Γ-convergence for incompressible elastic plates

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We derive a two-dimensional model for elastic plates as a Γ-limit of three-dimensional nonlinear elasticity with the constraint of incompressibility. The resulting model describes plate bending, and is determined from the isochoric elastic moduli of the three-dimensional problem.
Without the constraint of incompressibility, a plate theory was first derived by Friesecke, James and Müller [14]. We extend their result to the case of p growth at infinity with $p \in [1, 2)$, and to the case of incompressible materials. The main difficulty is the construction of a recovery sequence which satisfies the nonlinear constraint pointwise. One main ingredient is the density of smooth isometries in $W^{2,2}$ isometries, which was obtained by Pakzad [23] for convex domains and by Hornung [18, 17] for piecewise C^1 domains.

1 Introduction

The constraint of incompressibility is frequently encountered in the framework of nonlinear elasticity, for example in models of elastomeric materials whose entropic shear moduli are much smaller than the bulk modulus [3, 22, 2]. Recently both the mathematics and the physics community have directed considerable effort to the analytic derivation of elastic properties of thin sheets of these materials, e.g., for liquid-crystal elastomers, see [7, 29, 1] and references therein. A common feature in all these works is that the incompressibility of the material is incorporated as a rigid constraint on the determinant of the deformation gradient. This nonlinear constraint does not allow one to use standard techniques in the calculus of variation for the derivation of two-dimensional models from fully three dimensional models via Γ–convergence.

A two-dimensional theory for the elastic behavior of plates has been first proposed by Kirchhoff back in 1850 with heuristic arguments; a rigorous derivation was obtained in 2002 by Friesecke, James, and Müller [16, 14], see also [15]. These authors used the theory of Γ–convergence [10, 11] to identify a variational principle for the deformation of the cross-section ω as a limit of functionals for thin domains $\Omega_h = \omega \times (-h/2, h/2)$ in the limit $h \to 0$. The main ingredient in the proof is a rigidity estimate that gives a quantitative estimate of how close a mapping is to a rigid body motion in L^2. Previous work (see, e.g., [13, 24, 25] and the references therein) relied instead on ad hoc assumptions on the deformation that could not be verified rigorously.

In this paper we address the question of deriving a limiting theory for plates made out of incompressible materials. Incompressibility is expressed by the constraint that the determinant of the deformation gradient be equal to one a.e. We consider the functional $I_h : W^{1,p}(\Omega_h; \mathbb{R}^3) \to [0, \infty]$ given by

$$I_h[u] = \frac{1}{h^3} \int_{\Omega_h} W(\nabla u) \, \mathrm{d}x_p \, \mathrm{d}x_3 \tag{1.1}$$

where we write $x_p = (x_1, x_2)$ for the in-plane variables. The energy density $W : \mathbb{M}^{3\times 3} \to [0, \infty]$ incorporates the incompressibility of the material and is defined by

$$W(F) = \begin{cases} W_c(F) & \text{if } \det F = 1, \\ \infty & \text{otherwise.} \end{cases} \tag{1.2}$$

We suppose that W_c is invariant under composition with a rotation, in the sense that

$$W_c(RF) = W_c(F) \quad \text{for all } R \in \mathrm{SO}(3) \tag{1.3}$$

where $\mathrm{SO}(3)$ is the group of all proper rotations in $\mathbb{R}^3$, i.e., the set of all matrices $R \in \mathbb{M}^{3\times 3}$ with $R^T R = \mathrm{Id}$ and $\det R = 1$. We assume that $W_c : \mathbb{M}^{3\times 3} \to \mathbb{R}$ is nonnegative, and that $W_c(F) = 0$ if and only if $F \in \mathrm{SO}(3)$. Moreover we require as in [14] that W_c is twice continuously differentiable in a neighborhood of $\mathrm{SO}(3)$, that it has quadratic growth close to $\mathrm{SO}(3)$ but we only require p-growth, with $p \geq 1$, at infinity, in the sense of (1.7) below. Moreover we show by an example that compactness fails if $p < 1$. As in [14] we do not require growth conditions from above, it suffices that the energy is finite in a neighborhood of $\mathrm{SO}(3)$.

Membrane theories are obtained by considering the limit of the energy per unit volume, i.e., the limit of the functionals $h^2 I_h$. The limiting model allows for shearing and stretching of the body; a precise Γ–convergence result was first obtained by LeDret and Raoult [20, 21] for continuous energy densities. Their work was extended to energy densities which are infinite if the determinant of the deformation gradient is negative in [5, 4], based on an adaption of Whitney's ideas by Ben Belgacem and Bennequin.

For incompressible materials, i.e., for the much more rigid constraint $\det \nabla u = 1$, this was accomplished independently in [8] and in [26, 27], partly motivated by [6]. In all these cases the limiting problem is again a variational functional. The new energy density is obtained in two steps from the original density: first one optimizes in the out-of-plane direction, then one passes to the relaxation of this new energy. In view of this mechanism it is not hard to verify the lower bound in the definition of Γ–convergence, and the main difficulty is the construction of a recovery sequence. In the cases with a constraint on the determinant, the key ingredient is Whitney's characterization of the fundamental singularity of smooth mappings $u : \mathbb{R}^2 \to \mathbb{R}^3$. It allows one to approximate a given function in the energy norm by a smooth function whose gradient has full rank everywhere.

Plate theories arise instead if the energy per unit volume is scaled with an additional factor of h^{-2}, as we have done in our definition (1.1). Not surprisingly, the result is also related to geometric questions. The response of the plate is characterized by its bending rigidity and therefore the deformations of the thin body are expected to be close to isometries. In the limit it turns out that the energy is finite only on isometric immersions of the two-dimensional domain ω into $\mathbb{R}^3$ and that the elastic moduli are related to the moduli of the corresponding linear theory.

In order to state our main theorem we define

$$Q_3 = \nabla^2 W_c(\mathrm{Id})\,, \tag{1.4}$$

which we view as a quadratic map from $\mathbb{M}^{3\times 3}$ into $\mathbb{R}$, and

$$Q_2(\widehat{G}) = \min_{d\in\mathbb{R}^3:\mathrm{Tr}(\widetilde{G}|d)=0} Q_3\left(\widetilde{\pi}_0(\widehat{G})|d\right)\,, \quad \forall \widehat{G} \in \mathbb{M}^{2\times 2} \tag{1.5}$$

which is a quadratic map from $\mathbb{M}^{2\times 2}$ into $\mathbb{R}$. Here $\widetilde{\pi}_0(\widehat{G}) \in \mathbb{M}^{3\times 2}$ denotes the matrix with $\widetilde{\pi}_0(\widehat{G})_{ij} = \widehat{G}_{ij}$, $i, j = 1, 2$ and $\widetilde{\pi}_0(\widehat{G})_{3j} = 0$, $j = 1, 2$, that is, $\widetilde{\pi}_0$ is the natural embedding of $\mathbb{M}^{2\times 2}$ into $\mathbb{M}^{3\times 2}$. Here and in the following we emphasize the dimension of a matrix by using $\widetilde{(\cdot)}$ for matrices in $\mathbb{M}^{3\times 2}$. In particular, we write $\widetilde{\nabla} u$ for the in-plane gradient of a map u. Hence $\nabla u = \widetilde{\nabla} u$ for a function $u : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ that is defined on the cross-section of the domain, and $\widetilde{\nabla} u = (\partial_1 u|\partial_2 u)$ for a function $u : \Omega_h \subset \mathbb{R}^3 \to \mathbb{R}^3$ defined on the three-dimensional body.

Finally, the limiting energy is finite only if u has higher regularity and is given by

$$I_0[u] = \begin{cases} \dfrac{1}{24}\displaystyle\int_\omega Q_2(\mathrm{II}_u)\,\mathrm{d}x_p & \text{if } u(x) = u(x_p) \text{ and } \widetilde{\nabla} u \in W^{1,2}(\omega; \mathrm{O}(2,3))\,, \\ \infty & \text{otherwise.} \end{cases} \tag{1.6}$$

Here $\mathrm{II}_u : \omega \to \mathbb{M}^{2\times 2}$ is the second fundamental form of the isometry defined by u, given by

$$\mathrm{II}_u = (\widetilde{\nabla} u)^T \widetilde{\nabla} b_u\,, \qquad b_u = \partial_1 u \wedge \partial_2 u\,.$$

The unit vector field b_u is normal to the surface described by u. In particular, there is a gain of regularity in the limiting theory which is also reflected in the fact that the topology in the definition of the Γ–limit is the strong topology in $W^{1,p}$ and not the weak topology as in the case of membrane theories.

It is a remarkable feature that in both the membrane theory and the plate theory the constraint of incompressibility is relaxed in the limit to the extent that the surface area of the cross-section does not need to be conserved. In fact, the nonlinear constraint is reflected in the energy entering the limiting plate theory through its linearization, in the sense that the minimum in (1.5) is taken over all matrices with trace zero. Physically this corresponds to the fact that the volume constraint can be enforced by usage of the out-of-plane component of the deformation gradient (see Lemma 3.1). Precisely, for any in-plane deformation gradient $\widetilde{G} \in \mathbb{M}^{3\times 2}$ such that $\operatorname{rank} \widetilde{G} = 2$ one can find $d \in \mathbb{R}^3$ such that the matrix $(\widetilde{G}|d) \in \mathbb{M}^{3\times 3}$ has determinant one. A crucial observation is that, due to the thin-film geometry, if $\widetilde{G} = \widetilde{\nabla} u : \omega \to \mathbb{M}^{3\times 2}$ is an in-plane gradient then for any smooth d the matrix $(\widetilde{G}|d)$ can be well approximated by a gradient (see Lemma 3.1), and therefore any in-plane gradient field $\widetilde{G} : \omega \to \mathbb{M}^{3\times 2}$ can be well approximated by the first two columns of a volume-conserving gradient field $\nabla u_h : \Omega_h \to \mathbb{M}^{3\times 3}$. This observation is at the basis of both the result for membrane theories in [8, 26, 27] and the present one. In the case of the membrane theory an essential difficulty was dealing with singularities (i.e., with points where $\operatorname{rank} \widetilde{G}$ is less than 2). In the present case such singularities are excluded by the kinematics of the limiting problem; however the construction of the recovery sequence is more subtle since the derivation of a plate theory requires a much finer control on the energy, and therefore more attention to the details of the construction.

One key step in most Γ-convergence results is to prove density of a suitable class of "good" functions in the space defined by the limiting functional. In the present case, smooth functions constitute the natural class of "good" functions to consider, and one is confronted with the question of approximating $W^{2,2}$ isometries with C^∞ isometries (the main difficulty being the step from $W^{2,2}$ to C^2). This is a difficult geometric issue, since one is trying to construct smooth isometries, which are developable surfaces and hence very rigid.

In the following we say that a bounded Lipschitz domain $\omega \subset \mathbb{R}^2$ has the approximation property for isometries if $C^\infty(\bar{\omega};\mathbb{R}^3)$ isometries are dense in $W^{2,2}(\omega;\mathbb{R}^3)$ isometries with respect to the $W^{2,2}$ norm. As above, a map $u : \omega \to \mathbb{R}^3$ is an isometry if $\widetilde{\nabla} u \in \mathrm{O}(2,3)$ a.e. The approximation property for isometries was established by Pakzad [23] for convex domains, using a method developed by Kirchheim [19] to show that $W^{2,2}$ isometries are developable. In [23] only interior regularity is shown, but by a straightforward blow-up one can achieve regularity up to the boundary. The same arguments can be extended to strictly star-shaped domains. On domains ω whose boundary is C^1 away from a closed null set (including in particular piecewise C^1 boundaries) the approximation property for isometries was recently established by Hornung [18, 17]. Therefore Theorem 1.1 below holds in a large class of domains. In order to make a clear separation between the geometric argument entering the density estimate, and the analytic construction, we formulate our result in terms of Lipschitz domains which satisfy the approximation property.

Finally we recall that the convergence of sequences of functions $u_h : \Omega_h \to \mathbb{R}^3$ on varying domains Ω_h is understood as the convergence the rescaled functions defined on the domain Ω_1 of thickness one. The rescaling operator $\mathcal{T}_h$ is defined by

$$(\mathcal{T}_h u)(x_p, x_3) = u(x_p, hx_3) \qquad \text{for } x_p \in \omega \text{ and } x_3 \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

For the limiting map $v : \omega \to \mathbb{R}^3$ the rescaling reduces to the extension $(\mathcal{T}_0 v)(x_p, x_3) = v(x_p)$. For simplicity we write $\mathcal{T}u$ and $\mathcal{T}v$, if the appropriate index is clear from the context.

The following theorem summarizes the main results in this paper. It was announced in [9].

Theorem 1.1. *Let $\omega \subset \mathbb{R}^2$ be an open and bounded Lipschitz domain which satisfies the approximation property for isometries. Suppose that W_c is a stored energy density with*

$$W_c(F) \geq c \min\left\{\mathrm{dist}^2(F, \mathrm{SO}(3)), c\right\} \quad \text{and} \quad W_c(F) \geq c|F|^p - \frac{1}{c} \quad \forall F \in \mathbb{M}^{3\times 3}, \tag{1.7}$$

for some $p \geq 1$ and $c > 0$. Moreover, assume that $W_c(\mathrm{Id}) = 0$, that W_c is C^2 smooth in a neighborhood of Id *and that W_c is frame indifferent in the sense of (1.3). Let W be defined by (1.2). Then the functionals I_h given by (1.1) converge in the sense of Γ–convergence with respect to strong convergence in $W^{1,p}(\Omega_1;\mathbb{R}^3)$ to the limiting functional I_0 defined in (1.6). That is, the following assertions are true:*

(a) Compactness and lower bound: *For every sequence $u_h \in W^{1,p}(\Omega_h;\mathbb{R}^3)$ with $I_h[u_h] < C < \infty$ there exists a subsequence u_h and $u \in W^{2,2}(\omega;\mathbb{R}^3)$ with $\widetilde{\nabla} u \in \mathrm{O}(2,3)$ a.e., such that*

$$\mathcal{T}_h \left(u_h - \frac{1}{|\Omega_h|} \int_{\Omega_h} u_h \, \mathrm{d}x \right) \to \mathcal{T}_0 u$$

strongly in $W^{1,p}(\Omega_1;\mathbb{R}^3)$. Moreover, if the entire sequence converges in this sense, then

$$I_0[u] \leq \liminf_{h \to 0} I_h[u_h].$$

(b) Upper bound: *For any $u \in W^{2,2}(\omega;\mathbb{R}^3)$ with $\widetilde{\nabla} u \in \mathrm{O}(2,3)$ a.e. there exists a family $u_h \in C^1(\Omega_h;\mathbb{R}^3)$ such that $\mathcal{T}_h u_h \to \mathcal{T}_0 u$ strongly in $W^{1,p}(\Omega_1;\mathbb{R}^3)$ and*

$$\limsup_{h \to 0} I_h[u_h] \leq I_0[u].$$

Note that the constraint of incompressibility has been relaxed since changes in the surface area of the cross-section can be accommodated by changes in the thickness of the plate.

The idea of the proof for the compactness result and for the lower bound is to replace the constraint $\det \nabla u = 1$ by a penalization, and to use the corresponding assertions in the derivation of the plate theory by Friesecke, James and Müller [14]. The details are carried out in Section 2. The results in [14] require quadratic growth at infinity and we present in Section 4 an extension of their theory to the case $1 \leq p < 2$ (see Theorem 4.1).

The proof of the upper bound has two main ingredients. Since ω has the approximation property for isometries, it suffices to consider smooth isometries u (see also the discussion preceding the statement of the theorem). Then one needs an explicit construction for the recovery sequence in the case that u is smooth. The details are presented in Section 3. We enforce the determinant constraint by a suitable change of coordinates in the out-of-plane direction, following the approach developed for membrane theories in [8].

Notation: Throughout this paper we assume that $\omega \subset \mathbb{R}^2$ and we set $\Omega_h = \omega \times (-h/2, h/2)$. The points $x \in \Omega_h$ are also written as $x = (x_p, x_3)$ with $x_p = (x_1, x_2)$. The Euclidean norm of a matrix is denoted by $|F|^2 = \mathrm{Tr}(F^T F)$ where F^T is the transpose of the matrix F and $\mathrm{Tr}(F)$ the trace of a square matrix F. The symbol $\widetilde{\nabla}$ stands for the in-plane derivatives, i.e., $\widetilde{\nabla} = (\partial_1, \partial_2)$. Throughout the paper we write $\widehat{G}$ for 2×2 matrices and $\widetilde{G}$ for 3×2 matrices, respectively.

2 Compactness and lower bound

The first part in the proof of the Γ–convergence result is the verification of compactness of families of deformations u_h with $I_h[u_h] \leq C < \infty$ and of the lower bound: if $\mathcal{T} u_h \to \mathcal{T} u$ in $W^{1,p}(\Omega_h; \mathbb{R}^3)$, then

$$I[u] \leq \liminf_{h \to 0} I_h[u_h].$$

These properties will follow from the corresponding results for the plate theory in [14] by introducing an approximation of I_h from below. We define, for $k \geq 0$,

$$W^k(F) = W_c(F) + \frac{k}{2} (\det F - 1)^2$$

and

$$I_h^k[u] = \frac{1}{h^3} \int_{\Omega_h} W^k(\nabla u) \, \mathrm{d}x_p \, \mathrm{d}x_3.$$

In order to exploit the lower bound from [14] we consider the quadratic form associated with W^k given by

$$Q_3^k(F) = \nabla^2 W^k(\mathrm{Id})(F, F) = Q_3(F) + k(\mathrm{Tr}\, F)^2 \tag{2.1}$$

and the corresponding reduced forms

$$Q_2^k(\widehat{G}) = \min_{d \in \mathbb{R}^3} Q_3^k(\widetilde{G}|d)\,, \quad \text{with } \widetilde{G} = \widetilde{\pi}_0(\widehat{G}) \tag{2.2}$$

see the discussion following (1.5). The minimum on the right-hand side is attained and positive unless $\widehat{G}$ is a skew-symmetric matrix. The next lemma provides a quantitative comparison of Q_2 and Q_2^k.

Lemma 2.1. *Let Q_3 be the quadratic form on $\mathbb{M}^{3\times3}$ defined in (1.4). Then the forms Q_2 and Q_2^k, defined via (1.5), (2.1), and (2.2) satisfy*

$$Q_2(\widehat{G}) - \frac{c}{\sqrt{k}} \|\widehat{G}\|^2 \leq Q_2^k(\widehat{G}) \leq Q_2(\widehat{G})\,, \quad \forall \widehat{G} \in \mathbb{M}^{2\times2}\,.$$

Here c depends only on Q_3.

Proof. We observe that for any $\widehat{G} \in \mathbb{M}^{2\times 2}$

$$\begin{aligned} Q_2^k(\widehat{G}) = \min_{d\in\mathbb{R}^3} Q_3^k(\widetilde{G}|d) &\leq \min_{d\in\mathbb{R}^3\colon \mathrm{Tr}(\widetilde{G}|d)=0} Q_3^k(\widetilde{G}|d) \\ &= \min_{d\in\mathbb{R}^3\colon \mathrm{Tr}(\widetilde{G}|d)=0} Q_3(\widetilde{G}|d) = Q_2(\widehat{G}) \end{aligned}$$

and this establishes the second estimate. To prove the first one, fix $\widehat{G} \in \mathbb{M}^{2\times 2}$, and note that we may assume that $\|\widehat{G}\| = 1$ since the inequality is homogeneous of degree two. Let d, d^k be vectors that realize the minimum in the definition of Q_2 and Q_2^k (recall the notation following (1.5)),

$$\begin{aligned} Q_2(\widehat{G}) &= Q_3(\widetilde{G}|d) \qquad & \mathrm{Tr}(\widetilde{G}|d) &= 0, \\ Q_2^k(\widehat{G}) &= Q_3^k(\widetilde{G}|d^k) \qquad & \text{where } \widetilde{G} &= \widetilde{\pi}_0(\widehat{G})\,. \end{aligned}$$

Since Q_3 is strictly positive definite on symmetric matrices we conclude that

$$\|d\| \leq c\,\|\widehat{G}\| \leq c \tag{2.3}$$

where the constant c depends only on Q_3. The condition $\mathrm{Tr}(\widetilde{G}|d) = 0$ implies $d_3 = -\widehat{G}_{11} - \widehat{G}_{22}$ and thus

$$d_3^k - d_3 = \mathrm{Tr}(\widetilde{G}|d^k)\,.$$

Moreover,

$$k(\mathrm{Tr}(\widetilde{G}|d^k))^2 \leq Q_2^k(\widehat{G}) \leq Q_2(\widehat{G}) \leq c\,.$$

This proves that

$$|d_3^k - d_3| \leq \frac{c}{\sqrt{k}}\,. \tag{2.4}$$

We define $\overline{d} = (d_1^k, d_2^k, d_3)$ and obtain with $\widetilde{G} = \widetilde{\pi}_0(\widehat{G})$

$$\begin{aligned} Q_2^k(\widehat{G}) &= Q_3(\widetilde{G}|d^k) + k\big(\,\mathrm{Tr}(\widetilde{G}|d^k)\big)^2 \\ &\geq Q_3(\widetilde{G}|\overline{d}) + Q_3(\widetilde{G}|d^k) - Q_3(\widetilde{G}|\overline{d}) \\ &\geq Q_3(\widetilde{G}|\overline{d}) - c\,|d_3 - d_3^k|\,. \end{aligned} \tag{2.5}$$

Indeed, the terms in the difference $Q_3(\widetilde{G}|d^k) - Q_3(\widetilde{G}|\overline{d})$ that do not contain d_3^k and d_3 cancel. The terms that are linear in these variables are estimated by $c|d_3 - d_3^k|$, where as above the constant c depends only on Q_3. The quadratic term is bounded by the same expression in view of (2.3) and (2.4). These two inequalities imply that $|d_3 + d_3^k|$ is bounded by a constant that depends only on $\|\widehat{G}\|$. Finally, since $\mathrm{Tr}(\widetilde{G}|\overline{d}) = 0$, we obtain

$$Q_2(\widehat{G}) \leq Q_3(\widetilde{G}|\overline{d}) \leq Q_2^k(\widehat{G}) + c\,|d_3 - d_3^k|$$

which concludes the proof. □

After these preparations we are in a position to verify the lower bound in the Γ–convergence statement.

Proof of Theorem 1.1, Part (a). Let $h \to 0$ and $u_h \in W^{1,p}(\Omega_h;\mathbb{R}^3)$ be such that $I_h[u_h] \le C$ for all h. In order to prove compactness, we fix a value $k > 0$ ($k = 1$ will do) and observe that, for all h,

$$I_h^k[u_h] \le I_h[u_h] \le C\,.$$

In view of Theorem 4.1 in [14] (for $p \ge 2$) or of Theorem 4.1 below (for $1 \le p < 2$) there exists a subsequence (not relabeled) and a $u \in W^{2,2}(\omega;\mathbb{R}^3)$ with $\widetilde{\nabla} u \in \mathrm{O}(2,3)$ a.e., such that

$$\mathcal{T}_h\left(u_h - \frac{1}{|\Omega_h|}\int_{\Omega_h} u_h \,\mathrm{d}x\right) \to \mathcal{T}_0 u \;\text{ in } W^{1,p}(\Omega_1;\mathbb{R}^3)\,. \tag{2.6}$$

This concludes the proof of compactness.

In order to verify the lower bound, we can assume that the entire sequence converges in the sense of (2.6). Then we obtain from Theorem 4.1 in [14] (for $p \ge 2$) or from Theorem 4.1 below (for $1 \le p < 2$) that for all $k > 0$

$$I_0^k[u] \le \liminf_{h\to 0} I_h^k[u_h] \le \liminf_{h\to 0} I_h[u_h]\,,$$

where

$$I_0^k[u] = \begin{cases} \dfrac{1}{24}\displaystyle\int_\omega Q_2^k(\mathrm{II}_u)\,\mathrm{d}x_p & \text{if } \widetilde{\nabla} u \in W^{1,2}(\omega;\mathrm{O}(2,3)),\\ \infty & \text{otherwise.}\end{cases}$$

Therefore

$$\liminf_{h\to 0} I_h[u_h] \ge \sup_{k>0} I_0^k[u]\,.$$

To determine the supremum on the right-hand side, we observe that in view of the estimates in Lemma 2.1 we have

$$\begin{aligned} I_0^k[u] &= I_0[u] + \frac{1}{24}\int_\omega (Q_2^k - Q_2)\big(\mathrm{II}_u\big)\,\mathrm{d}x_p \\ &\ge I_0[u] - \frac{1}{24}\frac{c}{\sqrt{k}}\int_\omega |\mathrm{II}_u|^2\,\mathrm{d}x_p \ge I_0[u] - \frac{C}{\sqrt{k}} \end{aligned}$$

and taking the limit $k \to \infty$ establishes the assertion. $\square$

3 Upper bound

The construction of the recovery sequence is more subtle. Given an isometry $u : \omega \to \mathbb{R}^3$ we need to extend u to a function $u_h : \Omega_h \to \mathbb{R}^3$ with the following objectives in mind:

(a) ∇u_h is pointwise close to the set of rotations SO(3);

(b) the incompressibility condition $\det \nabla u_h = 1$ is satisfied pointwise;

(c) the leading-order correction in the expansion of $\nabla u_h - R$, $R \in \mathrm{SO}(3)$, in the plate thickness realizes asymptotically as $h \to 0$ the minimization in the passage from Q_3 to Q_2.

In order to achieve these goals we exploit a variation of the traditional construction technique for compressible materials, which goes back to Kirchhoff, coupled with a smoothing of the internal variables which respects the linearized incompressibility constraint and which allows us to enforce nonlinear incompressibility through the nonlinear change of the independent variables introduced in [8]. Since we are dealing with domains having the approximation property for isometries we can focus on smooth deformations u.

Lemma 3.1. *Let $\omega \subset \mathbb{R}^2$ be bounded and Lipschitz, $u \in C^3(\bar{\omega}; \mathbb{R}^3)$, with $\widetilde{\nabla} u \in \mathrm{O}(2,3)$, let $b = \partial_1 u \wedge \partial_2 u$, and suppose that $d \in C^2(\bar{\omega}; \mathbb{R}^3)$ is a vector field such that*

$$\mathrm{Tr}\left[(\widetilde{\nabla} u|b)^T(\widetilde{\nabla} b|d)\right] = 0\,. \tag{3.1}$$

Then there exists for every sequence $h_j \to 0$ a sequence $u_j \in C^1(\overline{\Omega}_{h_j}; \mathbb{R}^3)$ such that $\mathcal{T} u_j \to \mathcal{T} u$ strongly in $W^{1,p}$ and

$$\limsup_{j\to\infty} I_{h_j}[u_j] \le \frac{1}{24}\int_\omega Q_3\big((\widetilde{\nabla} u|b)^T(\widetilde{\nabla} b|d)\big)\,\mathrm{d}x_p\,.$$

Proof. We use the Kirchhoff construction, modified as in [8] in order to enforce the nonlinear constraint on the determinant. We define

$$v(x_p, x_3) = u(x_p) + x_3 b(x_p) + \frac{1}{2}x_3^2 d(x_p)\,,$$

and calculate

$$\nabla v = (\widetilde{\nabla} u|b) + x_3(\widetilde{\nabla} b|d) + \frac{1}{2}x_3^2(\widetilde{\nabla} d|0)\,.$$

By the definition of b and since $\widetilde{\nabla} u \in \mathrm{O}(2,3)$ we have $R = (\widetilde{\nabla} u|b) \in \mathrm{SO}(3)$. The determinant of the deformation gradient can be computed by

$$\begin{aligned}
\det \nabla v &= \det(R^T \nabla v) \\
&= \det\Big(\mathrm{Id} + x_3 R^T(\widetilde{\nabla} b|d) + \frac{1}{2}x_3^2 R^T(\widetilde{\nabla} d|0)\Big) \\
&= 1 + x_3 \operatorname{Tr}(R^T(\widetilde{\nabla} b|d)) + x_3^2 E(x_3, b, d, \widetilde{\nabla} u, \widetilde{\nabla} b, \widetilde{\nabla} d)
\end{aligned}$$

where E is a polynomial in its arguments. By (3.1) the linear term vanishes, and under our regularity assumptions we obtain

$$|\det \nabla v - 1| + |\partial_1 \det \nabla v| + |\partial_2 \det \nabla v| \le C|x_3|^2\,, \tag{3.2}$$

where C depends only on the C^2 norm of u, b and d. As in [8], we define $w = v \circ \Phi$, with $\Phi(x_p, x_3) = (x_p, \varphi(x_p, x_3))$, and we determine φ so that

$$\det \nabla w = (\det \nabla v \circ \Phi)\, \det \nabla \Phi = (\det \nabla v \circ \Phi)\, \partial_3 \varphi = 1\,.$$

This implies that φ has to be a solution of the ODE

$$\partial_3 \varphi(x_p, x_3) = \frac{1}{\det \nabla v(x_p, \varphi(x_p, x_3))}$$

subject to the initial condition $\varphi(x_p, 0) = 0$. By the theory of families of solutions of parameter dependent ordinary differential equations (see, e.g., [28]) there exists an $h_0 > 0$ such that one can find a unique solution $\varphi \in C^1(\overline{\Omega}_{h_0})$. The bounds (3.2) on $\det \nabla v$ imply that

$$\begin{aligned}
|\varphi(x_p, x_3) - x_3| &\le C x_3^3\,, \\
|\partial_3 \varphi(x_p, x_3) - 1| &\le C x_3^2\,, \\
|\partial_i \varphi(x_p, x_3)| &\le C x_3^3\,, \quad i = 1, 2\,.
\end{aligned}$$

Therefore the function w is C^1 smooth on $\overline{\Omega}_{h_0}$, and obeys $\det \nabla w = 1$. We now estimate how close $\nabla w = \nabla(v \circ \Phi)$ is to a rotation. By the chain rule,

$$\begin{aligned}
\nabla w &= \nabla\Big(u + \varphi b + \frac{1}{2}\varphi^2 d\Big) \\
&= (\widetilde{\nabla} u \,|\, b\, \partial_3 \varphi) + \varphi(\widetilde{\nabla} b \,|\, d\, \partial_3 \varphi) + \frac{1}{2}\varphi^2(\widetilde{\nabla} d|0) + (b + \varphi d) \otimes (\partial_1 \varphi, \partial_2 \varphi, 0)^T \\
&= (\widetilde{\nabla} u \,|\, b) + x_3(\widetilde{\nabla} b \,|\, d) \\
&\quad + (0 \,|\, b)(\partial_3 \varphi - 1) + (\varphi - x_3)(\widetilde{\nabla} b \,|\, d) + \varphi\big(0 \,|\, d\, (\partial_3 \varphi - 1)\big) \\
&\quad + \frac{1}{2}\,\varphi^2(\widetilde{\nabla} d|0) + (b + \varphi d) \otimes (\widetilde{\nabla} \varphi|0).
\end{aligned}$$

We conclude with $R = (\widetilde{\nabla} u \,|\, b)$ that

$$R^T \nabla w = \mathrm{Id} + x_3 R^T (\widetilde{\nabla} b \,|\, d) + O(x_3^2)$$

where $\mathrm{O}(x_3^2)$ represents terms which are uniformly bounded by constants (depending on u, b, d and their derivatives) times x_3^2. We conclude that

$$W_c(\nabla w) = W_c(R^T \nabla w) \leq \frac{1}{2} x_3^2 Q_3\big(R^T(\widetilde{\nabla} b|d)\big) + O(x_3^3) + o(x_3^2)$$

where $o(x_3^2)$ represents the remainder in the Taylor series of W_c at the identity. This implies

$$\begin{aligned} &\frac{1}{h^3} \int_{\Omega_h} W_c(\nabla w) \,\mathrm{d}x_p \,\mathrm{d}x_3 \\ &\qquad \leq \frac{1}{h^3} \int_{-h/2}^{h/2} \frac{1}{2} x_3^2 \,\mathrm{d}x_3 \int_\omega Q_3\big(R^T(\widetilde{\nabla} b|d)\big) \,\mathrm{d}x_p + \frac{O(h^3) + o(h^2)}{h^2} \,. \end{aligned}$$

Computing the integral in x_3 and taking the limit $h \to 0$ concludes the proof. □

The next step concerns the incorporation of the minimization step in the passage from Q_3 to Q_2.

Lemma 3.2. *Let $u \in C^4(\bar\omega; \mathbb{R}^3)$, with $\widetilde{\nabla} u \in \mathrm{O}(2,3)$, and suppose that $h_j \to 0$. Then there exists a sequence $u_j \in C^1(\overline{\Omega}_{h_j}; \mathbb{R}^3)$ with $\mathcal{T} u_j \to \mathcal{T} u$ such that*

$$\limsup_{j \to \infty} I_{h_j}[u_j] \leq I_0[u] \,.$$

Proof. Let $b = \partial_1 u \wedge \partial_2 u$, and $R = (\widetilde{\nabla} u | b)$. Since $\mathrm{II}_u = \widetilde{\nabla} u^T \widetilde{\nabla} b$, we have $\widetilde{\mathrm{II}}_u = R^T \widetilde{\nabla} b$. Let $d^0 : \bar\omega \to \mathbb{R}^3$ be a measurable vector field such that

$$Q_2(\mathrm{II}_u) = Q_3(R^T \widetilde{\nabla} b | d^0) \,, \qquad \mathrm{Tr}\left[R^T \widetilde{\nabla} b | d^0 \right] = 0 \,.$$

By (2.3) we have $\|d^0\| \leq C \|\widetilde{\nabla} b\|$, where C depends only on Q_3. The condition on the trace is equivalent to

$$d_3^0 = -\,\mathrm{Tr}\,\mathrm{II}_u = -\,\mathrm{Tr}(R^T \widetilde{\nabla} b | 0) = -\partial_1 u \cdot \partial_1 b - \partial_2 u \cdot \partial_2 b \,. \tag{3.3}$$

This implies that the third component d_3^0 is in $C^2(\bar\omega; \mathbb{R}^3)$. For $i = 1, 2$ let $d_i^j \in C^2(\bar\omega; \mathbb{R}^3)$ be a sequence which converges to d_i^0 in $L^2(\bar\omega; \mathbb{R}^3)$. We define $d_3^j = d_3^0$, and observe that d^j fulfills automatically (3.3) for any j. Thus

$$\lim_{j \to \infty} \int_\omega Q_3(R^T \widetilde{\nabla} b | d^j) \,\mathrm{d}x_p = \int_\omega Q_2(\mathrm{II}_u) \,\mathrm{d}x_p \,.$$

We now apply the previous Lemma to (u, b, Rd^j), and take a diagonal sequence. □

After these preparations we are in a position to establish the upper bound in the statement of the theorem.

Proof of Theorem 1.1, Part (b). Since the cross-section ω has the approximation property for isometries we may find a sequence of smooth functions u_k that approximate u in the $W^{2,2}$ norm. This implies that

$$u_k \to u\,, \qquad I_0[u_k] \to I_0[u]\,.$$

For each of these functions we apply Lemma 3.2 to obtain a sequence $u_{k,j}$ such that as $j \to \infty$

$$\mathcal{T}_{h_j} u_{k,j} \to \mathcal{T}_0 u_k\,, \qquad I_{h_j}[u_{k,j}] \to I_0[u_k]\,.$$

We finally choose a suitable diagonal sequence $u_{k(j),j} \to u$ such that $I_{h_j}[u_{k(j),j}] \to I_0[u]$, and conclude the proof. □

4 A plate theory for compressible materials with p growth at infinity

The derivation of a plate theory (without the constraint of incompressibility) in [14] assumed the energy density to grow at least quadratically at infinity. It is clear that this hypothesis covers the case of p growth with $p \geq 2$ as well. In this section we extend the argument to the case of p-growth with $p \in [1, 2)$, and show that the result is false for $p < 1$.

Theorem 4.1. *Let $\omega \subset \mathbb{R}^2$ be an open and bounded Lipschitz domain. Suppose that W_c is a stored energy density which satisfies (1.3), (1.7), $W_c(\mathrm{Id}) = 0$, and is C^2 smooth in a neighborhood of* Id. *Then the functionals*

$$I^c_h[u] = \frac{1}{h^3}\int_{\Omega_h} W_c(\widetilde{\nabla} u)\,\mathrm{d}x$$

converge in the sense of Γ-convergence with respect to strong convergence in $W^{1,p}(\Omega_1;\mathbb{R}^3)$ to

$$I^c_0[u] = \begin{cases} \dfrac{1}{24}\displaystyle\int_\omega Q^c_2(\mathrm{II}_u)\,\mathrm{d}x_p & \text{if } u(x) = u(x_p) \text{ and } \widetilde{\nabla} u \in W^{1,2}(\omega;\mathrm{O}(2,3))\,, \\ \infty & \text{otherwise.}\end{cases}$$

Here

$$Q^c_2(\widehat{G}) = \min_{d\in\mathbb{R}^3} Q_3\left(\widetilde{\pi}_0(\widehat{G})|d\right)\,, \tag{4.1}$$

and $Q_3 = \nabla^2 W_c(\mathrm{Id})$*, as in (1.4);* $\widetilde{\pi}_0$ *was defined after (1.5). More precisely,*

(a) Compactness and lower bound: *For every sequence* $u_h \in W^{1,p}(\Omega_h;\mathbb{R}^3)$ *with* $I_h^c[u_h] < C < \infty$ *there exist a subsequence* u_h *and* $u \in W^{2,2}(\omega;\mathbb{R}^3)$ *with* $\widetilde{\nabla} u \in \mathrm{O}(2,3)$ *a.e., such that*

$$\mathcal{T}_h\left(u_h - \frac{1}{|\Omega_h|}\int_{\Omega_h} u_h \,\mathrm{d}x\right) \to \mathcal{T}_0 u$$

strongly in $W^{1,p}(\Omega_1;\mathbb{R}^3)$*. Moreover, if the entire sequence converges in this sense, then*

$$I_0^c[u] \le \liminf_{h\to 0} I_h^c[u_h].$$

(b) Upper bound: *For any* $u \in W^{2,2}(\omega;\mathbb{R}^3)$ *with* $\widetilde{\nabla} u \in \mathrm{O}(2,3)$ *a.e. there exists a family* $u_h \in W^{1,2}(\Omega_h;\mathbb{R}^3)$ *such that* $\mathcal{T}_h u_h \to \mathcal{T}_0 u$ *strongly in* $W^{1,p}(\Omega_1;\mathbb{R}^3)$ *and*

$$\limsup_{h\to 0} I_h^c[u_h] \le I_0^c[u].$$

In particular, the form Q_3 is the same for both the compressible and the incompressible theory. The passage to Q_2 is however different, in the incompressible case (see (1.5)) the minimum is taken over all vectors d such that the resulting matrix has trace equal to zero, in the present compressible case (see (4.1)) one minimizes over all vectors $d \in \mathbb{R}^3$. Although in this section only compressible energies are used, we keep for consistency of notation with the previous section the apex c on W, Q_2 and I.

One key point in proving this theorem is to perform an appropriate truncation. This will be done using the following result.

Proposition 4.2 (See Sect. 6.6.2 of [12] and Prop. A.1 of [16])**.** *Let* m*,* $n \ge 1$ *and let* $1 \le p < \infty$*. Suppose that* $U \subset \mathbb{R}^n$ *is a bounded Lipschitz domain. Then there exists a constant* $C(U,m,n,p)$ *with the following property. For each* $u \in W^{1,p}(U;\mathbb{R}^m)$ *and each* $\lambda > 0$ *there exists a* $v \in W^{1,\infty}(U;\mathbb{R}^m)$ *such that*

(i) $\|\nabla v\|_{L^\infty(U;\mathbb{R}^m)} \le C\lambda$,

(ii) $\big|\{x \in U :\, u(x) \ne v(x)\}\big| \le \dfrac{C}{\lambda^p}\displaystyle\int_{\{x\in U:|\nabla u|(x)>\lambda\}} |\nabla u|^p \,\mathrm{d}x.$

Since the constant in Proposition 4.2 depends on the domain, we cannot use it directly with $U = \Omega_h$. An application of Proposition 4.2 to $\mathcal{T}_h u_h$ on the fixed domain Ω_1 is appropriate only for the tangential derivatives. Indeed, by chain rule, $\partial_3 \mathcal{T}_h u_h = h \mathcal{T}_h \partial_3 u_h$. Thus, if we replace $\mathcal{T}_h u_h$ by some Lipschitz continuous sequence $\mathcal{T}_h v_h$ we merely obtain a bound of the form c/h for $\partial_3 v_h$.

For these reasons, we use the truncation on subdomains. We consider the parallelepipeds of the type

$$T_{a,h} = (a_1 - \tfrac{3}{2}h, a_1 + \tfrac{3}{2}h) \times (a_2 - \tfrac{3}{2}h, a_2 + \tfrac{3}{2}h) \times (-\tfrac{1}{2}h, \tfrac{1}{2}h)\,, \quad a \in h\mathbb{Z}^2. \tag{4.2}$$

On the sets $T_{a,h}$ contained in Ω_h we can construct a Lipschitz approximation $v_{a,h}$ to u_h. Since all parallelepipeds have the same shape, the constants in the estimates do not depend on a and h. However, the function $\nabla v_{a,h}$ constructed in this way is only locally a gradient field, not globally. Therefore in the entire argument one has to keep track of both $v_{a,h}$, on which "good" (i.e., L^2) local estimates hold, and of ∇u_h, on which "bad" (i.e., L^p) global estimates hold.

Proof of Theorem 4.1. Part (b), i.e., the upper bound, follows directly from Th. 6.1(ii) of [14], since the growth from below is not relevant. We remark that for domains with the approximation property a straightforward adaptation of the proof of the construction used for Theorem 1.1(b) (eliminating the parts related to the isochoric constraint) allows one to construct directly sequences $u_h \in C^\infty$, avoiding the subtle truncation arguments used in [14].

We split the proof of Part (a) into two steps, dealing first with the compactness statement and then with the lower bound. In both proofs the key point is to define a suitable truncation of the sequence u_h; since the domain depends on h this has to be done locally in order to obtain uniform estimates. After the truncation the proof follows the outline of the proof of Theorem 6.1 in [14], which is carried out in parallel on the truncated sequence (on subdomains, but obtaining L^2 estimates) and on the original sequence (which is defined globally, but obeys only L^p estimates), taking care to keep track of the link between the two.

(i) Compactness. In order to obtain convergence of the whole sequence we have to normalize any additive constants. In order to be able to apply Poincaré's inequality on subdomains of ω we fix a ball $B(z, 2R) \subset \omega$. Without loss of generality we may assume that the integral of u_h on $B_h = B(z, R) \times (-h/2, h/2)$ is equal to zero.

In order to truncate u_h, we consider the squares

$$S_{a,h} = a + \left(-\frac{h}{2}, \frac{h}{2}\right)^2\,, \quad a \in h\mathbb{Z}^2\,.$$

Let λ be fixed such that $W_c(F) \geq c|F|^p$ for $|F| \geq \lambda$. This implies that $W_c(F) \geq c\,\mathrm{dist}^2(F, \mathrm{SO}(3))$ for all $|F| \leq \lambda$. In the following, all constants may depend on the choice of λ, as well as on the constants in the growth conditions for W_c, but not on h or u_h; the specific value may change from line to line. For each $a \in h\mathbb{Z}^2$ such that $S_{a,3h} \subset \omega$, we define $T_{a,h} = S_{a,3h} \times (-h/2, h/2)$ as in (4.2) and $v_{a,h} \in W^{1,\infty}(T_{a,h}; \mathbb{R}^3)$ as a truncation of u_h restricted to $T_{a,h}$ on a scale λ, in the sense of Proposition 4.2. By scaling the constants appearing in Proposition 4.2 do not depend on h. We estimate

$$\int_{T_{a,h}} \operatorname{dist}^2(\nabla v_{a,h}, \mathrm{SO}(3))\,\mathrm{d}x \\ \leq \int_{T_{a,h}\cap\{v_{a,h}=u_h\}} \operatorname{dist}^2(\nabla v_{a,h}, \mathrm{SO}(3))\,\mathrm{d}x + c\,|T_{a,h} \cap \{v_{a,h} \neq u_h\}|\,.$$

On the set where $v_{a,h} = u_h$, we have $\nabla v_{a,h} = \nabla u_h$ and $|\nabla u_h| \leq \lambda$ a.e., hence $\operatorname{dist}^2(\nabla u_h, \mathrm{SO}(3)) \leq c W_c(\nabla u_h)$ for some constant $c > 0$. From (ii) of Proposition 4.2 we have

$$|T_{a,h} \cap \{v_{a,h} \neq u_h\}| \leq c \int_{T_{a,h}\cap\{|\nabla u_h|>\lambda\}} |\nabla u_h|^p\,\mathrm{d}x \leq c \int_{T_{a,h}} W_c(\nabla u_h)\,\mathrm{d}x\,.$$

We conclude

$$\int_{T_{a,h}} \operatorname{dist}^2(\nabla v_{a,h}, \mathrm{SO}(3))\,\mathrm{d}x \leq c \int_{T_{a,h}} W_c(\nabla u_h)\,\mathrm{d}x\,.$$

Analogously, we have

$$\begin{aligned} \int_{T_{a,h}} |\nabla v_{a,h} - \nabla u_h|^p\,\mathrm{d}x &= \int_{T_{a,h}\cap\{v_{a,h}\neq u_h\}} |\nabla v_{a,h} - \nabla u_h|^p\,\mathrm{d}x \\ &\leq c\lambda^p\,|T_{a,h} \cap \{v_{a,h} \neq u_h\}| + c \int_{T_{a,h}\cap\{v_{a,h}\neq u_h\}\cap\{|\nabla u_h|\geq\lambda\}} |\nabla u_h|^p\,\mathrm{d}x \\ &\leq c \int_{T_{a,h}} W_c(\nabla u_h)\,\mathrm{d}x\,. \end{aligned} \tag{4.3}$$

By the quantitative Liouville rigidity theorem [14, Th. 3.1], for every $T_{a,h} \subset \Omega_h$ there is $R_{a,h} \in \mathrm{SO}(3)$ such that

$$\begin{aligned} \int_{T_{a,h}} |\nabla v_{a,h} - R_{a,h}|^2\,\mathrm{d}x &\leq c \int_{T_{a,h}} \operatorname{dist}^2(\nabla v_{a,h}, \mathrm{SO}(3))\,\mathrm{d}x \\ &\leq c \int_{T_{a,h}} W_c(\nabla u_h)\,\mathrm{d}x\,. \end{aligned} \tag{4.4}$$

Again, by scaling one sees that the constants do not depend on h. We now assert that for all a, $b \in h\mathbb{Z}^2$ such that $T_{a,h}, T_{b,h} \subset \Omega_h$ and $|a-b| < 2h$, one has

$$|T_{a,h}|\,|R_{a,h} - R_{b,h}|^2 \le c \int_{T_{a,h}\cup T_{b,h}} W_c(\nabla u_h)\,\mathrm{d}x\,. \tag{4.5}$$

To prove the assertion, we define

$$\mathcal{G}_{a,b,h} = \{x \in T_{a,h} \cap T_{b,h} : |\nabla u_h|(x) < \lambda\}\,.$$

We observe

$$\int_{\mathcal{G}_{a,b,h}} |\nabla u_h - \nabla v_{a,h}|^2\,\mathrm{d}x \le c|T_{a,h} \cap \{u_h \ne v_{a,h}\}| \le c \int_{T_{a,h}} W_c(\nabla u_h)\,\mathrm{d}x\,,$$

and the same for b. Therefore, recalling (4.3) and (4.4),

$$\begin{aligned}
c|\mathcal{G}_{a,b,h}|\,&|R_{a,h} - R_{b,h}|^2 \\
&\le \int_{\mathcal{G}_{a,b,h}} |R_{a,h} - \nabla v_{a,h}|^2\,\mathrm{d}x + \int_{\mathcal{G}_{a,b,h}} |R_{b,h} - \nabla v_{b,h}|^2\,\mathrm{d}x \\
&\quad + \int_{\mathcal{G}_{a,b,h}} |\nabla u_h - \nabla v_{a,h}|^2\,\mathrm{d}x + \int_{\mathcal{G}_{a,b,h}} |\nabla u_h - \nabla v_{b,h}|^2\,\mathrm{d}x \\
&\le c \int_{T_{a,h}\cup T_{b,h}} W_c(\nabla u_h)\,\mathrm{d}x\,.
\end{aligned}$$

If $|\mathcal{G}_{a,b,h}| \ge h^3/10$, the assertion is proven. In the converse case we have

$$\int_{T_{a,h}\cup T_{b,h}} W_c(\nabla u_h)\,\mathrm{d}x \ge c|(T_{a,h} \cup T_{b,h}) \setminus \mathcal{G}_{a,b,h}| \ge ch^3 \ge ch^3|R_{a,h} - R_{b,h}|^2\,,$$

since $R_{a,h}, R_{b,h} \in \mathrm{SO}(3)$. This concludes the proof of the assertion (4.5).

Equation (4.5) is a discrete $W^{1,2}$ estimate for $R_{a,h}$. It implies, in an appropriate sense, strong convergence of $R_{a,h}$ in L^2. To make this statement precise, we define functions $R_h : \omega \to \mathbb{M}^{3\times 3}$ by

$$R_h(x_p) = \begin{cases} R_{a,h} & \text{if } x_p \in S_{a,h} \subset S_{a,3h} \subset \omega \text{ for some } a \in h\mathbb{Z}^2\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then (4.5) implies that after taking a subsequence the sequence R_h converges strongly in $L^2(\omega; \mathbb{M}^{3\times 3})$ to a function $R : \omega \to \mathrm{SO}(3)$. This argument is completely analogous to the case of quadratic growth, we refer to the final part of the proof of Th. 4.1 in [14] (starting with the estimate (4.8) which corresponds to (4.5) here) for details.

It remains to show that there exists a $u \in W^{2,2}(\omega;\mathbb{R}^3)$ such that $Re_i = \partial_i u$ for $i=1,2$, and that $\mathcal{T}_h u_h \to \mathcal{T}_0 u$ strongly in $W^{1,p}(\Omega_1;\mathbb{R}^3)$. For $h>0$ and $j \in \mathbb{N}\setminus\{0\}$ let $C_j(h)$ be the set of centers defined by

$$C_j(h) = \left\{a \in jh\mathbb{Z}^2,\, S_{a,3h} \subset \omega\right\}\,.$$

We define the function $v_h \in L^p(\Omega_h;\mathbb{R}^3)$ by

$$v_h(x) = \begin{cases} v_{a,h}(x) & \text{if } x \in S_{a,h}\times(-h/2,h/2),\ a \in C_1(h)\,,\\ 0 & \text{otherwise.}\end{cases}$$

By definition, $v_h \in W^{1,p}(S_{a,h}\times(-h/2,h/2);\mathbb{R}^3)$ for all h and all $a \in C_1(h)$, and $v_h \in SBV(\Omega_h;\mathbb{R}^3)$. Let Dv_h be its distributional gradient, and ∇v_h be the part which is absolutely continuous with respect to the Lebesgue measure (the rest is concentrated on the boundaries of the $S_{a,h}\times(-h/2,h/2)$). Clearly

$$\nabla v_h(x) = \begin{cases} \nabla v_{a,h}(x) & \text{if } x \in S_{a,h}\times(-h/2,h/2),\ a \in C_1(h),\\ 0 & \text{otherwise}\,.\end{cases}$$

Analogous statements hold for the rescaling $\mathcal{T}_h v_h$. For any $\widetilde{\omega} \subset\subset \omega$ with $B(z,R)\subset\widetilde{\omega}$ there exists an $h_0>0$ such that $\widetilde{\omega}$ is contained in the union of the squares $S_{a,h}$ with $a \in C_1(h)$, for all $h<h_0$. We conclude from (4.3) that for $h\in(0,h_0)$ and $\widetilde{\Omega}_1 = \widetilde{\omega}\times(-1/2,1/2)$

$$\begin{aligned}\int_{\widetilde{\Omega}_1} |\partial_1\mathcal{T}_h u_h - \partial_1 \mathcal{T}_h v_h|^p\,\mathrm{d}x &= \frac{1}{h}\int_{\widetilde{\omega}\times(-h/2,h/2)} |\partial_1 u_h - \partial_1 v_h|^p\,\mathrm{d}x\\ &\le \frac{c}{h}\int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x \le ch^2 I_h^c[u_h]\,.\end{aligned}$$

Analogously, and using (4.4),

$$\begin{aligned}\int_{\widetilde{\Omega}_1} |\partial_1 \mathcal{T}_h v_h - R_h e_1|^p\,\mathrm{d}x &\le |\widetilde{\Omega}_1|^{(2-p)/2}\Big(\int_{\widetilde{\Omega}_1}|\partial_1\mathcal{T}_h v_h - R_h e_1|^2\,\mathrm{d}x\Big)^{p/2}\\ &\le c\Big(\frac{1}{h}\int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x\Big)^{p/2} \le ch^p\big(I_h^c[u_h]\big)^{p/2}\,.\end{aligned}$$

This implies that $\partial_1\mathcal{T}_h u_h \to Re_1$ in $L^p(\widetilde{\Omega}_1;\mathbb{R}^3)$, and the same for $\partial_2\mathcal{T}_h u_h$ and Re_2.

We now turn to the normal component. By definition, $\partial_3\mathcal{T}_h u_h = h\mathcal{T}_h\partial_3 u_h$ and $|\partial_3\mathcal{T}_h v_{a,h}| \le ch$, and we obtain recalling (4.3)

$$\begin{aligned}\int_{\widetilde{\Omega}_1}|\partial_3\mathcal{T}_h u_h|^p\,\mathrm{d}x &\le c\int_{\widetilde{\Omega}_1}|\partial_3\mathcal{T}_h v_h|^p\,\mathrm{d}x + ch^p\int_{\widetilde{\Omega}_1}|\mathcal{T}_h(\partial_3 u_h - \partial_3 v_h)|^p\,\mathrm{d}x\\ &\le ch^p + ch^{p-1}\int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x \le ch^p + ch^{p+2} I_h^c[u_h]\,.\end{aligned}$$

Therefore $\nabla \mathcal{T}_h u_h$ is uniformly bounded in $L^p(\widetilde{\Omega}_1;\mathbb{M}^{3\times 3})$, and since by assumption u_h (and hence $\mathcal{T}_h u_h$) have average zero on $B_h \subset \widetilde{\Omega}_h$, we get that $\mathcal{T}_h u_h$ is uniformly bounded in $W^{1,p}(\widetilde{\Omega}_1;\mathbb{R}^3)$. This implies that $\mathcal{T}_h u_h$ has a subsequence which converges strongly in L^p and weakly in $W^{1,p}$ to a limit U. At the same time the partial derivatives $\partial_{1,2}\mathcal{T}_h u_h$ converge strongly in L^p to $Re_{1,2}$, and $\partial_3 \mathcal{T}_h u_h$ to zero. By uniqueness of the weak limit we obtain $\partial_{1,2}U = Re_{1,2}$, $\partial_3 U = 0$, and $\mathcal{T}_h u_h \to U$ strongly in $W^{1,p}(\widetilde{\Omega}_1;\mathbb{R}^3)$. Since this holds for any $\widetilde{\omega} \subset\subset \omega$, and $R \in L^\infty(\omega)$, we conclude that there exists a $u \in W^{1,p}(\omega;\mathbb{R}^3)$ such that $U = \mathcal{T}u$. The higher regularity for u follows from (4.5) as in [14]. In order to prove strong convergence in $W^{1,p}(\Omega_1)$, we observe

$$
\begin{aligned}
\limsup_{h\to 0} \int_{\Omega_1} &|\mathcal{T}_h \nabla u_h - \mathcal{T}_h \nabla u|^p \,\mathrm{d}x \\
&\le c|\Omega_1 \setminus \widetilde{\Omega}_1| + c \limsup_{h\to 0} \int_{\Omega_1\setminus\widetilde{\Omega}_1} W_c(\mathcal{T}_h \nabla u_h)\,\mathrm{d}x \\
&\quad + \limsup_{h\to 0} \int_{\widetilde{\Omega}_1} |\mathcal{T}_h \nabla u_h - \mathcal{T}_h \nabla u|^p \,\mathrm{d}x \\
&\le c|\Omega_1 \setminus \widetilde{\Omega}_1| = c|\omega \setminus \widetilde{\omega}|
\end{aligned}
$$

for any $\widetilde{\omega} \subset\subset \omega$, which implies the assertion.

For future reference we observe that (4.3) and (4.4) imply additionally convergence for the rescaled gradient $h^{-1}\partial_3 \mathcal{T}_h u_h = \mathcal{T}_h \partial_3 u_h$. Indeed,

$$
\int_{\widetilde{\Omega}_1} \left|\mathcal{T}_h \nabla v_h - \mathcal{T}_h \nabla u_h\right|^p \mathrm{d}x \le \frac{c}{h} \int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x
$$

and

$$
\int_{\widetilde{\Omega}_1} \left|\mathcal{T}_h \nabla v_h - R_h\right|^2 \mathrm{d}x \le \frac{c}{h} \int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x\,.
$$

In particular, $\mathcal{T}_h \nabla u_h - R_h \to 0$ strongly in $L^p(\widetilde{\Omega}_1;\mathbb{M}^{3\times 3})$ and in view of the strong convergence of R_h to $R : \Omega_1 \to \mathrm{SO}(3)$ we deduce, arguing as above,

$$
\mathcal{T}_h \nabla u_h \to R = (\widetilde{\nabla} u, \partial_1 u \wedge \partial_2 u) \quad \text{in } L^p(\Omega_1;\mathbb{M}^{3\times 3}). \tag{4.6}
$$

(ii) Lower bound. The general strategy is to prove first a lower bound,

$$
\liminf_{h\to 0} \frac{1}{h^3} \int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x \ge \frac{1}{2} \int_{\Omega_1} Q_2(\widehat{G})\,\mathrm{d}x \tag{4.7}
$$

and to show that $\widehat{G}$ is affine in x_3. The assertion then follows from integration in x_3 using that Q_2 is quadratic and from a characterization of $\widehat{G}$ in terms of $\widetilde{\nabla} u$. In order to define G and prove (4.7) we consider the fields

$$g_h(x) = \begin{cases} \dfrac{R_{a,h}^T \nabla u_h(x) - \mathrm{Id}}{h} & \text{if } x \in T_{a,h},\ a \in C_3(h)\,, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\bar{g}_h(x) = \begin{cases} \dfrac{R_{a,h}^T \nabla v_{a,h}(x) - \mathrm{Id}}{h} & \text{if } x \in T_{a,h},\ a \in C_3(h)\,, \\ 0 & \text{otherwise} \end{cases}$$

where $R_{a,h}$ is the rotation in (4.4). Note that we choose $a \in 3h\mathbb{Z}^2$ in order to ensure that the $T_{a,h}$ are disjoint. We assert next that there exists a $G \in L^2(\Omega_1; \mathbb{M}^{3\times3})$ such that, up to a subsequence,

$$\mathcal{T}_h g_h \rightharpoonup G \text{ in } L^p(\Omega_1; \mathbb{M}^{3\times3}) \tag{4.8}$$

and

$$\mathcal{T}_h \bar{g}_h \rightharpoonup G \text{ in } L^2(\Omega_1; \mathbb{M}^{3\times3})\,. \tag{4.9}$$

To show this, note first that by (4.4),

$$\begin{aligned} \frac{1}{h}\int_{\Omega_h} |\bar{g}_h|^2 \,\mathrm{d}x &= \frac{1}{h^3} \sum_{a\in C_3(h)} \int_{T_{a,h}} |\nabla v_{a,h} - R_{a,h}|^2 \,\mathrm{d}x \\ &\le \frac{c}{h^3} \int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x \le C\,. \end{aligned} \tag{4.10}$$

Thus $\mathcal{T}_h \bar{g}_h$ is bounded in $L^2(\Omega_1; \mathbb{M}^{3\times3})$ and there exists a subsequence that converges weakly to a limit G, and (4.9) is proven. At the same time, by (4.3)

$$\begin{aligned} \frac{1}{h}\int_{\Omega_h} |\bar{g}_h - g_h|^p \,\mathrm{d}x &= \frac{1}{h^{p+1}} \sum_{a\in C_3(h)} \int_{T_{a,h}} |\nabla v_{a,h} - \nabla u_h|^p \,\mathrm{d}x \\ &\le \frac{c}{h^{p+1}} \int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x \le Ch^{2-p}\,. \end{aligned}$$

Therefore $\mathcal{T}_h \bar{g}_h - \mathcal{T}_h g_h \to 0$ in L^p. Since both weak convergence in L^2 and strong convergence in L^p imply weak convergence in L^p, (4.8) follows.

We now define

$$\chi_h(x) = \begin{cases} 1 & \text{if } |g_h|(x) \le h^{-1/2}\,, \\ 0 & \text{otherwise,} \end{cases}$$

and observe that $\mathcal{T}_h\chi_h \to 1$ in measure and $|\chi_h| \le 1$. Therefore

$$\mathcal{T}_h(\chi_h g_h) \rightharpoonup G \text{ in } L^p(\Omega_1;\mathbb{M}^{3\times 3})\,. \tag{4.11}$$

Indeed, for any $\psi \in L^{p'}$ (as usual $1/p' = 1 - 1/p$, $p' = \infty$ if $p = 1$) we have

$$\int_{\Omega_1} \mathcal{T}_h(\chi_h g_h)\psi \,\mathrm{d}x = \int_{\Omega_1} (\mathcal{T}_h\chi_h - 1)(\mathcal{T}_h g_h)\psi \,\mathrm{d}x + \int_{\Omega_1} (\mathcal{T}_h g_h)\psi \,\mathrm{d}x.$$

By (4.8) the weakly converging sequence g_h is equi-integrable, hence the first term can be made arbitrarily small. The second term converges by the definition of weak convergence, and (4.11) is proven.

We now estimate, recalling (4.3),

$$\begin{aligned}
\int_{\Omega_h} (\chi_h g_h - \chi_h \bar{g}_h)^2 \,\mathrm{d}x &\le \frac{1}{h^2} \sum_{a\in C_3(h)} \int_{T_{a,h}\cap\{u_h\ne v_{a,h}\}} \chi_h |\nabla u_h - \nabla v_{a,h}|^2 \,\mathrm{d}x \\
&\le \frac{c}{h^2} \sum_{a\in C_3(h)} \int_{T_{a,h}\cap\{u_h\ne v_{a,h}\}} |\nabla u_h - \nabla v_{a,h}|^p \,\mathrm{d}x \\
&\le \frac{c}{h^2} \int_{\Omega_h} W_c(\nabla u_h) \,\mathrm{d}x \le Ch\,.
\end{aligned}$$

By (4.9) $\mathcal{T}_h\chi_h\bar{g}_h$ is bounded in L^2, hence we conclude

$$\frac{1}{h}\int_{\Omega_h} |\chi_h g_h|^2 \,\mathrm{d}x \le C\,.$$

Recalling (4.11), by uniqueness of the weak limit we obtain

$$\mathcal{T}_h(\chi_h g_h) \rightharpoonup G \text{ in } L^2(\Omega_1;\mathbb{M}^{3\times 3})\,. \tag{4.12}$$

We now expand W_c in a Taylor series about the identity matrix, and obtain

$$W_c(\mathrm{Id} + A) \ge \frac{1}{2}Q_3(A) - \omega(|A|)\,,$$

with $\omega(t)/t^2 \to 0$ as $t \to 0$. Then by frame indifference

$$\begin{aligned}
\frac{1}{h^3}\int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x &\ge \frac{1}{h^3}\int_{\Omega_h} \chi_h W_c(R_h^T\nabla u_h)\,\mathrm{d}x \\
&\ge \frac{1}{h}\int_{\Omega_h} \frac{1}{2}\chi_h Q_3(g_h)\,\mathrm{d}x - \frac{1}{h^3}\int_{\Omega_h} \chi_h \omega(h|g_h|)\,\mathrm{d}x\,.
\end{aligned}$$

We first show that the last term converges to zero. On the support of χ_h we have $h|g_h| \le h^{1/2}$; therefore $\chi_h \omega(h|g_h|)/(h|g_h|)^2 \to 0$ in L^∞, and $h^{-1} \int_{\Omega_h} \chi_h |g_h|^2 \, \mathrm{d}x$ is bounded by (4.12). We conclude that

$$\begin{aligned} \liminf_{h\to 0} \frac{1}{h^3} \int_{\Omega_h} W_c(\nabla u_h)\, \mathrm{d}x &\ge \liminf_{h\to 0} \frac{1}{h} \int_{\Omega_h} \frac{1}{2} Q_3(\chi_h g_h)\, \mathrm{d}x \\ &= \liminf_{h\to 0} \int_{\Omega_1} \frac{1}{2} Q_3(\mathcal{T}_h \chi_h g_h)\, \mathrm{d}x. \end{aligned}$$

Since Q_3 is a positive semidefinite quadratic form it is convex, and hence weakly lower semicontinuous. Recalling (4.12) we obtain

$$\liminf_{h\to 0} \frac{1}{h^3} \int_{\Omega_h} W_c(\nabla u_h)\, \mathrm{d}x \ge \int_{\Omega_1} \frac{1}{2} Q_3(G)\, \mathrm{d}x\,. \tag{4.13}$$

The final part of the proof follows closely the lines of [14], with minor changes to take care of the weaker convergences. For the convenience of the reader we give a brief account of the argument in the present notation. Since Q_3 does not depend on the skew-symmetric part of G, we may assume that $G_{31} = G_{32} = 0$ for all x and we conclude from the definition of Q_2 that $Q_3(G) \ge Q_2(\widehat{G})$ where $\widehat{G} \in \mathbb{M}^{2\times 2}$ corresponds to the in-plane variables in G. This concludes the proof of (4.7).

We shall now prove that $\widehat{G}$ is affine in x_3. We define the difference quotient $\widetilde{H}_h$ of $\mathcal{T}_h \widetilde{g}_h$ in the x_3 direction,

$$\begin{aligned} \widetilde{H}_h(x_p, x_3) &= \frac{1}{z} \big(\mathcal{T}_h \widetilde{g}_h(x_p, x_3 + z) - \mathcal{T}_h \widetilde{g}_h(x_p, x_3) \big) \\ &= \frac{1}{hz} R_h^T \big(\widetilde{\nabla} \mathcal{T}_h u_h(x_p, x_3 + z) - \widetilde{\nabla} \mathcal{T}_h u_h(x_p, x_3) \big)\,, \end{aligned}$$

where the notation with a tilde indicates as before that we consider a 3×2 matrix, that is, $\widetilde{G} = (Ge_1, Ge_2)$ for the 3×2-matrix formed by the first two columns of any matrix $G \in \mathbb{M}^{3\times 3}$.

Let $\widetilde{\Omega} \subset\subset \Omega_1$ and choose $|z| < \mathrm{dist}(\widetilde{\Omega}_1, \partial\Omega_1)$ so that the difference quotients are well defined. By (4.8) we have

$$\widetilde{H}_h \rightharpoonup \widetilde{H} = \frac{1}{z} \big(\widetilde{G}(x_p, x_3 + z) - \widetilde{G}(x_p, x_3) \big) \quad \text{in } L^p(\widetilde{\Omega}_1; \mathbb{M}^{3\times 2}).$$

It follows from (4.6) that the bounded sequence R_h converges to R in measure and that the limit is given by $R = (\widetilde{\nabla} U | b) \in W^{1,2}(\Omega; \mathbb{M}^{3\times 3})$ with $b = \partial_1 U \wedge \partial_2 U$. Hence

$$\frac{1}{hz} \big(\widetilde{\nabla} \mathcal{T}_h u_h(x_p, x_3 + z) - \widetilde{\nabla} \mathcal{T}_h u_h(x_p, x_3) \big) = R_h \widetilde{H}_h \rightharpoonup (\widetilde{\nabla} U | b) \widetilde{H} \text{ in } L^p(\widetilde{\Omega}_1; \mathbb{M}^{3\times 2}).$$

In order to identify $\widetilde{H}$ we rewrite the left-hand side as

$$\widetilde{\nabla}\Big(\frac{1}{hz}\int_{x_3}^{x_3+z} \partial_3 \mathcal{T}_h u_h(x_p,s)\,\mathrm{d}s\Big).$$

In view of the strong convergence (4.6) we conclude that

$$\frac{1}{hz}\int_{x_3}^{x_3+z} \partial_3 \mathcal{T}_h u_h(x_p,s)\,\mathrm{d}s \to \frac{1}{z}\int_{x_3}^{x_3+z} b(x_p,s)\,\mathrm{d}s = b(x_p) \quad \text{in } L^p(\widetilde{\Omega}_1;\mathbb{R}^3)\,,$$

since b is independent of x_3. Hence

$$\frac{1}{hz}\big(\widetilde{\nabla}\mathcal{T}_h u_h(x_p,x_3+z) - \widetilde{\nabla}\mathcal{T}_h u_h(x_p,x_3)\big) \to \widetilde{\nabla} b \quad \in \mathcal{D}'(\Omega).$$

By the uniqueness of the distributional limit we deduce that $\widetilde{\nabla} b = (\widetilde{\nabla} U|b)\widetilde{H} \in L^p$, and hence $b \in W^{1,p}(\Omega;\mathbb{R}^3)$. Additionally we deduce the representation formulas

$$\widetilde{H} = (\widetilde{\nabla} U|b)^T \widetilde{\nabla} b, \quad \widetilde{G}(x_p,x_3) = \widetilde{G}(x_p,0) + x_3 \widetilde{H}(x_p).$$

This implies for the 2×2-matrix $\widehat{G}$ that

$$\widehat{G}(x_p,x_3) = \widehat{G}(x_p,0) + x_3 \mathrm{II}(x_p), \quad \mathrm{II}(x_p) = (\widetilde{\nabla} U)^T \widetilde{\nabla} b.$$

Since $\widetilde{\Omega}_1$ was an arbitrary subset, the foregoing identities are true in Ω_1.

Finally, from the inequality (4.13) and the trivial bound $Q_3(G) \geq Q_2^c(\widehat{G})$ we obtain

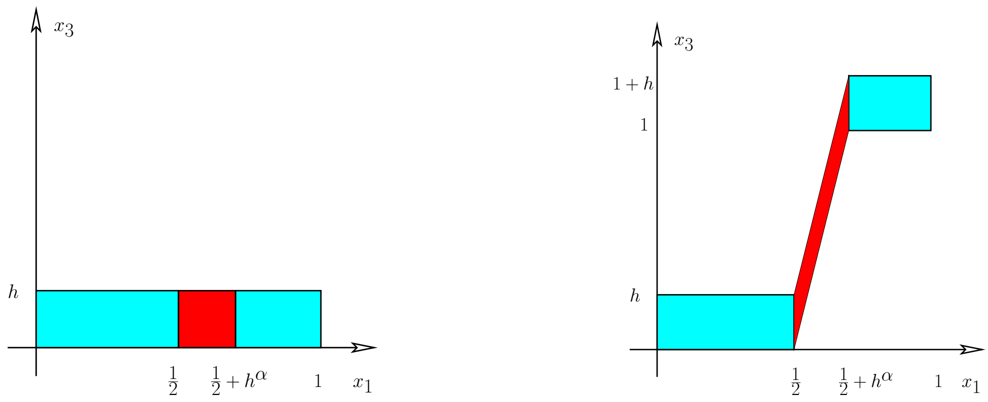
$$\liminf_{h\to 0} \frac{1}{h^3}\int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x \geq \frac{1}{2}\int_{\Omega_1} Q_3(G)\,\mathrm{d}x \geq \frac{1}{2}\int_{\Omega_1} Q_2^c(\widehat{G})\,\mathrm{d}x\,.$$

Since $\widehat{G}$ is affine in x_3, and Q_2^c is a quadratic form, we conclude that

$$\liminf_{h\to 0} \frac{1}{h^3}\int_{\Omega_h} W_c(\nabla u_h)\,\mathrm{d}x \geq \frac{1}{2}\int_{\Omega_1} \big(Q_2^c(\widehat{G}(x_p,0)) + x_3^2 Q_2^c(\mathrm{II}(x_p))\big)\,\mathrm{d}x$$

(the linear term vanishes after integration in x_3). This implies the assertion in Theorem 4.1 since the quadratic from Q_2^c is nonnegative. $\square$

It is interesting to note that the compactness statement fails if $p < 1$. This is due to the fact that one can approximate a discontinuous function, which corresponds to breaking the plate, by piecewise affine functions with



The unfractured bar *The almost fractured bar*

Figure 1: Sketch of the construction in (4.14).

bounded energy, see Figure 1 for a sketch of the situation. Precisely, let $p \in (0,1)$, and assume for simplicity

$$W_c(F) = \min\left\{\mathrm{dist}^2(F, \mathrm{SO}(3)), \mathrm{dist}^p(F, \mathrm{SO}(3))\right\} .$$

We work on $\omega = (0,1)^2$, and set, for a given $\alpha > 0$ to be chosen later,

$$u_h(x) = \begin{cases} x & \text{if } x_1 < 1/2\,, \\ x + \dfrac{x_1 - 1/2}{h^\alpha} e_2 & \text{if } 1/2 < x_1 < 1/2 + h^\alpha\,, \\ x + e_2 & \text{otherwise.} \end{cases} \tag{4.14}$$

The energy of the system is concentrated on the small part of width h^α (and volume $h^{\alpha+1}$) where the plate is sheared and the corresponding deformation gradient is given by

$$\nabla u_h = \begin{pmatrix} 1 & 0 & 0 \\ h^{-\alpha} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The total energy is thus bounded by

$$I_h[u_h] = I_h^c[u_h] \le \frac{1}{h^3} \int_{\Omega_h} \mathrm{dist}^p(\nabla u_h, \mathrm{SO}(3))\,\mathrm{d}x \le \frac{1}{h^3}\, h^\alpha\, h\, h^{-\alpha p} = h^{\alpha(1-p)-2}\,,$$

which is uniformly bounded provided $\alpha \ge 2/(1-p)$. It is clear that the limit configuration is discontinuous and thus not in $W^{2,2}$.

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