Fuzzy Tracking Control Design for Nonlinear Dynamic Systems via T–S Fuzzy Model

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***Abstract*—This study introduces a fuzzy control design method for nonlinear systems with a guaranteed H_∞ model reference tracking performance. First, the Takagi and Sugeno (TS) fuzzy model is employed to represent a nonlinear system. Next, based on the fuzzy model, a fuzzy observer-based fuzzy controller is developed to reduce the tracking error as small as possible for all bounded reference inputs. The advantage of proposed tracking control design is that only a simple fuzzy controller is used in our approach without feedback linearization technique and complicated adaptive scheme. By the proposed method, the fuzzy tracking control design problem is parameterized in terms of a linear matrix inequality problem (LMIP). The LMIP can be solved very efficiently using the convex optimization techniques. Simulation example is given to illustrate the design procedures and tracking performance of the proposed method.**

***Index Terms*—Fuzzy tracking control, LMIP.**

I. Introduction

THE tasks of stabilization and tracking are two typical control problems. In general, tracking problems are more difficult than stabilization problems especially for nonlinear systems. For nonlinear system design, various control schemes are introduced including exact feedback linearization, sliding mode control and adaptive control. The technique of exact feedback linearization needs perfect knowledge of the nonlinear system and uses that knowledge to cancel the nonlinearities of the system. Since perfect knowledge of the system is almost impossible, the technique of exact feedback linearization seems impractical for nonlinear system design [1], [32]. Recently, based on feedback linearization technique, H_∞ adaptive fuzzy control schemes have been introduced to deal with nonlinear systems [3], [4]. However, the complicated parameter update law and control algorithm make this control scheme impractical, especially in the case of considering the projection algorithm for the parameter update law to avoid the singularity of feedback linearization control [26], [27]. An advantage of sliding mode control is its robustness to uncertainties [5]. However, chattering phenomenon that results in low control accuracy and high heat loss in electrical power circuits is inevitable in the sliding mode control. It may also excite unmodeled high-frequency dynamics, which degrades the performance of the system and may even lead to instability.

Recently, Takagi–Sugeno (T–S) type fuzzy controllers have been successfully applied to the stabilization control design of nonlinear systems [9]–[13]. In most of these applications, the fuzzy systems were thought of as universal approximators for nonlinear systems. The T–S fuzzy model [9] has been proved to be a very good representation for a certain class of nonlinear dynamic systems. In their studies, a nonlinear plant was represented by a set of linear models interpolated by membership functions (T–S) fuzzy model) and then a model-based fuzzy controller was developed to stabilize the T–S fuzzy model. On the other hand, tracking control designs are also important issues for practical applications; for example, in robotic tracking control, missile tracking control and attitude tracking control of aircraft. However, there are very few studies concerning with tracking control design based on the T–S fuzzy model, especially for continuous-time systems.

In general, tracking control design is more general and more difficult than the stabilization control design. In [14], feedback linearization technique is proposed to systematically design a fuzzy tracking controller for discrete-time systems. As pointed out in [15], the controller derived by feedback linearization may not be bounded, i.e., the fuzzy controller is not guaranteed to be stable for nonminimum phase system. In this work, tracking control design based on the T–S fuzzy model is studied. First, the T–S fuzzy model is used to represent a nonlinear system. An H_∞ tracking performance, which is related to tracking error for all bounded reference inputs, is formulated, then a fuzzy observer-based fuzzy controller is developed to reduce tracking error as small as possible. Conventionally, the robust H_∞ performance design problems for nonlinear systems have to solve the Hamilton–Jacobi equation, which is a nonlinear partial differential equation. Only some very special nonlinear systems have a closed form solution [34]. To avoid solving nonlinear partial differential equation, the T–S fuzzy model is employed to represent a nonlinear system.

Based on the T–S fuzzy model, the outcome of the fuzzy tracking control problem is parameterized in terms of a linear matrix inequality problem (LMIP) [23]. The LMIP can be solved very efficiently by a convex optimization techniques [24], [25] to complete the fuzzy tracking control design. The difference between our approach and those in [14], [15] is that only a simple fuzzy controller is used in our approach without feedback linearization technique and complicated adaptive scheme. Furthermore, the stability of the proposed fuzzy controller is guaranteed for a general class of T–S fuzzy systems. As a special case when all state variables are available, the

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fuzzy tracking control using state feedback is also introduced in the Appendix.

The main contribution of this paper is as follows. Based on the proposed fuzzy approach, a simple algorithm based on linear matrix inequality (LMI) optimization techniques is developed systematically to solve the fuzzy tracking control design for nonlinear systems. Therefore, the proposed tracking control design is suitable for practical applications.

The paper is organized as follows. Section II presents the problem formulation. In Section III, a fuzzy observer-based tracking control is considered. In Section IV, simulation example is provided to demonstrate the design effectiveness. Finally, concluding remarks are made in Section V. As a special case, the fuzzy tracking control using state feedback is introduced in the Appendix.

II. Problem Formulation

A fuzzy dynamic model has been proposed by Takagi and Sugeno [9] to represent a nonlinear system. The T–S fuzzy model is a piecewise interpolation of several linear models through membership functions. The fuzzy model is described by fuzzy *If-Then* rules and will be employed here to deal with the control design problem for the nonlinear system. The ith rule of the fuzzy model for the nonlinear system is of the following form [9], [10], [12], [15]:

Plant Rule i.

If $z_1(t)$ is F_{i1} and $\cdots$ and $z_g(t)$ is F_{ig}

then
$$\begin{aligned}\dot{x}(t) &= A_i x(t) + B_i u(t) + w(t)\\ y(t) &= C_i x(t) + v(t), \qquad \text{for } i = 1,2,\ldots,L\end{aligned} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^{n\times 1}$ denotes the state vector, $u(t) = [u_1(t), u_2(t), \cdots, u_m(t)]^T \in R^{m\times 1}$ denotes the control input; $w(t) = [w_1(t), w_2(t), \cdots, w_n(t)]^T \in R^{n\times 1}$ denotes bounded external disturbance; $y(t)$ denotes output of the system; $v(t)$ denotes the measurement noise; F_{ij} is the fuzzy set, $A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$; L is the number of *If-Then* rules; and $z_1(t)$, $z_2(t), \cdots, z_g(t)$ are the premise variables.

The fuzzy system is inferred as follows [9], [10], [12]:

$$\begin{aligned}\dot{x}(t) &= \frac{\sum_{i=1}^{L}\mu_i(z(t))[A_i x(t) + B_i u(t)] + w(t)}{\sum_{i=1}^{L}\mu_i(z(t))}\\ &= \sum_{i=1}^{L} h_i(z(t))[A_i x(t) + B_i u(t)] + w(t)\end{aligned} \tag{2}$$

$$\begin{aligned}y(t) &= \frac{\sum_{i=1}^{L}\mu_i(z(t))[C_i x(t)] + v(t)}{\sum_{i=1}^{L}\mu_i(z(t))}\\ &= \sum_{i=1}^{L} h_i(z(t))[C_i x(t)] + v(t)\end{aligned} \tag{3}$$

where

$$\begin{aligned}\mu_i(z(t)) &= \prod_{j=1}^{g} F_{ij}(z_j(t))\\ h_i(z(t)) &= \frac{\mu_i(z(t))}{\sum_{i=1}^{L}\mu_i(z(t))}\\ z(t) &= [z_1(t), z_2(t), \cdots, z_g(t)]\end{aligned} \tag{4}$$

and where $F_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in F_{ij}.

We assume

$$\mu_i(z(t)) \geq 0$$

and

$$\sum_{i=1}^{L}\mu_i(z(t)) > 0, \qquad \text{for } i = 1,2,\ldots,L$$

for all t.

Therefore, we get [6]–[8], [10]

$$h_i(z(t)) \geq 0, \qquad \text{for } i = 1,2,\ldots,L \tag{5}$$

and

$$\sum_{i=1}^{L} h_i(z(t)) = 1. \tag{6}$$

The TS fuzzy model in (2) is a general nonlinear time-varying equation and has been used to model the behaviors of complex nonlinear dynamic systems [15].

Consider a reference model as follows [20]–[22]:

$$\dot{x}_r(t) = A_r x_r(t) + r(t) \tag{7}$$

where

- $x_r(t)$ reference state;
- A_r specific asymptotically stable matrix;
- $r(t)$ bounded reference input.

It is assumed that $x_r(t)$, for all $t \geq 0$, represents a desired trajectory for $x(t)$ to follow.

Let us consider the H_∞ tracking performance related to tracking error $x(t) - x_r(t)$ as follows [2], [3]

$$\frac{\int_0^{t_f}\{[x(t) - x_r(t)]^T Q[x(t) - x_r(t)]\}\,dt}{\int_0^{t_f}\tilde{w}(t)^T\tilde{w}(t)\,dt} \leq \rho^2 \tag{8}$$

or

$$\begin{aligned}&\int_0^{t_f}\{[x(t) - x_r(t)]^T Q[x(t) - x_r(t)]\}\,dt\\ &\qquad \leq \rho^2\int_0^{t_f}\tilde{w}(t)^T\tilde{w}(t)\,dt\end{aligned} \tag{9}$$

where

- $\tilde{w}(t)$ $= [v(t), w(t), r(t)]^T$ for all reference input $r(t)$, external disturbance $w(t)$ and measurement noise $v(t)$;
- t_f terminal time of control;
- Q positive definite weighting matrix;
- ρ prescribed attenuation level.

The physical meaning of (8) or (9) is that the effect of any $\tilde{w}(t)$ on tracking error $x(t)-x_r(t)$ must be attenuated below a desired level ρ from the viewpoint of energy, no matter what $\tilde{w}(t)$ is, i.e., the L_2 gain from $\tilde{w}(t)$ to $x(t)-x_r(t)$ must be equal to or less than a prescribed value ρ^2.

Suppose the following fuzzy observer is proposed to deal with the state estimation of nonlinear system (1).

Observer Rule i.

If $z_1(t)$ is F_{i1} and $\cdots$ and $z_g(t)$ is F_{ig},

then $\dot{\hat{x}}t) = A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t))$ (10)

where L_i is the observer gain for the ith observer rule and $\hat{y}(t) = \sum_{i=1}^{L} h_i(z(t))C_i\hat{x}(t)$.

The overall fuzzy observer is represented as follows:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{L} h_i(z(t))[A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t))]. \tag{11}$$

Let us denote the estimation errors as

$$e(t) = x(t) - \hat{x}(t). \tag{12}$$

By differentiating (12), we get

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t)\\ &= \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t))[A_ix(t) + B_iu(t) + w(t)]\\ &\quad - [A_i\hat{x}(t) + B_iu(t) + L_iC_j(x(t) - \hat{x}(t)) + L_iv(t)]\\ &= \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t))[(A_i - L_iC_j)e(t) - L_iv(t)]\\ &\quad + w(t).\end{aligned} \tag{13}$$

Suppose the following fuzzy controller is employed to deal with the above control system design.

Control Rule j.

If $z_1(t)$ is F_{j1} and $\cdots$ and $z_g(t)$ is F_{jg},

then $u(t) = K_j[\hat{x}(t) - x_r(t)]$, for $j = 1, 2, \cdots, L$. (14)

Hence, the overall fuzzy controller is given by

$$\begin{aligned}u(t) &= \frac{\sum_{j=1}^{L}\mu_j(z(t))[K_j(\hat{x}(t) - x_r(t))]}{\sum_{j=1}^{L}\mu_j(z(t))}\\ &= \sum_{j=1}^{L} h_j(z(t))[K_j(\hat{x}(t) - x_r(t))].\end{aligned} \tag{15}$$

Remark 1:

1) The premise variables $z(t)$ can be measurable state variables, outputs or combination of measurable state variables. For T–S type fuzzy model, using state variables as premise variables are common, but not always [10]–[13], [15]–[19]. The limitation of this approach is that some state variables must be measurable to construct the fuzzy observer and fuzzy controller. This is a common limitation for control system design of T–S fuzzy approach [17], [18]. If the premise variables of the fuzzy observer depend on the estimated state variables, i.e., $\hat{z}(t)$ instead of $z(t)$ in the fuzzy observer, the situation becomes more complicated. In this case, it is difficult to directly find control gains K_j and observer gains L_i. The problem has been discussed in [18].
2) The problem of constructing T–S fuzzy model for nonlinear systems can be found in [28]–[31]. ■

After manipulation, the augmented system can be expressed as the following form:

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))[\tilde{A}_{ij}\tilde{x}(t) + \tilde{E}_i\tilde{w}(t)]. \tag{16}$$

Let us denote

$$\begin{aligned}\tilde{A}_{ij} &= \begin{bmatrix} A_i - L_iC_j & 0 & 0\\ -B_iK_j & A_i + B_iK_j & -B_iK_j\\ 0 & 0 & A_r\end{bmatrix}\\ \tilde{x}(t) &= \begin{bmatrix} e(t)\\ x(t)\\ x_r(t)\end{bmatrix}, \quad \tilde{w}(t) = \begin{bmatrix} v(t)\\ w(t)\\ r(t)\end{bmatrix}\\ \tilde{E}_i &= \begin{bmatrix} -L_i & I & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}.\end{aligned} \tag{17}$$

Therefore, the augmented system defined in (16) can be expressed as the following form:

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))[\tilde{A}_{ij}\tilde{x}(t) + \tilde{E}_i\tilde{w}(t)]. \tag{18}$$

Hence, the H_∞ tracking performance in (9) can be modified as follows, if the initial condition is also considered

$$\begin{aligned}&\int_0^{t_f} \{(x(t) - x_r(t))^TQ(x(t) - x_r(t))\}\,dt\\ &\quad = \int_0^{t_f} \tilde{x}^T(t)\tilde{Q}\tilde{x}(t)\,dt\\ &\quad \le \tilde{x}^T(0)\tilde{P}\tilde{x}(0) + \rho^2\int_0^{t_f} \tilde{w}(t)^T\tilde{w}(t)\,dt\end{aligned} \tag{19}$$

where $\tilde{P}$ is a symmetric positive definite weighting matrix and

$$\tilde{Q} = \begin{bmatrix} 0 & 0 & 0\\ 0 & Q & -Q\\ 0 & -Q & Q\end{bmatrix}.$$

The purpose of this study is to determine a fuzzy controller in (15) for the augmented system in (18) with the guaranteed H_∞ tracking performance in (1) for all $\tilde{w}(t)$. Thereafter, the attenuation level ρ^2 can also be minimized so that the H_∞ tracking performance in (19) is reduced as small as possible. Furthermore, the closed-loop system

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\tilde{A}_{ij}\tilde{x}(t) \tag{20}$$

is quadratically stable.

III. Fuzzy Observer-Based Tracking Control Design

The design purpose in this study is to specify the fuzzy control in (15) to achieve the H_∞ tracking control performance in (19). Then, we obtain the following result:

Theorem 1: In the nonlinear system (18), if $\tilde{P} = \tilde{P} > 0$ is the common solution of the following matrix inequalities:

$$\tilde{A}_{ij}^T \tilde{P} + \tilde{P}\overline{A}_{ij} + \frac{1}{\rho^2}\tilde{P}\tilde{E}_i\tilde{E}_i^T\tilde{P} + \tilde{Q} < 0 \tag{21}$$

for $h_i(z(t))h_j(z(t)) \neq 0$ and $i, j = 1, 2, \cdots, L$, then the H_∞ tracking control performance in (19) is guaranteed for a prescribed ρ^2.

Proof: From (19), we obtain

$$\begin{aligned}
&\int_0^{t_f} \{(x(t)-x_r(t))^T Q (x(t)-x_r(t))\}\, dt \\
&\quad = \int_0^{t_f} \tilde{x}^T(t)\tilde{Q}\tilde{x}(t)\, dt \\
&\quad = \tilde{x}^T(0)\tilde{P}\tilde{x}(0) - \tilde{x}^T(t_f)\tilde{P}\tilde{x}(t_f) \\
&\qquad + \int_0^{t_f} \left\{ \tilde{x}^T(t)\tilde{Q}\tilde{x}(t) + \frac{d}{dt}(\tilde{x}^T(t)\tilde{P}\tilde{x}(t)) \right\} dt \\
&\quad \le \tilde{x}^T(0)\tilde{P}\tilde{x}(0) + \int_0^{t_f} \Bigg\{ \tilde{x}^T(t)\tilde{Q}\tilde{x}(t) \\
&\qquad + \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) [(\tilde{A}_{ij}\tilde{x}(t))^T \tilde{P}\tilde{x}(t) \\
&\qquad + \tilde{x}^T(t)\tilde{P}\tilde{A}_{ij}\tilde{x}(t) + \Big[(\tilde{E}_i\tilde{w}(t))^T\tilde{P}\tilde{x}(t) \\
&\qquad + \tilde{x}^T(t)\tilde{P}\tilde{E}_i\tilde{w}(t) - \rho^2\tilde{w}(t)^T\tilde{w}(t) \\
&\qquad - \frac{1}{\rho^2}\tilde{x}^T(t)\tilde{P}\tilde{E}_i\tilde{E}_i^T\tilde{P}\tilde{x}(t)\Big] \\
&\qquad + \frac{1}{\rho^2}\tilde{x}^T(t)\tilde{P}\tilde{E}_i\tilde{E}_i^T\tilde{P}\tilde{x}(t) + \rho^2\tilde{w}(t)^T\tilde{w}(t)] \Bigg\} dt \\
&\quad = \tilde{x}^T(0)\tilde{P}\tilde{x}(0) + \int_0^{t_f} \Bigg\{ \tilde{x}^T(t)\tilde{Q}\tilde{x}(t) \\
&\qquad + \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) [(\tilde{A}_{ij}\tilde{x}(t))^T \tilde{P}\tilde{x}(t) \\
&\qquad + \tilde{x}^T(t)\tilde{P}\tilde{A}_{ij}\tilde{x}(t) - \left[\frac{1}{\rho}\tilde{E}_i^T\tilde{P}\tilde{x}(t) - \rho\tilde{w}(t)\right]^T \\
&\qquad \cdot \left[\frac{1}{\rho}\tilde{E}_i^T\tilde{P}\tilde{x}(t) - \rho\tilde{w}(t)\right] \\
&\qquad + \frac{1}{\rho^2}\tilde{x}^T(t)\tilde{P}\tilde{E}_i\tilde{E}_i^T\tilde{P}\tilde{x}(t)\Big] + \rho^2\tilde{w}(t)^T\tilde{w}(t) \Bigg\} dt \\
&\quad \le \tilde{x}^T(0)\tilde{P}\tilde{x}(0) + \int_0^{t_f} \Bigg\{ \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t))\tilde{x}^T(t) \\
&\qquad \times \left[\tilde{Q} + \tilde{A}_{ij}^T\tilde{P} + \tilde{P}\tilde{A}_{ij} + \frac{1}{\rho^2}\tilde{P}\tilde{E}_i\tilde{E}_i^T\tilde{P}\right]\tilde{x}(t) \\
&\qquad + \rho^2\tilde{w}(t)^T\tilde{w}(t) \Bigg\} dt.
\end{aligned}$$

By (21), we obtain

$$\begin{aligned}
&\int_0^{t_f} \tilde{x}^T(t)\tilde{Q}\tilde{x}(t)\, dt \\
&\qquad \le \tilde{x}^T(0)\tilde{P}\tilde{x}(0) + \rho^2 \int_0^{t_f} \tilde{w}(t)^T\tilde{w}(t)\, dt.
\end{aligned} \tag{22}$$

Therefore, the H_∞ tracking control performance is achieved with a prescribed ρ^2. This completes the proof. ■

Remark 2: The adaptive fuzzy tracking design in [3], [4] lies in tuning a fuzzy logic system by an adaptive law to approximate the feedback linearization controller and using H_∞ control scheme to attenuate the effect of external disturbance. Furthermore, the system must be minimum phase. In this study, based on T–S fuzzy model, the H_∞ tracking performance is guaranteed for all bounded reference inputs without complex update law. Furthermore, the stability of closed loop system is guaranteed and the attenuation level ρ^2 is minimized. ■

To obtain better tracking performance, the tracking control problem can be formulated as the following minimization problem:

$$\begin{aligned}
&\min_{\tilde{\mathrm{P}}} \rho^2 \\
&\text{subject to } \tilde{P} > 0 \text{ and (21).}
\end{aligned} \tag{23}$$

To prove the closed-loop system in (20) is quadratically stable, let us define a Lyapunov function for the system of (20) as

$$V(t) = \tilde{x}^T(t)\tilde{P}\tilde{x}(t) \tag{24}$$

where the weighting matrix $\tilde{P}$ is the same as that in (19).

By differentiating (24), we obtain

$$\begin{aligned}
\dot{V}(t) &= \dot{\tilde{x}}^T(t)\tilde{P}\tilde{x}(t) + \tilde{x}^T(t)\tilde{P}\dot{\tilde{x}}(t) \\
&= \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t)) \\
&\qquad \times \tilde{x}^T(t)[\tilde{A}_{ij}^T\tilde{P} + \tilde{P}\tilde{A}_{ij}]\tilde{x}(t).
\end{aligned} \tag{25}$$

Then, we obtain the following result:

Theorem 2: In the nonlinear closed-loop system (20), if there exists the common solution $\tilde{P} = \tilde{P}^T > 0$ for the minimization problem in (23), then the closed-loop system in (20) is quadratically stable.

Proof: From (25), we obtain

$$\begin{aligned}
\dot{V}(t) = &\sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t)) \\
&\times \tilde{x}^T(t)[\tilde{A}_{ij}^T\tilde{P} + \tilde{P}\tilde{A}_{ij}]\tilde{x}(t).
\end{aligned} \tag{26}$$

By (21), we get

$$\dot{V}(t) < 0. \tag{27}$$

This completes the proof. ■

From the analysis above, the most important task of the fuzzy observer-based state feedback tracking control problem is how to solve the common solution $\tilde{P} = \tilde{P}^T > 0$ from the minimization problem (23). In general, it is not easy to analytically determine the common solution $\tilde{P} = \tilde{P}^T > 0$ for (23). Fortunately, (23) can be transferred into a minimization problem subject to some linear matrix inequalities (LMIs). The LMIP can be solved in a computationally efficient manner using a convex optimization technique such as the interior point method [23], [24].

For the convenience of design, we assume

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & 0 & 0 \\ 0 & \tilde{P}_{22} & 0 \\ 0 & 0 & \tilde{P}_{33} \end{bmatrix}. \tag{28}$$

By substituting (28) into (21), we obtain

$$\begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & S_{23} \\ 0 & S_{32} & S_{33} \end{bmatrix} < 0$$

where

$$\begin{aligned} S_{11} &= (A_i - L_iC_j)^T\tilde{P}_{11} + \tilde{P}_{11}(A_i - L_iC_j) \\ &\quad + \frac{1}{\rho^2}\tilde{P}_{11}(L_iL_i^T + I)\tilde{P}_{11} \\ S_{12} &= S_{21}^T = -\tilde{P}_{22}B_iK_j + \frac{1}{\rho^2}\tilde{P}_{11}\tilde{P}_{22} \\ S_{22} &= (A_i + B_iK_j)^T\tilde{P}_{22} + \tilde{P}_{22}(A_i + B_iK_j) \\ &\quad + \frac{1}{\rho^2}\tilde{P}_{22}\tilde{P}_{22} + Q \\ S_{23} &= S_{32}^T = -\tilde{P}_{22}B_iK_j - Q \\ S_{33} &= A_r^T\tilde{P}_{33} + \tilde{P}_{33}A_r + \frac{1}{\rho^2}\tilde{P}_{33}\tilde{P}_{33} + Q. \end{aligned}$$

With $Z_i = \tilde{P}_{11}L_i$, we obtain

$$\begin{bmatrix} M_{11}^* & \tilde{P}_{11} & Z_i & M_{41}^{*T} & 0 & 0 \\ \tilde{P}_{11} & -\rho^2 I & 0 & 0 & 0 & 0 \\ Z_i^T & 0 & -\rho^2 I & 0 & 0 & 0 \\ M_{41}^* & 0 & 0 & M_{44}^* & M_{45}^* & 0 \\ 0 & 0 & 0 & M_{45}^{*T} & M_{55}^* & \tilde{P}_{33} \\ 0 & 0 & 0 & 0 & \tilde{P}_{33} & -\rho^2 I \end{bmatrix} < 0 \tag{29}$$

where

$$\begin{aligned} M_{11}^* &= A_i^T\tilde{P}_{11} + \tilde{P}_{11}A_i - Z_iC_j - (Z_iC_j)^T \\ M_{41}^* &= -(B_iK_j)^T\tilde{P}_{22} + \frac{1}{\rho^2}\tilde{P}_{22}\tilde{P}_{11} \\ M_{44}^* &= (A_i + B_iK_j)^T\tilde{P}_{22} + \tilde{P}_{22}(A_i + B_iK_j) \\ &\quad + \frac{1}{\rho^2}\tilde{P}_{22}\tilde{P}_{22} + Q \\ M_{45}^* &= -\tilde{P}_{22}B_iK_j - Q \\ M_{55}^* &= A_r^T\tilde{P}_{33} + \tilde{P}_{33}A_r + Q. \end{aligned}$$

Since five parameters $\tilde{P}_{11}, \tilde{P}_{22}, \tilde{P}_{33}, K_j$, and L_i should be determined from (29), there are no effective algorithms for solving them simultaneously, until now. However, we can solve them by the following two-step procedures.

In the first step, note that (29) implies that $M_{44}^* < 0$

$$\begin{aligned} &(A_i + B_iK_j)^T\tilde{P}_{22} + \tilde{P}_{22}(A_i + B_iK_j) \\ &\quad + \frac{1}{\rho^2}\tilde{P}_{22}\tilde{P}_{22} + Q < 0. \end{aligned} \tag{30}$$

With $\tilde{W}_{22} = \tilde{P}_{22}^{-1}$ and $Y_j = K_j\tilde{W}_{22}$, (30) is equivalent to

$$\begin{aligned} &\tilde{W}_{22}A_i^T + A_i\tilde{W}_{22} + B_iY_j + (B_iY_j)^T \\ &\quad + \frac{1}{\rho^2}I + \tilde{W}_{22}Q\tilde{W}_{22} < 0. \end{aligned} \tag{31}$$

By the Schur complements, (31) is equivalent to the following LMIs:

$$\begin{bmatrix} H_{11} & \tilde{W}_{22} \\ \tilde{W}_{22} & -Q^{-1} \end{bmatrix} < 0 \tag{32}$$

where $H_{11} = \tilde{W}_{22}A_i^T + A_i\tilde{W}_{22} + B_iY_j + (B_iY_j)^T + (1/\rho^2)I$.

The parameters $\tilde{W}_{22}$ and Y_j (thus $\tilde{P}_{22} = \tilde{W}_{22}^{-1}$ and $K_j = Y_j\tilde{W}_{22}^{-1}$) can be obtained by solving the LMIP in (32) for a prescribed attenuation level ρ^2. In the second step, by substituting $\tilde{P}_{22}$ and K_j into (29), (29) becomes standard LMIs. Similarly, we can easily solve $\tilde{P}_{11}$, $\tilde{P}_{33}$ and Z_i (thus $L_i = \tilde{P}_{11}^{-1}Z_i$) from (29).

Recall that the attenuation level ρ^2 can be minimized so that the H_∞ tracking performance in (19) is reduced as small as possible

$$\min_{\{\tilde{P}_{11}, \tilde{P}_{22}, \tilde{P}_{33}\}} \rho^2$$

subject to $\tilde{P}_{11} > 0, \tilde{P}_{22} > 0, \tilde{P}_{33} > 0$ and (29).

According to the analysis above, the tracking control via fuzzy observer-based state feedback is summarized as follows.

Design Procedures:

1) Select membership functions and construct fuzzy plant rules in (1).
2) Given an initial attenuation level ρ^2.
3) Solve the LMIP in (32) to obtain $\tilde{W}_{22}$ and Y_j (thus $\tilde{P}_{22} = \tilde{W}_{22}^{-1}$ and $K_j = Y_j\tilde{W}_{22}^{-1}$ can also be obtained).
4) Substitute $\tilde{P}_{22}$ and K_j into (29) and then solve the LMIP in (29) to obtain $\tilde{P}_{11}$, $\tilde{P}_{33}$ and Z_i (thus $L_i = \tilde{P}_{11}^{-1}Z_i$ can also be obtained).
5) Decrease ρ^2 and repeat Steps 3–5 until $\tilde{W}_{22}$, $\tilde{P}_{11}$ and $\tilde{P}_{33}$ can not be found.
6) Construct the fuzzy observer (11).
7) Construct the fuzzy controller (15).

Remark 3: Software packages such as LMI optimization toolbox in Matlab [25] have been developed for easily solving the LMIP. ■

IV. Simulation Example

Consider a two-link robot system as shown in Fig. 1. The dynamic equation of the two-link robot system is given as follows [33]:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \tag{33}$$

where

$$\begin{aligned} M(q) &= \begin{bmatrix} (m_1 + m_2)l_1^2 & m_2l_1l_2(s_1s_2 + c_1c_2) \\ m_2l_1l_2(s_1s_2 + c_1c_2) & m_2l_2^2 \end{bmatrix} \\ C(q, \dot{q}) &= m_2l_1l_2(c_1s_2 - s_1c_2)\begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix} \\ G(q) &= \begin{bmatrix} -(m_1 + m_2)l_1gs_1 \\ -m_2l_2gs_2 \end{bmatrix} \end{aligned}$$

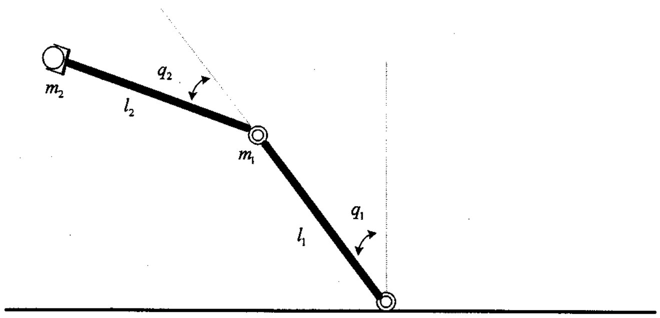

Fig. 1. The configuration of two-link robot systems.

and $q = [q_1, q_2]^T$, q_1, q_2 are generalized coordinates, $M(q)$ is the moment of inertia, $C(q, \dot{q})$ includes coriolis, centripetal forces, and $G(q)$ is the gravitational force. Other quantities are: link mass m_1, $m_2(kg)$, link length l_1, $l_2(m)$, angular position q_1, $q_2(rad)$, applied torques $\tau = [\tau_1, \tau_2]^T(N-m)$, the acceleration due to gravity $g = 9.8(m/s^2)$, and short-hand notation $s_1 = \sin(\mathrm{q}_1)$, $s_2 = \sin(\mathrm{q}_2)$, $c_1 = \cos(\mathrm{q}_1)$, and $c_2 = \cos(\mathrm{q}_2)$. Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, and $x_4 = \dot{q}_2$, then (33) can be written as the following state-space form including external disturbances and measurement noises:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + w_1 \\
\dot{x}_2 &= f_1(x) + g_{11}(x)\tau_1 + g_{12}\tau_2 + w_2 \\
\dot{x}_3 &= x_4 + w_3 \\
\dot{x}_4 &= f_2(x) + g_{21}(x)\tau_1 + g_{22}\tau_2 + w_4 \\
y_1 &= x_1 + v_1 \\
y_2 &= x_3 + v_2
\end{aligned}
\tag{34}
$$

where w_1, w_2, w_3, and w_4 denote external disturbances and v_1 and v_2 denote measurement noises and

$$
\begin{aligned}
f_1(x) &= \frac{(s_1c_2 - c_1s_2)}{l_1l_2[(m_1+m_2) - m_2(s_1s_2 + c_1c_2)^2]} \\
&\quad \times [m_2l_1l_2(s_1s_2 + c_1c_2)x_2^2 - m_2l_2^2x_4^2] \\
&\quad + \frac{1}{l_1l_2[(m_1+m_2) - m_2(s_1s_2 + c_1c_2)^2]} \\
&\quad \times [(m_1+m_2)l_2gs_1 - m_2l_2gs_2(s_1s_2 + c_1c_2)] \\
f_2(x) &= \frac{(s_1c_2 - c_1s_2)}{l_1l_2[(m_1+m_2) - m_2(s_1s_2 + c_1c_2)^2]} \\
&\quad \times [-(m_1+m_2)l_1^2x_2^2 + m_2l_1l_2(s_1s_2 + c_1c_2)x_4^2] \\
&\quad + \frac{1}{l_1l_2[(m_1+m_2) - m_2(s_1s_2 + c_1c_2)^2]} \\
&\quad \times [-(m_1+m_2)l_1gs_1(s_1s_2 + c_1c_2) \\
&\quad + (m_1+m_2)l_1gs_2], \\
g_{11}(x) &= \frac{m_2l_2^2}{m_2l_1^2l_2^2[(m_1+m_2) - m_2(s_1s_2 + c_1c_2)^2]} \\
g_{12}(x) &= \frac{-m_2l_1l_2(s_1s_2 + c_1c_2)}{m_2l_1^2l_2^2[(m_1+m_2) - m_2(s_1s_2 + c_1c_2)^2]} \\
g_{21}(x) &= \frac{-m_2l_1l_2(s_1s_2 + c_1c_2)}{m_2l_1^2l_2^2[(m_1+m_2) - m_2(s_1s_2 + c_1c_2)^2]} \\
g_{22}(x) &= \frac{(m_1+m_2)l_1^2}{m_2l_1^2l_2^2[(m_1+m_2) - m_2(s_1s_2 + c_1c_2)^2]}.
\end{aligned}
$$

Following the design procedures in the above section, the fuzzy tracking control design is given by the following steps:

Step 1: To use the fuzzy control approach, we must have a fuzzy model that represents the dynamics of the nonlinear plant. Assume that x_1 and x_3 are measurable through the optical encoder attached on the robot. In this example, link mass $m_1 = 1(kg)$, $m_2 = 1(kg)$, and link length $l_1 = 1(m)$, $l_2 = 1(m)$ are given, angular position q_1, q_2 are constrained within $[-(\pi/2), (\pi/2)]$ and external disturbances $w_1 = 0.1\sin(2t)$, $w_2 = 0.1\cos(2t)$, $w_3 = 0.1\cos(2t)$, and $w_4 = 0.1\sin(2t)$ are given and measurement noises v_1 and v_2 are assumed to be zero mean white noise with variance equals to 0.1%. To minimize the design effort and complexity, we try to use as few rules as possible. The T–S fuzzy model for the system in (34) is given by the following nine-rule fuzzy model:

Rule 1) If x_1 is about $-\frac{\pi}{2}$ and x_3 is about $\frac{\pi}{2}$,
then $\dot{x} = A_1x + B_1u + w$
$y = C_1x + v.$

Rule 2) If x_1 is about $-\frac{\pi}{2}$ and x_3 is about 0,
then $\dot{x} = A_2x + B_u + w$
$y = C_2x + v.$

Rule 3) If x_1 is about $-\frac{\pi}{2}$ and x_3 is about $\frac{\pi}{2}$,
then $\dot{x} = A_3x + B_3u + w$
$y = C_3x + v.$

Rule 4) If x_1 is about 0 and x_3 is about $-\frac{\pi}{2}$,
then $\dot{x} = A_4x + B_4u + w$
$y = C_4x + v.$

Rule 5) If x_1 is about 0 and x_3 is about 0,
then $\dot{x} = A_5x + B_5u + w$
$y = C_5x + v.$

Rule 6) If x_1 is about 0 and x_3 is about $\frac{\pi}{2}$,
then $\dot{x} = A_6x + B_6u + w$
$y = C_6x + v.$

Rule 7) If x_1 is about $\frac{\pi}{2}$ and x_3 is about $-\frac{\pi}{2}$,
then $\dot{x} = A_7x + B_7u + w$
$y = C_7x + v.$

Rule 8) If x_1 is about $\frac{\pi}{2}$ and x_3 is about 0,
then $\dot{x} = A_8x + B_8u + w$
$y = C_8x + v.$

Rule 9) If x_1 is about $\frac{\pi}{2}$ and x_3 is about $\frac{\pi}{2}$,
then $\dot{x} = A_9x + B_9u + w$
$y = C_9x + v$

where $x = [x_1, x_2, x_3, x_4]^T$, $u = [\tau_1, \tau_2]^T$, $w = [w_1, w_2, w_3, w_4]^T$, $v = [v_1, v_2]^T$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5.927 & -0.001 & -0.315 & -8.4 \times 10^{-6} \\ 0 & 0 & 0 & 1 \\ -6.859 & 0.002 & 3.155 & 6.2 \times 10^{-6} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.0428 & -0.0011 & 0.1791 & -0.0002 \\ 0 & 0 & 0 & 1 \\ 3.5436 & 0.0313 & 2.5611 & 1.14 \times 10^{-5} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.2728 & 0.0030 & 0.4339 & -0.0001 \\ 0 & 0 & 0 & 1 \\ 9.1041 & 0.0158 & -1.0574 & -3.2 \times 10^{-5} \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.4535 & 0.0017 & 1.2427 & 0.0002 \\ 0 & 0 & 0 & 1 \\ -3.1873 & -0.0306 & 5.1911 & -1.8 \times 10^{-5} \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 11.1336 & 0.0 & -1.8145 & 0.0 \\ 0 & 0 & 0 & 1 \\ -9.0918 & 0.0 & 9.1638 & 0.0 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1702 & -0.0010 & 1.6870 & -0.0002 \\ 0 & 0 & 0 & 1 \\ -2.3559 & 0.0314 & 4.5298 & 1.1 \times 10^{-5} \end{bmatrix}$$

$$A_7 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1206 & -0.0041 & 0.6205 & 0.0001 \\ 0 & 0 & 0 & 1 \\ 8.8794 & -0.0193 & -1.0119 & 4.4 \times 10^{-5} \end{bmatrix}$$

$$A_8 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.6421 & 0.0018 & 0.0721 & 0.0002 \\ 0 & 0 & 0 & 1 \\ 2.4290 & -0.0305 & 2.9832 & -1.9 \times 10^{-5} \end{bmatrix}$$

$$A_9 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.2933 & -0.0009 & -0.2188 & -1.2 \times 10^{-5} \\ 0 & 0 & 0 & 1 \\ -7.4649 & 0.0024 & 3.2693 & 9.2 \times 10^{-6} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} \quad B_4 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_5 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix} \quad B_6 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_7 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} \quad B_8 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_9 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}$$

$$C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{for } i = 1, \cdots, 9.$$

The reference model is given as

$$A_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix}$$

and

$$r(t) = [0, 8\sin(t), 0, 8\cos(t)]^T.$$

For the convenience of design, triangle type membership functions are adopted for Rule 1 through Rule 9.

Steps 2–5: Solve LMIP using the LMI optimization toolbox in Matlab. In this case

$$\tilde{P}_{11} = \begin{bmatrix} 1.5097 \times 10^0 & -5.6709 \times 10^{-1} & 1.6095 \times 10^{-1} & -9.1080 \times 10^{-2} \\ -5.6709 \times 10^{-1} & 2.1365 \times 10^{-1} & -6.2016 \times 10^{-2} & 3.4924 \times 10^{-2} \\ 1.6095 \times 10^{-1} & -6.2016 \times 10^{-2} & 4.3109 \times 10^{-1} & -1.4470 \times 10^{-1} \\ -9.1080 \times 10^{-2} & 3.4924 \times 10^{-2} & -1.4470 \times 10^{-1} & 5.0333 \times 10^{-2} \end{bmatrix}$$

$$\tilde{W}_{22} = \begin{bmatrix} 7.9577 \times 10^3 & -2.1030 \times 10^5 & -3.8664 \times 10^2 & 9.4840 \times 10^3 \\ -2.1030 \times 10^5 & 6.6342 \times 10^6 & 8.7428 \times 10^3 & -2.2568 \times 10^5 \\ -3.8664 \times 10^2 & 8.7428 \times 10^3 & 9.0528 \times 10^3 & -2.8448 \times 10^5 \\ 9.4840 \times 10^3 & -2.2568 \times 10^5 & -2.8448 \times 10^5 & 1.0629 \times 10^7 \end{bmatrix}$$

$$\tilde{P}_{33} = \begin{bmatrix} 2.6701 \times 10^0 & 8.9051 \times 10^{-1} & -1.8708 \times 10^{-3} & -8.3602 \times 10^{-4} \\ 8.9051 \times 10^{-1} & 7.6799 \times 10^{-1} & -8.5026 \times 10^{-4} & 1.7538 \times 10^{-5} \\ -1.8708 \times 10^{-3} & -8.5026 \times 10^{-4} & 2.6730 \times 10^0 & 8.9170 \times 10^{-1} \\ -8.3602 \times 10^{-4} & 1.7538 \times 10^{-5} & 8.9170 \times 10^{-1} & 7.6790 \times 10^{-1} \end{bmatrix}.$$

Step 6: The observer parameters are found to be

$$L_1 = \begin{bmatrix} 4.5110 \times 10^2 & -4.4693 \times 10^1 \\ 1.2168 \times 10^3 & -1.7120 \times 10^2 \\ -7.5851 \times 10^1 & 2.5735 \times 10^2 \\ -2.4621 \times 10^2 & 7.7910 \times 10^2 \end{bmatrix}$$

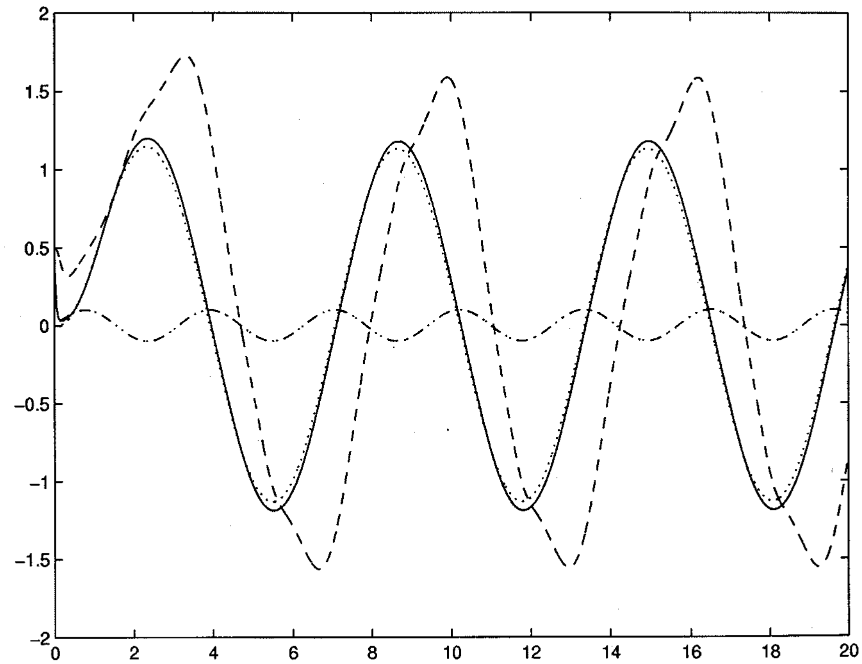

Fig. 2. The trajectories of the state variable x_1 (the proposed fuzzy control: solid line), reference state variable x_{r1} (dotted line), state variable x_1 (the linear control: dashed line) and external disturbance w_1 (dashdot line).

$$L_2 = \begin{bmatrix} 4.6963\times10^2 & -8.1664\times10^1 \\ 1.2667\times10^3 & -2.7067\times10^2 \\ -8.0699\times10^1 & 2.6313\times10^2 \\ -2.6021\times10^2 & 7.9779\times10^2 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 4.0385\times10^2 & -1.1052\times10^2 \\ 1.0888\times10^3 & -3.5054\times10^2 \\ -6.4433\times10^1 & 2.7854\times10^2 \\ -2.0810\times10^2 & 8.4484\times10^2 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 4.3523\times10^2 & -5.5508\times10^1 \\ 1.1773\times10^3 & -1.9855\times10^2 \\ -8.8939\times10^1 & 2.5128\times10^2 \\ -2.8464\times10^2 & 7.6144\times10^2 \end{bmatrix}$$

$$L_5 = \begin{bmatrix} 3.7229\times10^2 & -4.1800\times10^1 \\ 1.0041\times10^3 & -1.6094\times10^2 \\ -5.7890\times10^1 & 2.4382\times10^2 \\ -1.8940\times10^2 & 7.3892\times10^2 \end{bmatrix}$$

$$L_6 = \begin{bmatrix} 4.3828\times10^2 & -5.9589\times10^1 \\ 1.1861\times10^3 & -2.0975\times10^2 \\ -9.2605\times10^1 & 2.5321\times10^2 \\ -2.9571\times10^2 & 7.6731\times10^2 \end{bmatrix}$$

$$L_7 = \begin{bmatrix} 4.0793\times10^2 & -1.0870\times10^2 \\ 1.1001\times10^3 & -3.4554\times10^2 \\ -6.6944\times10^1 & 2.7792\times10^2 \\ -2.1585\times10^2 & 8.4292\times10^2 \end{bmatrix}$$

$$L_8 = \begin{bmatrix} 4.6273\times10^2 & -7.6872\times10^1 \\ 1.2481\times10^3 & -2.5757\times10^2 \\ -7.9199\times10^1 & 2.6137\times10^2 \\ -2.5558\times10^2 & 7.9235\times10^2 \end{bmatrix}$$

$$L_9 = \begin{bmatrix} 4.4698\times10^2 & -4.2650\times10^1 \\ 1.2059\times10^3 & -1.6560\times10^2 \\ -7.6254\times10^1 & 2.5663\times10^2 \\ -2.4734\times10^2 & 7.7684\times10^2 \end{bmatrix}.$$

Step 7: The control parameters are found to be

$$K_1 = \begin{bmatrix} -1.1409\times10^4 & -3.9188\times10^2 & -3.3955\times10^3 & -9.0411\times10^1 \\ -2.5707\times10^3 & -8.1172\times10^1 & -8.1895\times10^3 & -2.3703\times10^2 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} -1.1162\times10^4 & -3.9108\times10^2 & -1.0318\times10^3 & -2.6501\times10^1 \\ -7.6060\times10^2 & -2.2040\times10^1 & -8.1335\times10^3 & -2.3977\times10^2 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} -1.0428\times10^4 & -3.6347\times10^2 & -1.2792\times10^2 & -1.5470\times10^0 \\ -1.3571\times10^2 & -1.3056\times10^0 & -7.6424\times10^3 & -2.2424\times10^2 \end{bmatrix}$$

$$K_4 = \begin{bmatrix} -1.1163\times10^4 & -3.9114\times10^2 & -1.0279\times10^3 & -2.6391\times10^1 \\ -7.6200\times10^2 & -2.2068\times10^1 & -8.1345\times10^3 & -2.3981\times10^2 \end{bmatrix}$$

$$K_5 = \begin{bmatrix} -1.1853\times10^4 & -4.1892\times10^2 & -7.7869\times10^2 & -1.9598\times10^1 \\ -5.8800\times10^2 & -1.6314\times10^1 & -8.5878\times10^3 & -2.5501\times10^2 \end{bmatrix}$$

$$K_6 = \begin{bmatrix} -1.1162\times10^4 & -3.9110\times10^2 & -1.0290\times10^3 & -2.6424\times10^1 \\ -7.6159\times10^2 & -2.2062\times10^1 & -8.1337\times10^3 & -2.3978\times10^2 \end{bmatrix}$$

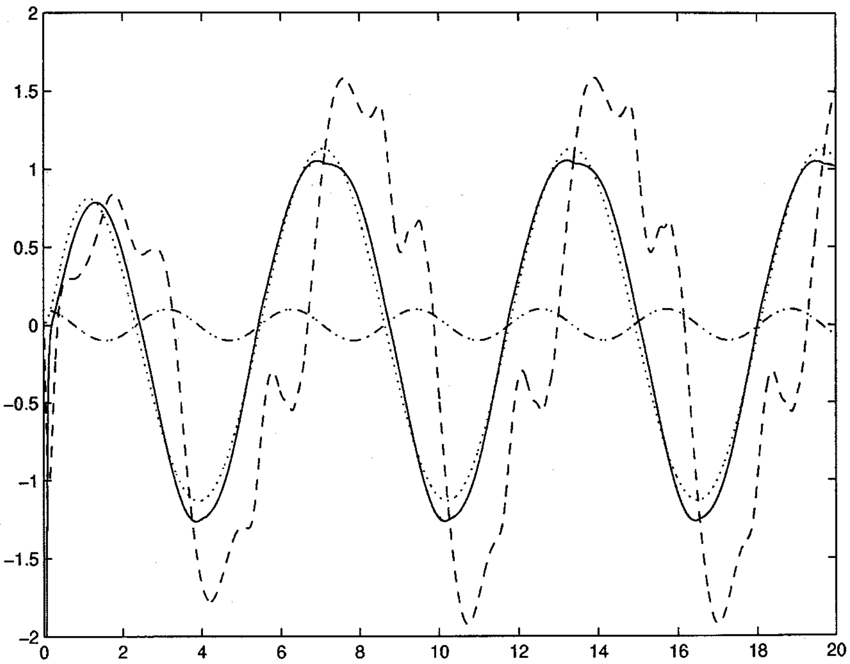

Fig. 3. The trajectories of the state variable x_2 (the proposed fuzzy control: solid line), reference state variable x_{r2} (dotted line), state variable x_2 (the linear control: dashed line) and external disturbance w_2 (dashdot line).

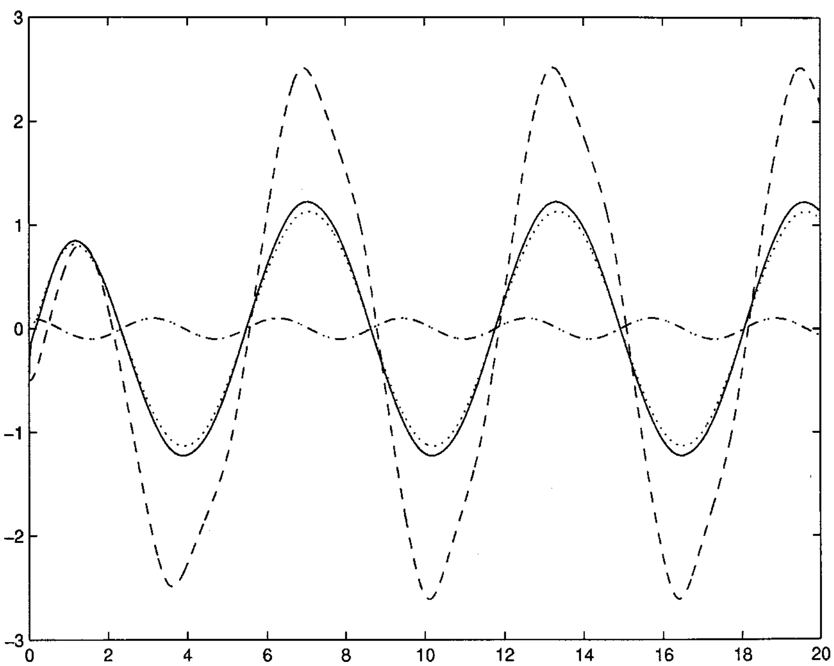

Fig. 4. The trajectories of the state variable x_3 (the proposed fuzzy control: solid line), reference state variable x_{r3} (dotted line), state variable x_3 (the linear control: dashed line) and external disturbance w_3 (dashdot line).

$$K_7 = \begin{bmatrix} -1.0430\times10^4 & -3.6353\times10^2 & -1.2540\times10^2 & -1.4751\times10^0 \\ -1.3574\times10^2 & -1.2959\times10^0 & -7.6436\times10^3 & -2.2428\times10^2 \end{bmatrix}$$

$$K_8 = \begin{bmatrix} -1.1164\times10^4 & -3.9113\times10^2 & -1.0308\times10^3 & -2.6470\times10^1 \\ -7.6024\times10^2 & -2.2023\times10^1 & -8.1344\times10^3 & -2.3980\times10^2 \end{bmatrix}$$

$$K_9 = \begin{bmatrix} -1.1409\times10^4 & -3.9188\times10^2 & -3.3969\times10^3 & -9.0449\times10^1 \\ -2.5697\times10^3 & -8.1138\times10^1 & -8.1900\times10^3 & -2.3705\times10^2 \end{bmatrix}.$$

Figs. 2–5 present the simulation results for the proposed fuzzy tracking control. The initial condition is assumed to be $(x_1(0),\ x_2(0),\ x_3(0),\ x_4(0),\ x_{r1}(0),\ x_{r2}(0),\ x_{r3}(0),\ x_{r4}(0),$

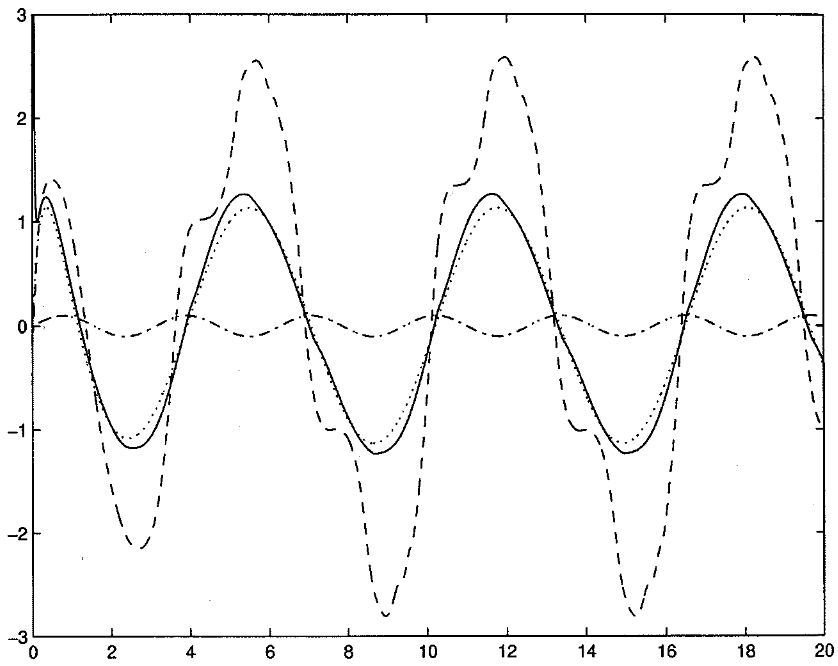

Fig. 5. The trajectories of the state variable x_4 (the proposed fuzzy control: solidline), reference state variable x_{r4} (dotted line), state variable x_4 (the linear control: dashed line) and external disturbance w_4 (dashdot line).

$\hat{x}_1(0)$, $\hat{x}_2(0)$, $\hat{x}_3(0)$, $\hat{x}_4(0))^T$ = (0.5, 0, −0.5, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T in the simulations. Fig. 2 (Figs. 3–5) show the trajectories of the states x_1 (x_2, x_3, and x_4) including reference state x_{r1} (x_{r2}, x_{r3}, and x_{r4}). For comparison, the control results obtained with a simple linear output feedback control designed by linearization around origin and pole placement located at (−5, −6, −15, −16) for both controller and observer are also shown in Figs. 2–5. From the simulation results, the performance of the proposed fuzzy controller is obviously better than that of the linear controller.

V. Conclusion

In this study, a fuzzy H_∞ model reference tracking control scheme has been proposed. Based on the T–S fuzzy model, a fuzzy observer based fuzzy controller is developed to reduce the tracking error as small as possible by minimizing the attenuation level ρ^2. Furthermore, the stability of the closed loop nonlinear systems is discussed in this study. If all state variables are available, a state feedback H_∞ tracking control design is also developed.

The advantage of proposed tracking control design is that only a simple fuzzy controller is used in our approach without feedback linearization technique and complicated adaptive scheme. In the proposed fuzzy control method, the outcome of the fuzzy H_∞ tracking control problem is parameterized in terms of a LMIP. The LMIP can be solved very efficiently by LMI optimization toolbox in Matlab [25] to complete the fuzzy tracking control design. A simple and systematic algorithm based on LMI optimization techniques is developed to solve the fuzzy H_∞ tracking control problem. A simulation example is given to illustrate the design procedures and tracking performance of the proposed method.

Appendix

If all state variables are available, the following fuzzy controller is employed to deal with the above control system design.

Control Rule j.

If $z_1(t)$ is F_{j1} and $\cdots$ and $z_g(t)$ is F_{jg},

then $u(t) = K_j[x(t) - x_r(t)]$ for $j = 1, 2, \cdots, L$. (35)

Hence, the overall fuzzy controller is given by

$$u(t) = \sum_{j=1}^{L} h_j(z(t))[K_j(x(t) - x_r(t))]. \tag{36}$$

After manipulation, the augmented system can be expressed as the following form:

$$\dot{\overline{x}}(t) = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \overline{A}_{ij} \overline{x}(t) + \overline{w}(t) \tag{37}$$

where

$$\overline{x}(t) = \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix}, \quad \overline{w}(t) = \begin{bmatrix} w(t) \\ r(t) \end{bmatrix}$$
$$\overline{A}_{ij} = \begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ 0 & A_r \end{bmatrix}. \tag{38}$$

Similarly, let us consider the H_∞ tracking performance related to tracking error $x(t) - x_r(t)$ as follows:

$$\begin{aligned} &\int_0^{t_f} \{[x(t) - x_r(t)]^T Q [x(t) - x_r(t)]\}\, dt \\ &\quad = \int_0^{t_f} \overline{x}^T(t) \overline{Q} \overline{x}(t)\, dt \\ &\quad \le \overline{x}^T(0) \overline{P} \overline{x}(0) + \rho^2 \int_0^{t_f} \overline{w}(t)^T \overline{w}(t)\, dt \end{aligned} \tag{39}$$

where $\overline{w}(t) = [w(t), r(t)]^T$, $\overline{P}$ is a symmetric positive definite weighting matrix

$$\overline{Q} = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix}. \tag{40}$$

The purpose of this study is to determine a fuzzy controller in (36) for the augmented system in (37) with the guaranteed H_∞ tracking performance in (39) for all $\overline{w}(t)$ and the closed-loop system

$$\dot{\overline{x}}(t) = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \overline{A}_{ij} \overline{x}(t) \tag{41}$$

is quadratically stable.

Then, we obtain the following results.

Theorem 3: In the nonlinear system (37), if $\overline{P} = \overline{P} > 0$ is the common solution of the following matrix inequalities:

$$\overline{A}_{ij}^T \overline{P} + \overline{PA}_{ij} + \frac{1}{\rho^2} \overline{PP} + \overline{Q} < 0 \tag{42}$$

for $i, j = 1, 2, \cdots, L$, then the H_∞ tracking control performance in (39) is guaranteed for a prescribed ρ^2.

Proof: The proof is similar to that of Theorem 1. ■

Theorem 4: In the nonlinear closed-loop system (41), if there exists the common solution $\overline{P} = \overline{P}^T > 0$ for (42), then the closed-loop system in (41) is quadratically stable.

Proof: The proof is similar to that of Theorem 2. ■

Similarly, to obtain better tracking performance, the tracking control problem can be formulated as the following minimization problem so that the H_∞ tracking performance in (39) is reduced as small as possible

$$\begin{aligned} &\min_{\overline{P}} \rho^2 \\ &\text{subject to } \overline{P} > 0 \text{ and (42).} \end{aligned} \tag{43}$$

For the convenience of design, we assume

$$\overline{P} = \begin{bmatrix} \overline{P}_{11} & 0 \\ 0 & \overline{P}_{22} \end{bmatrix}. \tag{44}$$

By substituting (44) into (42), we obtain

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} < 0 \tag{45}$$

where

$$\begin{aligned} F_{11} &= (A_i + B_i K_j)^T \overline{P}_{11} + \overline{P}_{11}(A_i + B_i K_j) \\ &\quad + \frac{1}{\rho^2} \overline{P}_{11} \overline{P}_{11} + Q \\ F_{12} &= F_{21}^T = -\overline{P}_{11} B_i K_j - Q \\ F_{22} &= A_r^T \overline{P}_{22} + \overline{P}_{22} A_r + \frac{1}{\rho^2} \overline{P}_{22} \overline{P}_{22} + Q. \end{aligned}$$

By the Schur complement [23], (45) is equivalent to

$$\begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22} & \overline{P}_{22} \\ 0 & \overline{P}_{22} & -\rho^2 I \end{bmatrix} < 0 \tag{46}$$

where

$$\begin{aligned} H_{11} &= (A_i + B_i K_j)^T \overline{P}_{11} + \overline{P}_{11}(A_i + B_i K_j) \\ &\quad + \frac{1}{\rho^2} \overline{P}_{11} \overline{P}_{11} + Q \\ H_{12} &= H_{21}^T = -\overline{P}_{11} B_i K_j - Q \\ H_{22} &= A_r^T \overline{P}_{22} + \overline{P}_{22} A_r + Q. \end{aligned}$$

By the same argument as above section, we can solve $\overline{P}_{11}$, $\overline{P}_{22}$, and K_j by the following two-step procedures.

In the first step, note that (46) implies that

$$\begin{aligned} &(A_i + B_i K_j)^T \overline{P}_{11} + \overline{P}_{11}(A_i + B_i K_j) \\ &\quad + \frac{1}{\rho^2} \overline{P}_{11} \overline{P}_{11} + Q < 0. \end{aligned} \tag{47}$$

With $\overline{W}_{11} = \overline{P}_{11}^{-1}$ and $Y_j = K_j \overline{W}_{11}$, (47) is equivalent to

$$\begin{aligned} &\overline{W}_{11} A_i^T + A_i \overline{W}_{11} + B_i Y_j + (B_i Y_j)^T \\ &\quad + \frac{1}{\rho^2} I + \overline{W}_{11} Q \overline{W}_{11} < 0. \end{aligned} \tag{48}$$

By the Schur complements, (48) is equivalent to the following LMIs.

$$\begin{bmatrix} \begin{pmatrix} \overline{W}_{11} A_i^T + A_i \overline{W}_{11} + B_i Y_j \\ +(B_i Y_j)^T + \frac{1}{\rho^2} I \end{pmatrix} & \overline{W}_{11} \\ \overline{W}_{11} & -Q^{-1} \end{bmatrix} < 0. \tag{49}$$

The parameters $\overline{W}_{11}$ and Y_j (thus $\overline{P}_{11} = \overline{W}_{11}^{-1}$ and $K_j = Y_j \overline{W}_{11}^{-1}$) can be obtained by solving the LMIP. (49).

In the second step, by substituting $\overline{P}_{11}$ and K_j into (46), (46) becomes standard linear matrix inequalities (LMI's). Similarly, we can easily solve $\overline{P}_{22}$ from (46). If there exist positive definite solutions $\overline{P}_{11}$ and $\overline{P}_{22}$ in (46), the closed loop system is stable and the H_∞ tracking performance in (39) can be achieved for a prescribed attenuation level ρ^2.

Recall that the attenuation level ρ^2 can be minimized so that the H_∞ tracking performance in (39) is reduced as small as possible

$$\begin{aligned} &\min_{\{\overline{P}_{11}, \overline{P}_{22}\}} \quad \rho^2 \\ &\text{subject to} \quad \overline{P}_{11} = \overline{P}_{11}^T > 0, \overline{P}_{22} = \overline{P}_{22}^T > 0 \text{ and (46).} \end{aligned} \tag{50}$$

This minimization problem can be solved by decreasing ρ^2 until solutions $\overline{P}_{11} = \overline{P}_{11}^T > 0$ and $\overline{P}_{22} = \overline{P}_{22}^T > 0$ in (46) can not be found.

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