G-invariant positive solutions for a quasilinear Schrödinger equation

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Abstract. We are concerned with a quasilinear elliptic equation of the form:

$$-\Delta u + a(x)u - \Delta(|u|^{\alpha})|u|^{\alpha-2}u = h(u) \quad \text{in } \mathbb{R}^{N},$$

where $\alpha > 1$ and $N \geq 1$. By using variational approaches, we prove the existence of at least one positive solution of the above equation under suitable conditions on $a(x)$ and h. Especially, we are interested in the situation that $a(x)$ is invariant under the finite group action G.

Keywords: quasilinear elliptic equation, G-invariant solution, interaction estimate.

1. Introduction

In this paper we consider the existence of positive solutions for the following quasilinear elliptic problem:

$$-\Delta u + a(x)u - \Delta(|u|^{\alpha})|u|^{\alpha-2}u = h(u) \quad \text{in } \mathbb{R}^{N}, \tag{1.1}$$

where $\alpha > 1$, $N \geq 1$ and h behaves like pure power nonlinearities. Solutions of (1.1) are related to standing wave solutions for the following Schrödinger equation:

$$i\frac{\partial}{\partial t}z = -\Delta z + V(x)z - h(|z|^2)z - \kappa\Delta(|z|^{\alpha})|z|^{\alpha-2}z, \quad (t,x) \in (0,\infty) \times \mathbb{R}^{N}, \tag{1.2}$$

where $\kappa > 0$ is a constant. Functions of the form $z(t,x) = u(x)e^{-i\lambda t}$ satisfy (1.2) if and only if $u(x)$ satisfies (1.1) (with $a(x) = V(x) - \lambda$, $\kappa = 1$). Quasilinear term $-\kappa\Delta(|z|^\alpha)|z|^{\alpha-2}z$ derives from a superfluid film equation in plasma physics, which was introduced in Kurihara [19]. See also [10, 11, 16, 20] for physical backgrounds. However, very few results are known about the existence of solutions for (1.1).

We are looking for positive solutions of (1.1) in the setting of G-invariant potential. We denote the orthogonal group in $\mathbb{R}^N$ by $O(N)$ and let $G \subset O(N)$ be a finite subgroup which satisfies for all $x \in S^{N-1} = \{x \in \mathbb{R}^N; |x| = 1\}$, there exists at least one $g \in G$ such that $gx \neq x$. We put

$$m = \min_{x \in S^{N-1}} \text{card}\{gx; g \in G\} \ (\geq 2)$$

and choose $x_0 \in S^{N-1}$ such that $\text{card}\{gx_0; g \in G\} = m$. We also put

$$\{gx_0; g \in G\} = \{e_1, \cdots, e_m\} \text{ and } \lambda_0 = \min_{i \neq j} |e_i - e_j| \in (0, 2]. \tag{1.3}$$

Our conditions on potential $a(x) \in C(\mathbb{R}^N, \mathbb{R})$ are as follows.

(a1) There exists $a_\infty > 0$ such that $\lim_{|x|\to\infty} a(x) = a_\infty$.

(a2) There exists $a_0 > 0$ such that $\inf_{x \in \mathbb{R}^N} a(x) \geq a_0$.

(a3) $a(gx) = a(x)$ for all $x \in \mathbb{R}^N$ and $g \in G$.

(a4) There exist $\lambda > \lambda_0$ and $c_0 > 0$ such that $a_\infty - a(x) \geq -c_0 e^{-\lambda|x|}$ for all $x \in \mathbb{R}^N$.

For nonlinear term $h \in C^1(\mathbb{R}_+, \mathbb{R})$, we assume $h(s) \equiv 0$ for $s \leq 0$ and

(h1) There exists $\eta > 0$ such that $\lim_{s\to 0+} \dfrac{h(s)}{s^{1+\eta}} = 0$.

(h2) There exist $c > 0$ and $2\alpha - 1 \leq p < \infty$ for $N = 1, 2$, $2\alpha - 1 \leq p < \dfrac{(2\alpha-1)N+2}{N-2}$ for $N \geq 3$ such that $h(s) \leq c(|s| + |s|^p)$ for all $s \geq 0$.

(h3) There exists $\theta \geq 2\alpha - 1$ such that $0 < \theta h(s) \leq h'(s)s$ for all $s \geq 0$.

Our main result is the following

Theorem 1.1. *Assume $\alpha \geq \dfrac{3}{2}$, (a1)–(a4) and (h1)–(h3). Further we assume either*

(i) *(h3) holds for $\theta > 2\alpha - 1$ or*

(ii) *(h3) holds for $\theta = 2\alpha - 1$ and (h2) holds for $2\alpha - 1 \leq p < \dfrac{(2\alpha-1)N+4}{N}$ if $N \geq 4$. If $N = 3$, (h2) holds for*

$$\begin{cases} 2\alpha - 1 \leq p \leq 5 & \text{if } \dfrac{3}{2} < \alpha \leq \dfrac{7}{3}, \\ 2\alpha - 1 \leq p < \dfrac{6\alpha+1}{3} & \text{if } \alpha > \dfrac{7}{3}. \end{cases}$$

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Then (1.1) *has at least one positive solution* $u_0 \in H^1(\mathbb{R}^N)$ *which is invariant under the group action* G *on* x, *that is,* $u_0(gx) = u_0(x)$ *for all* $x \in \mathbb{R}^N$ *and* $g \in G$.

Remark 1.2. (i) λ in $(a4)$ corresponds to a convergent rate from below in the setting of $(a1)$ and $\lambda > \lambda_0$ plays an essential role in our main result.

(ii) Since $\alpha \geq \frac{3}{2}$, condition $(h2)$ implies that h may have a supercritical growth in the sense of Sobolev embedding.

(iii) It follows from $(h3)$ that $\dfrac{H(s)}{s^{q+1}}$, $\dfrac{h(s)}{s^q}$ are non-decreasing for all $1 \leq q \leq \theta$ and $s \geq 0$.

(iv) When $\theta = 2\alpha - 1$ in $(h3)$, we require either $2\alpha - 1 \leq p \leq \frac{N+2}{N-2}$ or $2\alpha - 1 \leq p < \frac{(2\alpha-1)N+4}{N}$. Since we assume $\alpha \geq \frac{3}{2}$, it follows $\frac{N+2}{N-2} \leq \frac{(2\alpha-1)N+4}{N}$ for $N \geq 4$.

When $h(s) = s^p$, we need no restriction with respect to p (see Remark 4.6 below). More precisely, we have the following

Theorem 1.3. *Assume* $\alpha \geq \dfrac{3}{2}$, $(a1)$–$(a4)$ *and* $h(s) = s^p$, $2\alpha - 1 \leq p < \infty$ *for* $N = 1, 2$ *and*
$$2\alpha - 1 \leq p < \frac{(2\alpha-1)N+2}{N-2} \quad \text{for } N \geq 3. \text{ Then problem (1.1) has a G-invariant positive solution.}$$

Equation (1.1) has a variational structure, that is, one can obtain solutions of (1.1) as critical points of the associated functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + a(x)u^2 \, dx + \frac{\alpha}{2} \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha - 2} \, dx - \int_{\mathbb{R}^N} H(u) \, dx. \qquad (1.4)$$

We remark that nonlinear functional $\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha - 2} \, dx$ is not defined on all $H^1(\mathbb{R}^N)$ except for $N = 1$. Thus it is very difficult to handle (1.4) directly. In [**27**], which is first existence result for (1.1) up to our knowledge, they overcome this difficulty by restricting themselves to $N = 1$ or radially symmetric case. In [**21**], they used a constrained minimization argument and obtained a positive solution only up to unknown Lagrange multiplier. We also mention to the result of [**23**] in which they made a change of variables and worked on a suitable Orlicz space. On the other hand in [**13**], they used of a change of unknown function that they called *dual approach* and introduced an associated semilinear equation in the case of $\alpha = 2$. We also refer [**12**, **22**, **24**] for other results on (1.1). Especially in [**22**], they studied autonomous case of (1.1) on the annulus and obtained multi-bump positive solutions whose bumps have group symmetries.

In this paper, we adapt dual approach introduced in [**13**] to our setting $\alpha > 1$. More precisely, we will show that for some suitable function $f(t)$ ($f(t)$ depends on $\alpha > 1$), if $v \in H^1(\mathbb{R}^N)$ is a solution of the following semilinear problem:

$$-\Delta v + a(x)f(v)f'(v) = h(f(v))f'(v) \quad \text{in } \mathbb{R}^N, \qquad (1.5)$$

then $u = f(v)$ is a solution of (1.1). Thus to obtain solutions of quasilinear equation (1.1), it suffices to show the existence of solutions of semilinear equation (1.5). The functional associated to (1.5) is defined by

$$I(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + a(x)f(v)^2 \, dx - \int_{\mathbb{R}^N} H(f(v)) \, dx. \qquad (1.6)$$

As we will see in Section 2, $I(v)$ is well-defined for all $v \in H^1(\mathbb{R}^N)$ under $(h2)$. Not only we can convert our quasilinear problem into a semilinear problem, but also we have the following adequacy of our dual approach. By the form of $J(u)$, a natural function space which corresponds to (1.1) seems to be

$$X := \{ u \in H^1(\mathbb{R}^N) \,; \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha - 2} dx < \infty \}. \qquad (1.7)$$

We will see that X is equal to

$$\begin{aligned} Y &:= \{ u \in H^1(\mathbb{R}^N) \,; \text{ there exists } v \in H^1(\mathbb{R}^N) \text{ such that } u = f(v) \} \\ &= \{ f(v) \,; v \in H^1(\mathbb{R}^N) \}. \end{aligned} \qquad (1.8)$$

We will also observe if $u = f(v)$, then

$$J(u) = I(v) \text{ and } J'(u)u = I'(v)\frac{f(v)}{f'(v)}. \qquad (1.9)$$

These relations tell us two facts: Firstly the transformation f does not change the energy level. Secondly since $\frac{f(v)}{f'(v)} = 0$ if and only if $v = 0$ (see Section 2 for the definition of f), any nontrivial critical point u of $J(u)$ can be written by $u = f(v)$ for some $v \in H^1(\mathbb{R}^N)$ with $I'(v) = 0$. Moreover relation (1.9) enables us to use both $J(u)$ and $I(v)$ to find a positive solution of (1.1). We also mention that another *dual approach* has been used in the study of periodic solutions of nonlinear wave equations. For this topic, we refer [**9**, **14**, **26**].

A further difficulty is caused by a lack of compactness for corresponding functional $J(u)$ or $I(v)$. In the earlier results mentioned above, they assumed

$(a5)$ $a(x) \leq a_\infty$ for all $x \in \mathbb{R}^N$.

It is well-known that in the semilinear case, the mountain pass minimax value for corresponding functional is attained under the assumption $(a5)$. However without any order relation between $a(x)$ and a_∞, the mountain pass minimax value is not attained in general. We remark that in some sense, this is still valid in our quasilinear case. We make a

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combination of concentration compactness principle and interaction estimate to overcome this difficulty. These arguments appear, for instance, in [**1**, **4**, **5**, **17**] and the references there in. Such kinds of arguments also appear when we study inhomogeneous elliptic problems (see [**2**, **3**, **28**, **29**]). We modify these arguments according to our setting and obtain the existence of a positive solution without assuming $(a5)$. To make use of concentration compactness principle and interaction estimate, it is important to study the following autonomous equation as a limit case of (1.1):

$$-\Delta u + a_\infty u - \Delta(|u|^\alpha)|u|^{\alpha-2}u = h(u) \quad \text{in } \mathbb{R}^N. \tag{1.10}$$

As to (1.10), we have the following result.

Theorem 1.4. *Assume $\alpha > 1$ and $(h1)$–$(h3)$. Then (1.10) has at least one solution $w(x)$ having the following properties:*

(i) $w \in C^2(\mathbb{R}^N, \mathbb{R})$ *and* $w(x) > 0$ *for all* $x \in \mathbb{R}^N$.

(ii) w *is radially symmetric:* $w(x) = w(|x|)$ *and decreases with respect to* $r = |x|$.

(iii) *There exist* c, $c' > 0$ *such that*

$$\lim_{|x|\to\infty} e^{\sqrt{a_\infty}|x|}(|x|+1)^{\frac{N-1}{2}} w(x) = c, \quad \lim_{r\to\infty} e^{\sqrt{a_\infty}r}(r+1)^{\frac{N-1}{2}} \frac{\partial w}{\partial r} = -c'.$$

(iv) w *is a least energy solution of (1.10), that is,*

$$J_\infty(w) = c_\infty := \inf\{J_\infty(u); J_\infty'(u) = 0, u \in X \setminus \{0\}\},$$

where J_∞ is the associated functional defined by

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + a_\infty u^2 \, dx + \frac{\alpha}{2}\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha-2} \, dx - \int_{\mathbb{R}^N} H(u) \, dx. \tag{1.11}$$

We remark that [**13**] obtained a similar result as Theorem 1.4 in the case $\alpha = 2$. However, they do not mention to the property (iv). As to the proof of Theorem 1.4, we also use dual approach and consider the corresponding semilinear equation:

$$-\Delta v + a_\infty f(v)f'(v) = h(f(v))f'(v) \quad \text{in } \mathbb{R}^N \tag{1.12}$$

for some suitable $f(t)$. To study (1.12), we define the associated functional to (1.12) by

$$I_\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + a_\infty f^2(v) \, dx - \int_{\mathbb{R}^N} H(f(v)) \, dx. \tag{1.13}$$

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In the following sections, we prove Theorem 1.1 and Theorem 1.4 by variational approaches. Our paper is organized as follows. In Section 2, we explain the dual approach introduced in [**13**]. We also give some properties on $f(t)$. In Section 3, we mention some known results of (1.12) and prove Theorem 1.4. In Section 4, we show several properties on $I(v)$ to find a nontrivial critical point. In Section 5, we establish an interaction estimate which is a key of our existence result. Finally in Section 6, we complete the proof of Theorem 1.1.

Notation. Throughout this paper, C, C', C_1, C_2, *etc.* denote various positive constants which is not essential to our problem.

2. Dual approach and functional framework

2.1. Dual approach

From now on, we always assume $\alpha > 1$. Let f be a solution of the following ODE:

$$f'(t) = \frac{1}{\sqrt{1 + \alpha f(t)^{2\alpha-2}}} \quad \text{on } t \in [0, \infty), \quad f(0) = 0. \tag{2.1}$$

We put $f(t) = -f(-t)$ on $t \in (-\infty, 0]$. By the standard theory of ODE, we can easily see that f is uniquely determined, of class $C^\infty(\mathbb{R}, \mathbb{R})$ and invertible on all $\mathbb{R}$. We notice that $f''(t) = -\alpha(\alpha-1)|f(t)|^{2\alpha-4}f(t)(f'(t))^4$. Especially f is concave for $t \geq 0$. Using function $f(t)$, we have the following lemma.

Lemma 2.1. *Let $v \in H^1(\mathbb{R}^N)$ be a nontrivial critical point of $I(v)$ and put $u = f(v)$. Then u is a positive solution of (1.1).*

Proof. We can easily see that if $v \in H^1(\mathbb{R}^N)$ is a nontrivial critical point of $I(v)$, then v is a solution of (1.5). For $v = f^{-1}(u)$ (f^{-1} denotes the inverse of f), we have

$$\nabla v = (f^{-1})'(u)\nabla u, \quad \Delta v = (f^{-1})''(u)|\nabla u|^2 + (f^{-1})'(u)\Delta u.$$

Moreover direct calculations yield

$$(f^{-1})'(t) = \frac{1}{f'(f^{-1}(t))} = \sqrt{1 + \alpha|f(f^{-1}(t))|^{2\alpha-2}} = \sqrt{1 + \alpha|t|^{2\alpha-2}},$$

$$(f^{-1})''(t) = \frac{\alpha(\alpha-1)|t|^{2\alpha-4}t}{\sqrt{1 + \alpha|t|^{2\alpha-2}}}.$$

Thus we have

$$\Delta v = \frac{\alpha(\alpha-1)|u|^{2\alpha-4}u}{\sqrt{1 + \alpha|u|^{2\alpha-2}}}|\nabla u|^2 + \sqrt{1 + \alpha|u|^{2\alpha-2}}\Delta u.$$

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Consequently from (1.5), we can observe that u satisfies

$$-\sqrt{1+\alpha|u|^{2\alpha-2}}\Delta u - \frac{\alpha(\alpha-1)|u|^{2\alpha-4}u}{\sqrt{1+\alpha|u|^{2\alpha-2}}}|\nabla u|^2 + \frac{a(x)u}{\sqrt{1+\alpha|u|^{2\alpha-2}}} = \frac{h(u)}{\sqrt{1+\alpha|u|^{2\alpha-2}}},$$

or equivalently,

$$-\Delta u - \alpha|u|^{2\alpha-2}\Delta u - \alpha(\alpha-1)|u|^{2\alpha-4}u|\nabla u|^2 + a(x)u = h(u). \tag{2.2}$$

On the other hand, it follows from

$$\begin{aligned}
\Delta(|u|^\alpha) &= \operatorname{div}\left(\alpha|u|^{\alpha-2}u\nabla u\right) \\
&= \alpha|u|^{\alpha-2}u\Delta u + \nabla u \cdot \nabla(\alpha|u|^{\alpha-2}u) \\
&= \alpha|u|^{\alpha-2}u\Delta u + \nabla u \cdot (\alpha(\alpha-1)|u|^{\alpha-2}\nabla u) \\
&= \alpha|u|^{\alpha-2}u\Delta u + \alpha(\alpha-1)|u|^{\alpha-2}|\nabla u|^2
\end{aligned}$$

that

$$\Delta(|u|^\alpha)|u|^{\alpha-2}u = \alpha|u|^{2\alpha-2}\Delta u + \alpha(\alpha-1)|u|^{2\alpha-4}u|\nabla u|^2. \tag{2.3}$$

Thus from (2.2) and (2.3), we see that if v is a solution of (1.5), then $u = f(v)$ is a solution of (1.1).

Finally we show that u is positive. Testing $u_- = \max\{-u, 0\}$ in (1.1), we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} |\nabla u_-|^2 + a(x)u_-^2 \, dx + \alpha(\alpha-1)\int_{\mathbb{R}^N} |\nabla u|^2|u|^{2\alpha-2}|u_-|^2 \, dx \\
&\quad + \alpha\int_{\mathbb{R}^N} |\nabla u_-|^2|u|^{2\alpha-2} \, dx - \int_{\mathbb{R}^N} h(u)u_- \, dx \\
&\geq \int_{\mathbb{R}^N} |\nabla u_-|^2 + a_0 u_-^2 \, dx \geq 0.
\end{aligned}$$

Thus it follows $u_- \equiv 0$. By the maximum principle, we obtain $u > 0$. ∎

As to asymptotic behaviours on the unique solution f of (2.1), we have the following

Lemma 2.2. *It follows*

(i) $\displaystyle\lim_{t\to 0} \frac{f(t)}{t} = 1.$

(ii) $\displaystyle\lim_{t\to\infty} \frac{f(t)}{t^{\frac{1}{\alpha}}} = \alpha^{\frac{1}{2\alpha}}.$

(iii) $|f(t)| \leq |t|$, $|f'(t)| \leq 1$, $|f''(t)| \leq \alpha(\alpha-1)|t|^{2\alpha-3}$ *for all $t \in \mathbb{R}$.*

Proof. We have from (2.1)

$$t = \int \sqrt{1+\alpha f(t)^{2\alpha-2}} \, df + C_0, \tag{2.4}$$

where C_0 is a constant which satisfies $f(0) = 0$. By the l'Hôspital's rule,

$$\lim_{t \to 0+} \frac{f(t)}{t} = \lim_{t \to 0+} \frac{f(t)}{\int \sqrt{1 + \alpha f(t)^{2\alpha-2}} \, df + C_0} = \lim_{t \to 0+} \frac{f'(t)}{\sqrt{1 + \alpha f(t)^{2\alpha-2}} f'(t)} = 1.$$

Since $f(t) = -f(-t)$ on $t \in (-\infty, 0]$, we obtain $\lim_{t \to 0} \frac{f(t)}{t} = 1$. Similarly we have from (2.4)

$$\lim_{t \to \infty} \frac{f(t)}{t^{\frac{1}{\alpha}}} = \lim_{t \to \infty} \frac{f(t)}{\left(\int \sqrt{1 + \alpha f(t)^{2\alpha-2}} \, df + C_0 \right)^{\frac{1}{\alpha}}}$$

$$= \lim_{t \to \infty} \frac{f'(t)}{\frac{1}{\alpha} \left(\int \sqrt{1 + \alpha f(t)^{2\alpha-2}} \, df + C_0 \right)^{\frac{1-\alpha}{\alpha}} \sqrt{1 + \alpha f(t)^{2\alpha-2}} f'(t)}$$

$$= \lim_{t \to \infty} \alpha \left(\frac{\int \sqrt{1 + \alpha f(t)^{2\alpha-2}} \, df + C_0}{(1 + \alpha f(t)^{2\alpha-2})^{\frac{\alpha}{2(\alpha-1)}}} \right)^{\frac{\alpha-1}{\alpha}}. \tag{2.5}$$

Moreover

$$\lim_{t \to \infty} \frac{\int \sqrt{1 + \alpha f(t)^{2\alpha-2}} \, df + C_0}{(1 + \alpha f(t)^{2\alpha-2})^{\frac{\alpha}{2(\alpha-1)}}} = \lim_{t \to \infty} \frac{\sqrt{1 + \alpha f(t)^{2\alpha-2}}}{\alpha^2 f(t)^{2\alpha-3}(1 + \alpha f(t)^{2\alpha-2})^{\frac{2-\alpha}{2(\alpha-1)}}}$$

$$= \lim_{t \to \infty} \frac{1}{\alpha^2} \left(\frac{1 + \alpha f(t)^{2\alpha-2}}{f(t)^{4\alpha-6}(1 + \alpha f(t)^{2\alpha-2})^{\frac{2-\alpha}{(\alpha-1)}}} \right)^{\frac{1}{2}}$$

$$= \lim_{t \to \infty} \frac{1}{\alpha^2} \left(\frac{f(t)^{-2\alpha+2} + \alpha}{(f(t)^{-2\alpha+2} + \alpha)^{\frac{2-\alpha}{\alpha-1}}} \right)^{\frac{1}{2}}$$

$$= \frac{1}{\alpha^2} \left(\frac{\alpha}{\alpha^{\frac{2-\alpha}{\alpha-1}}} \right)^{\frac{1}{2}} = \alpha^{\frac{1-2\alpha}{2(\alpha-1)}}. \tag{2.6}$$

Thus from (2.5) and (2.6), we have

$$\lim_{t \to \infty} \frac{f(t)}{t^{\frac{1}{\alpha}}} = \alpha \alpha^{\frac{1-2\alpha}{2(\alpha-1)} \frac{\alpha-1}{\alpha}} = \alpha^{\frac{1}{2\alpha}}.$$

The proof of (iii) is standard. Thus we omit the proof here. ∎

Lemma 2.3. *It follows*

$$\frac{1}{\alpha} |f(t)| \le |t| |f'(t)| \le |f(t)| \quad \text{for all } t \in \mathbb{R}.$$

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Proof. Since $f(t)$ is an odd function, it suffices to show that

$$\frac{1}{\alpha}f(t) \leq \frac{t}{\sqrt{1 + \alpha f(t)^{2\alpha-2}}} \leq f(t) \quad \text{for all } t \geq 0.$$

We claim that

$$F(t) := f(t)\sqrt{1 + \alpha f(t)^{2\alpha-2}} - t \geq 0 \quad \text{for all } t \geq 0.$$

Indeed we see $F(0) = 0$ and by (2.1)

$$F'(t) = \frac{\alpha(\alpha-1)f(t)^{2\alpha-2}}{1 + \alpha f(t)^{2\alpha-2}} > 0.$$

Thus $F(t) \geq 0$ for all $t \geq 0$, that is,

$$\frac{t}{\sqrt{1 + \alpha f(t)^{2\alpha-2}}} \leq f(t) \quad \text{for all } t \geq 0$$

is proved. The other inequality is also shown in a similar way. ∎

2.2. Functional framework.

In what follows, we denote the unique global solution of (2.1) by f. We also denote the inverse of f by f^{-1}. Hereafter in this paper, we use the following notation:

$$\|v\|_{H^1}^2 = \int_{\mathbb{R}^N} |\nabla v|^2 + v^2 \, dx, \ v \in H^1(\mathbb{R}^N).$$

Lemma 2.4. *Assume $\alpha > 1$, (a1), (a2) and (h2). Then $I(v)$ ($I_\infty(v)$) is well-defined on $H^1(\mathbb{R}^N)$ and of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$.*

Proof. By (iii) of Lemma 2.2 and (h2), it follows

$$I(v) \leq \int_{\mathbb{R}^N} |\nabla v|^2 + a(x)f(v)^2 \, dx + \int_{\mathbb{R}^N} cf(v)^2 + cf(v)^{p+1} \, dx \leq C\|v\|_{H^1} + c\int_{\mathbb{R}^N} f(v)^{p+1} \, dx.$$

By Lemma 2.2 (i) and (ii), we have

$$f(v) \leq C_1\chi_{|v|\leq 1}v + C_2\chi_{|v|\geq 1}|v|^{\frac{1}{\alpha}}, \ f(v)^{p+1} \leq C_1v^2 + C_2|v|^{\frac{p+1}{\alpha}}$$

where χ is the characteristic function. Since $\frac{p+1}{\alpha} < \frac{2N}{N-2}$, $I(v)$ is well-defined on $H^1(\mathbb{R}^N)$. In a standard way, we can show $I(v) \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$. ∎

By Lemma 2.1, it suffices to show the existence of a nontrivial critical point of $I(v)$ ($I_\infty(v)$) to obtain a positive solution of (1.1) ((1.10) respectively). As introduced in Section 1, the natural function space associated to (1.1) is X defined in (1.7). The next lemma gives another characterization of the function space.

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Lemma 2.5. *It follows $X = Y$, where Y is defined in (1.8).*

Proof. First we show $Y \subset X$. For $v \in H^1(\mathbb{R}^N)$, we put $u = f(v)$. Then we have

$$|\nabla f(v)|^2 = |f'(v)|^2 |\nabla v|^2 = \frac{1}{1 + \alpha|f(v)|^{2\alpha-2}}|\nabla v|^2.$$

By (iii) of Lemma 2.2 and (2.1), we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx + \alpha \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha-2} \, dx = \int_{\mathbb{R}^N} |\nabla v|^2 + f(v)^2 \, dx \leq C\|v\|_{H^1}^2 < \infty. \quad (2.7)$$

Thus it follows $Y \subset X$.

To show $X \subset Y$, it suffices to show $f^{-1}(u) \in H^1(\mathbb{R}^N)$ for all $u \in X$. For $u \in X$, we put $v = f^{-1}(u)$. Then it follows

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, dx = \int_{\mathbb{R}^N} |(f^{-1})'(u)|^2 |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} (1 + \alpha|u|^{2\alpha-2})|\nabla u|^2 \, dx < \infty.$$

Next by (i) and (ii) of Lemma 2.2, it follows

$$\lim_{s \to 0} \frac{f^{-1}(s)}{s} = 1, \quad \lim_{s \to \infty} \frac{f^{-1}(s)}{s^\alpha} = c$$

for some $c > 0$. Thus there exist constants C_1, $C_2 > 0$ such that

$$|f^{-1}(s)| \leq C_1 \chi_{|s| \leq 1}|s| + C_2 \chi_{|s| \geq 1}|s|^\alpha \quad \text{for all } s \in \mathbb{R}.$$

Then we have

$$|v|^2 \leq C_1 \chi_{|u| \leq 1}|u|^2 + C_2 \chi_{|u| \geq 1}|u|^{2\alpha} \leq C_1|u|^2 + C_2|u|^{\frac{2N\alpha}{N-2}}.$$

By Sobolev's inequality, we obtain

$$\int_{\mathbb{R}^N} |v|^2 \, dx \leq C_1 \int_{\mathbb{R}^N} |u|^2 \, dx + C_2 \int_{\mathbb{R}^N} |u|^{\frac{2N\alpha}{N-2}} \, dx$$

$$\leq C_1 \int_{\mathbb{R}^N} |u|^2 \, dx + C_2' \left(\int_{\mathbb{R}^N} \alpha^2 |\nabla u|^2 |u|^{2\alpha-2} \, dx \right)^{\frac{N}{N-2}} < \infty.$$

Thus it follows $X \subset Y$ and hence $X = Y$. ∎

Remark 2.6. It follows from Lemma 2.5 that $J(u)$ (and $J_\infty(u)$) is well-defined on all Y.

Lemma 2.7. For any $v \in H^1(\mathbb{R}^N)$, we put $u = f(v)$. Then we have

(i) $J(u) = I(v)$,

(ii) $J'(u)u = I'(v)\dfrac{f(v)}{f'(v)}$.

Proof. Firstly we prove that $\dfrac{f(v)}{f'(v)} \in H^1(\mathbb{R}^N)$ for all $v \in H^1(\mathbb{R}^N)$. Since

$$\nabla\left(\frac{f(v)}{f'(v)}\right) = \frac{1 + \alpha^2|f(v)|^{2\alpha-2}}{1 + \alpha|f(v)|^{2\alpha-2}}\nabla v \le \alpha\nabla v,$$

we see that

$$\int_{\mathbb{R}^N}\left|\nabla\left(\frac{f(v)}{f'(v)}\right)\right|^2 dx \le \alpha^2\int_{\mathbb{R}^N}|\nabla v|^2\, dx \quad \text{for all } v \in H^1(\mathbb{R}^N).$$

Moreover from Lemma 2.3, we have $\dfrac{|f(v)|}{|f'(v)|} \le \alpha|v|$. Thus we observe that

$$\int_{\mathbb{R}^N}\left|\frac{f(v)}{f'(v)}\right|^2 dx \le \alpha^2\int_{\mathbb{R}^N}|v|^2\, dx \quad \text{for all } v \in H^1(\mathbb{R}^N).$$

Therefore we obtain

$$\left\|\frac{f(v)}{f'(v)}\right\|_{H^1} \le \alpha\|v\|_{H^1} \quad \text{for all } v \in H^1(\mathbb{R}^N). \tag{2.8}$$

Next we substitute $u = f(v)$ into $J(u)$. Then it follows

$$
\begin{aligned}
J(u) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla f(v)|^2 + a(x)f(v)^2\, dx + \frac{\alpha}{2}\int_{\mathbb{R}^N}|\nabla f(v)|^2|f(v)|^{2\alpha-2}\, dx - \int_{\mathbb{R}^N}H(f(v))\, dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2(1 + \alpha|f(v)|^{2\alpha-2})|f'(v)|^2 + a(x)f(v)^2\, dx - \int_{\mathbb{R}^N}H(f(v))\, dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + a(x)f(v)^2\, dx - \int_{\mathbb{R}^N}H(f(v))\, dx \\
&= I(v)
\end{aligned}
$$

and we obtain (i).

Finally we are going to show (ii). We observe that for $\phi \in Y$,

$$J'(u)\phi = \int_{\mathbb{R}^N}\nabla u \cdot \nabla\phi + a(x)u\phi\, dx + \alpha(\alpha-1)\int_{\mathbb{R}^N}|\nabla u|^2|u|^{2\alpha-4}u\phi\, dx$$

$$+ \alpha\int_{\mathbb{R}^N}\nabla u \cdot \nabla\phi|u|^{2\alpha-2}\, dx - \int_{\mathbb{R}^N}h(u)\phi\, dx,$$

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and for $\phi \in H^1(\mathbb{R}^N)$,

$$I'(v)\phi = \int_{\mathbb{R}^N} \nabla v \cdot \nabla \phi + a(x)f(v)f'(v)\phi \, dx - \int_{\mathbb{R}^N} h(f(v))f'(v)\phi \, dx.$$

We substitute $u = f(v)$ into $J'(u)u$. Then it follows

$$J'(u)u = \int_{\mathbb{R}^N} |\nabla v|^2 (1 + \alpha^2 f^{2\alpha-2}(v))|f'(v)|^2 + a(x)f^2(v) \, dx - \int_{\mathbb{R}^N} h(f(v))f(v) \, dx$$

Since

$$\nabla v \cdot \nabla \left(\frac{f(v)}{f'(v)} \right) = \frac{1 + \alpha^2 |f(v)|^{2\alpha-2}}{1 + \alpha|f(v)|^{2\alpha-2}} |\nabla v|^2 = |\nabla v|^2 (1 + \alpha^2 |f(v)|^{2\alpha-2})|f'(v)|^2,$$

we have $J'(u)u = I'(v)\dfrac{f(v)}{f'(v)}$ for all $v \in H^1(\mathbb{R}^N)$. ∎

Remark 2.8. (i) We can easily see that $J_\infty(u)$ and $I_\infty(v)$ satisfy the same relation as in Lemma 2.7, that is, for $u = f(v)$,

$$J_\infty(u) = I_\infty(v), \quad J'_\infty(u)u = I'_\infty(v)\frac{f(v)}{f'(v)}.$$

(ii) As we have shown in Lemma 2.1, if $I'(v) = 0$, then $u = f(v)$ is a critical point of $J(u)$. Lemma 2.5 and 2.7 imply the converse holds, that is, if $u \in X$ satisfies $J'(u) = 0$, then $v = f^{-1}(u)$ is a critical point of $I(v)$.

3. Some properties on the limit equation and proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. To this aim, we define a least energy level for $I_\infty(v)$ by

$$\tilde{c}_\infty = \inf\{I_\infty(v) \, ; \, I'_\infty(v) = 0, \, v \in H^1(\mathbb{R}^N) \backslash \{0\}\}.$$

Proposition 3.1. *Assume* (h1) *and* (h2). *Then semilinear problem* (1.12) *has at least one solution* $\tilde{w}(x)$ *which has the following properties:*

(i) $\tilde{w} \in C^2(\mathbb{R}^N, \mathbb{R})$ *and* $\tilde{w}(x) > 0$ *for all* $x \in \mathbb{R}^N$.
(ii) $\tilde{w}(x) = \tilde{w}(|x|)$ *and* $\dfrac{\partial \tilde{w}}{\partial r} < 0$.
(iii) *There exist* $c, \delta > 0$ *such that* $|D^k \tilde{w}(x)| \leq ce^{-\delta|x|}$ *for all* $x \in \mathbb{R}^N$ *and* $|k| \leq 2$.
(iv) $\tilde{w}$ *is a least energy solution of* (1.12), *that is,* $I_\infty(\tilde{w}) = \tilde{c}_\infty$.

Proof. We claim that

$$\lim_{s \to \infty} \frac{h(f(s))f'(s)}{s^{\frac{N+2}{N-2}}} = 0.$$

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Indeed from $(h2)$ and (ii) of Lemma 2.2, we see

$$\lim_{s\to\infty}\frac{h(f(s))f'(s)}{s^{\frac{N+2}{N-2}}}=\lim_{s\to\infty}\left(\frac{s^{\frac{2\alpha-2}{\alpha}}}{1+\alpha f(s)^{2\alpha-2}}\right)^{\frac{1}{2}}\frac{h(f(s))}{f(s)^{\frac{(2\alpha-1)N+2}{N-2}}}\left(\frac{f(s)}{s^{\frac{1}{\alpha}}}\right)^{\frac{(2\alpha-1)N+2}{N-2}}=0.$$

Hence $h(f(s))f'(s)$ has a subcritical growth as $s\to\infty$. Then we can easily see that the existence of a least energy solution $\tilde{w}$ having the above properties (i)–(iv) follows directly from fundamental results due to Berestycki-Lions [**8**] and Berestycki-Gallouët-Kavian [**7**]. We omit the details here. ∎

Remark 3.2. As in [**13**], we can prove Proposition 3.1 under very weaker conditions on $h(s)$. More precisely, we only require

$(h1')$ $\displaystyle\lim_{s\to 0+}\frac{h(s)}{s}=0.$

$(h2')$ There exist $c>0$ and $1<p+1<\infty$ for $N=1,\ 2,\ 1<p<\dfrac{(2\alpha-1)N+2}{N-2}$ for $N\geq 3$ such that $h(s)\leq c(1+|s|^p)$ for all $s\geq 0$.

We remark that it follows from Lemma 2.1 that $w=f(\tilde{w})$ is a positive solution of (1.10). Moreover we can easily see that $w=f(\tilde{w})$ also satisfies (i)–(iii) of Proposition 3.1 (see Lemma 3.6 below). At this stage, to prove Theorem 1.4, we only show $w=f(\tilde{w})$ satisfies (iii) and (iv) of Theorem 1.4. As we have observed in Lemma 2.7, we know that $I_\infty(\tilde{w})=J_\infty(w)$. This implies $c_\infty\leq\tilde{c}_\infty$. In order to prove the reverse inequality, we need the following lemma.

Lemma 3.3. *Assume $(h1)$–$(h3)$. Let $J'_\infty(u)=0$ and $u\not\equiv 0$. Then $J_\infty(su)\leq J_\infty(u)$ for all $s\geq 0$. Moreover for any $T>1$, there exists $c=c(T)>0$ such that*

$$J_\infty(u)-J_\infty(su)\geq c(s-1)^2\quad\text{for all }s\in[0,T].$$

Proof. We put

$$k(s):=J_\infty(su)=\frac{s^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2+a_\infty u^2\,dx+\frac{\alpha}{2}s^{2\alpha}\int_{\mathbb{R}^N}|\nabla u|^2|u|^{2\alpha-2}\,dx-\int_{\mathbb{R}^N}H(su)\,dx.$$

Then we have

$$k'(s)=s\int_{\mathbb{R}^N}|\nabla u|^2+a_\infty u^2\,dx+\alpha^2 s^{2\alpha-1}\int_{\mathbb{R}^N}|\nabla u|^2|u|^{2\alpha-2}\,dx-\int_{\mathbb{R}^N}h(su)u\,dx.$$

Since $J'_\infty(u)=0$, it follows

$$k'(s)=\alpha^2(s-s^{2\alpha-1})\int_{\mathbb{R}^N}|\nabla u|^2|u|^{2\alpha-2}\,dx+\int_{\mathbb{R}^N}h(u)su-h(su)u\,dx.$$

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From $(h3)$, it follows

$$\frac{h(su)}{su} < \frac{h(u)}{u} \text{ for } s < 1, \quad \frac{h(su)}{su} > \frac{h(u)}{u} \text{ for } s > 1.$$

Thus we have $k'(s) = 0$ if and only if $s = 0,\ 1$, $k'(s) > 0$ for $0 < s < 1$ and $k'(s) < 0$ for $s > 1$. Thus we obtain $k(s) \leq k(1)$ for all $s \geq 0$ and hence $J_\infty(su) \leq J_\infty(u)$.

Next it follows from

$$0 = J'_\infty(u)u = \int_{\mathbb{R}^N} |\nabla u|^2 + a_\infty u^2 \, dx + \alpha^2 \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha-2} \, dx - \int_{\mathbb{R}^N} h(u)u \, dx$$

that

$$
\begin{aligned}
J_\infty(u) - J_\infty(su) &= \frac{1-s^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + a_\infty u^2 \, dx + \frac{\alpha}{2}(1-s^{2\alpha}) \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha-2} \, dx \\
&\quad + \int_{\mathbb{R}^N} H(su) - H(u) \, dx \\
&= \left(\frac{\alpha-1}{2\alpha} - \frac{s^2}{2} + \frac{s^{2\alpha}}{2\alpha} \right) \int_{\mathbb{R}^N} |\nabla u|^2 + a_\infty u^2 \, dx \\
&\quad + \int_{\mathbb{R}^N} H(su) - H(u) + \frac{1}{2\alpha} h(u)u - \frac{s^{2\alpha}}{2\alpha} h(u)u \, dx.
\end{aligned}
$$

We put

$$k_1(s) := \frac{\alpha-1}{2\alpha} - \frac{s^2}{2} + \frac{s^{2\alpha}}{2\alpha}.$$

We claim that for any $T > 1$, there exists $c(T) > 0$ such that

$$k_1(s) \geq c(s-1)^2 \quad \text{for all } s \in [0, T]. \tag{3.1}$$

We take $\delta > 0$ so that $[1-\delta, 1+\delta] \subset [0, T]$. By Taylor's expansion, we have

$$k_1(s) = \frac{\alpha-1}{2}(s-1)^2 + O((s-1)^3), \quad s \in [1-\delta, 1+\delta].$$

Thus taking $\delta > 0$ smaller if necessary, it follows $k_1(1-\delta) > 0$ and $k_1(1+\delta) > 0$. Choosing small $c > 0$, we obtain

$$k_1(s) \geq c(s-1)^2 \quad \text{for all } s \in [1-\delta, 1+\delta]. \tag{3.2}$$

Moreover we observe that $k'_1(s) = -s + s^{2\alpha-1}$. Then $k'_1(s) \leq 0$ for $s \in [0, 1-\delta]$ and $k'_1(s) > 0$ for $s \in [1+\delta, T]$. Thus we have $k_1(s) \geq \min\{k_1(1-\delta), k_1(1+\delta)\}$. Finally we take $c(T) > 0$ so that

$$\min\{k_1(1-\delta), k_1(1+\delta)\} \geq \max\{c(T), c(T)(T-1)^2\}.$$

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Then we see

$$k_1(s) \geq c(s-1)^2 \quad \text{for all } s \in [0, 1 - \delta] \cup [1 + \delta, T]. \tag{3.3}$$

Thus (3.1) follows from (3.2) and (3.3).

Next we put

$$k_2(s) := H(su) - H(u) + \frac{1}{2\alpha}h(u)u - \frac{s^{2\alpha}}{2\alpha}h(u)u.$$

We claim that $k_2(s) \geq 0$ for all $s \geq 0$. From $(h3)$, we can see that

$$k_2'(s) = h(su)u - s^{2\alpha-1}h(u)u = s^{2\alpha-1}u^{\theta+1}\left(s^{\theta-2\alpha+1}\frac{h(su)}{(su)^\theta} - \frac{h(u)}{u^\theta}\right)\begin{cases} \geq 0 & \text{for } s > 1 \\ \leq 0 & \text{for } s < 1. \end{cases}$$

Since $k_2(0) = -H(u) + \frac{1}{2\alpha}h(u)u > 0$ and $k_2(1) = 0$, we have $k_2(s) \geq 0$ for all $s \geq 0$. ∎

Lemma 3.4. *Assume $(h1)$–$(h3)$. Then it follows $c_\infty = \tilde{c}_\infty$. Especially $\tilde{w}$ is a least energy solution of (1.12) if and only if $w = f(\tilde{w})$ is a least energy solution of (1.10).*

Proof. Firstly we show that c_∞ is well-defined and positive. In fact for any $u \not\equiv 0$ with $J_\infty'(u) = 0$, it follows from $(h3)$ that

$$J_\infty(u) \geq \left(\frac{1}{2} - \frac{1}{\theta+1}\right)\int_{\mathbb{R}^N}|\nabla u|^2 + a_\infty u^2\,dx + \left(\frac{\alpha}{2} - \frac{\alpha^2}{\theta+1}\right)\int_{\mathbb{R}^N}|\nabla u|^2|u|^{2\alpha-2}\,dx > 0.$$

Now let $\tilde{w}(x)$ be a least energy solution of (1.12). We put $w = f(\tilde{w})$. Then by Lemma 2.7, we have $c_\infty \leq J_\infty(w) = I_\infty(\tilde{w}) = \tilde{c}_\infty$.

On the other hand, $\tilde{c}_\infty$ has a *Mountain Pass characterization*, that is,

$$\tilde{c}_\infty = \inf_{\gamma \in \Gamma}\max_{t \in [0,1]}I_\infty(\gamma(t)), \quad \Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)); \gamma(0) = 0, \ I_\infty(\gamma(1)) < 0\}$$

(see [13] and the references there in). Let u_n be a minimizing sequence such that $J_\infty(u_n) \to c_\infty$ and $J_\infty'(u_n) = 0$ for all $n \in \mathbb{N}$. We choose $t_n > 1$ so that $I_\infty(f^{-1}(t_n u_n)) < 0$ for each n. Finally we define $\gamma_n(t) = f^{-1}(tt_n u_n)$. Then $\gamma_n(t) \in \Gamma$ and

$$\tilde{c}_\infty \leq \max_{t \in [0,1]}I_\infty(f^{-1}(tt_n u_n)) = \max_{t \in [0,1]}J_\infty(tt_n u_n).$$

Since $J_\infty'(u_n) = 0$, the maximum value is achieved at $t = \frac{1}{t_n}$ by Lemma 3.3. Thus we obtain $\tilde{c}_\infty \leq J_\infty(u_n) \to c_\infty$ and hence $\tilde{c}_\infty = c_\infty$. ∎

Remark 3.5. A similar result as Lemma 3.4 has been obtained in [12] where another type of quasilinear Schrödinger equations was studied.

Next we prove precise decay estimates of the least energy solution of (1.10) at infinity.

Lemma 3.6. *Assume (h1)–(h2). Let $w(x)$ be a least energy solution of (1.10). Then $w \in C^2(\mathbb{R}^N, \mathbb{R})$, $w(x) = w(|x|)$ and $\frac{\partial w}{\partial r} < 0$. Moreover there exist $c, c' > 0$ such that*

$$\lim_{|x| \to \infty} e^{\sqrt{a_\infty}|x|}(|x| + 1)^{\frac{N-1}{2}} w(x) = c, \quad \lim_{r \to \infty} e^{\sqrt{a_\infty}r}(r + 1)^{\frac{N-1}{2}} \frac{\partial w}{\partial r} = -c'. \quad (3.4)$$

Proof. Let $\tilde{w}$ be a least energy solution of (1.12). Then by Lemma 3.4, $w = f(\tilde{w})$ is a least energy solution of (1.10). By (i), (ii) of Proposition 3.1 and the fact $f \in C^\infty(\mathbb{R}, \mathbb{R})$, we have $w \in C^2(\mathbb{R}^N, \mathbb{R})$ and $w(x) = w(|x|)$. Moreover since $\frac{\partial w}{\partial r} = f'(\tilde{w})\frac{\partial \tilde{w}}{\partial r}$, $f'(\tilde{w}) > 0$ and $\frac{\partial \tilde{w}}{\partial r} < 0$, it follows $\frac{\partial w}{\partial r} < 0$.

Next by Lemma 2.2 and (h1), we can apply the standard comparison principle to obtain

$$|D^k \tilde{w}(x)| \le c e^{-\delta|x|} \quad \text{for all } \delta \in (0, \sqrt{a_\infty}), \ x \in \mathbb{R}^N, \ |k| \le 2 \text{ and some } c > 0.$$

By Lemma 2.2, we observe that

$$|w| = |f(\tilde{w})| \le |\tilde{w}| \le c e^{-\delta|x|},$$
$$\left| \frac{\partial w}{\partial x_i} \right| = \left| f'(\tilde{w}) \frac{\partial \tilde{w}}{\partial x_i} \right| \le \left| \frac{\partial \tilde{w}}{\partial x_i} \right| \le c e^{-\delta|x|},$$
$$\left| \frac{\partial^2 w}{\partial x_i^2} \right| = \left| f''(\tilde{w})(\frac{\partial \tilde{w}}{\partial x_i})^2 + f'(\tilde{w}) \frac{\partial^2 \tilde{w}}{\partial x_i^2} \right| \le \alpha(\alpha - 1)|\tilde{w}|^{2\alpha - 3} \left| \frac{\partial \tilde{w}}{\partial x_i} \right|^2 + \left| \frac{\partial^2 \tilde{w}}{\partial x_i^2} \right| \le c e^{-\delta|x|}.$$

Thus for $|x| \gg 1$, we have

$$-\Delta w + a_\infty w = h(w) + \alpha w^{2\alpha - 2} \Delta w + \alpha(\alpha - 1) w^{2\alpha - 3}|\nabla w|^2 \le c e^{-(1+\eta)\delta|x|} + c e^{-(2\alpha - 1)\delta|x|}.$$

Since $2\alpha - 1 > 1$ and we can take δ arbitrary close to $\sqrt{a_\infty}$, it follows

$$-\Delta w + a_\infty w = o(G(x)) \text{ as } |x| \to \infty$$

where $G(x)$ is the fundamental solution of $-\Delta + a_\infty I$. Then by Gidas-Ni-Nirenberg's asymptotic result [**15**], we obtain (3.4). ∎

Proof of Theorem 1.4. Theorem 1.4 follows directly from Proposition 3.1, Lemma 3.4 and 3.6. ∎

Remark 3.7. From Lemma 3.6 and $\frac{\partial w}{\partial r} < 0$, there exists $c > 1$ such that

$$\frac{1}{c} e^{-\sqrt{a_\infty}|x|}(|x| + 1)^{-\frac{N-1}{2}} \le w(x) \le c e^{-\sqrt{a_\infty}|x|}(|x| + 1)^{-\frac{N-1}{2}}, \quad (3.5)$$

$$-c e^{-\sqrt{a_\infty}r}(r + 1)^{-\frac{N-1}{2}} \le \frac{\partial w}{\partial r} \le -\frac{1}{c} e^{-\sqrt{a_\infty}r}(r + 1)^{-\frac{N-1}{2}}, \quad (3.6)$$

$$|\nabla w(x)| \le c w(x) \quad \text{for all } x \in \mathbb{R}^N. \quad (3.7)$$

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Hereafter in this paper, we may assume that $a_\infty = 1$ without loss of generality.

4. Properties of energy functional I

In this section, we prepare some lemmas to find a positive solution of (1.1) and (1.5). First we show $I(v)$ has the Mountain Pass Geometry. Next we apply concentration compactness principle and give a global compactness type result for $I(v)$. In order to prove these properties, it is rather convenient to consider $I(v)$ rather than $J(u)$ because $J(u)$ has the quasilinear term. Finally we will show any Cerami sequences are bounded in $H^1(\mathbb{R}^N)$.

Lemma 4.1. *Assume* $(a1)$, $(a2)$ *and* $(h1)$–$(h3)$. *Then* $I(v)$ *has the Mountain Pass Geometry.*

Proof. First we observe that

$$I(v) \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + a_0 f(v)^2 dx - \int_{\mathbb{R}^N} H(f(v)) \, dx.$$

We put $\tilde{H}(s) = -\frac{a_0}{2} f(s)^2 + H(f(s))$. Then by Lemma 2.2, we have

$$\lim_{s \to 0} \frac{\tilde{H}(s)}{s^2} = -\frac{a_0}{2}, \quad \lim_{s \to \infty} \frac{\tilde{H}(s)}{s^{\frac{2N}{N-2}}} = 0.$$

Thus for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\tilde{H}(s) \leq -\frac{a_0 - \epsilon}{2} s^2 + C_\epsilon |s|^{\frac{2N}{N-2}} \quad \text{for all } s \in \mathbb{R}.$$

We take $\epsilon = \frac{a_0}{2}$. Then we obtain

$$I(v) \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{a_0}{4} \int_{\mathbb{R}^N} v^2 \, dx - C_{\frac{a_0}{2}} \int_{\mathbb{R}^N} |v|^{\frac{2N}{N-2}} \, dx.$$

Thus there exist c, $\rho > 0$ such that $I(v) \geq c$ for all $v \in H^1(\mathbb{R}^N)$ with $\|v\|_{H^1} = \rho$.

Next we choose $\phi \in C_0^\infty(\mathbb{R}^N)$ so that $\phi(x) \geq 0$ for $x \in \mathbb{R}^N$ and

$$\frac{\alpha}{2} \int_{\mathbb{R}^N} |\nabla \phi|^2 \phi^{2\alpha-2} \, dx < \frac{1}{2} \int_{\mathbb{R}^N} \phi^{\theta+1} \, dx.$$

Then from $(h3)$, we have for $s > 1$,

$$I(f^{-1}(s\phi)) = J(s\phi) \leq \frac{s^2}{2} \int_{\mathbb{R}^N} |\nabla \phi|^2 + \|a\|_{L^\infty} \phi^2 \, dx + (\frac{s^{2\alpha}}{2} - s^{\theta+1}) \int_{\mathbb{R}^N} \phi^{\theta+1} \, dx.$$

Since $\theta \geq 2\alpha - 1$, it follows $I(f^{-1}(s\phi)) \to -\infty$ as $s \to \infty$. Thus $I(v)$ has the Mountain Pass Geometry. ∎

Lemma 4.2. *Assume $(a1)$, $(a2)$, $(h1)$ and $(h2)$. For $c > 0$, let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a sequence such that*

$$I(v_n) \to c, \ I'(v_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^N) \text{ and } \|v_n\|_{H^1} \text{ is bounded.}$$

Then passing to a subsequence, there exist $v_0 \in H^1(\mathbb{R}^N)$, $k \in \mathbb{N} \cup \{0\}$, $\{y_n^i\} \subset \mathbb{R}^N$, $i = 1, \cdots, k$ and $w^i \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$I(v_n) \to I(v_0) + \sum_{i=1}^{k} I_\infty(w^i),$$

$$\left\| v_n - v_0 - \sum_{i=1}^{k} w^i(\cdot - y_n^i) \right\|_{H^1} \to 0,$$

$$I'(v_0) = 0,$$

$$I'_\infty(w^i) = 0,$$

$$|y_n^i| \to \infty, \ |y_n^i - y_n^{i'}| \to \infty \text{ as } n \to \infty.$$

Proof. We show that if $v_n \rightharpoonup v_0$ in $H^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} f(v_n) f'(v_n) \phi \, dx \to \int_{\mathbb{R}^N} f(v_0) f'(v_0) \phi \, dx,$$

$$\int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) \phi \, dx \to \int_{\mathbb{R}^N} h(f(v_0)) f'(v_0) \phi \, dx \tag{4.1}$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$. We prove (4.1). We fix $\phi \in C_0^\infty(\mathbb{R}^N)$. Since $v_n \rightharpoonup v_0$ in $H^1(\mathbb{R}^N)$, passing to a subsequence, we may assume that

$$v_n \to v_0 \text{ in } L_{loc}^q(\mathbb{R}^N) \quad \text{for all } q \in [2, \frac{2N}{N-2}),$$

$$v_n \to v_0 \text{ a.e. in } \mathbb{R}^N .$$

Moreover we see for all $R > 0$, there exists $\psi \in L^q(B_R(0))$ such that $|v_n| \le \psi$ a.e. $B_R(0)$. Since $f, \ f' \in C^\infty(\mathbb{R})$, it follows $h(f(v_n)) f'(v_n) \phi \to h(f(v_0)) f'(v_0) \phi$ a.e. in $\mathbb{R}^N$. From $(h2)$, Lemma 2.2 and 2.3, we obtain

$$|h(f(v_n))| |f'(v_n)| |\phi| \le c|f(v_n)| |f'(v_n)| |\phi| + c|f(v_n)|^p |f'(v_n)| |\phi|$$

$$\le c \frac{|f(v_n)|^2}{|v_n|} |\phi| + c \frac{|f(v_n)|^{p+1}}{|v_n|} |\phi|$$

$$\le C|v_n| |\phi| + C'|v_n|^{\frac{p+1}{\alpha} - 1} |\phi|$$

$$\le C|\psi| |\phi| + C'|\psi|^{\frac{p+1}{\alpha} - 1} |\phi|. \tag{4.2}$$

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We take $R > 0$ large enough so that $\operatorname{supp} \phi \subset B_R(0)$. Then by (4.2), $\frac{p+1}{\alpha} < \frac{2N}{N-2}$ and Hölder's inequality, we have

$$
\int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) \phi \, dx \leq C \int_{\operatorname{supp} \phi} |\psi| |\phi| \, dx + C' \int_{\operatorname{supp} \phi} |\psi|^{\frac{p+1}{\alpha}-1} |\phi| \, dx
$$

$$
\leq C \left(\int_{B_R(0)} |\psi|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\operatorname{supp} \phi} |\phi|^2 \, dx \right)^{\frac{1}{2}}
$$

$$
+ C' \left(\int_{B_R(0)} |\psi|^{\frac{p+1}{\alpha}} \, dx \right)^{\frac{p+1-\alpha}{p+1}} \left(\int_{\operatorname{supp} \phi} |\phi|^{\frac{p+1}{\alpha}} \, dx \right)^{\frac{\alpha}{p+1}}
$$

$$
< \infty.
$$

Thus by Lebesgue's convergence theorem, we obtain (4.1).

Arguing similarly, we can see if $v_n \rightharpoonup v_0$ in $H^1(\mathbb{R}^N)$, then

$$
\int_{\mathbb{R}^N} f(v_n)^2 dx \to \int_{\mathbb{R}^N} f(v_0)^2 \, dx, \quad \int_{\mathbb{R}^N} H(f(v_n)) dx \to \int_{\mathbb{R}^N} H(f(v_0)) \, dx.
$$

Then Lemma 4.2 follows from a similar argument as in [18]. ∎

Now we define

$$
H_G^1(\mathbb{R}^N) := \{ u \in H^1(\mathbb{R}^N) \, ; u(gx) = u(x) \text{ for all } g \in G \}.
$$

By the principle of symmetric criticality due to Palais [25], we see that if the restriction $I|_{H_G^1(\mathbb{R}^N)}(v)$ has a critical point, then it is in fact a critical point of $I(v)$. Thus to find a G-invariant solution, it suffices to work in the restricted space $H_G^1(\mathbb{R}^N)$. As a consequence of Lemma 4.2, we can obtain the following corollary.

Corollary 4.3. *Assume* (a1)–(a3), (h1) *and* (h2). *For* $c < mc_\infty$, *let* $\{v_n\} \subset H_G^1(\mathbb{R}^N)$ *be a sequence such that*

$$
I(v_n) \to c, \ I'(v_n) \to 0 \text{ in } (H_G^1)^{-1} \text{ and } \|v_n\|_{H^1} \text{ is bounded.}
$$

Then $\{v_n\}$ *has a convergent subsequence.*

Proof. Corollary 4.3 follows from a similar argument in [1]. ∎

Next we show the boundedness of any Cerami sequences of $I(v)$.

Lemma 4.4. *Assume* $(a1)$, $(a2)$ *and* $(h1)$–$(h3)$. *For* $c > 0$, *let* $\{v_n\}$ *be a Cerami sequence, that is,*

$$I(v_n) \to c \text{ and } I'(v_n)(1 + \|v_n\|_{H^1}) \to 0 \text{ in } H^{-1} \text{as } n \to \infty. \tag{4.3}$$

Then $\|v_n\|_{H^1}$ *is bounded.*

Proof. First we observe that $\|v_n\|_{H^1}$ is controlled by $\int_{\mathbb{R}^N} |\nabla v_n|^2 + a(x)f(v_n)^2 \, dx$. In fact from $I(v_n) \to c$, we have

$$\int_{\mathbb{R}^N} H(f(v_n)) \, dx \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 + a(x)f(v_n)^2 dx + c + o(1).$$

From $(h3)$ and Lemma 2.2, it follows $H(f(s)) \geq Cs^2$ for all $s \geq 1$. Then we have

$$\int_{\mathbb{R}^N} a(x)v_n^2 \, dx \leq \int_{|v_n| \leq 1} a(x)v_n^2 \, dx + \int_{|v_n| \geq 1} a(x)v_n^2 \, dx$$

$$\leq C \int_{|v_n| \leq 1} a(x)f(v_n)^2 \, dx + C' \int_{|v_n| \geq 1} H(f(v_n)) \, dx$$

$$\leq C \int_{\mathbb{R}^N} a(x)f(v_n)^2 \, dx + C' \int_{\mathbb{R}^N} H(f(v_n)) \, dx.$$

Thus it follows

$$\|v_n\|_{H^1}^2 \leq \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + C \int_{\mathbb{R}^N} a(x)f(v_n)^2 \, dx + C' \int_{\mathbb{R}^N} H(f(v_n)) \, dx$$

$$\leq C \int_{\mathbb{R}^N} |\nabla v_n|^2 + a(x)f(v_n)^2 dx + C' + o(1).$$

Next we prove $\int_{\mathbb{R}^N} |\nabla v_n|^2 + a(x)f(v_n)^2 \, dx$ is bounded. We put $u_n = f(v_n)$ and recall that

$$\left\| \frac{f(v_n)}{f'(v_n)} \right\|_{H^1} \leq \alpha \|v_n\|_{H^1}, \quad I'(v_n)\frac{f(v_n)}{f'(v_n)} = J'(u_n)u_n, \quad I(v_n) = J(u_n).$$

From (4.3), we have

$$J(u_n) = c + o(1), \quad J'(u_n)u_n \leq \alpha I'(v_n)\|v_n\|_{H^1} = o(1). \tag{4.4}$$

It follows from $(h3)$ that

$$\left(\frac{1}{2} - \frac{1}{\theta + 1}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 + a(x)u_n^2 \, dx + \left(\frac{\alpha}{2} - \frac{\alpha^2}{\theta + 1}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 |u_n|^{2\alpha - 2} \, dx \leq c + o(1). \tag{4.5}$$

Since

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + a(x)f(v_n)^2 \, dx = \int_{\mathbb{R}^N} (1 + \alpha|u_n|^{2\alpha - 2})|\nabla u_n|^2 + a(x)u_n^2 \, dx, \tag{4.6}$$

it follows from (4.5) that $\int_{\mathbb{R}^N} |\nabla v_n|^2 + a(x)f(v_n)^2\,dx$ is bounded for $\theta > 2\alpha - 1$. We suppose $\theta = 2\alpha - 1$. Then we have from (4.5)

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + a(x)u_n^2\,dx \leq c + o(1). \tag{4.7}$$

By (4.6) and (4.7), it suffices to show

$$K(u_n) := \int_{\mathbb{R}^N} |u_n|^{2\alpha-2}|\nabla u_n|^2\,dx \tag{4.8}$$

is bounded. From (4.4) and $(h2)$, we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 |u_n|^{2\alpha-2}\,dx \leq c + o(1) + \int_{\mathbb{R}^N} H(u_n)dx$$

$$\leq c + o(1) + C\int_{\mathbb{R}^N} u_n^2 dx + C'\int_{\mathbb{R}^N} |u_n|^{p+1}\,dx$$

$$\leq c + o(1) + C'\int_{\mathbb{R}^N} |u_n|^{p+1}dx. \tag{4.9}$$

Now we suppose $N = 1$, 2 or $N = 3$ and $p \leq 5$. Then by Sobolev's inequality, it follows

$$\int_{\mathbb{R}^N} |u_n|^{p+1}\,dx \leq C\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 + a(x)u_n^2\,dx\right)^{\frac{p+1}{2}} \leq C'.$$

Thus $K(u_n)$ is bounded.

Next we consider the case $N \geq 3$ and $p < \dfrac{(2\alpha-1)N+4}{N}$. Using Hölder's and Sobolev's inequalities, it follows

$$\int_{\mathbb{R}^N} |u_n|^{p+1}\,dx \leq \left(\int_{\mathbb{R}^N} u_n^2\,dx\right)^{\frac{2N\alpha-(N-2)(p+1)}{2(N\alpha-N+2)}}\left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N\alpha}{N-2}}\,dx\right)^{\frac{(N-2)(p-1)}{2(N\alpha-N+2)}}$$

$$\leq C\left(\int_{\mathbb{R}^N} u_n^2\,dx\right)^{\frac{2N\alpha-(N-2)(p+1)}{2(N\alpha-N+2)}}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2|u_n|^{2\alpha-2}\,dx\right)^{\frac{N(p-1)}{2(N\alpha-N+2)}}$$

$$\leq CK(u_n)^{\frac{N(p-1)}{2(N\alpha-N+2)}}. \tag{4.10}$$

Here we used (4.7). From (4.9) and (4.10), we obtain

$$K(u_n) \leq c + o(1) + CK(u_n)^{\frac{N(p-1)}{2(N\alpha-N+2)}}.$$

Since $p < \frac{(2\alpha-1)N+4}{N}$, or equivalently $\frac{N(p-1)}{2(N\alpha-N+2)} < 1$, we have $K(u_n)$ is bounded and hence $\|v_n\|_{H^1}$ is bounded. ∎

Corollary 4.5. *Suppose $h(u) = u^p$, $2\alpha \leq p + 1 < \infty$ for $N = 1, 2$ and $2\alpha - 1 \leq p < \frac{(2\alpha-1)N+2}{N-2}$ for $N \geq 3$. Let $\{v_n\}$ be a Cerami sequence of $I(v)$. Then $\|v_n\|_{H^1}$ is bounded.*

Proof. If $p + 1 > 2\alpha$ or $p + 1 = 2\alpha$ and $N = 1$ or 2, the claim follows similarly. We suppose $N \geq 3$ and $p + 1 = 2\alpha$. It is sufficient to show $K(u_n)$ defined in (4.8) is bounded. Arguing as above, we have

$$K(u_n) \leq c + o(1) + \frac{1}{2\alpha} \int_{\mathbb{R}^N} |u_n|^{2\alpha}\, dx$$

$$\leq c + o(1) + \frac{1}{2\alpha} \left(\int_{\mathbb{R}^N} |u_n|^2\, dx \right)^{\frac{2\alpha}{N\alpha - N + 2}} \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N\alpha}{N-2}}\, dx \right)^{\frac{(\alpha-1)(N-2)}{N\alpha - N + 2}}$$

$$\leq c + o(1) + C \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 |u_n|^{2\alpha - 2}\, dx \right)^{\frac{N(\alpha-1)}{N\alpha - N + 2}}$$

$$= c + o(1) + CK(u_n)^{\frac{N(\alpha-1)}{N\alpha - N + 2}}.$$

Since $\frac{N(\alpha-1)}{N\alpha - N + 2} < 1$, $K(u_n)$ is bounded. ∎

Remark 4.6. As we have observed above, we require only $2\alpha - 1 \leq p < \frac{(2\alpha-1)N+2}{N-2}$ if $h(s) = s^p$. This is because of the growth order at infinity when $\theta = 2\alpha - 1$ in $(h3)$. In the case $h(s) = s^p$ and $p = 2\alpha - 1$, the growth order of $H(s)$ at infinity is exactly equal to $s^{2\alpha}$. However $(2\alpha - 1)h(s) \leq h'(s)s$ does not implies $H(s)$ behaves like $s^{2\alpha}$ at infinity in general. For example, $h(s) = s^{2\alpha-1} + s^q$ for some $q > 2\alpha - 1$ fulfills $(h3)$ with $\theta = 2\alpha - 1$. Thus we need to restrict the range of p for general nonlinearities.

5. Interaction estimate

In this section, we establish an interaction estimate which plays an important role in the existence result. To this aim, it seems to be better to use energy functional $J(u)$ rather than $I(v)$ because function $f(t)$ is nonlinear and concave. Even if we consider $J(u)$, we need delicate estimates since $J(u)$ has the quasilinear term.

Now let $\tilde{w}$ be a least energy solution of (1.12). For $l > 0$, we put

$$\gamma(s) := f^{-1}\left(s \sum_{i=1}^{m} f(\tilde{w}(x - le_i))\right) \in H_G^1(\mathbb{R}^N)$$

where $\{e_i\} \subset S^{N-1}$ are vectors defined in (1.3). In this setting, we have the following.

Proposition 5.1. *Assume $\alpha \geq \frac{3}{2}$, $(a1)$–$(a4)$ and $(h1)$–$(h3)$. Then there exists $l_0 > 0$ such that if $l \geq l_0$, then*

$$\sup_{s \geq 0} I(\gamma(s)) < mc_\infty. \tag{5.1}$$

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For simplicity, we write $w = f(\tilde{w})$, $\tilde{w}_i = \tilde{w}(x - le_i)$ and $u_i = f(\tilde{w}_i)$ for $i = 1, \cdots, m$. To prove Proposition 5.1, we need some preliminaries. For the proof of next lemma, we refer to [1, 9].

Lemma 5.2. *Assume (h3). For any integer $m \geq 2$ and $K > 0$, it follows:*

(i) *If $t_i \in [0, K]$ for all $i = 1, \cdots, m$, then*

$$H\left(\sum_{i=1}^{m} t_i\right) - \sum_{i=1}^{m} H(t_i) - \frac{1}{2} \sum_{i,j=1,\ i\neq j}^{m} h(t_i)t_j \geq 0.$$

(ii) *For any $\epsilon > 0$, there exists $\delta_{\epsilon,K} > 0$ such that if $t_i \in (0, \frac{K}{2})$ and $t_j \in (0, \delta_{\epsilon,K}]$ for $i \neq j$, then*

$$H\left(\sum_{i=1}^{m} t_i\right) - \sum_{i=1}^{m} H(t_i) - \frac{1}{2} \sum_{i,j=1,\ i\neq j}^{m} h(t_i)t_j$$

$$\geq \frac{1}{2} \sum_{i,j=1,\ i\neq j}^{m} h(t_i)t_j - \epsilon \sum_{j=1,\ j\neq i}^{m} \left(t_j + \frac{1}{2}t_j^2 + \frac{1}{2}t_i t_j\right).$$

For an interaction estimate of u_i and u_j, we have the following

Lemma 5.3. *Let i, $j \in [1, m]$ and $i \neq j$.*

(i) *For any $q > 0$ and $\delta \in (0, 1)$, there exists $C > 0$ such that*

$$\int_{\mathbb{R}^N} u_i^q u_j^q + |\nabla u_i|^q |\nabla u_j|^q \, dx \leq C e^{-q(1-\delta)|e_i - e_j|l} \quad \text{for all } l \geq 0.$$

(ii) *For any $R > 1$ and $q > 0$, there exist $C_R > 0$ and $l_1(R) > 0$ such that*

$$\int_{B_R(le_i)} u_j^q \, dx \leq C_R e^{-q|e_i - e_j|l} l^{-\frac{q(N-1)}{2}} \quad \text{for all } l \geq l_1.$$

(iii) *For any $R > 1$, there exist $M_R > 0$ and $l_2(R) > 0$ such that*

$$\int_{\partial B_R(le_i)} (\nabla u_i \cdot \nu)u_j - (\nabla u_j \cdot \nu)u_i \, dS \leq -M_R e^{-|e_i - e_j|l} l^{-\frac{N-1}{2}} \quad \text{for all } l \geq l_2$$

and $\lim_{R\to\infty} M_R = M^* > 0$, where ν is the outer unit normal vector on $\partial B_R(le_i)$.

(iv) *For any $R > 1$ and r, $s > 0$, $r \neq s$, there exist $K_R > 0$ and $l_3(R) > 0$ such that*

$$\int_{\mathbb{R}^N \setminus (\cup_{i=1}^{m} B_R(le_i))} u_i^r u_j^s \, dx \leq K_R e^{-\min\{r,s\}|e_i - e_j|l} l^{-\frac{\min\{r,s\}(N-1)}{2}} \quad \text{for all } l \geq l_3$$

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and $\lim\limits_{R \to \infty} K_R = 0$.

Proof. (i) and (ii) are rather standard.

(iii) First we observe that for every $x \in \mathbb{R}^N$,

$$-|x - l(e_j - e_i)| + l|e_j - e_i| \to \frac{x \cdot (e_j - e_i)}{|e_j - e_i|}, \quad \frac{|x - l(e_j - e_i)|}{l|e_j - e_i|} \to 1 \text{ as } l \to \infty. \tag{5.2}$$

Thus from (3.5), there exist $c > 1$ and $\tilde{l} > 0$ such that if $l \geq \tilde{l}$,

$$\frac{1}{c} e^{\frac{x \cdot (e_j - e_i)}{|e_j - e_i|}} \leq \frac{w(x - l(e_j - e_i))}{e^{-|e_j - e_i|l}(|e_j - e_i|l + 1)^{-\frac{N-1}{2}}} \leq c e^{\frac{x \cdot (e_j - e_i)}{|e_j - e_i|}} \quad \text{for all } x \in B_R(0). \tag{5.3}$$

Next we recall that $\nabla u \cdot \nu < 0$ by Lemma 3.6. Then from (3.5), we have

$$\int_{\partial B_R(le_i)} (\nabla u_i \cdot \nu) u_j \, dS \leq -c \int_{\partial B_R(0)} R^{-\frac{N-1}{2}} e^{-R} e^{\frac{x \cdot (e_j - e_i)}{|e_j - e_i|}} \, dS \times e^{-|e_j - e_i|l} l^{-\frac{N-1}{2}}$$

for some $c > 0$ independent of l and R. Next we claim that

$$\lim_{R \to \infty} \int_{\partial B_R(0)} R^{-\frac{N-1}{2}} e^{-R} e^{\frac{x \cdot (e_j - e_i)}{|e_j - e_i|}} \, dS > 0. \tag{5.4}$$

Indeed, up to rotation if necessary, we may assume that $\frac{e_j - e_i}{|e_j - e_i|} = (1, 0, \cdots, 0)$. Changing the variable $x = Ry$, we have

$$\int_{\partial B_R(0)} R^{-\frac{N-1}{2}} e^{-R} e^{\frac{x \cdot (e_j - e_i)}{|e_j - e_i|}} \, dS = \int_{\partial B_R(0)} R^{-\frac{N-1}{2}} e^{-R} e^{x_1} \, dS = R^{\frac{N-1}{2}} e^{-R} \int_{\partial B_1(0)} e^{Ry_1} \, dS_y.$$

Using polar coordinates

$$y_1 = \cos \theta_1,$$
$$y_2 = \sin \theta_1 \cos \theta_2,$$
$$\vdots$$
$$y_N = \sin \theta_1 \cdots \sin \theta_{N-2} \sin \theta_{N-1},$$

we have

$$R^{\frac{N-1}{2}} e^{-R} \int_{\partial B_1(0)} e^{Ry_1} \, dS_y$$
$$= R^{\frac{N-1}{2}} e^{-R} \int_0^\pi \int_0^\pi \cdots \int_0^{2\pi} e^{R \cos \theta_1} (\sin \theta_1)^{N-2} (\sin \theta_2)^{N-3} \cdots \sin \theta_{N-2} \, d\theta_1 \cdots d\theta_{N-1}$$
$$= c R^{\frac{N-1}{2}} \int_0^\pi e^{-R(1 - \cos \theta_1)} (\sin \theta_1)^{N-2} \, d\theta_1.$$

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We put $R(1 - \cos\theta_1) = t$. Then it follows

$$R^{\frac{N-1}{2}} \int_0^\pi e^{-R(1-\cos\theta_1)}(\sin\theta_1)^{N-2}\,d\theta_1$$

$$= \int_0^R e^{-t}(2t - \frac{t^2}{R})^{\frac{N-3}{2}}\,dt + \int_R^{2R} e^{-t}(-2t + \frac{t^2}{R})^{\frac{N-3}{2}}\,dt$$

$$\to \int_0^\infty e^{-t}(2t)^{\frac{N-3}{2}}\,dt \in (0,\infty) \text{ as } R \to \infty.$$

Thus we obtain (5.4). Next we observe that

$$-\int_{\partial B_R(le_i)} (\nabla u_j \cdot \nu)u_i\,dS = -\int_{\partial B_R(0)} (\nabla w(x - l(e_j - e_i)) \cdot \nu)w\,dS.$$

Since w is radially symmetric, we have

$$\nabla w(x - l(e_j - e_i)) \cdot \nu = w'(|x - l(e_j - e_i)|)\frac{x - l(e_j - e_i)}{|x - l(e_j - e_i)|} \cdot \frac{x}{|x|},$$

where we write $w' = \frac{\partial w}{\partial r}$. Now it follows

$$\frac{x - l(e_j - e_i)}{|x - l(e_j - e_i)|} \cdot \frac{x}{|x|} \to -\frac{x_1}{|x|} \text{ as } l \to \infty \text{ uniformly in } x \in \partial B_R(0).$$

Moreover by Lemma 3.6, there exists $c > 0$ such that

$$\frac{w'(|x - l(e_j - e_i)|)}{w(x - l(e_j - e_i))} \to -c \text{ as } l \to \infty \text{ uniformly in } x \in \partial B_R(0).$$

Thus from (5.3), there exist $\bar{l} > 0$ and $c > 0$ independent of l, R such that if $l \geq \bar{l}$,

$$\nabla w(x - l(e_j - e_i)) \cdot \nu \geq c\frac{x_1}{|x|}e^{x_1}e^{-|e_j-e_i|l}(|e_j - e_i|l + 1)^{-\frac{N-1}{2}}.$$

Next we observe that

$$\int_{\partial B_R(0)} we^{x_1}\frac{x_1}{|x|}\,dS = \frac{1}{R}\int_{\partial B_R(0)\cup\{x_1>0\}} we^{x_1}x_1\,dS + \frac{1}{R}\int_{\partial B_R(0)\cup\{x_1<0\}} we^{x_1}x_1\,dS$$

$$= \frac{1}{R}\int_{\partial B_R(0)\cup\{x_1>0\}} we^{x_1}x_1\,dS - \frac{1}{R}\int_{\partial B_R(0)\cup\{x_1>0\}} we^{-x_1}x_1\,dS$$

$$= \frac{1}{R}\int_{\partial B_R(0)\cup\{x_1>0\}} w(e^{x_1} - e^{-x_1})x_1\,dS > 0.$$

Arguing similarly as above, we have

$$\lim_{R\to\infty} \frac{1}{R}\int_{\partial B_R(0)\cup\{x_1>0\}} w(e^{x_1} - e^{-x_1})x_1\,dS \in (0,\infty).$$

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Thus there exist $M_R > 0$ and $l_2 > 0$ such that for $l \geq l_2$,

$$\int_{\partial B_R(le_i)} (\nabla u_i \cdot \nu) u_j - (\nabla u_j \cdot \nu) u_i \, dS \leq -M_R e^{-|e_j - e_i|l} l^{-\frac{N-1}{2}}$$

and $\lim_{R \to \infty} M_R = M^* > 0$.

(iv) We may assume without loss of generality that $r > s > 0$. From (3.5), we have

$$\int_{\mathbb{R}^N \setminus (\cup_{i=1}^m B_R(le_i))} u_i^r u_j^s \, dx \leq c \int_{|x| \geq R} |x|^{-\frac{r(N-1)}{2}} e^{-r|x|} w(x - l(e_j - e_i))^s \, dx.$$

We show that

$$\lim_{l \to \infty} e^{s|e_i - e_j|l} l^{\frac{s(N-1)}{2}} \int_{|x| \geq R} |x|^{-\frac{r(N-1)}{2}} e^{-r|x|} w(x - l(e_j - e_i))^s \, dx$$

$$= c \int_{|x| \geq R} e^{-sx_1} |x|^{-\frac{r(N-1)}{2}} e^{-r|x|} dx \qquad (5.5)$$

for some $c > 0$ independent of R. Indeed from (5.2) and Lemma 3.6, we have

$$\lim_{l \to \infty} e^{s|e_j - e_i|l} l^{\frac{s(N-1)}{2}} w(x - l(e_j - e_i))^s = ce^{-sx_1} \quad \text{for all } x \in \mathbb{R}^N.$$

Moreover we have

$$|e_j - e_i|l - |x - l(e_j - e_i)| \leq |x|,$$

$$\left(\frac{l}{|x - l(e_j - e_i)| + 1} \right)^{\frac{N-1}{2}} \leq \left(\frac{2}{|e_j - e_i|} \right)^{\frac{N-1}{2}} \max\{1, |x|^{\frac{N-1}{2}}\}.$$

Thus we obtain

$$e^{s|e_j - e_i|l} l^{\frac{s(N-1)}{2}} |x|^{-\frac{r(N-1)}{2}} e^{-r|x|} w(x - l(e_j - e_i))^s$$

$$\leq c \max\{1, |x|^{\frac{N-1}{2}}\} e^{-(r-s)|x|} |x|^{-\frac{r(N-1)}{2}} \in L^1(\mathbb{R}^N).$$

By the Lebesgue's convergence theorem, (5.5) follows. Finally we observe that

$$K_R := \int_{|x| \geq R} e^{-sx_1} |x|^{-\frac{r(N-1)}{2}} e^{-r|x|} \, dx \leq c \int_{|x| \geq R} e^{-(r-s)|x|} |x|^{-\frac{r(N-1)}{2}} \, dx.$$

Since $r - s > 0$, it follows $\lim_{R \to \infty} K_R = 0$. ∎

Proof of Proposition 5.1. For simplicity, we prove Proposition 5.1 for the case $m = 2$. Then it follows $|e_1 - e_2| = \lambda_0 = 2$. We can similarly prove (5.1) for the case $m > 2$.

First we observe that $I(\gamma(s)) \to -\infty$ as $s \to \infty$ uniformly in $l > 1$. Moreover by the continuity of $I(\gamma(s))$ with respect to s, there exists $0 < s_1 < 1 < s_2$ such that

$$I(\gamma(s)) < mc_\infty \quad \text{for all } s \in [0, s_1] \cup [s_2, \infty) \text{ and } l \geq 1. \tag{5.6}$$

Thus to prove (5.1), it suffices to show $I(\gamma(s)) < mc_\infty$ for $s \in [s_1, s_2]$.

Step 1: [Decomposition of the energy]. By direct calculations, it follows

$$I(\gamma(s)) = J(su_1 + su_2)$$

$$= J_\infty(su_1) + J_\infty(su_2) + s^2 \int_{\mathbb{R}^N} \nabla u_1 \cdot \nabla u_2 + a_\infty u_1 u_2 \, dx$$

$$+ \frac{s^2}{2} \int_{\mathbb{R}^N} (a(x) - a_\infty)(u_1 + u_2)^2 \, dx$$

$$+ \frac{\alpha s^{2\alpha}}{2} \int_{\mathbb{R}^N} (u_1 + u_2)^{2\alpha-2} |\nabla u_1 + \nabla u_2|^2 - |\nabla u_1|^2 u_1^{2\alpha-2} - |\nabla u_2|^2 u_2^{2\alpha-2} \, dx$$

$$- \int_{\mathbb{R}^N} H(su_1 + su_2) - H(su_1) - H(su_2) \, dx.$$

Using $\frac{s^2}{2} J_\infty'(u_1)u_2 = 0$, $\frac{s^2}{2} J_\infty'(u_2)u_1 = 0$ and Lemma 3.3, we obtain

$$I(\gamma(s)) \leq 2J_\infty(w) - 2c(s-1)^2 + \frac{s^2}{2} \int_{\mathbb{R}^N} (a(x) - a_\infty)(u_1 + u_2)^2 \, dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}^N} s^2 h(u_1)u_2 - h(su_1)su_2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} s^2 h(u_2)u_1 - h(su_2)su_1 \, dx$$

$$- \int_{\mathbb{R}^N} H(su_1 + su_2) - H(su_1) - H(su_2) - \frac{1}{2} h(su_1)su_2 - \frac{1}{2} h(su_2)su_1 \, dx$$

$$+ \frac{\alpha s^{2\alpha}}{2} \int_{\mathbb{R}^N} (u_1 + u_2)^{2\alpha-2} |\nabla u_1 + \nabla u_2|^2 - |\nabla u_1|^2 u_1^{2\alpha-2} - |\nabla u_2|^2 u_2^{2\alpha-2} \, dx$$

$$- \frac{s^2}{2} \int_{\mathbb{R}^N} \alpha \nabla u_1 \cdot \nabla u_2 (u_1^{2\alpha-2} + u_2^{2\alpha-2})$$

$$+ \alpha(\alpha - 1)(|\nabla u_1|^2 u_1^{2\alpha-3} u_2 + |\nabla u_2|^2 u_2^{2\alpha-3} u_1) \, dx$$

$$=: 2J_\infty(w) - 2c(s-1)^2 + \frac{s^2}{2} \int_{\mathbb{R}^N} (a(x) - a_\infty)(u_1 + u_2)^2 \, dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}^N} s^2 h(u_1)u_2 - h(su_1)su_2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} s^2 h(u_2)u_1 - h(su_2)su_1 \, dx$$

$$- L_1(\mathbb{R}^N) + L_2(\mathbb{R}^N) - L_3(\mathbb{R}^N).$$

By $(a4)$, Lemma 2.4 in [1] and (3.5), there exists $C_0 > 0$ such that

$$\int_{\mathbb{R}^N} (a(x) - a_\infty)(u_1 + u_2)^2 \, dx \leq C_0 \max\{e^{-\lambda l}, e^{-2l} l^{-(N-1)}\} \quad \text{for all } l \geq 1.$$

Next by Taylor's expansion, it follows

$$s^2 h(u_1)u_2 - h(su_1)su_2 = s^2 h(u_1)u_2 - sh(u_1)u_2 + sh(u_1)u_2 - h(su_1)su_2$$
$$= s(s-1)h(u_1)u_2 - s(s-1)h'((1+\theta(s-1))u_1)u_1 u_2$$

for some $0 < \theta < 1$. From (3.5), there exists $C > 0$ independent of $s \in [s_1, s_2]$, $l > 0$ and $x \in \mathbb{R}^N$ such that

$$|s^2 h(u_1)u_2 - h(su_1)su_2| \leq C|s-1|u_1 u_2.$$

Thus by (i) of Lemma 5.3, we have

$$\int_{\mathbb{R}^N} s^2 h(u_1)u_2 - h(su_1)su_2 dx \leq C|s-1| \int_{\mathbb{R}^N} u_1 u_2 \, dx \leq C|s-1|e^{-(1-\delta)\lambda_0 l} \tag{5.7}$$

for any $l \geq 0$ and $0 < \delta < 1$. Thus we obtain

$$I(\gamma(s)) \leq 2J_\infty(w) - 2c(s-1)^2 + C_0 \max\{e^{-\lambda l}, e^{-2l}l^{-(N-1)}\} + C|s-1|e^{-(1-\delta)\lambda_0 l}$$
$$- L_1(\mathbb{R}^N) + L_2(\mathbb{R}^N) - L_3(\mathbb{R}^N). \tag{5.8}$$

Now let $R > 0$ be arbitrary given. Taking large $l > 0$ enough so that $B_R(le_1) \cap B_R(le_2) = \emptyset$. We decompose

$$\mathbb{R}^N = B_R(le_1) \cup B_R(le_2) \cup (\mathbb{R}^N \setminus (B_R(le_1) \cup B_R(le_2))).$$

For simplicity, we write $B_i = B_R(le_i)$, $i = 1, 2$, $\Omega = \mathbb{R}^N \setminus (B_1 \cup B_2)$. Then we have

$$L_i(\mathbb{R}^N) = L_i(B_1) + L_i(B_2) + L_i(\Omega) \quad \text{for } i = 1, 2, 3.$$

Step 2: [Estimate of $L_1(B_1)$]. We observe that u_2 is exponentially small on B_1. Then by (ii) of Lemma 5.2 and (5.7), we have

$$-L_1(B_1) \leq -\frac{1}{2} \int_{B_1} h(su_1)su_2 \, dx - \frac{1}{2} \int_{B_1} h(su_2)su_1 \, dx$$
$$+ \epsilon \int_{B_1} su_2 + \frac{s^2}{s} u_2^2 + \frac{s^2}{2} u_1 u_2 \, dx$$
$$\leq -\frac{s^2}{2} \int_{B_1} h(u_1)u_2 \, dx + \frac{1}{2} \int_{B_1} s^2 h(u_1)u_2 - h(su_1)su_2 \, dx$$
$$+ \epsilon \int_{B_1} su_2 + \frac{s^2}{s} u_2^2 + \frac{s^2}{2} u_1 u_2 \, dx$$
$$\leq -\frac{s^2}{2} \int_{B_1} h(u_1)u_2 \, dx + C|s-1|e^{-(1-\delta)\lambda_0 l} + C'\epsilon \int_{B_1} u_2 \, dx.$$

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It follows from (ii) of Lemma 5.3 that

$$\int_{B_1} u_2 \, dx \le C_R e^{-\lambda_0 l} l^{-\frac{N-1}{2}}.$$

Thus we obtain

$$-L_1(B_1) \le -\frac{s^2}{2} \int_{B_1} h(u_1) u_2 \, dx + C|s-1| e^{-(1-\delta)\lambda_0 l} + C' \epsilon e^{-\lambda_0 l} l^{-\frac{N-1}{2}}, \qquad (5.9)$$

where $\epsilon > 0$ is arbitrary and we choose a suitable $\epsilon > 0$ later. Next we estimate $-\frac{s^2}{2} \int_{B_1} h(u_1) u_2 \, dx$ as follows. Since u_1 is a solution of (1.10), it follows

$$-\int_{B_1} h(u_1) u_2 dx = \int_{\partial B_1} (\nabla u_1 \cdot \nu) u_2 (1 + \alpha u_1^{2\alpha-2}) \, dS - \int_{B_1} \nabla u_1 \cdot \nabla u_2 + a_\infty u_1 u_2 \, dx$$
$$- \alpha \int_{B_1} u_1^{2\alpha-2} \nabla u_1 \cdot \nabla u_2 \, dx - \alpha(\alpha-1) \int_{B_1} u_1^{2\alpha-3} |\nabla u_1|^2 u_2 \, dx.$$

Since u_2 is also a solution of (1.10), we have

$$-\int_{B_1} \nabla u_1 \cdot \nabla u_2 + a_\infty u_1 u_2 \, dx = -\int_{\partial B_1} (\nabla u_2 \cdot \nu) u_1 (1 + \alpha u_2^{2\alpha-2}) \, dS$$
$$+ \alpha(\alpha-1) \int_{B_1} u_2^{2\alpha-3} |\nabla u_2|^2 u_1 \, dx$$
$$+ \alpha \int_{B_1} u_2^{2\alpha-2} \nabla u_1 \cdot \nabla u_2 \, dx - \int_{B_1} h(u_2) u_1 \, dx.$$

Thus we obtain from (3.7) and $-\int_{B_1} h(u_2) u_1 \, dx \le 0$,

$$-\frac{s^2}{2} \int_{B_1} h(u_1) u_2 \, dx$$
$$\le \frac{s^2}{2} \int_{\partial B_1} (\nabla u_1 \cdot \nu) u_2 (1 + \alpha u_1^{2\alpha-2}) - (\nabla u_2 \cdot \nu) u_1 (1 + \alpha u_2^{2\alpha-2}) \, dS$$
$$- \frac{\alpha(\alpha-1)}{2} s^2 \int_{B_1} u_1^{2\alpha-3} |\nabla u_1|^2 u_2 \, dx - \frac{\alpha}{2} s^2 \int_{B_1} u_1^{2\alpha-2} \nabla u_1 \cdot \nabla u_2 \, dx$$
$$+ \frac{\alpha(\alpha-1)}{2} s^2 \int_{B_1} u_2^{2\alpha-3} |\nabla u_2|^2 u_1 \, dx + \frac{\alpha}{2} s^2 \int_{B_1} u_2^{2\alpha-2} \nabla u_1 \cdot \nabla u_2 \, dx$$
$$\le \frac{s^2}{2} \int_{\partial B_1} (\nabla u_1 \cdot \nu) u_2 (1 + \alpha u_1^{2\alpha-2}) - (\nabla u_2 \cdot \nu) u_1 (1 + \alpha u_2^{2\alpha-2}) \, dS$$
$$- \frac{\alpha(\alpha-1)}{2} s^2 \int_{B_1} u_1^{2\alpha-3} |\nabla u_1|^2 u_2 \, dx - \frac{\alpha}{2} s^2 \int_{B_1} u_1^{2\alpha-2} \nabla u_1 \cdot \nabla u_2 \, dx$$
$$+ C \int_{B_1} u_2^{2\alpha-1} \, dx. \qquad (5.10)$$

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From (5.9), (5.10), (ii) and (iii) of Lemma 5.3, we obtain

$$
-L_1(B_1) \le -\frac{\alpha(\alpha-1)}{2}s^2 \int_{B_1} u_1^{2\alpha-3}|\nabla u_1|^2 u_2\, dx - \frac{\alpha}{2}s^2 \int_{B_1} u_1^{2\alpha-2}\nabla u_1 \cdot \nabla u_2\, dx
$$
$$
+ C(-M_R e^{-\lambda_0 l}l^{-\frac{N-1}{2}} + |s-1|e^{-(1-\delta)\lambda_0 l} + \epsilon e^{-\lambda_0 l}l^{-\frac{N-1}{2}}
$$
$$
+ e^{-(2\alpha-1)\lambda_0 l}) \tag{5.11}
$$

for all $l \ge \max\{l_1, l_2\}$.

Step 3: [Estimate of $L_2(B_1)$ and $L_3(B_1)$]. By Taylor's expansion, there exists $c > 0$ independent of l and R such that

$$
L_2(B_1) \le \frac{\alpha s^{2\alpha}}{2} \int_{B_1} (2\alpha-2)u_1^{2\alpha-3}u_2|\nabla u_1|^2 + 2u_1^{2\alpha-2}\nabla u_1 \cdot \nabla u_2\, dx
$$
$$
+ cs^{2\alpha} \int_{B_1} u_1^{2\alpha-4}u_2^2|\nabla u_1|^2 + u_1^{2\alpha-2}|\nabla u_2|^2
$$
$$
+ u_2^{2\alpha-2}|\nabla u_2|^2 + u_1^{2\alpha-3}u_2|\nabla u_1||\nabla u_2|\, dx. \tag{5.12}
$$

We also have

$$
-L_3(B_1) \le -\frac{\alpha s^2}{2}\int_{B_1} \nabla u_1 \cdot \nabla u_2 u_1^{2\alpha-2}\, dx - \frac{\alpha(\alpha-1)}{2}s^2 \int_{B_1} |\nabla u_1|^2 u_1^{2\alpha-3}u_2\, dx
$$
$$
+ cs^2 \int_{B_1} |\nabla u_1||\nabla u_2|u_2^{2\alpha-2} + |\nabla u_2|^2 u_2^{2\alpha-3}u_1\, dx. \tag{5.13}
$$

From (3.7), (5.12) and (5.13), we obtain

$$
L_2(B_1) - L_3(B_1) \le \frac{\alpha}{2}(2s^{2\alpha} - s^2)\int_{B_1} \nabla u_1 \cdot \nabla u_2 u_1^{2\alpha-2}\, dx
$$
$$
+ \frac{\alpha(\alpha-1)}{2}(2s^{2\alpha} - s^2)\int_{B_1} |\nabla u_1|^2 u_1^{2\alpha-3}u_2\, dx
$$
$$
+ C\int_{B_1} u_2^2 + u_2^{2\alpha} + u_2^{2\alpha-1}\, dx.
$$

It follows from (ii) of Lemma 5.3 and $\alpha \ge \dfrac{3}{2}$ that

$$
L_2(B_1) - L_3(B_1) \le \frac{\alpha}{2}(2s^{2\alpha} - s^2)\int_{B_1} \nabla u_1 \cdot \nabla u_2 u_1^{2\alpha-2}\, dx
$$
$$
+ \frac{\alpha(\alpha-1)}{2}(2s^{2\alpha} - s^2)\int_{B_1} |\nabla u_1|^2 u_1^{2\alpha-3}u_2\, dx
$$
$$
+ Ce^{-2\lambda_0 l}. \tag{5.14}
$$

Step 4: [Decay estimate on B_1]. From (5.11) and (5.14), we have

$$-L_1(B_1) + L_2(B_1) - L_3(B_1)$$
$$\leq \alpha(s^{2\alpha} - s^2) \int_{B_1} \nabla u_1 \cdot \nabla u_2 u_1^{2\alpha-2}\, dx + \alpha(\alpha-1)(s^{2\alpha} - s^2) \int_{B_1} u_1^{2\alpha-3} |\nabla u_1|^2 u_2\, dx$$
$$+ C(-M_R e^{-\lambda_0 l} l^{-\frac{N-1}{2}} + |s-1| e^{-(1-\delta)\lambda_0 l} + \epsilon e^{-\lambda_0 l} l^{-\frac{N-1}{2}} + e^{-2\lambda_0 l}).$$

Arguing as Lemma 3.3, there exists $c > 0$ such that $|s^{2\alpha} - s^2| \leq c|s-1|$ for all $s \in [s_1, s_2]$. Then we have from (3.7) and (i) of Lemma 5.3

$$\alpha(s^{2\alpha} - s^2) \int_{B_1} \nabla u_1 \cdot \nabla u_2 u_1^{2\alpha-2}\, dx + \alpha(\alpha-1)(s^{2\alpha} - s^2) \int_{B_1} u_1^{2\alpha-3} |\nabla u_1|^2 u_2\, dx$$
$$\leq C|s-1| \int_{B_1} u_1^{2\alpha-1} u_2\, dx$$
$$\leq C'|s-1| \int_{B_1} u_1 u_2\, dx$$
$$\leq C''|s-1| e^{-(1-\delta)\lambda_0 l}.$$

Consequently we obtain

$$-L_1(B_1) + L_2(B_1) - L_3(B_1)$$
$$\leq C(-M_R e^{-\lambda_0 l} l^{-\frac{N-1}{2}} + |s-1| e^{-(1-\delta)\lambda_0 l} + \epsilon e^{-\lambda_0 l} l^{-\frac{N-1}{2}} + e^{-2\lambda_0 l}) \quad (5.15)$$

for all $l \geq \max\{l_1, l_2\}$. Similarly we can estimate $L_1(B_2)$, $L_2(B_2)$ and $L_3(B_2)$, that is,

$$-L_1(B_2) + L_2(B_2) - L_3(B_2)$$
$$\leq C(-M_R e^{-\lambda_0 l} l^{-\frac{N-1}{2}} + |s-1| e^{-(1-\delta)\lambda_0 l} + \epsilon e^{-\lambda_0 l} l^{-\frac{N-1}{2}} + e^{-2\lambda_0 l}) \quad (5.16)$$

for all $l \geq \max\{l_1, l_2\}$.

Step 5: [Decay estimate on Ω]. We notice that both u_1 and u_2 are exponentially small on Ω. First by (i) of Lemma 5.2, we have $-L_1(\Omega) \leq 0$. Next since

$$(u_1 + u_2)^{2\alpha-2} - u_1^{2\alpha-2} = \int_0^{u_2} (2\alpha-2)(u_1 + \tau)^{2\alpha-3}\, d\tau \leq c(u_1 + u_2)^{2\alpha-3} u_2,$$

it follows

$$L_2(\Omega) \leq C \int_\Omega |\nabla u_1|^2 u_2 (u_1^{2\alpha-3} + u_2^{2\alpha-3})$$
$$+ |\nabla u_2|^2 u_1 (u_2^{2\alpha-3} + u_1^{2\alpha-3}) + |\nabla u_1||\nabla u_2|(u_1^{2\alpha-2} + u_2^{2\alpha-2})\, dx.$$

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Finally we have

$$-L_3(\Omega) \leq C \int_\Omega |\nabla u_1||\nabla u_2|(u_1^{2\alpha-2} + u_2^{2\alpha-2}) + |\nabla u_1|^2 u_1^{2\alpha-3} u_2 + |\nabla u_2|^2 u_2^{2\alpha-3} u_1 \, dx.$$

Thus we obtain from (3.7), (iv) of Lemma 5.3 and $\alpha \geq \dfrac{3}{2}$,

$$\begin{aligned}
-L_1(\Omega) + L_2(\Omega) - L_3(\Omega) &\leq C \int_\Omega u_1^{2\alpha-2} u_2^2 + u_1^2 u_2^{2\alpha-2} + u_1^{2\alpha-1} u_2 + u_2^{2\alpha-1} u_1 \, dx \\
&\leq C(K_R e^{-\min\{2,2\alpha-2\}\lambda_0 l} l^{-\frac{\min\{2,2\alpha-2\}(N-1)}{2}} + K_R e^{-\lambda_0 l} l^{-\frac{N-1}{2}}) \\
&\leq C' K_R e^{-\lambda_0 l} l^{-\frac{N-1}{2}} \tag{5.17}
\end{aligned}$$

for all $l \geq l_3$.

Step 6: [Conclusion]. From (5.8), (5.15), (5.16) and (5.17) we have

$$\begin{aligned}
I(\gamma(s)) &\leq 2J_\infty(w) - 2c(s-1)^2 + C_0 \max\{e^{-\lambda l}, e^{-2l} l^{-(N-1)}\} + C_1 K_R e^{-\lambda_0 l} l^{-\frac{N-1}{2}} \\
&\quad + C_2 e^{-2\lambda_0 l} + C_3 |s-1| e^{-(1-\delta)\lambda_0 l} + C_4 \epsilon e^{-\lambda_0 l} l^{-\frac{N-1}{2}} - C_5 M_R e^{-\lambda_0 l} l^{-\frac{N-1}{2}}
\end{aligned}$$

for all $l \geq \max\{l_1, l_2, l_3\}$, where $0 < \delta < 1$ and $\epsilon > 0$ are arbitrary. Now we fix large $R > 0$ so that

$$C_1 K_R \leq \frac{C_5}{2} M_R.$$

Next using Young's inequality, we have for all $\eta > 0$, there exists $C_\eta > 0$ such that

$$C_3 |s-1| e^{-(1-\delta)\lambda_0 l} \leq \eta(s-1)^2 + C_\eta e^{-2(1-\delta)\lambda_0 l}.$$

Thus choosing $\eta > 0$ so small that

$$-2c(s-1)^2 + \eta(s-1)^2 \leq 0,$$

we have

$$\begin{aligned}
I(\gamma(s)) &\leq 2J_\infty(w) + C_0 \max\{e^{-\lambda l}, e^{-2l} l^{-(N-1)}\} \\
&\quad + C_2 e^{-2\lambda_0 l} + C_3' e^{-2(1-\delta)\lambda_0 l} - \left(\frac{C_5}{2} M_R - C_4 \epsilon\right) e^{-\lambda_0 l} l^{-\frac{N-1}{2}}
\end{aligned}$$

for all $l \geq \max\{l_1, l_2, l_3\}$. Finally we choose $\epsilon > 0$ small enough so that $\frac{C_5}{2} M_R - C_4 \epsilon > 0$. We also choose $0 < \delta < \frac{1}{2}$. Since $2 = \lambda_0 < \lambda$ and $2(1-\delta) > 1$, there exists $l_0 > \max\{l_1, l_2, l_3\}$ such that for $l \geq l_0$,

$$I(\gamma(s)) < 2J_\infty(w) \quad \text{for all } s \in [s_1, s_2].$$

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From (5.6) and Lemma 3.4, we obtain

$$\sup_{s \geq 0} I(\gamma(s)) < 2J_\infty(w) = 2I_\infty(\tilde{w}) = 2\tilde{c}_\infty.$$

Thus we complete the proof of Proposition 5.1. ∎

6. Proof of Theorem 1.1

By Lemma 4.1, we can define the Mountain Pass value c_{MP} by

$$c_{MP} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \ \Gamma = \{\gamma \in C([0,1], H_G^1(\mathbb{R}^N)) \,; \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

By Proposition 5.1, we have $c_{MP} < mc_\infty$. By Corollary 4.3 and Lemma 4.4, we can apply the variant of the Mountain Pass Theorem [6]. Then there exists $v_0 \in H_G^1(\mathbb{R}^N)$ such that $I(v_0) = c_{MP}$ and $I'(v_0) = 0$. Putting $u_0 = f(v_0)$, then u_0 is a G-invariant positive solution of (1.1). Similarly Theorem 1.3 follows from Lemma 4.1, Corollary 4.3, 4.5 and Proposition 5.1. ∎

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