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G-expectation weighted Sobolev spaces, backward SDE and path dependent PDE

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Abstract. Beginning from a space of smooth, cylindrical and non-anticipative processes defined on a Wiener probability space $(\Omega, \mathcal{F}, P)$, we introduce a P-weighted Sobolev space, or "P-Sobolev space", of non-anticipative path-dependent processes $u = u(t, \omega)$ such that the corresponding Sobolev derivatives $\mathcal{D}_t + (1/2)\Delta_x$ and $\mathcal{D}_x u$ of Dupire's type are well defined on this space. We identify each element of this Sobolev space with the one in the space of classical L^p_P integrable Itô's process. Consequently, a new path-dependent Itô's formula is applied to all such Itô processes.

It follows that the path-dependent nonlinear Feynman–Kac formula is satisfied for most L^p_P-solutions of backward SDEs: each solution of such BSDE is identified with the solution of the corresponding quasi-linear path-dependent PDE (PPDE). Rich and important results of existence, uniqueness, monotonicity and regularity of BSDEs, obtained in the past decades can be directly applied to obtain their corresponding properties in the new fields of PPDEs.

In the above framework of P-Sobolev space based on the Wiener probability measure P, only the derivatives $\mathcal{D}_t + (1/2)\Delta_x$ and $\mathcal{D}_x u$ are well-defined and well-integrated. This prevents us from formulating and solving a fully nonlinear PPDE. We then replace the linear Wiener expectation E_P by a sublinear G-expectation $\mathbb{E}^G$ and thus introduce the corresponding G-expectation weighted Sobolev space, or "G-Sobolev space", in which the derivatives $\mathcal{D}_t u$, $\mathcal{D}_x u$ and $\mathcal{D}^2_x u$ are all well defined separately. We then formulate a type of fully nonlinear PPDEs in the G-Sobolev space and then identify them to a type of backward SDEs driven by G-Brownian motion.

1. Introduction.

Recently Dupire [**7**] introduced the notion of horizontal derivative (time-derivative) and vertical derivative (space derivative) for smooth and non-anticipative process of paths which was further extended by Cont and Fournie [**2**]. He then derived his functional Itô's formula and hence formulated the corresponding functional Feynman–Kac formula.

On the other hand, in the theory of nonlinear expectation introduced in [**20**], [**21**], the conditional expectation $\mathbb{E}^G_t[\xi](\omega)$ is a cylindrical solution $u(t, \omega)$ of a path-dependent

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fully nonlinear path-dependent PDE (called G-heat equation, see Remark 3.5 for more details). A general G-martingale is in fact the limit of such solutions. This idea can be also applied to solve a backward stochastic differential equation (BSDE) as a limit of cylindrical PPDEs derived from the nonlinear Feynman–Kac formula of [**16**], [**17**]. Peng [**25**] then proposed to define a general solutions of BSDEs (resp. G-martingales) as the solutions to the corresponding nonlinear path-dependent PDEs.

Since rich and important results in the fields of BSDE and nonlinear martingales have been obtained in the past decades, how to combine Peng's cylindrical construction and the idea of Dupire's derivatives to rigorously interpret a general BSDE as a well-posed path-dependent PDE becomes a very interesting problem.

Directly using Dupire's derivative and combing with techniques based on the classical nonlinear Feynman–Kac formula, Peng and Wang [**27**] has proved the existence and uniqueness for systems of quasi-linear PDE. But inherent from a drawback of Dupire's definition, a solution of the PPDE is still defined on the cadlag paths thus we still cannot directly interpret its solution as that of the corresponding BSDE.

Introducing a new notion of viscosity solution via a dynamical nonlinear expectation, Ekren et al. [**8**], [**10**] has proved the comparison stability and existence theorem for a type of quasi-linear (and fully nonlinear) path-dependent PDE. But the results are based on the assumption of continuity of $u(t, \omega)$ with respect to ω, which is not the case for many well-posed BSDEs and nonlinear martingales.

In this paper we will introduce a quite simple and very fundamental framework of Sobolev space based on probability measures (resp. nonlinear expectations) define on the space of functions of continuous paths. Many important solutions in BSDEs can be interpreted as the unique solution of the corresponding PPDEs.

A classical Sobolev space, say $W^{1,p}(\mathbb{R}^d)$ for a given $p \geq 1$, is a completion of $C_0^\infty(\mathbb{R}^d)$, the space of all infinitely differentiable real functions $u = u(x)$ with compact supports, under the norm $\|u\|_{L^p(\mathbb{R}^d)} + \|\nabla u\|_{L^p(\mathbb{R}^d)}$. The gradient operator ∇ is extended as a continuous mapping: $W^{1,p}(\mathbb{R}^d) \mapsto (L^p(\mathbb{R}^d))^d$. This framework plays an important role in the study of various types of PDEs.

In this paper, inspired by Malliavin's derivative and Dupire's derivative, we introduce some of Sobolev spaces weighted by the Wiener probability measure P, called P-Sobolev space, or by G-expectation, called G-Sobolev space. An element of the first one corresponds exactly to an L_P^p-integrable Itô process and, similarly the second one corresponds to an L_G^p-integrable G-Itô process. We define the solutions to a type of quasi-linear (resp. fully nonlinear) *path dependent PDEs* (Abbreviated by PPDE) in the P-Sobolev space (resp. G-Sobolev space) and establish a 1-1 correspondence between them and the BSDEs (resp. BSDEs driven by a G-Brownian motion). It turns out that rich results of existence, uniqueness, monotonicity and regularity obtained in the theory of BSDEs (resp. G-BSDEs) in the past decades can be directly applied to obtain the corresponding results in the quasi-linear (resp. fully nonlinear) PPDEs.

1.1. Sobolev solutions of quasi-linear path dependent PDEs.

A classical backward SDE is defined on a Wiener probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The problem is to find a pair of $\mathbb{F}$-progressively measurable processes (Y, Z) such that

$$Y_t = \xi + \int_t^T f(s, \omega, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \tag{1.1}$$

where $f : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \mapsto \mathbb{R}^n$ is a given function and $\xi : \Omega \mapsto \mathbb{R}^n$ is a given $\mathcal{F}_T$-measurable random vector. BSDE provides a probabilistic interpretation of a system of quasi-linear parabolic PDE ([**16**], [**17**]). Let us consider the Markovian situation of BSDE (1.1) in which $\xi = \varphi(B_T)$ and $f(t, \omega, y, z) = g(t, B_t(\omega), y, z)$ for deterministic and continuous functions $\varphi(x)$ and $g(t, x, y, z)$ satisfying some regularity conditions. Assume that (Y, Z) is the solution to the BSDE(ξ, f), i.e.,

$$Y_t = \varphi(B_T) + \int_t^T g(s, B_s, Y_s, Z_s) ds - \int_t^T Z_s dB_s.$$

By the classical arguments in the BSDE theory, we know that Y is a Markovian solution, namely, there exists a deterministic function $u(t,x)$ such that $Y_t = u(t, B_t)$. Assume $u(t,x)$ is smooth. Applying Itô's formula we get

$$u(t, B_t) = \varphi(B_T) - \int_t^T \left[\partial_s u(s, B_s) + \frac{1}{2}\Delta u(s, B_s)\right] ds - \int_t^T \partial_x u(s, B_s) dB_s.$$

Since the decomposition of a continuous semimartingale is unique, we have $Z_s = \partial_x u(s, B_s)$ and

$$\partial_s u(s, B_s) + \frac{1}{2}\Delta u(s, B_s) + g(s, B_s, u(s, B_s), \partial_x u(s, B_s)) = 0, \ P\text{-a.s.},$$

which is just the following quasi-linear PDE

$$\partial_s u(s, x) + \frac{1}{2}\Delta u(s, x) + g(s, x, u(s, x), \partial_x u(s, x)) = 0, \tag{1.2}$$

with terminal condition $u(T, T) = \varphi(x)$. This is a typical case of the nonlinear Feynman–Kac formula introduced in Peng [**17**] and Pardoux-Peng [**16**].

It is known that, in general, the solution to a typical backward SDE is a functional of Brownian paths. A very interesting and long-standing problem is to interpret the BSDEs as a path dependent counterpart of the above type of PDEs. In Section 2 we introduce a P-Sobolev space $W_{\mathcal{A}}^{1/2,1;p}(0,T)$ in the Wiener probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ on which the differential operators $\mathcal{A} = \mathcal{D}_t + (1/2)\Delta_x$ and $\mathcal{D}_x$ are well-defined. We have proved (see Theorem 2.9) that u is an element of $W_{\mathcal{A}}^{1/2,1;p}(0,T)$ if and only if it is an L^p-integrable Itô process of the following type

$$u(t, \omega) = u_0 + \int_0^t \eta(s, \omega) ds + \int_0^t \zeta(s, \omega) dB_s, \quad \zeta \in H_P^p(0,T), \ \eta \in M_P^p(0,T). \tag{1.3}$$

Moreover, we have

$$\mathcal{A}u = \eta, \quad \mathcal{D}_x u = \zeta.$$

We thus have a very general result of functional Itô's formula. It follows that the corresponding PPDE (1.2) can be rigorously formulated as: to find $u \in W_{\mathcal{A}}^{1/2,1;p}(0,T)$ such that

$$\mathcal{A}u + f(t, u, \mathcal{D}_x u) = 0. \tag{1.4}$$

In Theorem 2.11 we establish the 1-1 correspondence between backward BSDEs (1.1) and the quasi-linear PPDEs (1.4). The rich research results in the well-developed BSDE theory can be directly interpreted as the corresponding results in the new research area of path-dependent PDEs. On the other hand, the rich results in the Sobolev solutions of parabolic PDEs can also be applied to the BSDE theory.

1.2. Sobolev solutions of fully non-linear path dependent PDEs.
In the above formulation the time derivative $\mathcal{D}_t$ and the second order space derivative $\mathcal{D}_x^2$ are 'mixed' together as $\mathcal{A} = \mathcal{D}_t + (1/2)\Delta_x$ and only $\mathcal{A}$ and $\mathcal{D}_x$ are well-defined and well-integrated in this framework. This prevent us from formulating and solving a fully non-linear PPDEs. We therefore replace the Wiener expectation E_P by a G-expectation $\mathbb{E}^G$ to define a G-expectation-weighted Sobolev space or 'G-Sobolev space' in Section 4 and Section 5.

Recall that, just similar to the classical Feynman–Kac formula, G-Brownian motion corresponds to the following well-posed PDE (called G-heat equation)

$$\begin{gathered}\partial_t u(t,x) + G(\partial_x^2 u(t,x)) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}, \\ u(T,x) = \varphi(x),\end{gathered} \tag{1.5}$$

where $G(a) = (1/2)(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$, for some $0 \le \underline{\sigma} \le \overline{\sigma} < \infty$, is a given sublinear function. Let φ be a bounded Lipschitz function and let B_t be a G-Brownian motion in the G-expectation space $(\Omega_T, L_G^1(\Omega_T), \mathbb{E}^G)$. Then the G-martingale $\mathbb{E}^G[\varphi(B_T)]$ is equal to $u(t, B_t)$, where u is the solution of the PDE (1.5).

Now for a given $\xi \in L_G^1(\Omega_T)$, the G-martingale $u(t,\omega) = \mathbb{E}_t^G[\xi](\omega)$ should be a reasonable candidate of path dependent solutions of equation (1.5) with terminal condition $u(T,\omega) = \xi(\omega)$. In this paper we formulate it as the unique solution of path-dependent PDE of (1.5) in a G-expectation-weighted Sobolev space (see Corollary 5.6).

In Section 4 and Section 5 we will define two types of G-Sobolev spaces: $W_G^{1,2;p}(0,T)$ and $W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$. All derivatives $\mathcal{D}_t$, $\mathcal{D}_x$ and $\mathcal{D}_x^2$ are well-defined continuous operators on $W_G^{1,2;p}(0,T)$. Moreover in Theorem 4.5 we have proved that the space $W_G^{1,2;p}(0,T)$ consists of the totality of L_G^p-integrable G-Itô processes:

$$u(t,\omega) = u(0,\omega) + \int_0^t \eta(s,\omega)ds + \int_0^t \zeta(s,\omega)dB_s + \frac{1}{2}\int_0^t \gamma(s,\omega)d\langle B\rangle_s, \tag{1.6}$$

where $\eta, \gamma \in M_G^p(0,T)$ and $\zeta \in H_G^p(0,T)$. Furthermore, we have

$$\mathcal{D}_s u(s,\omega) = \eta(s,\omega), \quad \mathcal{D}_x u(s,\omega) = \zeta(s,\omega), \quad \mathcal{D}_x^2 u(s,\omega) = \gamma(s,\omega). \tag{1.7}$$

These differential operators are well-defined since the decomposition of a G-Itô process is unique:

$$\int_0^t \eta(s)ds + \int_0^t \zeta(s)dB_s + \frac{1}{2}\int_0^t \gamma(s)d\langle B\rangle_s = 0 \iff \eta, \gamma \equiv 0 \text{ and } \zeta \equiv 0.$$

This is mainly due to the distinguishability property of G-Itô's processes (see Lemma 4.3).

Observing that when we study the path-derivatives $\mathcal{D}_t$, $\mathcal{D}_x$ and $\mathcal{D}_x^2$ in the Sobolev spaces $W_G^{1,2;p}(0,T)$, some very deep phenomena happen. For example, for the path process $v(t,\omega) := \langle B\rangle_t(\omega)$ we have

$$\mathcal{D}_t v(t,\omega) \equiv 0, \quad \mathcal{D}_x v(t,\omega) \equiv 0, \text{ but } \mathcal{D}_x^2 v(t,\omega) \equiv 2.$$

In the linear case where $G(a) = a/2$, the space $(\Omega_T, L_G^p(\Omega_T), \mathbb{E}^G)$ coincides with the classical Wiener probability space $(\Omega, \mathcal{F}, P)$. The quadratic variation process $\langle B\rangle_t$ coincides with t. Therefore, corresponding to (1.6), the Itô process becomes

$$u(t,\omega) = u(0,\omega) + \int_0^t \beta(s,\omega)ds + \int_0^t \zeta(s,\omega)dB_s,$$

where $\beta = \eta + \gamma/2$. We thus have

$$\beta_s = \mathcal{D}_s u + \frac{1}{2}\mathcal{D}_x^2 u = \mathcal{A}u, \quad \zeta(s,\omega) = \mathcal{D}_x u(s,\omega).$$

This explains why the derivatives $\mathcal{D}_t$ and $\mathcal{D}_x^2$ are 'obliged' to be mixed together to become $\mathcal{A}$ in the P-Sobolev space.

Consider the following fully nonlinear path dependent PDEs: to find $u \in W_G^{1,2;p}(0,T)$ satisfying

$$\begin{gathered} \mathcal{D}_t u + G(\mathcal{D}_x^2 u) + f(u, \mathcal{D}_x u, \mathcal{D}_x^2 u) = 0, \quad t \in [0,T), \\ u(T,\omega) = \xi(\omega). \end{gathered} \tag{1.8}$$

We provide a 1-1 correspondence between the PPDE (1.8) and the following backward SDEs driven by G-Brownian motion: to find (Y, Z, η) satisfying

$$Y_t = \xi + \int_t^T f(Y_s, Z_s, \eta_s)ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

where $K_t = (1/2)\int_0^t \eta_s d\langle B\rangle_s - \int_0^t G(\eta_s)ds$.

In Equation (1.8), if f is independent of $\mathcal{D}_x^2 u$, the derivatives $\mathcal{D}_t u$, $\mathcal{D}_x^2 u$ appear as $\mathcal{A}_G u := \mathcal{D}_t u + G(\mathcal{D}_x^2 u)$, which is similar to the quasi-linear PPDE (1.4). In this case we can introduce a G-Sobolev space $W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$ as the nonlinear counterpart of the P-Sobolev space $W_{\mathcal{A}}^{1/2,1;p}(0,T)$. $W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$ is the completion of $W_G^{1,2;p}(0,T)$ under a weaker Sobolev norm. In this space, we give a weaker formulation of PPDE (1.8): to find $u \in W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$ such that

$$\begin{aligned} \mathcal{A}_G u + f(u(t,\omega), \mathcal{D}_x u(t,\omega)) = 0, \quad t \in [0,T), \\ u(T,\omega) = \xi(\omega). \end{aligned} \tag{1.9}$$

This formulation corresponds exactly to G-BSDEs studied in [**12**]. Consequently, the existence and uniqueness of weak solutions to the path dependent PDEs (1.9) have been obtained via the result of G-BSDEs.

The above results give a perfect answer to a suggestion proposed in [**25**] mentioned at beginning of this introduction.

The rest of the paper is organized as follows. In Section 2 we define the P-Sobolev space $W_{\mathcal{A}}^{1/2,1;p}(0,T)$ in the Wiener probability space. In the space $W_{\mathcal{A}}^{1/2,1;p}(0,T)$, we define the solutions to a type of quasi-linear PPDEs and establish the 1-1 correspondence between them and BSDEs. In Section 3, we present some basic notions and properties related to our framework of G-Sobolev spaces. In Section 4, we introduce the notion of the Sobolev space $W_G^{1,2;p}(0,T)$ weighted by the G-expectation and prove the 1-1 correspondence between BSDEs driven by G-Brownian motion and a type of fully nonlinear PPDEs. In Section 5, we introduce another G-Sobolev space $W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$ which is an expansion of $W_G^{1,2;p}(0,T)$. The 1-1 correspondence between the solutions of PPDEs defined in this space and the solutions of BSDEs driven by Brownian motions is then established.

2. Sobolev spaces on path space under Wiener expectation.

2.1. Smooth functions and processes of paths.

We recall that a classical Sobolev space, say $W^{1,p}(\mathbb{R}^d)$ for a given $p \geq 1$, is a completion of $C_0^\infty(\mathbb{R}^d)$, the space of all infinitely differentiable real functions $u = u(x)$ with compact supports, under the norm $\|u\|_{L^p(\mathbb{R}^d)} + \|\nabla u\|_{L^p(\mathbb{R}^d)}$.

Let $\Omega = C_0(\mathbb{R}_+; \mathbb{R}^d)$ be the space of all $\mathbb{R}^d$-valued continuous paths $\omega = (\omega(t))_{t \geq 0} \in \Omega$ with $\omega(0) = 0$ equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} \Big[\Big(\max_{t \in [0,i]} |\omega_t^1 - \omega_t^2| \Big) \wedge 1 \Big],$$

and let $B_t(\omega) = \omega(t)$, $t \in \mathbb{R}_+$ be the canonical process. It is clear that (Ω, ρ) is a complete separable metric space. We also denote $\Omega_T = \{\omega_{\cdot \wedge T} : \omega \in \Omega\}$ for each fixed $T \in [0, \infty)$.

DEFINITION 2.1 (Cylinder function of paths). A function $\xi : \Omega_T \to \mathbb{R}$ is called a cylinder function of paths on $[0,T]$ if it can be represented by

$$\xi(\omega) = \varphi(\omega(t_1), \dots, \omega(t_n)), \quad \omega \in \Omega_T,$$

for some $0 = t_0 < t_1 < \cdots < t_n = T$, where $\varphi : (\mathbb{R}^d)^n \to \mathbb{R}$ is a C^∞-function with at most polynomial growth. We denote by $C^\infty(\Omega_T)$ the collection of all cylinder functions of paths on $[0, T]$.

DEFINITION 2.2 (Cylinder step process). A function $\eta(t, \omega) : [0, T] \times \Omega_T \to \mathbb{R}$ is called a cylinder step process if there exists a time partition $\{t_i\}_{i=0}^n$ with $0 = t_0 < t_1 < \cdots < t_n = T$, such that

$$\eta(t, \omega) = \sum_{k=0}^{n-1} \xi_{t_k} 1_{(t_k, t_{k+1}]}(t). \tag{2.1}$$

Here $\xi_{t_k} := \varphi_k(\omega(t_1), \ldots, \omega(t_k))$ is a bounded cylinder function of paths on $[0, T]$. We denote by $M^0(0, T)$ the collection of all step processes.

The following space of functions of paths plays a similar role as $C_0^\infty(\mathbb{R}^d)$ in the construction of the classical Sobolev space $W^{1,p}(\mathbb{R}^d)$.

DEFINITION 2.3 (Cylinder process of paths). A function $u(t, \omega) : [0, T] \times \Omega_T \to \mathbb{R}$ is called a cylinder path process if there exists a time partition $\{t_i\}_{i=0}^n$ with $0 = t_0 < t_1 < \cdots < t_n = T$, such that for each $k = 0, 1, \ldots, n-1$ and $t \in (t_k, t_{k+1}]$,

$$u(t, \omega) = u_k(t, \omega(t); \omega(t_1), \ldots, \omega(t_k)).$$

Here for each k, the function $u_k : [t_k, t_{k+1}] \times (\mathbb{R}^d)^{k+1} \to \mathbb{R}$ is a C^∞-function with

$$u_k(t_k, x; x_1, \ldots, x_{k-1}, x) = u_{k-1}(t_k, x; x_1, \ldots, x_{k-1})$$

such that, all derivatives of u_k have at most polynomial growth. We denote by $\mathcal{C}^\infty(0, T)$ for the collection of all cylinder path processes.

For $\zeta_t = \sum_{i=0}^{n-1} \zeta_{t_i} 1_{(t_i, t_{i+1}]}(t) \in M^0(0, T)$, set

$$\int_0^t \zeta_s dB_s := \sum_{i=0}^{n-1} \zeta_{t_i}(B_{t_{i+1} \wedge t} - B_{t_i \wedge t}).$$

The following proposition follows directly from the definitions.

PROPOSITION 2.4. *Let η, ζ be cylinder step processes. Then*

$$u(t, \omega) := \int_0^t \eta(s, \omega) ds + \int_0^t \zeta(s, \omega) dB_s$$

belongs to $\mathcal{C}^\infty(0, T)$.

It is clear that $\mathcal{C}^\infty(0, T) \subset \mathcal{C}^\infty(0, \bar{T})$ for $\bar{T} \geq T$. We also set

$$\mathcal{C}^\infty(0,\infty) := \bigcup_{n=1}^{\infty} \mathcal{C}^\infty(0,n).$$

For $t \in (t_k, t_{k+1}]$, $n \in \mathbb{N}$, we denote

$$\mathcal{D}_t u(t,\omega) := \partial_{t+} u_k(t,x;x_1,\ldots,x_k)|_{x=\omega(t),x_1=\omega(t_1),\ldots,x_k=\omega(t_k)}.$$

For $t \in (t_k, t_{k+1}]$, we denote

$$\mathcal{D}_x u(t,\omega) := \partial_x u_k(t,x;x_1,\ldots,x_k)|_{x=\omega(t),x_1=\omega(t_1),\ldots,x_k=\omega(t_k)}, \tag{2.2}$$

$$\mathcal{D}_x^2 u(t,\omega) := \partial_x^2 u_k(t,x;x_1,\ldots,x_k)|_{x=\omega(t),x_1=\omega(t_1),\ldots,x_k=\omega(t_k)}, \tag{2.3}$$

$$\Delta_x u(t,\omega) := \mathrm{tr}[\mathcal{D}_x^2 u(t,\omega)]. \tag{2.4}$$

Let us indicate the relation between $\mathcal{D}_x u$ with the well-known Malliavin calculus. Let D be the Malliavin derivative operator. Then for each $u \in \mathcal{C}^\infty(0,T)$, we have

$$D_t u(t,\omega) = \mathcal{D}_x u(t,\omega).$$

But since the notion of $\mathcal{D}_x u(t,\omega)$ corresponds much more like the classical derivative of $\mathcal{D}_x u(t,x)$, emphasizing simply the perturbation of state point than the Malliavin's one emphasizing the perturbation of the whole path, thus we prefer to use the denotation $\mathcal{D}_x u(t,\omega)$.

In fact, the above definition of derivatives corresponds perfectly with Dupire's one, introduced originally in his insightful paper ([**7**]) (see also [**2**]). An advantage of our new formulation in this paper is that we do not need to define our derivatives on a larger space of right continuous paths with left limit.

In the sequel, we shall give the definitions of P-Sobolev and G-Sobolev spaces. For readers' convenience, we divide the discussions into two parts. First we consider this problem in the framework of the classical Wiener probability space, which presents a quite new point of view of Itô processes.

2.2. *P*-Sobolev spaces of path functions.

Let P be the Wiener probability measure on Ω_T. The canonical process $B_t(\omega) = \omega_t$ is a standard Brownian motion under P. Let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be the augmented filtration generated by $(B_t)_{t\in[0,T]}$.

For a $\mathcal{F}_T$-measurable random variable ξ, set $\|\xi\|_{L_P^p} := (E_P[|\xi|^p])^{1/p}$. Let $L_P^p(\Omega_T)$ be the space of $\mathcal{F}_T$-measurable random variables ξ with $\|\xi\|_{L_P^p} < \infty$, which coincides with the completion of $C^\infty(\Omega_T)$ with respect to the norm $\|\cdot\|_{L_P^p}$.

For a $\mathbb{F}$-progressively measurable process η, set $\|\eta\|_{M_P^p} := (E_P[\int_0^T |\eta_s|^p ds])^{1/p}$ (respectively, $\|\eta\|_{H_P^p} := (E_P[(\int_0^T |\eta_s|^2 ds)^{p/2}])^{1/p}$). Let $M_P^p(0,T)$ (respectively, $H_P^p(0,T)$) be the space of $\mathbb{F}$-progressively measurable processes η with $\|\eta\|_{M_P^p} < \infty$ (respectively, $\|\eta\|_{H_P^p} < \infty$), which coincides with the completion of $M^0(0,T)$ with respect to the norm $\|\cdot\|_{M_P^p}$ (respectively, $\|\cdot\|_{H_P^p}$). All equalities and inequalities in the following two subsections are in the sense of the corresponding spaces. In particular they hold P-almost surely.

The following proposition can be directly obtained from the Itô's formula.

PROPOSITION 2.5. *Given any $u \in \mathcal{C}^\infty(0,\infty)$, we have, for each $t \in [0,T]$,*

$$u(t,\omega) = u(0,\omega) + \int_0^t \mathcal{A}u(s,\omega)ds + \int_0^t \mathcal{D}_x u(s,\omega)dB_s,$$

where

$$\mathcal{A}u(s,\omega) := \left(\mathcal{D}_s + \frac{1}{2}\Delta_x\right)u(s,\omega) = \mathcal{D}_s u(s,\omega) + \frac{1}{2}\Delta_x u(s,\omega).$$

DEFINITION 2.6. For a process $u \in \mathcal{C}^\infty(0,T)$, we define the following two norms

$$\|u\|_{S_P^p} = \left\{E_P\left[\sup_{s\in[0,T]} |u_s|^p\right]\right\}^{1/p}$$

and

$$\|u\|_{W_{\mathcal{A}}^{1/2,1;p}} = \left\{E_P\left[\sup_{s\in[0,T]} |u_s|^p + \int_0^T |\mathcal{A}u_s|^p ds + \left\{\int_0^T |\mathcal{D}_x u_s|^2 ds\right\}^{p/2}\right]\right\}^{1/p}.$$

We denote by $S_P^p(0,T)$ the completion of $\mathcal{C}^\infty(0,T)$ with respect to the norm $\|\cdot\|_{S_P^p}$.

PROPOSITION 2.7. *The norm $\|\cdot\|_{W_{\mathcal{A}}^{1/2,1;p}}$ is closable in the space $S_P^p(0,T)$ in the following sense: Let $\{u^n\}_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{C}^\infty(0,T)$ with respect to the norm $\|\cdot\|_{W_{\mathcal{A}}^{1/2,1;p}}$. If $\|u^n\|_{S_P^p} \to 0$, we have $\|u^n\|_{W_{\mathcal{A}}^{1/2,1;p}} \to 0$.*

PROOF. The limit u of the Cauchy sequence $\{u^n\}_{n=1}^\infty$ under $\|\cdot\|_{W_{\mathcal{A}}^{1/2,1;p}}$ is of the following form

$$u(t,\omega) = u_0 + \int_0^t \eta(s,\omega)ds + \int_0^t \nu(s,\omega)dB_s, \quad \eta \in M_P^p(0,T), \quad \eta \in H_P^p(0,T).$$

But since $u \equiv 0$, then according to the uniqueness of the decomposition for the classical Itô processes, we must have $u_0 = 0$, $\eta \equiv 0$, $\nu \equiv 0$. □

DEFINITION 2.8. We denote by $W_{\mathcal{A}}^{1/2,1;p}(0,T)$ the completion of $\mathcal{C}^\infty(0,T)$ with respect to the norm $\|\cdot\|_{W_{\mathcal{A}}^{1/2,1;p}}$.

We call $W_{\mathcal{A}}^{1/2,1;p}(0,T)$ a P-weighted Sobolev space, or simply P-Sobolev space. From the above proposition, it is a subspace of $S_P^p(0,T)$. The differential operators $\mathcal{D}_x$ and $\mathcal{A}$, defined respectively in (2.2) and Proposition 2.5, can be continuously extended to this space:

$$\mathcal{D}_x : W_{\mathcal{A}}^{1/2,1;p}(0,T) \mapsto H_P^p(0,T),$$
$$\mathcal{A} : W_{\mathcal{A}}^{1/2,1;p}(0,T) \mapsto M_P^p(0,T).$$

The following proposition presents a quite new point of view to a classical Itô processes.

THEOREM 2.9. *For a given $u \in S_P^p(0,T)$, the following two conditions are equivalent*:

(i) $u \in W_{\mathcal{A}}^{1/2,1;p}(0,T)$;
(ii) *There exist $u_0 \in \mathbb{R}$, $\eta \in M_P^p(0,T)$ and $v \in H_P^p(0,T)$ such that*

$$u(t,\omega) = u_0 + \int_0^t \eta(s,\omega)ds + \int_0^t v(s,\omega)dB_s. \tag{2.5}$$

Moreover, we have

$$\left(\mathcal{D}_t + \frac{1}{2}\Delta_x\right)u(t,\omega) = \eta(t,\omega), \quad \mathcal{D}_x u(t,\omega) = v(t,\omega). \tag{2.6}$$

PROOF. The part of (i) $\Longrightarrow$ (ii) is directly from Proposition 2.5 and the definition of $W_{\mathcal{A}}^{1/2,1;p}(0,T)$. To prove (ii) $\Longrightarrow$ (i), we choose two sequences $\{\eta^n\}_{n=1}^\infty$ and $\{v^n\}_{n=1}^\infty$ in $M^0(0,T)$ such that $\|\eta^n - \eta\|_{M_P^p} \to 0$ and $\|v^n - v\|_{H_P^p} \to 0$. Set

$$u^n(t,\omega) := u_0 + \int_0^t \eta^n(s,\omega)ds + \int_0^t v^n(s,\omega)dB_s.$$

By Proposition 2.4 u^n belongs to $\mathcal{C}^\infty(0,T)$. It follows from Proposition 2.5 and the uniqueness of the decomposition of Itô processes that

$$\left(\mathcal{D}_t + \frac{1}{2}\Delta_x\right)u^n = \eta^n, \quad \mathcal{D}_x u^n = v^n.$$

Moreover, $\{u^n\}_{n=1}^\infty$ is a Cauchy sequence in $W_{\mathcal{A}}^{1/2,1;p}(0,T)$ with the limit u under $\|\cdot\|_{W_{\mathcal{A}}^{1/2,1;p}}$. We thus have (i) and (2.6). $\square$

REMARK 2.10. Theorem 2.9 means that each Itô's process u of form (2.5) with any given $u_0 \in \mathbb{R}$, $\eta \in M_P^p(0,T)$ and $v \in H_P^p(0,T)$ gives us a generalized path-dependent Itô's formula:

$$u(t,\omega) = u_0 + \int_0^t \left(\mathcal{D}_s + \frac{1}{2}\Delta_x\right)u(s,\omega)ds + \int_0^t \mathcal{D}_x u(s,\omega)dB_s.$$

2.3. Backward SDEs in Wiener space and related PPDEs.

Recall that a classical backward SDE is defined on a Wiener probability space $(\Omega, \mathcal{F}, P)$. The problem is to find a pair of processes $(Y, Z) \in S_P^p(0,T) \times H_P^p(0,T)$ such that

$$Y_t = \xi + \int_t^T f(s, \omega, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \tag{2.7}$$

where $f : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \mapsto \mathbb{R}^n$ is a given function and $\xi : \Omega \mapsto \mathbb{R}^n$ is a given $\mathcal{F}_T$-measurable random vector. The well-posedness of such BSDE were introduced in [**1**] for the linear case and by [**15**] for the general nonlinear case. BSDE provides a probabilistic interpretation of a system of quasilinear PDEs of parabolic and elliptic types. A typical example is when $\xi = \varphi(B_T)$ and $f(s,\omega,y,z) = g(t, B_t, y, z)$ for given functions $\varphi(x)$, $g(t,x,y,z)$ with some regularity conditions. In this case, the solution can be represented as $(Y_t, Z_t) = (u(t, B_t), \mathcal{D}_x u(t, B_t))$, where $u(t,x)$ is the solution of the following parabolic PDE

$$\begin{gathered} \left(\partial_t + \frac{1}{2}\Delta\right) u(t,x) + g(t,x,u(t,x), \mathcal{D}_x u(t,x)) = 0, \quad t \in [0,T), \\ u(T,x) = \varphi(x). \end{gathered} \tag{2.8}$$

It is known that, in general, the solution to Equation (2.7) is a functional of Brownian paths. A very interesting and longtime standing problem is to interpret the BSDEs as a generalized form of the above type of PDEs. In the P-Sobolev space $W_{\mathcal{A}}^{1/2,1;p}(0,T)$ the path dependent counterpart of Equation (2.8) is formulated as: to find $u \in W_{\mathcal{A}}^{1/2,1;p}(0,T)$ such that

$$\begin{gathered} \left(\mathcal{D}_t + \frac{1}{2}\Delta_x\right) u(t,\omega) + f(t, u(t,\omega), \mathcal{D}_x u(t,\omega)) = 0, \quad t \in [0,T), \\ u(T,\omega) = \xi(\omega). \end{gathered} \tag{2.9}$$

We will show that the well-posedness of backward SDEs (2.7) is equivalent to that of the path dependent PDEs (2.9). We make the following assumption:

Assumption 1. $(f(t,\omega,Y_t,Z_t))_{t\in[0,T]} \in M_P^p(0,T)$ for any $(Y,Z) \in S_P^p(0,T) \times H_P^p(0,T)$.

Theorem 2.11. *Let (Y, Z) be a solution to the backward SDE* (2.7). *Then we have $u(t,\omega) := Y_t(\omega) \in W_{\mathcal{A}}^{1/2,1;p}(0,T)$ with $\mathcal{D}_x u(t,\omega) = Z_t(\omega)$.*

Moreover, given $u(t,\omega) \in W_{\mathcal{A}}^{1/2,1;p}(0,T)$, the following (i) *and* (ii) *are equivalent*:

(i) *$(u, \mathcal{D}_x u)$ is a solution to the backward SDE* (2.7);
(ii) *u is a solution to the path dependent PDE* (2.9).

Remark 2.12. By this theorem, we can directly apply the result of existence and uniqueness of backward SDEs to get that of path dependent PDE (2.9). We recall the result from Pardoux and Peng [**15**] of existence and uniqueness to the backward SDE (2.7) under the following standard conditions: $\xi \in L^p_P(\Omega_T)$ and the function f satisfies Lipschitz condition in (y, z), namely, there exists a constant $C > 0$, such that, for all $\omega \in \Omega$,

$$\left|f(t,\omega,y,z) - f(t,\omega,y',z')\right| \le C\big(|y-y'| + |z-z'|\big), \quad y, y' \in \mathbb{R}^n,\ z, z' \in \mathbb{R}^{n\times d}.$$

This backward SDE can be directly seen as a well-posed path dependent PDE (2.9).

PROOF. (i) $\Longrightarrow$ (ii). Assume that (Y, Z) is a solution to the backward SDE (2.7). By Theorem 2.9, we have $u \in W^{1/2,1;p}_{\mathcal{A}}(0,T)$, $\mathcal{D}_x u = Z$ and

$$\mathcal{A}u(t,\omega) + f(t, u(t,\omega), \mathcal{D}_x u(t,\omega)) = 0.$$

(ii) $\Longrightarrow$ (i). Assume that $u \in W^{1/2,1;p}_{\mathcal{A}}(0,T)$ is a solution to the path dependent PDE (2.9). By Theorem 2.9 we have

$$\begin{aligned}
u(t,\omega) &= u(0,\omega) + \int_0^t \mathcal{A}u(s,\omega)ds + \int_0^t \mathcal{D}_x u(s,\omega)dB_s \\
&= u(0,\omega) - \int_0^t f(s, u(s,\omega), \mathcal{D}_x u(s,\omega))ds + \int_0^t \mathcal{D}_x u(s,\omega)dB_s \\
&= \xi(\omega) + \int_t^T f(s, u(s,\omega), \mathcal{D}_x u(s,\omega))ds - \int_t^T \mathcal{D}_x u(s,\omega)dB_s.
\end{aligned}$$

It follows that $(Y, Z) = (u, \mathcal{D}_x u) \in S^p_P(0,T) \times H^p_P(0,T)$ is a solution of (2.7). $\square$

REMARK 2.13. An advantage of the above formulation is that the path dependent PDE can be a system of PDEs, namely $u(t,\omega)$ can be $\mathbb{R}^n$-valued, or even H-valued for a Hilbert space H.

3. Some definitions and notations on G-expectation.

In the remaining sections we need to introduce the framework of G-expectation for the formulation of G-Sobolev space so that the corresponding fully nonlinear PPDE can be treated. For readers' convenience, in this section we present some main results of G-expectation theory related to our objective. More details with proofs and historical remarks can be found in a book of Peng [**24**].

Let Ω be a given set and $\mathcal{H}$ be a linear space of real functions define on a set Ω containing constants and satisfying $|\xi| \in \mathcal{H}$ for each $\xi \in \mathcal{H}$.

DEFINITION 3.1. A functional $\hat{\mathbb{E}} : \mathcal{H} \mapsto \mathbb{R}$ is called a sublinear expectation if

1. *Monotonicity*: for all X, Y in $\mathcal{H}$, $X \ge Y$, $\Longrightarrow$ $\hat{\mathbb{E}}[X] \ge \hat{\mathbb{E}}[Y]$.
2. *Constant translatability*: for all $c \in \mathbb{R}$, $X \in \mathcal{H}$, $\hat{\mathbb{E}}[X + c] = \hat{\mathbb{E}}[X] + c$.
3. *Sub-additivity*: for all X, Y in $\mathcal{H}$, $\hat{\mathbb{E}}[X + Y] \le \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$.

4. *Positive homogeneity*: for all $\lambda \geq 0$, $X \in \mathcal{H}$, $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$.

$\hat{\mathbb{E}}$ is called a linear expectation if 4. is replaced by the homogeneity, namely $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ holds for all $\lambda \in \mathbb{R}$.

The following result is well-known as representation theorem. It is a direct consequence of Hahn–Banach theorem (see Delbaen [**5**], Föllmer and Schied [**9**], or Peng [**24**]).

THEOREM 3.2. *Let $\hat{\mathbb{E}}$ be a sublinear expectation defined on $(\Omega, \mathcal{H})$. Then there exists a family of linear expectations $\{E_\theta : \theta \in \Theta\}$ on $(\Omega, \mathcal{H})$ such that*

$$\hat{\mathbb{E}}[\xi] = \max_{\theta \in \Theta} E_\theta[\xi].$$

A sublinear expectation $\hat{\mathbb{E}}$ on $(\Omega, \mathcal{H})$ is said to be regular if for each sequence $\{\xi_n\}_{n=1}^\infty \subset \mathcal{H}$ such that $\xi_n(\omega) \downarrow 0$, for each ω, we have $\hat{\mathbb{E}}[\xi_n] \downarrow 0$.

REMARK 3.3. If $\hat{\mathbb{E}}$ is regular then from the above representation we have $E_\theta[\xi_n] \downarrow 0$ for each $\theta \in \Theta$. It follows from Daniell–Stone theorem that there exists a unique (σ-additive) probability measure P_θ defined on $(\Omega, \sigma(\mathcal{H}))$ such that

$$E_\theta[\xi] = \int_\Omega \xi(\omega) dP_\theta(\omega), \quad \xi \in \mathcal{H}.$$

3.1. *G*-expectations and *G*-Brownian motion.

Let us recall the definitions of G-Brownian motion and its corresponding G-expectation introduced by Shige Peng (see [**21**], [**22**] and [**23**] for more details). Given a linear space of functions of paths:

$$L_{ip}(\Omega_T) := \{\varphi(\omega(t_1), \ldots, \omega(t_n)) : t_1, \ldots, t_n \in [0, T],\ \varphi \in C_{l,Lip}((\mathbb{R}^d)^n),\ n \in \mathbb{N}\},$$

where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|,$$

for any $x, y \in \mathbb{R}^n$ and for some $C > 0$, $m \in \mathbb{N}$ depending on φ.

Denote by $\mathbb{S}_d$ the space of $d \times d$ symmetric matrices. We are given a function $G : \mathbb{S}_d \mapsto \mathbb{R}$ by

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Theta} \mathrm{tr}[\gamma \gamma^T A], \quad A \in \mathbb{S}_d, \tag{3.1}$$

where Θ is a given non empty, bounded subset of $\mathbb{R}^{d \times d}$. Here $\mathbb{R}^{d \times d}$ is the space of all $d \times d$ matrices. When $d = 1$, we have $G(a) := (1/2)(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$, for $0 \leq \underline{\sigma}^2 \leq \overline{\sigma}^2$. We are also interested in the linear function $G(a) = a/2$ for the case $\underline{\sigma}^2 = \overline{\sigma}^2 = 1$. It's easily seen that the function G satisfies the following monotonicity and sublinearity:

a) $G(A) \geq G(B)$, if $A, B \in \mathbb{S}_d$ and $A \geq B$;
b) $G(A+B) \leq G(A) + G(B)$, $G(\lambda A) = \lambda G(A)$, for each $A, B \in \mathbb{S}_d$ and $\lambda \geq 0$.

For each $\varphi \in C_{l,Lip}(\mathbb{R}^d)$, set

$$N_G[\varphi] := u^{\varphi}(0,0),$$

where u is the solution of the following PDE (G-heat equation) defined on $[0,1) \times \mathbb{R}$:

$$\partial_t u + G(\partial_x^2 u) = 0 \tag{3.2}$$

with the terminal condition $u(1,x) = \varphi(x)$. If $G(A) = (1/2)\mathrm{tr}[A]$, $A \in \mathbb{S}_d$ is linear, $N_G[\cdot]$ is none other than the standard normal distribution. Here we call $N_G[\cdot]$ G-normal distribution.

Generally, for each $\xi(\omega) \in L_{ip}(\Omega_T)$ of the form

$$\xi(\omega) = \varphi(\omega(t_1), \omega(t_2), \ldots, \omega(t_n)), \quad 0 = t_0 < t_1 < \cdots < t_n = T,$$

we define the following conditional G-expectation

$$\mathbb{E}_t^G[\xi] := u_k(t, \omega(t); \omega(t_1), \ldots, \omega(t_{k-1}))$$

for each $t \in (t_{k-1}, t_k]$, $k = 1, \ldots, n$. Here, for each $k = 1, \ldots, n$, $u_k = u_k(t, x; x_1, \ldots, x_{k-1})$ is a function of (t,x) parameterized by $(x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1}$, which is the solution of the G-heat equation defined on $(t_{k-1}, t_k] \times \mathbb{R}$ with terminal conditions

$$u_k(t_k, x; x_1, \ldots, x_{k-1}) = u_{k+1}(t_k, x; x_1, \ldots, x_{k-1}, x), \quad \text{for } k < n$$

and $u_n(t_n, x; x_1, \ldots, x_{n-1}) = \varphi(x_1, \ldots, x_{n-1}, x)$.

The G-expectation of $\xi(\omega)$ is defined by $\mathbb{E}^G[\xi] = \mathbb{E}_0^G[\xi]$.

REMARK 3.4. The above G-heat equation has a unique viscosity solution. We refer to [**4**] for the definition, existence, uniqueness and comparison theorem of this type of parabolic PDEs (see also [**22**] for our specific situation). If G is non-degenerate, i.e., there exists a constant $\beta > 0$ such that $G(A) - G(B) \geq \beta \mathrm{Tr}[A-B]$ for each A, $B \in \mathbb{S}_d$ with $A \geq B$, then the above G-heat equation has a unique $C^{1,2}$-solution (see e.g. [**14**]).

REMARK 3.5. The above G-martingale is

$$u(t,\omega) := \mathbb{E}_t^G[\xi] = \sum_{k=0}^{n-1} u_k(t, \omega(t); \omega(t_1), \ldots, \omega(t_k)) 1_{[t_k, t_{k+1})}(t), \quad t \in [0,T].$$

u in fact the viscosity solution of the path-dependent PDE

$$\mathcal{D}_t u(t,\omega) + \frac{1}{2} G(\mathcal{D}_x^2 u(t,\omega)) = 0, \quad u(T,\omega) = \xi(\omega) \in L_{ip}(\Omega_T).$$

For the linear function $G(A) = (1/2)\mathrm{tr}[A]$, $\mathbb{E}^G$ is just the Wiener probability measure, under which the canonical process $B_t(\omega) = \omega_t$ is a standard Brownian motion. By the construction of G-expectation we can verify that under $\mathbb{E}^G$ the process B_t is still a process with stationary and independent increments in the following sense. We call it a G-Brownian motion.

(SI): For any $0 \le t_0 \le t_1 \le \cdots \le t_n$ and $s > 0$ the random vector $\xi := (B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}})$ is identically distributed with $\eta := (B_{s+t_1} - B_{s+t_0}, \ldots, B_{s+t_n} - B_{s+t_{n-1}})$, i.e., for any $\varphi \in C_{l,Lip}((\mathbb{R}^d)^n)$

$$\mathbb{E}^G[\varphi(\xi)] = \mathbb{E}^G[\varphi(\eta)].$$

(II): For any $0 \le t_0 \le t_1 \le \cdots \le t_n \le s_0 \le s_1 \le \cdots \le s_m$, the random vector $\eta := (B_{s_1} - B_{s_0}, \ldots, B_{s_m} - B_{s_{m-1}})$ is independent from $\xi := (B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}})$, i.e., for any $\varphi \in C_{l,Lip}((\mathbb{R}^d)^{m+n})$

$$\mathbb{E}^G[\varphi(\xi, \eta)] = \mathbb{E}^G[\mathbb{E}^G[\varphi(x, \eta)]|_{x=\xi}].$$

It is easy to check from the properties of the G-heat equation that $\mathbb{E}^G : L_{ip}(\Omega_T) \mapsto \mathbb{R}$ is a *sublinear expectation.*

Due to the above properties we obtain a natural norm $\|\xi\|_{L_G^p} := \mathbb{E}^G[|\xi|^p]^{1/p}$. The completion of $L_{ip}(\Omega_T)$ under $\|\cdot\|_{L_G^p}$ is a Banach space, denoted by $L_G^p(\Omega_T)$. Since $\mathbb{E}_t^G[\xi] : L_{ip}(\Omega_T) \mapsto L_{ip}(\Omega_t)$ is a contracting mapping under $\|\cdot\|_{L_G^1}$, it can be continuously extended to $\mathbb{E}_t^G[\xi] : L_G^1(\Omega_T) \mapsto L_G^1(\Omega_t)$. In particular $\mathbb{E}^G[\cdot]$ is a sublinear expectation on $L_G^1(\Omega_T)$.

Definition 3.6. A process Y with values in $L_G^1(\Omega_T)$ is called a G-martingale if $\mathbb{E}_s^G[Y_t] = Y_s$ for all $0 \le s < t \le T$. If both Y and $-Y$ are G-martingales then Y is called a symmetric G-martingale.

3.2. Elements of $L_G^p(\Omega_T)$ as functions of path.

A very interesting question is how to formulate $u(t, \omega) = \mathbb{E}_t^G[\xi](\omega)$ as a well-defined path-dependent solution of the G-heat equation. For this we need firstly to know how to understand u as a real function defined on $[0, T] \times \Omega_T$.

Denis, Hu, and Peng [**6**] proved that $\mathbb{E}^G$ is regular and obtained the following representation. We denote by $\mathcal{M}_1(\Omega_T)$ the collection of all probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$.

Theorem 3.7 ([**6**]). *There exists a tight subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ such that*

$$\mathbb{E}^G[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all} \quad \xi \in L_{ip}(\Omega_T).$$

$\mathcal{P}$ is called a set that represents $\mathbb{E}^G$.

Remark 3.8. Let W_t be a d-dimensional standard Brownian motion in the probability space $(\Omega^0, \mathcal{F}^0, P^0)$ and let $\mathbb{F} := (\mathcal{F}_t^0)_{t \ge 0}$ be the augmented filtration generated by

$(W_t)_{t\geq 0}$. Denote by $\mathcal{L}_{\mathbb{F}}^{\Theta}$ the set of $\mathbb{F}$-progressively measurable processes with values in $\Theta \subset \mathbb{R}^{d\times d}$, which is just the set in (3.1). [**6**] showed that

$$\mathcal{P}_\Theta := \left\{ P_h \,\middle|\, P_h := P^0 \circ \left(\int_0^\cdot h_s dW_s \right)^{-1}, h \in \mathcal{L}_{\mathbb{F}}^{\Theta} \right\}$$

is a set that represents $\mathbb{E}^G$.

REMARK 3.9. For a subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ that represents $\mathbb{E}^G$, set $\mathbb{E}^{\mathcal{P}}[\xi] := \sup_{P\in\mathcal{P}} E_P[\xi]$ for all $\xi \in L_{ip}(\Omega_T)$ and $c_{\mathcal{P}}(A) = \sup_{P\in\mathcal{P}} P(A)$ for all $A \in \mathcal{B}(\Omega_T)$. Denote by $\bar{\mathcal{P}}$ the closure of $\mathcal{P}$ with respect to the weak topology on $\mathcal{M}_1(\Omega_T)$. Clearly we have $\mathbb{E}^{\bar{\mathcal{P}}}[\xi] = \mathbb{E}^{\mathcal{P}}[\xi]$ for all $\xi \in L_{ip}(\Omega_T)$. For the capacity $c_{\mathcal{P}}$, the following properties hold (see ii) and iii) of Remark 2.7 in [**31**]).

Let $\mathcal{P}, \mathcal{P}'$ be two subsets that represent $\mathbb{E}^G$. Then

i) $c_{\mathcal{P}}(O) = c_{\mathcal{P}'}(O)$ for any open set $O \subset \Omega_T$;
ii) $c_{\bar{\mathcal{P}}}(A) = c_{\bar{\mathcal{P}}'}(A)$ for set $A \in \mathcal{B}(\Omega_T)$.

So the capacity $c_{\bar{\mathcal{P}}}$ is uniquely determined by the expectation $\mathbb{E}^G$, which is in turn uniquely determined by the function G. We denote it by c_G or simply by c.

Set $\mathcal{N}_T^c = \{A \subset \Omega_T |$ there is $B \in \mathcal{B}(\Omega_T)$ s.t. $A \subset B$ and $c(B) = 0\}$ and $\mathcal{B}_t^c = \sigma\{\mathcal{B}(\Omega_t), \mathcal{N}_T^c\}$, $t \in [0,T]$, which can be equivalently defined as

$$\mathcal{B}_t^c = \{A \subset \Omega_T | \text{ there is } B \in \mathcal{B}(\Omega_t) \text{ s.t. } A \triangle B \in \mathcal{N}_T^c\}.$$

For $A \in \mathcal{B}_T^c$, let $B_i, N_i \in \mathcal{B}(\Omega_T)$ s.t. $A\triangle B_i \subset N_i$ and $c(N_i) = 0$, $i = 1, 2$. Then

$$|c(B_1) - c(B_2)| \leq c(B_1 \triangle B_2) \leq c(N_1) + c(N_2) = 0.$$

So we can defined $c(A) := c(B_1)$. For $A \in \mathcal{B}_T^c$, we call A a polar set if $c(A) = 0$.

In the sequel, a random variable is assumed to be $\mathcal{B}_T^c$-measurable unless explicitly stated otherwise. We write $\xi \in \mathcal{B}_t^c$ if ξ is $\mathcal{B}_t^c$-measurable.

Let $\mathcal{P}$ be a weakly compact set that represents $\mathbb{E}^G$. Set $\mathbb{L}_G^p(\Omega_T) := \{\xi | \ \mathbb{E}^{\mathcal{P}}[|\xi|^p] < \infty\}$. So $L_G^p(\Omega_T)$ can be considered as the closure of $L_{ip}(\Omega_T)$ in the space $\mathbb{L}_G^p(\Omega_T)$, which, however, is strictly bigger than $L_G^p(\Omega_T)$ if G is nonlinear.

DEFINITION 3.10. A map $\xi : \Omega_T \to \mathbb{R}$ is said to be quasi-continuous with respect to c if for any $\varepsilon > 0$, there exists an open set O with $c(O) < \varepsilon$ such that $\xi|_{O^c}$ is continuous.

REMARK 3.11. i) A quasi-continuous mapping ξ is $\mathcal{B}_T^c$-measurable. Actually, for $n \in \mathbb{N}$, choose open set O_n such that $c(O_n) < 1/2^n$ and $\xi_n := \xi|_{O_n^c}$ is continuous. Then $\bar{\xi} := \overline{\lim}_{n\to\infty} \xi^n$ is $\mathcal{B}_T^c$-measurable and $[\bar{\xi} \neq \xi] \subset \cap_{m=1}^\infty \cup_{n=m}^\infty O_n$, which is a polar set.

ii) For a random variable ξ, if there is a quasi-continuous random variable η on Ω_t for some $t \in [0,T]$ such that $\xi = \eta$ except on a polar set, we say ξ has a quasi-continuous version on Ω_t. However, in this case, we don't know whether ξ is also quasi-continuous on Ω_t.

[6] gave a characterization for the spaces $L_G^p(\Omega_T)$, $p \geq 1$.

THEOREM 3.12. *For $p \geq 1$*

$$L_G^p(\Omega_t) = \Big\{ \xi \;\Big|\; \xi \text{ has a q.c. version on } \Omega_t,\; \lim_{n\to\infty} \mathbb{E}^G\big[|\xi|^p 1_{\{|\xi|>n\}}\big] = 0 \Big\}.$$

REMARK 3.13. i) Clearly, the canonical process $B_t(\omega) := \omega(t)$ belongs to $L_G^p(\Omega_t)$, which is called a G-Brownian motion in this sublinear expectation space $(\Omega, L_G^p(\Omega), \mathbb{E}^G)$.

ii) For $\xi, \eta \in L_G^1(\Omega_T)$, we say $\xi = \eta$ (resp. $\xi \geq \eta$), c-q.s., if $\xi(\omega) = \eta(\omega)$ ($\xi(\omega) \geq \eta(\omega)$) except on a polar set $N \subset \Omega$, which is equivalent to

$$\xi = \eta, \; P\text{-a.s., for all } P \in \mathcal{P},$$

where $\mathcal{P}$ is a set (NOT necessarily weakly compact) that represents $\mathbb{E}^G$. In the rest of the paper, except further specifications, all equalities and inequalities hold c-quasi surely.

3.3. Quadratic variation process.

In the rest of this section we only consider the one-dimensional G-Brownian motion for simplicity of notations. We assume from now on that G is non degenerate. For 1-dimensional case this condition is $\underline{\sigma} > 0$.

The quadratic variation process of a G-Brownian motion is a particularly important process. Its definition is quite classical: Let π_t^N, $N = 1, 2, \ldots$, be a sequence of partitions of $[0, t]$ such that $|\pi_t^N| \to 0$. We can easily prove that, in the space $L_G^2(\Omega)$,

$$\sum_{j=0}^{N-1} \big(B_{t_{j+1}^N} - B_{t_j^N}\big)^2$$

is a Cauchy sequence, whose limit, denoted by $\langle B\rangle_t$, is independent of the choice of the partitions π_t^N. From the above construction, $\{\langle B\rangle_t\}_{t\geq 0}$ is an increasing process with $\langle B\rangle_0 = 0$. We call it the *quadratic variation process* of the G-Brownian motion B. It characterizes the statistical uncertainty of the G-Brownian motion B. It is important to keep in mind that $\langle B\rangle_t$ is not a deterministic process unless $\underline{\sigma}^2 = \overline{\sigma}^2$, i.e., when B is a classical Brownian motion.

A very interesting point of the quadratic variation process $\langle B\rangle$ is, just like the G-Brownian motion B itself, the increment $\langle B\rangle_{t+s} - \langle B\rangle_s$ is independent from $(\langle B\rangle_{t_1}, \ldots, \langle B\rangle_{t_n})$ for all $t_1, \ldots, t_n \in [0, s]$ and identically distributed with $\langle B\rangle_t$. Furthermore, the distribution of $\langle B\rangle_t$ is given by $\mathbb{E}^G[\varphi(\langle B\rangle_t)] = \max_{v\in[\underline{\sigma}^2,\overline{\sigma}^2]} \varphi(vt)$ and we can also prove that, c-quasi-surely,

$$\underline{\sigma}^2 t \leq \langle B\rangle_{t+s} - \langle B\rangle_s \leq \overline{\sigma}^2 t.$$

3.4. Itô integral of G-Brownian motion.

Itô integral with respect to a G-Brownian motion is defined in an analogous way as the classical one, but in a language of "c-quasi-surely", or in other words, under L_G^p-norm.

Definition 3.14. For each $p \geq 1$, we denote by $M_G^p(0,T)$ the completion of the space of $M^0(0,T)$ under the norm

$$\|\eta\|_{M_G^p} := \left\{ \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p dt \right] \right\}^{1/p},$$

and by $H_G^p(0,T)$ the completion of the space of $M^0(0,T)$ under the norm

$$\|\eta\|_{H_G^p} := \left[\hat{\mathbb{E}}\left\{ \int_0^T |\eta_t|^2 dt \right\}^{p/2} \right]^{1/p}.$$

Just as the classical Itô's calculus, for an $\eta \in M^0(0,T)$ with the form of (2.1), we define its Itô integral by

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_{t_j} \big(B_{t_{j+1}} - B_{t_j} \big).$$

It is easy to check that $I : M^0(0,T) \longmapsto L_G^p(\Omega_T)$ is a linear mapping and, by Burkholder–Davis–Gundy inequality, there are universal constants $C_p > c_p$ such that

$$\begin{aligned}
\mathbb{E}^G[|I(\eta)|^p] &= \sup_{P \in \mathcal{P}_\Theta} E_P[|I(\eta)|^p] \\
&\leq C_p \sup_{P \in \mathcal{P}_\Theta} E_P\left[\left(\int_0^T |\eta_s|^2 d\langle B\rangle_s \right)^{p/2} \right] \leq C_{p,G}^p \|\eta\|_{H_G^p}^p; \qquad (3.3) \\
\mathbb{E}^G[|I(\eta)|^p] &= \sup_{P \in \mathcal{P}_\Theta} E_P[|I(\eta)|^p] \\
&\geq c_p \sup_{P \in \mathcal{P}_\Theta} E_P\left[\left(\int_0^T |\eta_s|^2 d\langle B\rangle_s \right)^{p/2} \right] \geq c_{p,G}^p \|\eta\|_{H_G^p}^p. \qquad (3.4)
\end{aligned}$$

Thus the linear mapping I can be continuously extended to $H_G^p(0,T)$. Moreover, this extension of I satisfies *Burkholder-Davis-Gundy inequality*

$$c_{p,G} \|\eta\|_{H_G^p} \leq \|I(\eta)\|_{L_G^p} \leq C_{p,G} \|\eta\|_{H_G^p}, \quad \eta \in H_G^p(0,T).$$

Therefore we can define, for a fixed $\eta \in H_G^p(0,T)$, the stochastic integral

$$\int_0^T \eta(s) dB_s := I(\eta).$$

We list some main properties of the Itô integral with respect to a G-Brownian motion.

Proposition 3.15. *Let $\eta, \theta \in H_G^p(0,T)$ and $0 \leq s \leq r \leq t \leq T$. Then we have*

(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$,

(ii) $\int_s^t (\alpha\eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u$, *if* α *is bounded and in* $L_G^1(\Omega_s)$,

(iii) $\hat{\mathbb{E}}_t^G[X + \int_t^T \eta_u dB_u] = \hat{\mathbb{E}}_t^G[X]$, *for* $X \in L_G^1(\Omega_T)$.

Now we shall list some useful results which can be derived directly from the classical ones. Consider an Itô process

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \eta_s^\nu d\langle B\rangle_s + \int_0^t \beta_s^\nu dB_s.$$

PROPOSITION 3.16 (Itô's formula). *Let* α^ν, $\eta^\nu \in M_G^1(0,T)$ *and* $\beta^\nu \in M_G^2(0,T)$, $\nu = 1, \ldots, n$. *Then for each* $t \geq 0$ *and each function* Φ *in* $C^{1,2}([0,t]\times\mathbb{R}^n)$, *with polynomial growth derivatives, we have*

$$\begin{aligned}\Phi(t, X_t) - \Phi(s, X_s) = \sum_{\nu=1}^n \int_s^t \partial_{x^\nu}\Phi(u, X_u)\beta_u^\nu dB_u + \int_s^t \big[\partial_u \Phi(u, X_u) + \partial_{x_\nu}\Phi(u, X_u)\alpha_u^\nu\big] du \\ + \int_s^t \left[\sum_{\nu=1}^n \partial_{x^\nu}\Phi(u, X_u)\eta_u^\nu + \frac{1}{2}\sum_{\nu,\mu=1}^n \partial^2_{x^\mu x^\nu}\Phi(u, X_u)\beta_u^\mu\beta_u^\nu\right] d\langle B\rangle_u.\end{aligned}$$

PROOF. Note that both sides of the equality are well-defined in $L_G^1(\Omega_T)$ and, by classical Itô's formula, the equality holds P-a.s., for each $P \in \mathcal{P}_\Theta$. So the equality holds q.s. □

Let $X_t = \int_0^t Z_s dB_s$, $Z \in H_G^p(0,T)$ for some $p \geq 1$. By arguments similar to those in (3.3) and (3.4), we get Doob's maximal inequality for the process X.

PROPOSITION 3.17 (Doob's Maximal Inequality). *Assume* $p > 1$,

$$\Big\| \sup_{t\in[0,T]} |X_t| \Big\|_{L_G^p} \leq \frac{p}{p-1} \|X_T\|_{L_G^p}.$$

3.5. On the spaces $M_G^p(0,T)$ and $H_G^p(0,T)$.

Let $\mathcal{P}$ be a weakly compact set that represents $\mathbb{E}^G$. Set

$$\mathbb{M}_G^p(0,T) := \left\{\eta \;\middle|\; \eta \text{ is a } \mathcal{B}_t^c\text{-progressively measurable process with } \mathbb{E}^{\mathcal{P}}\left[\int_0^T |\eta_s|^p ds\right] < \infty\right\},$$

$$\mathbb{H}_G^p(0,T) := \left\{\eta \;\middle|\; \eta \text{ is a } \mathcal{B}_t^c\text{-progressively measurable process with } \mathbb{E}^{\mathcal{P}}\left[\left(\int_0^T |\eta_s|^2 ds\right)^{p/2}\right] < \infty\right\}.$$

Then $M_G^p(0,T)$ (resp. $H_G^p(0,T)$) can be considered as the closure of $M^0(0,T)$ in the space $\mathbb{M}_G^p(0,T)$ (resp. $\mathbb{H}_G^p(0,T)$).

REMARK 3.18. Generally $\mathbb{M}_G^p(0,T)$ (resp. $\mathbb{H}_G^p(0,T)$) is strictly bigger than $M_G^p(0,T)$ (resp. $H_G^p(0,T)$) if G is non linear. Actually, as is pointed in [**33**] and [**34**], obviously we have

$$\theta_s = \overline{\lim}_{\delta\downarrow 0}\frac{\langle B\rangle_s - \langle B\rangle_{s-\delta}}{\delta} \in \mathbb{M}_G^p(0,T),$$

but θ does NOT belong to $M_G^p(0,T)$, where $\langle B\rangle$ is the quadratic variation process of the one dimensional G-Brownian motion B.

4. Sobolev spaces on path space under *G*-expectation.

In the above framework of P-Sobolev space $W_{\mathcal{A}}^{1/2,1;p}(0,T)$ based on the Wiener probability measure P, only the derivatives $\mathcal{A} = \mathcal{D}_t + (1/2)\mathcal{D}_x^2$ and $\mathcal{D}_x$ are well-defined and well-integrated. This prevents us from formulating and solving a fully nonlinear PPDE. This is mainly due to the fact that the process $\langle B\rangle_t$ coincides with t in a Wiener probability space. In this section we replace the linear Wiener expectation E_P by the sublinear G-expectation $\mathbb{E}^G$ and thus introduce the corresponding G-expectation weighted Sobolev space, or "G-Sobolev space". The derivatives $\mathcal{D}_t$, $\mathcal{D}_x$ and $\mathcal{D}_x^2$ are all well-defined and well-integrated in this framework. We then formulate a type of fully nonlinear PPDEs in the G-Sobolev space and identify them with a type of backward SDEs driven by G-Brownian motion.

4.1. *G*-Sobolev spaces of path functions.

In the G-expectation space, by G-Itô's formula, for $u \in \mathcal{C}^\infty(0,\infty)$ we immediately obtain the following decomposition.

PROPOSITION 4.1. *For each given $u \in \mathcal{C}^\infty(0,\infty)$ we have,*

$$\begin{aligned} u(t,\omega) &= u(0,\omega) + \int_0^t \mathcal{D}_s u(s,\omega)ds + \int_0^t \mathcal{D}_x u(s,\omega)dB_s + \frac{1}{2}\int_0^t \mathcal{D}_x^2 u(s,\omega)d\langle B\rangle_s \\ &= u(0,\omega) + \int_0^t \mathcal{A}_G u(s,\omega)ds + \int_0^t \mathcal{D}_x u(s,\omega)dB_s + K_t, \end{aligned}$$

where

$$\mathcal{A}_G u(s,\omega) := \mathcal{D}_s u(s,\omega) + G(\mathcal{D}_x^2 u(s,\omega)),$$

and K_t is a non-increasing G-martingale:

$$K_t := \frac{1}{2}\int_0^t \mathcal{D}_x^2 u(s,\omega)d\langle B\rangle_s - \int_0^t G(\mathcal{D}_x^2 u(s,\omega))ds.$$

Definition 4.2. 1) For $u \in \mathcal{C}^\infty(0,T)$, we set

$$\|u\|^p_{S^p_G} = \mathbb{E}^G\Big[\sup_{s\in[0,T]} |u_s|^p\Big].$$

We denote by $S^p_G(0,T)$ the completion of $u \in \mathcal{C}^\infty(0,T)$ with respect to the norm $\|\cdot\|_{S^p_G}$.

2) For $u \in \mathcal{C}^\infty(0,T)$, we set

$$\|u\|^p_{W^{1,2;p}_G} = \mathbb{E}^G\Big[\sup_{s\in[0,T]} |u_s|^p + \int_0^T (|\mathcal{D}_s u_s|^p + |\mathcal{D}^2_x u_s|^p) ds + \Big\{\int_0^T |\mathcal{D}_x u_s|^2 ds\Big\}^{p/2}\Big].$$

To define the G-Sobolev space, a key point is to obtain the uniqueness of the decomposition for G-Itô processes, which was actually solved by Song (2012) in the one-dimensional G-expectation space and by Peng, Song and Zhang (2014) for the multi-dimensional case.

For the simplification of notations, in the rest of this paper we only consider the 1-dimensional G-expectation space with $\bar{\sigma}^2 := \mathbb{E}^G[B_1^2] > \underline{\sigma}^2 := -\mathbb{E}^G[-B_1^2]$.

Lemma 4.3. *If*

$$u(t,\omega) = \int_0^t \zeta(s,\omega)ds + \int_0^t v(s,\omega)dB_s + \frac{1}{2}\int_0^t w(s,\omega)d\langle B\rangle_s = 0, \quad t \in [0,T],$$

with $\zeta, w \in M^p_G(0,T)$ and $v \in H^p_G(0,T)$, then we have $\zeta = v = w = 0$.

Proof. By the uniqueness of the decomposition for continuous semimartingales we have $v = 0$ and $\int_0^t \zeta(s,\omega)ds + (1/2)\int_0^t w(s,\omega)d\langle B\rangle_s = 0$. By Corollary 3.5 in Song (2012) we conclude that $\zeta = w = 0$. □

Proposition 4.4. *The norm $\|\cdot\|_{W^{1,2;p}_G}$ is closable in the space $S^p_G(0,T)$: Let $u^n \in \mathcal{C}^\infty(0,T)$ be a Cauchy sequence with respect to the norm $\|\cdot\|_{W^{1,2;p}_G}$. If $\|u^n\|_{S^p_G} \to 0$, we have $\|u^n\|_{W^{1,2;p}_G} \to 0$.*

Proof. The limit u of the Cauchy sequence $\{u^n\}_{n=1}^\infty$ under $\|\cdot\|_{W^{1,2;p}_G}$ is of the following form

$$u(t,\omega) = u_0 + \int_0^t \zeta(s,\omega)ds + \int_0^t v(s,\omega)dB_s + \frac{1}{2}\int_0^t w(s,\omega)d\langle B\rangle_s,$$
$$u_0 \in \mathbb{R}, \quad \zeta, w \in M^p_G(0,T), \quad v \in H^p_G(0,T).$$

But since $u \equiv 0$, then by Lemma 4.3, we must have $u_0 = 0$, $\zeta = w \equiv 0$, $v \equiv 0$. □

Denote by $W^{1,2;p}_G(0,T)$ the completion of $\mathcal{C}^\infty(0,T)$ with respect to the norm $\|\cdot\|_{W^{1,2;p}_G}$. By the above proposition, $W^{1,2;p}_G(0,T)$ is a subspace of $S^p_G(0,T)$. Now the differential operators $\mathcal{D}_t$, $\mathcal{D}^2_x$ (resp. $\mathcal{D}_x$), can be all continuously extended as continuous linear operators from $W_G^{1,2;p}(0,T)$ to $M_G^p(0,T)$ (resp. to $H_P^p(0,T)$).

THEOREM 4.5. *Assume* $u \in S_G^p(0,T)$. *Then the following two conditions are equivalent*:

(i) $u \in W_G^{1,2;p}(0,T)$;
(ii) *There exists* $u_0 \in \mathbb{R}$, $\zeta, w \in M_G^p(0,T)$ *and* $v \in H_G^p(0,T)$ *such that*

$$u(t,\omega) = u_0 + \int_0^t \zeta(s,\omega)ds + \int_0^t v(s,\omega)dB_s + \frac{1}{2}\int_0^t w(s,\omega)d\langle B\rangle_s. \qquad (4.1)$$

Moreover, we have

$$\mathcal{D}_t u(t,\omega) = \zeta(t,\omega), \quad \mathcal{D}_x u(t,\omega) = v(t,\omega), \quad \mathcal{D}_x^2 u(t,\omega) = w(t,\omega).$$

REMARK 4.6. Just like the remark for Theorem 2.9, the above theorem also gives us a general path-dependent Itô's formula: for each Itô process u of the form (4.1) with any given $\zeta, w \in M_G^p(0,T)$ and $v \in H_G^p(0,T)$, we have

$$u(t,\omega) = u_0 + \int_0^t \mathcal{D}_s u(s,\omega)ds + \int_0^t \mathcal{D}_x u(s,\omega)dB_s + \frac{1}{2}\int_0^t \mathcal{D}_x^2 u(s,\omega)d\langle B\rangle_s. \qquad (4.2)$$

PROOF. (i) $\Longrightarrow$ (ii) is obvious. For the part (ii) $\Longrightarrow$ (i) it suffices to prove it for the case that ζ, v, w in (4.1) are cylinder step processes. Set $t_k^n = kT/2^n$ and

$$Q^n(t,\omega) := \sum_{k=0}^{2^n-1} \left(B_{t_{k+1}^n \wedge t} - B_{t_k^n \wedge t}\right)^2 = \int_0^t \lambda^n(s,\omega)dB_s + \langle B\rangle_t,$$

where $\lambda^n(t,\omega) = \sum_{k=0}^{2^n-1} 2(B_t - B_{t_k})1_{(t_k,t_{k+1}]}(t)$. Set

$$\begin{aligned} u^n(t,\omega) &:= u(0,\omega) + \int_0^t \zeta(s,\omega)ds + \int_0^t v(s,\omega)dB_s + \frac{1}{2}\int_0^t w_s dQ^n(s,\omega) \\ &= u(0,\omega) + \int_0^t \zeta(s,\omega)ds + \int_0^t \left(v(s,\omega) + \frac{1}{2}w(s,\omega)\lambda^n(s,\omega)\right)dB_s \\ &\quad + \int_0^t \frac{1}{2}w(s,\omega)d\langle B\rangle_s. \end{aligned}$$

Clearly u^n belongs to $\mathcal{C}^\infty(0,T)$. By Proposition 4.1 and Lemma 4.3, we have

$$\mathcal{D}_t u^n(t,\omega) = \zeta(t,\omega), \quad \mathcal{D}_x u^n(t,\omega) = v(t,\omega) + \frac{1}{2}w(t,\omega)\lambda^n(t,\omega), \quad \mathcal{D}_x^2 u^n(t,\omega) = w(t,\omega).$$

It's easy to check that $\mathbb{E}^G[(\int_0^T |\mathcal{D}_x u^n(t,\omega) - v(t,\omega)|^2 dt)^{p/2}] \to 0$. So u belongs to $W_G^{1,2;p}(0,T)$ with

$$\mathcal{D}_t u(t,\omega) = \zeta(t,\omega), \quad \mathcal{D}_x u(t,\omega) = v(t,\omega), \quad \mathcal{D}_x^2 u(t,\omega) = w(t,\omega). \qquad \square$$

PROPOSITION 4.7. *For each $u, v \in W_G^{1,2;p}(0,T)$, we have $c_1 u + c_2 v \in W_G^{1,2;p}(0,T)$ and*

$$\mathcal{D}_t(c_1 u + c_2 v) = c_1 \mathcal{D}_t u + c_2 \mathcal{D}_t v,$$
$$\mathcal{D}_x(c_1 u + c_2 v) = c_1 \mathcal{D}_x u + c_2 \mathcal{D}_x v, \quad \mathcal{D}_x^2(c_1 u + c_2 v) = c_1 \mathcal{D}_x^2 u + c_2 \mathcal{D}_x^2 v.$$

Moreover, if their product uv is also in $W_G^{1,2;p}(0,T)$, then

$$\mathcal{D}_t(uv) = v\mathcal{D}_t u + u\mathcal{D}_t v, \quad \mathcal{D}_x(uv) = v\mathcal{D}_x u + u\mathcal{D}_x v,$$
$$\mathcal{D}_x^2(uv) = v\mathcal{D}_x^2 u + u\mathcal{D}_x^2 v + 2\mathcal{D}_x u \mathcal{D}_x v.$$

Applying Itô's formula for uv, the proof follows directly from Theorem 4.5.

REMARK 4.8. 1) By Theorem 4.5 we note that the equality

$$\mathcal{D}_x^2 u(t,\omega) = \mathcal{D}_x(\mathcal{D}_x u)(t,\omega)$$

does NOT always hold for general $u \in W_G^{1,2;p}(0,T)$ although it holds for $u \in \mathcal{C}^\infty(0,T)$. Let us see how this can happen from a simple example: Let $u(t,\omega) = \langle B\rangle_t$, $t \in [0,1]$. By the definition we have

$$\mathcal{D}_t u(t,\omega) = 0, \quad \mathcal{D}_x u(t,\omega) = 0, \quad \mathcal{D}_x^2 u(t,\omega) = 2.$$

Set $t_k^n = k/2^n$ and $u^n(t,\omega) = \sum_{k=0}^{2^n-1}(B_{t_{k+1}^n \wedge t} - B_{t_k^n \wedge t})^2$. By the definition we have

$$\mathcal{D}_t u^n(t,\omega) = 0, \quad \mathcal{D}_x u^n(t,\omega) = \sum_k 2(B_t - B_{t_k})1_{(t_k,t_{k+1}]}(t), \quad \mathcal{D}_x^2 u^n(t,\omega) = 2.$$

It is easily seen that $u^n \to u$ in $W_G^{1,2;2}(0,T)$. Particularly, $\mathcal{D}_x u^n \to \mathcal{D}_x u \equiv 0$ in $H_G^2(0,T)$. However,

$$\mathcal{D}_x(\mathcal{D}_x u^n)(t,\omega) = \mathcal{D}_x^2 u^n(t,\omega) = 2.$$

It does NOT converge to

$$\mathcal{D}_x(\mathcal{D}_x u)(t,\omega) = 0.$$

2) Compared to Theorem 2.9, here the derivatives $\mathcal{D}_t u$, $\mathcal{D}_x u$, $\mathcal{D}_x^2 u$ can be distinguished clearly.

4.2. Backward SDEs driven by *G*-Brownian motion.

In this section we show that a fully nonlinear backward stochastic differential equation is in fact the corresponding fully nonlinear path dependent PDE.

4.2.1. One-to-one correspondence.

Let us consider backward SDEs driven by G-Brownian motion of the following from: to find $Y \in S_G^p(0,T)$, $Z \in H_G^p(0,T)$, $\eta \in M_G^p(0,T)$ such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \eta_s)ds - \int_t^T Z_s dB_s - (K_T - K_t), \tag{4.3}$$

where $K_t = (1/2)\int_0^t \eta_s d\langle B\rangle_s - \int_0^t G(\eta_s)ds$, $f : [0,T]\times\mathbb{R}\times\mathbb{R}^d \times \mathbb{S}(d) \mapsto \mathbb{R}$ is a given function and $\xi \in L_G^p(\Omega_T)$ is a given random variable.

Similar to the framework of P-Sobolev space in Subsection 2.3, the corresponding problem of path dependent PDEs is to find a path-dependent $u \in W_G^{1,2;p}(0,T)$ such that

$$\begin{aligned} \mathcal{D}_t u + G(\mathcal{D}_x^2 u) + f(t, u, \mathcal{D}_x u, \mathcal{D}_x^2 u) = 0, \quad t \in (0,T], \\ u(T,\omega) = \xi(\omega). \end{aligned} \tag{4.4}$$

We call u a $W_G^{1,2;p}$-solution of the path dependent PDE (4.4).

ASSUMPTION 2. We assume that $f(t,\omega,Y_t,Z_t,\eta_t) \in M_G^p(0,T)$ for any $(Y,Z,\eta) \in S_G^p(0,T) \times H_G^p(0,T) \times M_G^p(0,T)$.

THEOREM 4.9. *Let (Y,Z,η) be a solution to the backward SDE* (4.3). *Then we have $u(t,\omega) := Y_t(\omega) \in W_G^{1,2;p}(0,T)$ with $\mathcal{D}_x u(t,\omega) = Z_t(\omega)$ and $\mathcal{D}_x^2 u(t,\omega) = \eta_t(\omega)$.*

Moreover, Given a $u \in W_G^{1,2;p}(0,T)$, the following conditions are equivalent:

(i) $(u, \mathcal{D}_x u, \mathcal{D}_x^2 u)$ *is a solution to the backward SDE* (4.3);
(ii) u *is a $W_G^{1,2;p}$-solution to the path dependent PDE* (4.4).

PROOF. (i) $\Rightarrow$ (ii). Let (Y,Z,η) be a solution to the backward SDE (4.3):

$$Y_t = Y_0 - \int_0^t [f(s,Y_s,Z_s,\eta_s) + G(\eta_s)]ds + \int_0^t Z_s dB_s + \frac{1}{2}\int_0^t \eta_s d\langle B\rangle_s,$$

and let $u(t,\omega) := Y_t(\omega)$. By Theorem 4.5, we have $u \in W_G^{1,2;p}(0,T)$ with $\mathcal{D}_x u = Z$, $\mathcal{D}_x^2 u = \eta$ and

$$\mathcal{D}_t u + G(\mathcal{D}_x^2 u) + f(t, u, \mathcal{D}_x u, \mathcal{D}_x^2 u) = 0.$$

(ii) $\Rightarrow$ (i). If u is a $W_G^{1,2;p}$-solution to the path dependent PDE (4.4), we have, by Theorem 4.5,

$$\begin{aligned} u(t) = u_0 - \int_0^t \left[G(\mathcal{D}_x^2 u(s)) + f(s, u, \mathcal{D}_x u(s), \mathcal{D}_x^2 u(s))\right]ds \\ + \int_0^t \mathcal{D}_x u(s) dB_s + \frac{1}{2}\int_0^t \mathcal{D}_x^2 u(s) d\langle B\rangle_s \end{aligned}$$

$$
\begin{aligned}
&= \xi + \int_t^T f\big(s, u, \mathcal{D}_x u(s), \mathcal{D}_x^2 u(s)\big) ds - \int_t^T \mathcal{D}_x u(s) dB_s \\
&\quad - \Big(\frac{1}{2}\int_t^T \mathcal{D}_x^2 u(s) d\langle B\rangle_s - \int_t^T G(\mathcal{D}_x^2 u(s)) ds\Big). \qquad \square
\end{aligned}
$$

4.2.2. Solutions of path dependent PDEs defined by G-BSDEs.

Now we consider a typical case of the path dependent PDE (4.4): f *is independent of* $\mathcal{D}_x^2 u$.

$$
\begin{aligned}
\mathcal{D}_t u + G(\mathcal{D}_x^2 u) + f(t, u, \mathcal{D}_x u) = 0, \quad t \in [0, T), \\
u(T, \omega) = \xi(\omega).
\end{aligned}
\tag{4.5}
$$

Let $u \in W_G^{1,2;p}(0,T)$ be a solution to the path dependent PDE (4.5). By Theorem 4.9, the processes

$$
Y_t := u(t,\omega), \quad Z_t := \mathcal{D}_x u(t,\omega), \quad K_t := \frac{1}{2}\int_0^t \mathcal{D}_x^2 u(s,\omega) d\langle B\rangle_s - \int_0^t G(\mathcal{D}_x^2 u(s,\omega)) ds
$$

satisfy the following backward SDE:

$$
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t), \tag{4.6}
$$

which is a type of BSDE driven by G-Brownian motion (G-BSDE) studied in [**12**] (see Appendix).

Let (Y, Z, K) be a solution of backward SDE (4.6) considered in Hu et al. (2014). Notice that, although we have many interesting examples, but to give reasonable conditions on ξ and f under which Y is in the Sobolev space $W_G^{1,2;p}(0,T)$ is still a very interesting and challenging problem.

In Section 5, we shall formulate $u = Y$ as the unique solution in a Sobolev space $W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$, which can be considered as the fully nonlinear counterpart of the P-Sobolev space $W_{\mathcal{A}}^{1/2,1;p}(0,T)$ introduced in Subsection 2.3 based on the Wiener probability P.

4.2.3. Examples and applications.

EXAMPLE 4.10. Let $\eta \in M_G^p(0,T)$. To find $u \in W_G^{1,2;p}(0,T)$ such that

$$
\begin{aligned}
\mathcal{D}_t u + G(\mathcal{D}_x^2 u + \eta_t) = 0, \\
u(T, \omega) = 0.
\end{aligned}
\tag{4.7}
$$

Assume that $u \in W_G^{1,2;p}(0,T)$ is a solution to (4.7). Then

$$
u(t,\omega) = \mathbb{E}_t^G\Big[\frac{1}{2}\int_t^T \eta_s d\langle B\rangle_s\Big].
$$

In fact,

$$u_t = -\int_t^T \mathcal{D}_s u_s ds - \int_t^T \mathcal{D}_x u_s dB_s - \frac{1}{2}\int_t^T \mathcal{D}_x^2 u_s d\langle B\rangle_s$$
$$= -(M_T - M_t) + \frac{1}{2}\int_t^T \eta_s d\langle B\rangle_s,$$

where $M_t := \int_0^t \mathcal{D}_x u_s dB_s + (1/2)\int_0^t (\mathcal{D}_x^2 u_s + \eta_s) d\langle B\rangle_s - \int_0^t G(\mathcal{D}_x^2 u_s + \eta_s) ds$ is a G-martingale. So

$$u_t = \mathbb{E}_t^G\left[\frac{1}{2}\int_t^T \eta_s d\langle B\rangle_s\right].$$

EXAMPLE 4.11. We consider the following problem for a given $\eta \in M_G^p(0,T)$: to find a solution $v \in W_G^{1,2;p}(0,T)$ of the following path dependent PDE:

$$\begin{aligned} \mathcal{D}_t v + G^\eta(\mathcal{D}_x^2 v) = 0, \\ v(T,\omega) = 0, \end{aligned} \tag{4.8}$$

where $G^\eta(\zeta_s) = (1/2)[G(\zeta_s + \eta_s) + G(\zeta_s - \eta_s)]$.

Assume that $v \in W_G^{1,2;p}(0,T)$ is a solution to (4.8). Then

$$v_t = \limsup_{n\to\infty} \mathbb{E}_t^G\left[\frac{1}{2}\int_t^T \delta_n(s)\eta_s d\langle B\rangle_s\right],$$

where $\delta_n(s) = \sum_{i=0}^{n-1}(-1)^i 1_{(iT/n,(i+1)T/n]}$.

Actually,

$$v_t = -\int_t^T \mathcal{D}_s v_s ds - \int_t^T \mathcal{D}_x v_s dB_s - \frac{1}{2}\int_t^T \mathcal{D}_x^2 v_s d\langle B\rangle_s$$
$$= -\int_t^T \mathcal{D}_x v_s dB_s - \frac{1}{2}\int_t^T (\mathcal{D}_x^2 v_s + \delta_n(s)\eta_s) d\langle B\rangle_s + \int_t^T G^\eta(\mathcal{D}_x^2 v_s) ds$$
$$+ \frac{1}{2}\int_t^T \delta_n(s)\eta_s d\langle B\rangle_s.$$

So

$$v_t + \limsup_{n\to\infty} \mathbb{E}_t^G\left[\frac{1}{2}\int_t^T (\mathcal{D}_x^2 v_s + \delta_n(s)\eta_s) d\langle B\rangle_s - \int_t^T G^\eta(\mathcal{D}_x^2 v_s) ds\right]$$
$$= \limsup_{n\to\infty} \mathbb{E}_t^G\left[\frac{1}{2}\int_t^T \delta_n(s)\eta_s d\langle B\rangle_s\right].$$

Noting that

$$\limsup_{n\to\infty} \mathbb{E}_t^G\left[\frac{1}{2}\int_t^T (\mathcal{D}_x^2 v_s + \delta_n(s)\eta_s)d\langle B\rangle_s - \int_t^T G^\eta(\mathcal{D}_x^2 v_s)ds\right] = 0,$$

we get

$$v_t = \limsup_{n\to\infty} \mathbb{E}_t^G\left[\frac{1}{2}\int_t^T \delta_n(s)\eta_s d\langle B\rangle_s\right].$$

EXAMPLE 4.12. Let $\eta \in M_G^p(0,T)$ and $\varepsilon \in [0, (\overline{\sigma}^2 - \underline{\sigma}^2)/2]$. To find $w \in W_G^{1,2;p}(0,T)$ such that

$$\begin{aligned} \mathcal{D}_t w + G_\varepsilon(\mathcal{D}_x^2 w) + \frac{1}{2}\eta_t &= 0, \\ w(T,\omega) &= 0, \end{aligned} \tag{4.9}$$

where $G_\varepsilon(a) = (1/2)[(\overline{\sigma}^2 - \varepsilon)a^+ - (\underline{\sigma}^2 + \varepsilon)a^-]$.

Assume that $w \in W_G^{1,2;p}(0,T)$ is a solution to (4.9). Then

$$w(t,\omega) = \mathbb{E}_t^{G_\varepsilon}\left[\frac{1}{2}\int_t^T \eta_s ds\right].$$

In fact,

$$\begin{aligned} w_t &= -\int_t^T \mathcal{D}_s w_s ds - \int_t^T \mathcal{D}_x w_s dB_s - \frac{1}{2}\int_t^T \mathcal{D}_x^2 w_s d\langle B\rangle_s \\ &= -(M_T^\varepsilon - M_t^\varepsilon) + \frac{1}{2}\int_t^T \eta_s ds, \end{aligned}$$

where $M_t^\varepsilon := \int_0^t \mathcal{D}_x w_s dB_s + (1/2)\int_0^t \mathcal{D}_x^2 w_s d\langle B\rangle_s - \int_0^t G_\varepsilon(\mathcal{D}_x^2 w_s)ds$ is a G_ε-martingale. So

$$w_t = \mathbb{E}_t^{G_\varepsilon}\left[\frac{1}{2}\int_t^T \eta_s ds\right].$$

Set $\beta = \overline{\sigma}^2/\underline{\sigma}^2$ and $\gamma = (\beta-1)/(\beta+1)$. For any $a, \alpha \in R$ and $\varepsilon \in [0, (\overline{\sigma}^2 - \underline{\sigma}^2)/2]$, it's easy to check that

$$G(a + \gamma|\alpha|) \geq G^\alpha(a) \geq G_\varepsilon(a) + \frac{1}{2}\varepsilon|\alpha|. \tag{4.10}$$

Denote by $u^\eta, v^\eta, w^{\eta,\varepsilon}$ the solutions to equations (4.7), (4.8) and (4.9), respectively. By (4.10) and the comparison theorem for the (path dependent) PDEs, we have

$$u_0^{\gamma|\eta|} \geq v_0^{\eta} \geq w_0^{\varepsilon|\eta|,\varepsilon},$$

which recovers the estimates obtained in [**33**] and [**26**].

COROLLARY 4.13. *For any $\eta \in M_G^1(0,T)$, we have*

$$\gamma \mathbb{E}^G\left[\int_0^T |\eta_s| d\langle B\rangle_s\right] \geq \limsup_{n\to\infty} \mathbb{E}^G\left[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s\right] \geq \varepsilon \mathbb{E}^{G_\varepsilon}\left[\int_0^T |\eta_s| ds\right].$$

5. Fully nonlinear PPDEs in G-Sobolev space $W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$ and BSDEs driven by G-Brownian motion.

5.1. G-Sobolev space $W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$.

5.1.1. Definitions.

For each $u \in W_G^{1,2;p}(0,T)$ with $\mathcal{D}_t u = \lambda$, $\mathcal{D}_x u = \zeta$ and $\mathcal{D}_x^2 u = \gamma$, we have

$$\begin{aligned} u(t,\omega) &= u_0 + \int_0^t \mathcal{D}_s u(s,\omega)ds + \int_0^t \mathcal{D}_x u(s,\omega)dB_s + \frac{1}{2}\int_0^t \mathcal{D}_x^2 u(s,\omega)d\langle B\rangle_s \\ &= u_0 + \int_0^t \mathcal{A}_G u(s,\omega)ds + \int_0^t \mathcal{D}_x u(s,\omega)dB_s + K_t^\gamma \end{aligned}$$

where we denote

$$K_t^\gamma = \frac{1}{2}\int_0^t \gamma(s,\omega)d\langle B\rangle_s - \int_0^t G(\gamma(s,\omega))d\langle B\rangle_s$$

and

$$\mathcal{A}_G u = \mathcal{D}_t u + G\big(\mathcal{D}_x^2 u\big).$$

We define the following distance for elements u, v in $W_G^{1,2;p}(0,T)$:

$$\begin{aligned} d_{W_{\mathcal{A}_G}^{1/2,1;p}}(u,v) = \Bigg\{\mathbb{E}^G\Bigg[\sup_{s\in[0,T]} |u_s - v_s|^p + \left(\int_0^T |\mathcal{D}_x(u_s - v_s)|^2 ds\right)^{p/2} \\ + \int_0^T |\mathcal{A}_G u_s - \mathcal{A}_G v_s|^p ds\Bigg]\Bigg\}^{1/p}. \end{aligned}$$

We denote by $W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$ the collection of all processes $u \in S_G^p(0,T)$ with the following property: there exists a Cauchy sequence $\{u^n\} \subset W_G^{1,2;p}(0,T)$ with respect to the metric $d_{W_{\mathcal{A}_G}^{1/2,1;p}}(\cdot,\cdot)$ such that $\|u^n - u\|_{S_G^p} \to 0$.

We denote by $\mathcal{K}^p$ the closure of $\mathcal{K}^0 := \{K^\gamma_\cdot : \gamma \in M_G^p(0,T)\}$ in the space $S_G^p(0,T)$. Obviously, we have

$$W^{1/2,1;p}_{\mathcal{A}_G}(0,T)=\left\{u=u_0+\int_0^t\beta(s)ds+\int_0^t\zeta(s)dB_s+K_t:\right.$$
$$\left.\beta\in M^p_G(0,T),\ \zeta\in H^p_G(0,T),\ K\in\mathcal{K}^p\right\}.$$

For each given $\eta\in M^p_G(0,T)$, we also denote

$$W^{1/2,1;p}_{\mathcal{A}_G}(\eta):=\left\{u=u_0+\int_0^t\eta(s)ds+\int_0^t\zeta(s)dB_s+K_t:\zeta\in H^p_G(0,T),\ K\in\mathcal{K}^p\right\}.$$

5.1.2. A review of the structure of G-martingales.

In order to understand the spaces $W^{1/2,1;p}_{\mathcal{A}_G}(0,T)$ and $W^{1/2,1;p}_{\mathcal{A}_G}(\eta)$, we recall the structure of G-martingales. [**22**] proved that for any $\xi\in C^\infty(\Omega_T)$, the G-martingale $X_t=\mathbb{E}^G_t[\xi]$ has the following representation:

$$X_t=\mathbb{E}^G[\xi]+\int_0^t Z_sdB_s+\frac{1}{2}\int_0^t\eta_sd\langle B\rangle_s-\int_0^t G(\eta_s)ds \tag{5.1}$$

for some $Z\in H^p_G(0,T)$, $\eta\in M^p_G(0,T)$ and conjectured that for any $\xi\in L^p_G(\Omega_T)$ the representation (5.1) holds. Besides, [**22**] showed that for any $\eta\in M^p_G(0,T)$,

$$K_t:=\frac{1}{2}\int_0^t\eta_sd\langle B\rangle_s-\int_0^t G(\eta_s)ds$$

is a non-increasing G-martingale. So any process $K\in\mathcal{K}^p$ is a non-increasing G-martingale with $K_T\in L^p_G(\Omega_T)$. By Theorem 5.4 in [**32**], the converse statement is also right.

For $p\geq 1$ and $\xi\in C^\infty(\Omega_T)$, set $\|\xi\|^p_{\mathbb{L}^p_G}=\mathbb{E}^G[\sup_{t\in[0,T]}|\mathbb{E}^G_t[\xi]|^p]$. Denote by $\mathbb{L}^p_G(\Omega_T)$ the closure of $C^\infty(\Omega_T)$ with respect to the norm $\|\cdot\|_{\mathbb{L}^p_G}$ in $L^p_G(\Omega_T)$. [**29**] showed that for any $\xi\in\mathbb{L}^2_G(\Omega_T)$ the G-martingale $X_t:=\mathbb{E}^G_t[\xi]$ has the following decomposition:

$$X_t=\mathbb{E}^G[\xi]+\int_0^t Z_sdB_s+K_t, \tag{5.2}$$

where K_t is a non-increasing G-martingale.

[**31**] showed that $\mathbb{L}^p_G(\Omega_T)\supset L^q_G(\Omega_T)$ for any $1\leq p<q$. Moreover, [**31**] proved that the decomposition (5.2) holds for any $\xi\in L^p_G(\Omega_T)$ with $p>1$. Independently, [**29**] showed that $\mathbb{L}^2_G(\Omega_T)\supset L^q_G(\Omega_T)$ for any $q>2$.

5.1.3. A probabilistic characterization of the G-Sobolev spaces.

PROPOSITION 5.1. *Assume $u\in S^p_G(0,T)$. Then the following two conditions are equivalent*:

1. $u\in W^{1/2,1;p}_{\mathcal{A}_G}(0,T)$;

2. *There exist $\eta \in M_G^p(0,T)$ and $\zeta \in H_G^p(0,T)$ such that*

$$K_t := u(t,\omega) - \int_0^t \eta_s ds - \int_0^t \zeta_s dB_s$$

is a non-increasing G-martingale.

The following proposition provides closability of the metric $d_{W_{\mathcal{A}_G}^{1/2,1;p}}$ in $W_G^{1,2;p}(\eta)$.

PROPOSITION 5.2. *The metric $d_{W_{\mathcal{A}_G}^{1/2,1;p}}(\cdot,\cdot)$ defined on the space $W_G^{1,2;p}(\eta)$ is closable in $S_G^p(0,T)$ in the following sence: Let $\{u^n\}_{n=1}^\infty$ and $\{\bar{u}^n\}_{n=1}^\infty$ be two Cauchy sequences in $W_G^{1,2;p}(\eta)$ with respect to the metric $d_{W_{\mathcal{A}_G}^{1/2,1;p}}(\cdot,\cdot)$. If $\|u^n - \bar{u}^n\|_{S_G^p} \to 0$, then we also have $d_{W_{\mathcal{A}_G}^{1/2,1;p}}(u^n, \bar{u}^n) \to 0$.*

PROOF. We denote the Cauchy limits of $\{u^n\}$ and $\{\bar{u}^n\}$ in $W_G^{1,2;p}(0,T)$ by

$$u(t,\omega) = u(0,\omega) - \int_0^t \eta(s,\omega)ds + \int_0^t v(s,\omega)dB_s + K_t$$

and

$$\bar{u}(t,\omega) = \bar{u}(0,\omega) - \int_0^t \eta(s,\omega)ds + \int_0^t \bar{v}(s,\omega)dB_s + \bar{K}_t$$

respectively, with $u(t,\omega) \equiv \bar{u}(t,\omega)$. Thus $\int_0^t v(s,\omega)dB_s + K_t \equiv \int_0^t \bar{v}(s,\omega)dB_s + \bar{K}_t$. Applying Itô's formula to $(u(t,\omega) - \bar{u}(t,\omega))^2 \equiv 0$, and then taking the G-expectation, we obtain

$$0 = \mathbb{E}^G \int_0^T |v(s,\omega) - \bar{v}(s,\omega)|^2 ds.$$

It follows that $v \equiv \bar{v}$ in $H_G^p(0,T)$ and thus $K \equiv \bar{K}$ in $\mathcal{K}^p$. □

PROPOSITION 5.3. *Let $\eta \in M_G^p(0,T)$ be given. Then $u \in W_{\mathcal{A}_G}^{1/2,1;p}(\eta)$ if and only if $u \in W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$ and $u_t - \int_0^t \mathcal{D}_x u(s)dB_s - \int_0^t \eta_s ds$ is a non-increasing G-martingale.*

5.2. Fully nonlinear path dependent PDEs.

We define a solution of the path dependent PDE (4.5) in the G-Sobolev space $W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$.

DEFINITION 5.4. An element $u \in W_{\mathcal{A}_G}^{1/2,1;p}(0,T)$ is called a $W_{\mathcal{A}_G}^{1/2,1;p}$-solution to the path dependent PDE (4.5) if

$$u(T,\omega) = \xi(\omega) \text{ and } u \in W_{\mathcal{A}_G}^{1/2,1;p}(\eta) \text{ with } \eta_t(\omega) = g(t,\omega,u(t,\omega),\mathcal{D}_x u(t,\omega)).$$

The following theorem asserts that a $W^{1/2,1;p}_{\mathcal{A}_G}$-solution of the PPDE (4.5) corresponds to a solution of G-BSDE (4.6) studied in [**12**].

THEOREM 5.5. (i) *Assume* (Y,Z,K) *is a solution to the backward SDE* (4.6) *and* $g(t,\omega,Y_t,Z_t)\in M^p_G(0,T)$. *Then* $u(t,\omega):=Y_t(\omega)$, $t\in[0,T]$ *is in* $W^{1/2,1;p}_{\mathcal{A}_G}(0,T)$ *with* $\mathcal{D}_x u(t,\omega)=Z_t(\omega)$, $t\in[0,T]$, *in* $H^p_G(0,T)$. *Moreover, we have* $u(T,\omega)=\xi(\omega)$, $u\in W^{1/2,1;p}_{\mathcal{A}_G}(\eta)$ *with* $\eta_t(\omega)=g(t,\omega,u(t,\omega),\mathcal{D}_x u(t,\omega))$, *namely,* u *is a* $W^{1/2,1;p}_{\mathcal{A}_G}$*-solution to the path dependent PDE* (4.5).

(ii) *Let* $u\in W^{1/2,1;p}_{\mathcal{A}_G}(0,T)$ *be a* $W^{1/2,1;p}_{\mathcal{A}_G}$*-solution to the path dependent PDE* (4.5). *Set*

$$K_t=u(t,\omega)+\int_0^t g(s,\omega,u(s,\omega),\mathcal{D}_x u(s,\omega))ds-\int_0^t \mathcal{D}_x u(s,\omega)dB_s.$$

Then the triple $(Y,Z,K)=(u,\mathcal{D}_x u,K)\in S^p_G(0,T)\times H^p_G(0,T)$ *is a solution to the backward SDE* (4.6).

Assume that the function $g(t,\omega,y,z):[0,T]\times\Omega_T\times R\times R\to R$ satisfies the following assumptions: there exists some $\beta>1$ such that

(H1) for any y,z, $g(t,\omega,y,z)\in M^\beta_G(0,T)$;
(H2) $|g(t,\omega,y,z)-g(t,\omega,y',z')|\le L(|y-y'|+|z-z'|)$ for some constant $L>0$.

COROLLARY 5.6. *Assume* $\xi\in L^\beta_G(\Omega_T)$ *and* g *satisfies* (H1) *and* (H2) *for some* $\beta>1$. *Then, for each* $p\in(1,\beta)$, *the path dependent PDE* (4.5) *has a unique* $W^{1/2,1;p}_{\mathcal{A}_G}$*-solution* u. *In particular,* $u(t,\omega):=\mathbb{E}^G_t[\xi](\omega)$ *is the unique* $W^{1/2,1;p}_{\mathcal{A}_G}$*-solution of*

$$\mathcal{D}_t u(t,\omega)+G(\mathcal{D}^2_x u(t,\omega))=0,\quad u(T,\omega)=\xi(\omega).$$

PROOF. The uniqueness is straightforward from Theorem 5.5 and Theorem 6.2.

We now prove the existence. By Theorem 6.2 we know that the backward SDE (4.6) has a solution (Y,Z,K). By the assumption (H1) and (H2), we conclude $g(t,\omega,Y_t(\omega),Z_t(\omega))\in M^p_G(0,T)$. So we get the existence from Theorem 5.5.

By the G-martingale decomposition theorem, $u\in S^p_G(0,T)$ is a G-martingale if and only if u is a solution of backward SDE (4.6) with $f=0$. □

REMARK 5.7. It is an interesting question whether a solution of a 2BSDE introduced by Soner, Touzi and Zhang [**30**] can be also interpreted as a Sobolev solution of the corresponding PPDE. A direct obstacle is that, by the definition, the solution (Y,Z,K) may not be within a completion of certain space of cylindrical functions of paths.

6. Appendix: Backward SDEs driven by *G*-BM.

In [**12**] the authors studied the backward stochastic differential equations driven by a G-Brownian motion $(B_t)_{t\ge0}$ in the following form:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t), \tag{6.1}$$

where K is a non-increasing G-martingale.

The main result in [**12**] is the existence and uniqueness of a solution (Y, Z, K) for Equation (6.1) in the G-framework under the following assumption: there exists some $\beta > 1$ such that (H1) and (H2) are satisfied.

DEFINITION 6.1. Let $\xi \in L_G^\beta(\Omega_T)$ and g satisfy (H1) and (H2) for some $\beta > 1$. A triplet of processes (Y, Z, K) is called a solution of Equation (6.1) if for some $1 < \alpha \leq \beta$ the following properties hold:

(a) $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T)$, K is a non-increasing G-martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$;
(b) $Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t)$.

The main result in [**12**] is the following theorem:

THEOREM 6.2. *Assume that $\xi \in L_G^\beta(\Omega_T)$ and f satisfies* (H1) *and* (H2) *for some $\beta > 1$. Then Equation* (6.1) *has a unique solution (Y, Z, K). Moreover, for any $1 < \alpha < \beta$ we have $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T)$ and $K_T \in L_G^\alpha(\Omega_T)$.*

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