n-1} Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right), \frac{t}{8} \right) \right\} \end{aligned}$$

$$\geq \min \left\{ N(x, 2^{(2q-1)n-3q}t^q), N(y, 2^{(2q-1)n-3q}t^q), \right. \\ \left. N(x, 2^{(1-q)n-3q}t^q), N(y, 2^{(1-q)n-3q}t^q) \right\}$$

for each $x, y \in X$ and $t > 0$. Observe that all the terms on the right hand side of the above inequality tend to 1 as $n \rightarrow \infty$, since $\frac{1}{2} < q < 1$. Hence, together with the similar argument after (2.6), we can say that $DF(x, y) = 0$ for all $x, y \in X$. Recall, in Case 1, the inequality (2.2) follows from (2.5). By the same reasoning, we get (2.2) from (2.8) in this case. Now to prove the uniqueness of F , let F' be another quadratic additive mapping satisfying (2.2). Then, together with (N4), (2.2), and (2.7), we have

$$\begin{aligned} & N'(F(x) - F'(x), t) \\ &= N'(J_n F(x) - J_n F'(x), t) \\ &\geq \min \left\{ N' \left(J_n F(x) - J_n f(x), \frac{t}{2} \right), N' \left(J_n f(x) - J_n F'(x), \frac{t}{2} \right) \right\} \\ &\geq \min \left\{ N' \left(\frac{(F - f)(2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), \left(\frac{(f - F')(2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), \right. \\ &\quad N' \left(\frac{(F - f)(-2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), N' \left(\frac{(f - F')(-2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), \\ &\quad N' \left(2^{n-1} \left((F - f) \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(2^{n-1} \left((f - F') \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), \\ &\quad \left. N' \left(2^{n-1} \left((F - f) \left(\frac{-x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(2^{n-1} \left((f - F') \left(\frac{-x}{2^n} \right) \right), \frac{t}{8} \right) \right\} \\ &\geq \min \left\{ \sup_{t' < t} N \left(x, 2^{(2q-1)n-2q} \left(\frac{(4 - 2^p)(2^p - 2)}{2} \right)^q t'^q \right), \right. \\ &\quad \left. \sup_{t' < t} N \left(x, 2^{(1-q)n-2q} \left(\frac{(4 - 2^p)(2^p - 2)}{2} \right)^q t'^q \right) \right\} \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} 2^{(2q-1)n-2q} = \lim_{n \rightarrow \infty} 2^{(1-q)n-2q} = \infty$ in this case, both terms on the right hand side of the above inequality tend to 1 as $n \rightarrow \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ and so $F(x) = F'(x)$ for all $x \in X$ by (N2).

Case 3. Finally, we take $0 < q < \frac{1}{2}$ and define $J_n f : X \rightarrow Y$ by

$$J_n f(x) = \frac{1}{2} \left(4^n (f(2^{-n}x) + f(-2^{-n}x)) + 2^n \left(f \left(\frac{x}{2^n} \right) - f \left(-\frac{x}{2^n} \right) \right) \right)$$

for all $x \in X$. Then we have $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = (4^{j-1} + 2^{j-2}) Df \left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}} \right)$$

$$+ (4^{j-1} - 2^{j-2})Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right)$$

which implies that if $n + m > m \geq 0$, then

$$\begin{aligned} & N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{4}{2^p}\right)^j \frac{t^p}{2^p}\right) \\ & \geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N'\left((4^{j-1} + 2^{j-2})Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), \frac{(2^{2j-1} + 2^{j-1})t^p}{2^{(j+1)p}}\right), \right. \right. \\ & \quad \left. \left. N'\left((4^{j-1} - 2^{j-2})Df\left(-\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}\right), \frac{(2^{2j-1} - 2^{j-1})t^p}{2^{(j+1)p}}\right)\right\} \right\} \\ & \geq \min \bigcup_{j=m}^{n+m-1} \left\{ N\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right) \right\} \\ & = N(x, t) \end{aligned}$$

for all $x \in X$ and $t > 0$. Similar to the previous cases, it leads us to define the mapping $F : X \rightarrow Y$ by $F(x) := N' - \lim_{n \rightarrow \infty} J_n f(x)$. Putting $m = 0$ in the above inequality, we have

$$(2.9) \quad N'(f(x) - J_n f(x), t) \geq N\left(x, \frac{t^q}{\left(\frac{1}{2^p} \sum_{j=0}^{n-1} \left(\frac{4}{2^p}\right)^j\right)^q}\right)$$

for all $x \in X$ and $t > 0$. Notice that

$$\begin{aligned} & N'\left(DJ_n f(x, y), \frac{t}{2}\right) \\ & \geq \min \left\{ N'\left(\frac{4^n}{2} Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{t}{8}\right), N'\left(\frac{4^n}{2} Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right), \frac{t}{8}\right), \right. \\ & \quad \left. N'\left(2^{n-1} Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{t}{8}\right), N'\left(2^{n-1} Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right), \frac{t}{8}\right) \right\} \\ & \geq \min \left\{ N\left(x, 2^{(1-2q)n-3qt^q}\right), N\left(y, 2^{(1-2q)n-3qt^q}\right), \right. \\ & \quad \left. N\left(x, 2^{(1-q)n-3qt^q}\right), N\left(y, 2^{(1-q)n-3qt^q}\right) \right\} \end{aligned}$$

for each $x, y \in X$ and $t > 0$. Since $0 < q < \frac{1}{2}$, all terms on the right hand side tend to 1 as $n \rightarrow \infty$, which implies that the last term of (2.6) tends to 1 as $n \rightarrow \infty$. Therefore, we can say that $DF \equiv 0$. Moreover, using the similar argument after (2.6) in Case 1, we get the inequality (2.2) from (2.9) in this case. To prove the uniqueness of F , let $F' : X \rightarrow Y$ be another quadratic additive function satisfying (2.2). Then by (2.7), we get

$$N'(F(x) - F'(x), t)$$

$$\begin{aligned}
 &\geq \min \left\{ N' \left(J_n F(x) - J_n f(x), \frac{t}{2} \right), N' \left(J_n f(x) - J_n F'(x), \frac{t}{2} \right) \right\} \\
 &\geq \min \left\{ N' \left(\frac{4^n}{2} \left((F - f) \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(\frac{4^n}{2} \left((f - F') \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), \right. \\
 &\quad N' \left(\frac{4^n}{2} \left((F - f) \left(-\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(\frac{4^n}{2} \left((f - F') \left(-\frac{x}{2^n} \right) \right), \frac{t}{8} \right), \\
 &\quad N' \left(2^{n-1} \left((F - f) \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(2^{n-1} \left((f - F') \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), \\
 &\quad \left. N' \left(2^{n-1} \left((F - f) \left(-\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(2^{n-1} \left((f - F') \left(-\frac{x}{2^n} \right) \right), \frac{t}{8} \right) \right\} \\
 &\geq \sup_{t' < t} N \left(x, 2^{(1-2q)n-2q} (2^p - 4)^q t'^q \right)
 \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Observe that, for $0 < q < \frac{1}{2}$, the last term tends to 1 as $n \rightarrow \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ and $F(x) = F'(x)$ for all $x \in X$ by (N2). \square

Remark 2.3. Consider a mapping $f : X \rightarrow Y$ satisfying (2.1) for all $x, y \in X$ and a real number $q < 0$. Take any $t > 0$. If we choose a real number s with $0 < 2s < t$, then we have

$$N'(Df(x, y), t) \geq N'(Df(x, y), 2s) \geq \min\{N(x, s^q), N(y, s^q)\}$$

for all $x, y \in X$. Since $q < 0$, we have $\lim_{s \rightarrow 0^+} s^q = \infty$. This implies that

$$\lim_{s \rightarrow 0^+} N(x, s^q) = \lim_{s \rightarrow 0^+} N(y, s^q) = 1$$

and so

$$N'(Df(x, y), t) = 1$$

for all $x, y \in X$ and $t > 0$. By (N2), it allows us to get $Df(x, y) = 0$ for all $x, y \in X$. In other words, f is itself a quadratic additive mapping if f is a fuzzy q -almost quadratic-additive mapping for the case $q < 0$.

Corollary 2.4. *Let f be an even mapping satisfying all of the conditions of Theorem 2.2. Then there is a unique quadratic mapping $F : X \rightarrow Y$ such that*

$$(2.10) \quad N'(F(x) - f(x), t) \geq \sup_{t' < t} N \left(x, (|4 - 2^p|t')^q \right)$$

for all $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. Let $J_n f$ be defined as in Theorem 2.2. Since f is an even mapping, we obtain

$$J_n f(x) = \begin{cases} \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} & \text{if } 0 < q < \frac{1}{2}, \\ \frac{1}{2} (4^n (f(2^{-n} x) + f(-2^{-n} x))) & \text{if } q > \frac{1}{2} \end{cases}$$

for all $x \in X$. Notice that $J_0f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \begin{cases} \frac{-1}{4^{j+2}} (Df(2^j x, 2^j x) + Df(-2^j x, -2^j x)) & \text{if } 0 < q < \frac{1}{2}, \\ 4^{j-1} (Df(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}) + Df(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}})) & \text{if } q > \frac{1}{2} \end{cases}$$

for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$. From these, using the similar method in Theorem 2.2, we obtain the quadratic-additive mapping F satisfying (2.10). Notice that F is also even, $F(x) := N' - \lim_{n \rightarrow \infty} J_n f(x)$ for all $x \in X$, and $DF(x, y) = 0$ for all $x, y \in X$. Hence, we get

$$F(x + y) + F(x - y) - 2F(x) - 2F(y) = \frac{1}{2}DF(x, y) = 0$$

for all $x, y \in X$. This means that F is a quadratic mapping. □

Corollary 2.5. *Let f be an odd mapping satisfying all of the conditions of Theorem 2.2. Then there is a unique additive mapping $F : X \rightarrow Y$ such that*

$$(2.11) \quad N'(F(x) - f(x), t) \geq \sup_{t' < t} N(x, (|2 - 2^p|t')^q)$$

for all $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. Let $J_n f$ be defined as in Theorem 2.2. Since f is an odd mapping, we obtain

$$J_n f(x) = \begin{cases} \frac{f(2^n x) + f(-2^n x)}{2^{n+1}} & \text{if } 0 < q < 1, \\ 2^{n-1} (f(2^{-n} x) + f(-2^{-n} x)) & \text{if } q > 1 \end{cases}$$

for all $x \in X$. Notice that $J_0f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \begin{cases} \frac{1}{2^{j+3}} (Df(-2^j x, -2^j x) - Df(2^j x, 2^j x)) & \text{if } 0 < q < 1 \\ 2^{j-2} (Df(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}) - Df(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}})) & \text{if } q > 1 \end{cases}$$

for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$. From these, using the similar method in Theorem 2.2, we obtain the quadratic-additive mapping F satisfying (2.11). Notice that F is also odd, $F(x) := N' - \lim_{n \rightarrow \infty} J_n f(x)$ for all $x \in X$, and $DF(x, y) = 0$ for all $x, y \in X$. Hence, we get

$$F(x + y) - F(x) - F(y) = \frac{1}{2}DF(x, y) = 0$$

for all $x, y \in X$. This means that F is an additive mapping. □

We can use Theorem 2.2 to get a classical result in the framework of normed spaces. Let $(X, \|\cdot\|)$ be a normed linear space. Then we can define a fuzzy norm N_X on X by following

$$N_X(x, t) = \begin{cases} 0, & t \leq \|x\| \\ 1, & t > \|x\| \end{cases}$$

where $x \in X$ and $t \in \mathbb{R}$, see [17]. Suppose that $f : X \rightarrow Y$ is a mapping into a Banach space $(Y, \|\cdot\|)$ such that

$$\|Df(x, y)\| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$. Let N_Y be a fuzzy norm on Y . Then we get

$$N_Y(Df(x, y), s + t) = \begin{cases} 0, & s + t \leq |||Df(x, y)||| \\ 1, & s + t > |||Df(x, y)||| \end{cases}$$

for all $x, y \in X$ and $s, t \in \mathbb{R}$. Consider the case $N_Y(Df(x, y), s + t) = 0$. This implies that

$$\|x\|^p + \|y\|^p \geq |||Df(x, y)||| \geq s + t$$

and so either $\|x\|^p \geq s$ or $\|y\|^p \geq t$ in this case. Hence, for $q = \frac{1}{p}$, we have

$$\min\{N_X(x, s^q), N_X(y, t^q)\} = 0$$

for all $x, y \in X$ and $s, t > 0$. Therefore, in every case, the inequality

$$N_Y(Df(x, y), s + t) \geq \min\{N_X(x, s^q), N_X(y, t^q)\}$$

holds. It means that f is a fuzzy q -almost quadratic additive mapping, and by Theorem 2.2, we get the following stability result.

Corollary 2.6. *Let $(X, \|\cdot\|)$ be a normed linear space and let $(Y, |||\cdot|||)$ be a Banach space. If*

$$|||Df(x, y)||| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$, then there is a unique quadratic-additive mapping $F : X \rightarrow Y$ such that

$$|||F(x) - f(x)||| \leq \begin{cases} \frac{\|x\|^p}{2-2^p} & \text{if } 0 < p < 1, \\ \frac{2\|x\|^p}{(2-2^p)(2^p-4)} & \text{if } 1 < p < 2, \\ \frac{\|x\|^p}{2^p-4} & \text{if } 2 < p \end{cases}$$

for all $x \in X$.

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