

γ -Connectedness in fuzzy topological spaces

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Abstract

The aim of this paper is to introduce the concept γ -connectedness in fuzzy topological spaces. We also investigate some interrelations between this types of fuzzy connectedness together with the preservation properties under some types of fuzzy continuity. A comparison between some types of connectedness in fuzzy topological spaces is of interest.

Key words : Fuzzy topological spaces, fuzzy γ -continuity, γ -separated fuzzy sets, γ -connected fuzzy sets, δ -connected fuzzy sets, θ -connected fuzzy sets.

1. Introduction and preliminaries

In [5], we introduce a new class of fuzzy functions called fuzzy γ -continuous functions, which contains the class of fuzzy continuous functions. We investigated several properties concerning such functions. The purpose of this paper is to introduce and study the concept of γ -connected fuzzy sets by the help of fuzzy γ -closure [5], and study the relationships between γ -connected fuzzy sets and such subsets in fuzzy topological spaces.

Throughout the paper, X and Y always represent fuzzy topological spaces (fts, for short) in Chang's sense [1]. For a fuzzy set U in X , clU , δclU , θclU and U' will respectively denote, the closure, δ -closure, θ -closure and complement of U . A fuzzy set U in a fts X is said to be quasi-coincident with a fuzzy set V , denoted by UqV , if there exists $x \in X$ such that $U(x)+V(x)>1$ [6]. It is known that $U \leq V$ iff U and V' are not quasi-coincident, denoted by $U\bar{q}V'$ [6].

Definition 1.1 [5]. A fuzzy set U in a fts X is called γ -open (γ -closed) if $U \leq cl int U \vee int cl U$ ($U \geq cl int U \wedge int cl U$).

Definition 1.2 [5]. For a fuzzy set U in a fts X , fuzzy γ -closure (γcl , for short) is defined as follows: $\gamma cl U = \wedge \{G : G \text{ is } \gamma\text{-closed fuzzy set, } U \leq G\}$.

Definition 1.3 [5]. A function $f: X \rightarrow Y$ is called fuzzy γ -continuous if for each open fuzzy set V in Y , $f^{-1}(V)$ is γ -open fuzzy set in X .

Definition 1.4 [5]. A function $f: X \rightarrow Y$ is called fuzzy γ -closed if the image of each closed fuzzy set in X is γ -closed fuzzy set in Y .

Definition 1.5 [3,4]. Two non-empty fuzzy sets U and V in a fts X are called separated (resp. δ -separated, θ -separated) fuzzy sets if $U\bar{q}clV$ and $V\bar{q}clU$ (resp. $U\bar{q}\delta clV$ and $V\bar{q}\delta clU$, $U\bar{q}\theta clV$ and $V\bar{q}\theta clU$).

Remark 1.6 [8]. For any two non-empty fuzzy sets U and V the following implications hold:

θ -separated fuzzy set \Rightarrow δ -separated fuzzy set \Rightarrow separated fuzzy set.

Definition 1.7 [3,4]. A fuzzy set U in a fts X is called connected (resp. δ -connected, θ -connected) fuzzy set if U cannot be expressed as the union of two separated (resp. δ -separated, θ -separated) fuzzy sets.

Remark 1.8 [8]. For a fuzzy set U , the following implications hold:

connected fuzzy set \Rightarrow δ -connected fuzzy set \Rightarrow θ -connected fuzzy set.

Definition 1.9 [7]. A fts is called fuzzy regular if for each open fuzzy set V and each fuzzy point pqV , there exists an open fuzzy set U such that $pqU \leq clU \leq V$.

2. γ -separated fuzzy sets

Definition 2.1. Two non-empty fuzzy sets U and V in a fts X are called γ -separated fuzzy sets if $U\bar{q}\gamma clV$ and $V\bar{q}\gamma clU$.

Remark 2.2. For any two non-empty fuzzy sets U and V , since $\gamma clU \leq clU \leq \delta clU \leq \theta clU$ and so V , the following implications hold:

θ - separated fuzzy set \Rightarrow δ - separated fuzzy set \Rightarrow separated fuzzy set \Rightarrow γ -separated fuzzy set.

Theorem 2.3. Let U, V be non-empty fuzzy sets in a fts X .
(i) if U and V are γ -separated fuzzy sets and U_1, V_1 are non-empty fuzzy sets such that $U_1 \leq U$ and $V_1 \leq V$, then U_1 and V_1 are also γ -separated fuzzy sets.

(ii) if $U \bar{q} V$ and either both are γ -open fuzzy sets or both are γ -closed fuzzy sets, then U and V are γ -separated fuzzy sets.

(iii) if U and V are either both γ -open fuzzy sets or both are γ -closed fuzzy sets and if $A = U \cap V'$ and $B = V \cap U'$, then A and B are γ -separated fuzzy sets.

Proof. (i) Since $U_1 \leq U$, we have $\gamma cl U_1 \leq \gamma cl U$, then $V \bar{q} \gamma cl U \Rightarrow V_1 \bar{q} \gamma cl U \Rightarrow V_1 \bar{q} \gamma cl U_1$. Similarly $U_1 \bar{q} \gamma cl V_1$. Hence U_1 and V_1 are γ -separated fuzzy sets.

(ii) When U and V are γ -closed fuzzy sets, then $U = \gamma cl U$ and $V = \gamma cl V$. Since $U \bar{q} V$, we have $\gamma cl U \bar{q} V$. When U and V are γ -open fuzzy sets, U' and V' are γ -closed fuzzy sets. Then $U \bar{q} V \Rightarrow U \leq V' \Rightarrow \gamma cl U \leq \gamma cl V' = V' \Rightarrow \gamma cl U \bar{q} V$. Similarly, $\gamma cl V \bar{q} U$. Hence U and V are γ -separated fuzzy sets.

(iii) When U and V are γ -open fuzzy sets, U' and V' are γ -closed fuzzy sets. Since $A \leq V'$, $\gamma cl A \leq \gamma cl V' = V'$ and so $\gamma cl A \bar{q} V$. Thus $B \bar{q} \gamma cl A$. Similarly, $A \bar{q} \gamma cl B$. Hence A and B are γ -separated fuzzy sets.

When U and V are γ -open fuzzy sets, $U = \gamma cl U$ and $V = \gamma cl V$. Since $A \leq V'$, $\gamma cl V \bar{q} A$ and hence $\gamma cl B \bar{q} A$. Similarly $\gamma cl A \bar{q} B$. Hence A and B are γ -separated fuzzy sets.

Theorem 2.4. Let U and V be non-empty fuzzy sets of a fts X , then U and V are γ -separated fuzzy sets iff there exist two γ -open fuzzy sets A and B such that $U \leq A$, $V \leq B$, $U \bar{q} B$ and $V \bar{q} A$.

Proof. Let U and V be γ -separated fuzzy sets. Putting $B = (\gamma cl U)'$ and $A = (\gamma cl V)'$, then A and B are γ -open fuzzy sets such that $U \leq A$, $V \leq B$, $U \bar{q} B$ and $V \bar{q} A$.

Conversely, let A and B be γ -open fuzzy sets such that $U \leq A$, $V \leq B$, $U \bar{q} B$ and $V \bar{q} A$. Since B' and A' are γ -closed fuzzy sets, we have $\gamma cl U \leq B' \leq V'$ and $\gamma cl V \leq A' \leq U'$. Thus $\gamma cl U \bar{q} V$ and $\gamma cl V \bar{q} U$. Hence U and V are γ -separated fuzzy sets.

3. γ -connected fuzzy sets

Definition 3.1. A fuzzy set which cannot be expressed as the union of two γ -separated fuzzy sets is said to be a γ -connected fuzzy set.

Remark 3.2. From the above definition and Remark 1.8, the following implications hold:

γ -connected fuzzy set \Rightarrow connected fuzzy set \Rightarrow
 δ -connected fuzzy set \Rightarrow θ -connected fuzzy set.

Lemma 3.3. A fts X is fuzzy regular iff $\gamma cl U = cl U = \delta cl U = \theta cl U$, for each fuzzy set U in X .

Proof. Obvious (see [2]).

Theorem 3.4. For a fuzzy set U in a fuzzy regular space X , the following are equivalent:

- (i) U is γ -connected fuzzy set .
- (ii) U is connected fuzzy set .
- (iii) U is δ -connected fuzzy set .
- (iv) U is θ -connected fuzzy set .

Proof. Follows directly by virtue of Lemma 3.3.

Theorem 3.5. In a fts X , if U is γ -connected fuzzy set, then $\gamma cl U$ is so.

Proof. Suppose that $\gamma cl U$ is not γ -connected fuzzy set, then there are two non-empty γ -separated fuzzy sets A and B in X such that $\gamma cl U = A \cup B$. Now, from $U = (A \cap U) \cup (B \cap U)$ and $\gamma cl (A \cap U) \leq \gamma cl A$, $\gamma cl (B \cap U) \leq \gamma cl B$ and $A \bar{q} B$, we obtain $\gamma cl (A \cap B) \bar{q} B$. Hence $\gamma cl (A \cap U) \bar{q} (B \cap U)$. Similarly $\gamma cl (B \cap U) \bar{q} (A \cap U)$. Therefore U is not γ -connected fuzzy set. Hence the result.

Theorem 3.6. Let U be a non-empty γ -connected fuzzy set in a fts X . If U is contained in the union of two γ -separated fuzzy sets A and B , then exactly one of the following conditions (i) and (ii) holds:

- (i) $U \leq A$ and $U \cap B = 0_X$.
- (ii) $U \leq B$ and $U \cap A = 0_X$.

Proof. We first note that when $U \cap B = 0_X$, Then $U \leq A$, since $U \leq A \cup B$. Similarly, when $U \cap A = 0_X$, we have $U \leq B$. Since $U \leq A \cup B$, both $U \cap A = 0_X$ and $U \cap B = 0_X$ cannot hold simultaneously. Again if $U \cap A \neq 0_X$ and $U \cap B \neq 0_X$, then by Theorem 2.3 (i), $U \cap A$ and $U \cap B$ are γ -separated fuzzy sets such that $U = (U \cap A) \cup (U \cap B)$,

contradicting the γ -connectedness of a fuzzy set U . Hence exactly one of the conditions (i) and (ii) must hold.

Theorem 3.7. Let $\{U_j : j \in J\}$ be a collection of γ -connected fuzzy sets in a fts X . If there exists $k \in J$ such that $U_j \cap U_k \neq 0_x$ for each $j \in J$, then $U = \{U_j : j \in J\}$ is γ -connected fuzzy set.

Proof. Suppose that U is not γ -connected fuzzy set. Then there exist γ -separated fuzzy sets A and B such that $U = A \cup B$. By Theorem 3.4, we have either (i) $U_j \leq A$ with $U_j \cap B = 0_x$ or (ii) $U_j \leq B$ with $U_j \cap A = 0_x$ for each $j \in J$. Similarly, either (iii) $U_k \leq B$ with $U_k \cap A = 0_x$ or (iv) $U_k \leq A$ with $U_k \cap B = 0_x$. We may assume, without loss of generality, that U_j is non-empty for each $j \in J$, and hence exactly one of (iii) and (iv) will hold. Since $U_j \cap U_k \neq 0_x$ for each $j \in J$, the conditions (i) and (iii) cannot hold, and similarly (ii) and (iv) cannot hold simultaneously. If (i) and (iv) hold, then $U_j \leq A$ with $U_j \cap B = 0_x$ for each $j \in J$. Then $U \leq A$ and $U \cap B = 0_x$ and thus $B = 0_x$, a contradiction. Similarly, if (ii) and (iii) hold, then we have $A = 0_x$, a contradiction. Hence the result.

Theorem 3.8. If a function $f: X \rightarrow Y$ is fuzzy γ -continuous and U is γ -connected fuzzy set relative to X , then $f(U)$ is connected fuzzy set relative to Y .

Proof. Suppose that $f(U)$ is not connected fuzzy set in Y , there exists two separated fuzzy sets A and B relative to Y such that $f(U) = A \cup B$. Then there exist open fuzzy sets G and H in Y such that $A \leq G, B \leq H, A \bar{q} H$ and $B \bar{q} G$. Since f is fuzzy γ -continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are γ -open fuzzy sets in X such that $f^{-1}(A) \leq f^{-1}(G), f^{-1}(B) \leq f^{-1}(H), f^{-1}(A) \bar{q} f^{-1}(H)$ and $f^{-1}(B) \bar{q} f^{-1}(G)$. Thus $f^{-1}(A)$ and $f^{-1}(B)$ are γ -separated fuzzy sets in X from Theorem 2.4, and $U = f^{-1}(f(U)) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. Hence this is contrary to the fact that U is γ -connected fuzzy set. Therefore $f(U)$ must be connected fuzzy set.

Theorem 3.9. If a bijective function $f: X \rightarrow Y$ is fuzzy γ -closed and U is γ -connected fuzzy set relative to Y , then $f^{-1}(U)$ is connected fuzzy set relative to X .

Proof. Suppose that $f^{-1}(U)$ is not connected fuzzy set in X , there exists two separated fuzzy sets A and B relative to X such that $f^{-1}(U) = A \cup B$. Then there exist open fuzzy sets G and H in X such that $A \leq G, B \leq H, A \bar{q} H$ and $B \bar{q} G$. Since f is fuzzy γ -closed function, $f(clG)$ and $f(clH)$ are γ -closed fuzzy sets in Y such that $f(clA) \leq f(clG), f(clB) \leq f(clH), f(clA) \bar{q} f(clH)$ and

$f(clB) \bar{q} f(clG)$. Since $f(clB) \bar{q} f(clG)$ implies $f(B) \bar{q} f(G)$. Thus $f(A)$ and $f(B)$ are γ -separated fuzzy sets in Y from Theorem 2.3 (ii), and $U = f(f^{-1}(U)) = f(A \cup B) = f(A) \cup f(B)$. Hence this is contrary to the fact that U is γ -connected fuzzy set. Therefore $f^{-1}(U)$ must be connected fuzzy set.

Definition 3.10. A function $f: X \rightarrow Y$ is called fuzzy γ -irresolute (fuzzy strongly γ -irresolute) if the inverse image of each γ -open fuzzy set in Y is γ -open (open) in X .

Lemma 3.11. Let $f: X \rightarrow Y$ be a function then the following are equivalent:

- (i) f is fuzzy γ -irresolute.
- (ii) The inverse image of each γ -closed fuzzy set in Y is γ -closed in X .
- (iii) $\gamma cl f^{-1}(V) \leq f^{-1}(\gamma cl V) \leq f^{-1}(cl V)$.

Proof. Obvious.

Theorem 3.12. [7]. If a function $f: X \rightarrow Y$ is fuzzy γ -irresolute and U is γ -connected fuzzy set relative to X , then $f(U)$ is γ -connected relative to Y .

Proof. By using Definition 3.10 and Lemma 3.11, it is a direct consequence of Theorem 3.8.

Definition 3.13. A function $f: X \rightarrow Y$ is called fuzzy $M\gamma$ -open (fuzzy $M\gamma$ -closed) if the image of each γ -open (γ -closed) fuzzy set in X is γ -open (γ -closed) in Y .

Definition 3.14. Two fts's X and Y are called fuzzy γ -homeomorphic if there exists a bijective function $f: X \rightarrow Y$ such that f is fuzzy γ -irresolute and fuzzy $M\gamma$ -open. Such function f is called a fuzzy γ -homeomorphism.

Remark 3.15. It is clear that each fuzzy homeomorphism is fuzzy γ -homeomorphism while the converse may not be true.

Example 3.16. An identity fuzzy function f from any fts X to a Discrete fts Y is fuzzy γ -homeomorphism but not fuzzy homeomorphism.

Definition 3.17. A function $f: X \rightarrow Y$ is called fuzzy strongly $M\gamma$ -open (fuzzy strongly $M\gamma$ -closed) if the image of each γ -open (γ -closed) fuzzy set in X is open (closed) in Y .

From the above definitions one can easily obtain the following theorems.

Theorem 3.18. Let a bijective function $f: X \rightarrow Y$ be a fuzzy $M\gamma$ -closed and U be a γ -connected fuzzy set relative to Y , then $f^{-1}(U)$ is γ -connected relative to X .

Proof. Obvious.

Theorem 3.19. Let a function $f: X \rightarrow Y$ be a fuzzy γ -homeomorphism, then G is γ -connected fuzzy set relative to X (G is γ -connected fuzzy set relative to Y) iff $f(G)$ is γ -connected fuzzy set relative to Y ($f^{-1}(G)$ is γ -connected fuzzy set relative to X).

Proof. Obvious.

Corollary 3.20. Let a function $f: X \rightarrow Y$ be a fuzzy homeomorphism, then G is γ -connected fuzzy set relative to X (G is γ -connected fuzzy set relative to Y) iff $f(G)$ is γ -connected fuzzy set relative to Y ($f^{-1}(G)$ is γ -connected fuzzy set relative to X).

Proof. It is obvious from Remark 3.15.

Theorem 3.21. Let X and Y be fuzzy γ -homeomorphic, then X is fuzzy γ -connected iff Y is fuzzy γ -connected.

Proof. Obvious.

Theorem 3.22. If a function $f: X \rightarrow Y$ is a fuzzy strongly $M\gamma$ -continuous and G is connected fuzzy set relative to X , then $f(G)$ is γ -connected fuzzy set relative to Y .

Proof. Obvious.

Theorem 3.23. Let a bijective function $f: X \rightarrow Y$ be a fuzzy strongly $M\gamma$ -closed and G is connected fuzzy set relative to Y , then $f^{-1}(G)$ is γ -connected fuzzy set relative to X .

References

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