U.P.B. Sci. Bull., ISSN 1223-7027

G-NEUTROSOPHIC SPACE

Mumtaz Ali[1], Florentin Smarandache[2], Munazza Naz[3], Muhammad Shabir[4]

In this article we give an extension of group action theory to neutrosophic theory and develop G-neutrosophic spaces by certain valuable techniques. Every G-neutrosophic space always contains a G-space. A G-neutrosophic space has neutrosophic orbits as well as strong neutrosophic orbits. Then we give an important theorem for orbits which tells us that how many orbits of a G-neutrosophic space. We also introduce new notions called pseudo neutrosophic space and ideal space and then give the important result that the transitive property implies to ideal property

.

Keywords: Group action, G-space, orbit, stabilizer, G-neutrosophic space, neutrosophic orbit, neutrosophic stabilizer.

1. Introduction

The Concept of a G-space came into being as a consequence of Group action on an ordinary set. Over the history of Mathematics and Algebra, theory of group action has emerged and proven to be an applicable and effective framework for the study of different kinds of structures to make connection among them. The applications of group action in different areas of science such as physics, chemistry, biology, computer science, game theory, cryptography etc has been worked out very well. The abstraction provided by group actions is a powerful one, because it allows geometrical ideas to be applied to more abstract objects. Many objects in Mathematics have natural group actions defined on them. In particular, groups can act on other groups, or even on themselves. Despite this generality, the theory of group actions contains wide-reaching theorems, such as the orbit stabilizer theorem, which can be used to prove deep results in several fields. Neutrosophy is a branch of neutrosophic philosophy which handles the origin and stages of neutralities. Neutrosophic science is a newly emerging science which has been firstly introduced by Florentin Smarandache in 1995. This is quite a general phenomenon which can be found almost everywhere in the nature. Neutrosophic approach provides a generosity to absorbing almost all the corresponding algebraic structures open heartedly. This tradition is also maintained in our work here. The combination of neutrosophy and group action gives some extra ordinary excitement while forming this new structure called G-neutrosophic space. This is a generalization of all the work of the past and some new notions are also raised due to this approach. Some new types of spaces and their core properties have been discovered here for the first time. Examples and counter examples have been illustrated wherever required. In this paper we have also coined a new term called pseudo neutrosophic spaces and a new property called ideal property. The link of transitivity with ideal property and the corresponding results are established.

2. Basic Concepts

Group Action

Definition 1: Let Ω be a non empty set and G be a group. Let $\mu:\Omega\times G\to\Omega$ be a mapping. Then μ is called an action of G on Ω such that for all $\omega\in\Omega$ and $g,h\in G$.

1) $\mu(\mu(\omega,g),h)=\mu(\omega,gh)$

2) $\mu(\omega,1)=\omega$, where 1 is the identity element in G.

Usually we write ω^g instead of $\mu(\omega,g)$. Therefore (1) and (2) becomes as

1) $(\omega^g)^h=(\omega)^{gh}$. For all $\omega\in\Omega$ and $g,h\in G$.

2) $\omega^1=\omega$.

Definition 2: Let Ω be a G-space. Let $\Omega_1\neq\phi$ be a subset of Ω. Then Ω_1 is called G-subspace of Ω if $\omega^g\in\Omega_1$ for all $\omega\in\Omega_1$ and $g\in G$.

Definition 3: We say that Ω is transitive G-space if for any $\alpha,\beta\in G$, there exist $g\in G$ such that $\alpha^g=\beta$.

Definition 4: Let $\alpha\in\Omega$, then α^G or αG is called G-orbit and is defined as $\alpha^G=\{\alpha^g:g\in G\}$.

A transitive G-subspace is also called an orbit.

Remark 1: A transitive G space has only one orbit.

Definition 5: Let G be a group acting on Ω and if $\alpha\in\Omega$, we denote stabilizer of α by G_α and is define as $G_\alpha=stab_G(\alpha)=\{g\in G:\alpha^g=\alpha\}$.

Lemma 1: Let Ω be a G-space and $\alpha\in\Omega$. Then

1) $G_\alpha\leq G$ and
2) There is one-one correspondence between the right cosets of G_α and the G-orbit α^G in G.

Corollary 1: If G is finite, then $|G|=|G_\alpha|.|\alpha^G|$

Definition 6: Let Ω be a G-space and $g \in G$. Then

$$fix_{\Omega}(g) = \{\alpha \in \Omega : \alpha^{g} = \alpha\}.$$

Theorem 1: Let Ω and G be finite. Then

$$|Orb_{\Omega}(G)| = \frac{1}{|G|}\sum_{g \in G} |fix_{\Omega}(g)|,$$

where $|Orb_{\Omega}(G)|$ is the number of orbits of G in Ω.

3. Neutrosophic Spaces

Definition 10: Let Ω be a G-space. Then $N(\Omega)$ is called G-neutrosophic space if $N(\Omega) = \langle \Omega \cup I \rangle$ which is generated by Ω and I.

Example 1: Let $\Omega = \{e, x, x^2, y, xy, x^2y\}$ =S_3 and $G = \{e, y\}$. Let $\mu : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\mu(\omega, g) = g(\omega)$, for all $\omega \in \Omega$ and $g \in G$. Then Ω be a G-space under this action. Let $N(\Omega)$ be the corresponding G-neutrosophic space, where

$$N(\Omega) = \langle \Omega \cup I \rangle = \{e, x, x^2, y, xy, x^2y, I, Ix, Ix^2, Iy, Ixy, Ix^2y\}$$

Theorem 3: $N(\Omega)$ always contains Ω.

Definition 11: Let $N(\Omega)$ be a neutrosophic space and $N(\Omega_1)$ be a subset of $N(\Omega)$. Then $N(\Omega_1)$ is called neutrosophic subspace of $N(\Omega)$ if $x^{g} \in N(\Omega_1)$ for all $x \in N(\Omega_1)$ and $g \in G$.

Example 2: In the above example (1). Let $N(\Omega_1) = \{x, xy\}$ and $N(\Omega_2) = \{Ix^2, Ix^2y\}$ are subsets of $N(\Omega)$. Then clearly $N(\Omega_1)$ and $N(\Omega_2)$ are neutrosophic subspaces of $N(\Omega)$.

Theorem 4: Let $N(\Omega)$ be a G-neutrosophic space and Ω be a G-space. Then Ω is always a neutrosophic subspace of $N(\Omega)$.

Proof: The proof is straightforward.

Definition 12: A neutrosophic subspace $N(\Omega_1)$ is called strong neutrosophic subspace or pure neutrosophic subspace if all the elements of $N(\Omega_1)$ are neutrosophic elements.

Example 3: In example (1), the neutrosophic subspace $N(\Omega_2) = \langle Ix^2, Ix^2y \rangle$ is a strong neutrosophic subspace or pure neutrosophic subspace of $N(\Omega)$.

Remark 2: Every strong neutrosophic subspace or pure neutrosophic subspace is trivially neutrosophic subspace.

The converse of the above remark is not true.

Example 4: In previous example $N(\Omega_1) = \langle x, xy \rangle$ is a neutrosophic subspace but it is not strong neutrosophic subspace or pure neutrosophic subspace of $N(\Omega)$.

Definition 13: Let $N(\Omega)$ be a G-neutrosophic space. Then $N(\Omega)$ is said to be transitive G-neutrosophic space if for any $x, y \in N(\Omega)$, there exists $g \in G$ such that $x^g = y$.

Example 5: Let $\Omega = G = (Z_4, +)$, where Z_4 is a group under addition modulo 4. Let $\mu : \Omega \times G \to \Omega$ be an action of G on itself defined by $\mu(\omega, g) = \omega + g$, for all $\omega \in \Omega$ and $g \in G$. Then Ω is a G-space and $N(\Omega)$ be the corresponding G-neutrosophic space, where

$$N(\Omega) = \{0,1,2,3,I,2I,3I,4I,1+I,2+I,3+I,1+2I,2+2I,2+3I,3+2I,3+3I\}$$

Then $N(\Omega)$ is not transitive neutrosophic space.

Theorem 5: All the G-neutrosophic spaces are intransitive G-neutrosophic spaces.

Definition 14: Let $n \in N \;\; \Omega$, the neutrosophic orbit of n is denoted by NO_n and is defined as $NO_n = \;\; n^g : g \in G \;\;$.

Equivalently neutrosophic orbit is a transitive neutrosophic subspace.

Example 6: In example 1 , the neutrosophic space $N \;\; \Omega$ has 6 neutrosophic orbits which are given below

$$NO_e = \;\; e, y \;\; , NO_x = \;\; x, xy \;\; ,$$

$$NO_{x^2} = \;\; x^2, x^2 y \;\; , NO_I = \;\; I, Iy \;\; ,$$

$$NO_{Ix} = \;\; Ix, Ixy \;\; , NO_{Ix^2} = \;\; Ix^2, Ix^2 y \;\; .$$

Definition 15: A neutrosophic orbit NO_n is called strong neutrosophic orbit or pure neutrosophic orbit if it has only neutrosophic elements.

Example 7: In example 1 ,

$$NO_I = \;\; I, Iy \;\; ,$$

$$NO_{Ix} = \;\; Ix, Ixy \;\; ,$$

$$NO_{Ix^2} = \;\; Ix^2, Ix^2 y \;\; .$$

are strong neutrosophic orbits or pure neutrosophic orbits of $N \;\; \Omega$.

Theorem 7: All strong neutrosophic orbits or pure neutrosophic orbits are neutrosophic orbits.

Proof: Straightforward

To show that the converse is not true, let us check the following example.

Example 8: In example 1

$$NO_e = \;\; e, y \;\; ,$$

$$NO_x = \;\; x, xy \;\; ,$$

$$NO_{x^2} = \;\; x^2, x^2 y \;\; .$$

are neutrosophic orbits of $N(\Omega)$ but they are not strong or pure neutrosophic orbits.

Definition 16: Let G be a group acting on Ω and $x \in N(\Omega)$. The neutrosophic stabilizer of x is defined as $G_x = stab_G(x) = \{g \in G : x^g = x\}$.

Example 9: Let $\Omega = \{e, x, x^2, y, xy, x^2y\}$ and $G = \{e, x, x^2\}$. Let $\mu : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\mu(\omega, g) = g(\omega)$, for all $\omega \in \Omega$ and $g \in G$. Then Ω is a G-space under this action. Now $N(\Omega)$ be the G-neutrosophic space, where

$$N(\Omega) = \{e, x, x^2, y, xy, x^2y, I, Ix, Ix^2, Iy, Ixy, Ix^2y\}$$

Let $x \in N(\Omega)$, then the neutrosophic stabilizer of x is $G_x = \{e\}$ and also let $I \in N(\Omega)$, so the neutrosophic stabilizer of I is $G_I = \{e\}$.

Lemma 2: Let $N(\Omega)$ be a neutrosophic space and $x \in N(\Omega)$, then

1) $G_x \leq G$.
2) There is also one-one correspondence between the right cosets of G_x and the neutrosophic orbit NO_x.

Corollary 2: Let G is finite and $x \in N(\Omega)$, then $|G| = |G_x|.|NO_x|$.

Definition 17: Let $x \in N(\Omega)$, then the neutrosophic stabilize of x is called strong neutrosophic stabilizer or pure neutrosophic stabilizer if and only if x is a neutrosophic element of $N(\Omega)$.

Example 10: In above example (9), $G_I = \{e\}$ is a strong neutrosophic or pure neutrosophic stabilizer of neutrosophic element I, where $I \in N(\Omega)$.

Remark 3: Every strong neutrosophic stabilizer or pure neutrosophic stabilizer is always a neutrosophic stabilizer

but the converse is not true.

Example 11: Let $x \in N(\Omega)$, where

$$N(\Omega) = \{e, x, x^2, y, xy, x^2y, I, Ix, Ix^2, Iy, Ixy, Ix^2y\}$$

Then $G_x = \{e\}$ is the neutrosophic stabilize of x but it is not strong neutrosophic stabilizer or pure neutrosophic stabilizer as x is not a neutrosophic element of $N(\Omega)$.

Definition 18: Le $N(\Omega)$ be a neutrosophic space and G be a finite group acting on Ω. For $g \in G$, $fix_{N(\Omega)}(g) = \{x \in N(\Omega) : x^g = x\}$

Example 12: In example 11,

$fix_{N(\Omega)}(e) = \{e, x, x^2, y, xy, x^2y, I, Ix, Ix^2, Iy, Ixy, Ix^2y\}$

$fix_{N(\Omega)}(g) = \phi$, where $g \neq e$.

Theorem 8: Let $N(\Omega)$ be a finite neutrosophic space, then

$$\left|NO_{N(\Omega)}(G)\right| = \frac{1}{|G|}\sum_{g \in G}\left|fix_{N(\Omega)}(g)\right|.$$

Proof: The proof is same as in group action.

Example 13: Consider example (1), only identity element of G fixes all the elements of $N(\Omega)$. Hence $fix_{N(\Omega)}(e) = \{e, x, x^2, y, xy, x^2y, I, Ix, Ix^2, Iy, Ixy, Ix^2y\}$ and hence $\left|fix_{N(\Omega)}(e)\right| = 12$.

The number of neutrosophic orbits of $N(\Omega)$ are given by above theorem

$$\left|NO_{N(\Omega)}(G)\right| = \frac{1}{2}(12) = 6$$

Hence $N(\Omega)$ has 6 neutrosophic orbits.

4. Pseudo Neutrosophic Space

Definition 19: A neutrosophic space $N(\Omega)$ is called pseudo neutrosophic space which does not contain a proper set which is a G-space.

Example 14: Let $\Omega = G = Z_2$ where Z_2 is a group under addition modulo 2. Let $\mu : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\mu(\omega, g) = \omega + g$, for all

$\omega \in \Omega$ and $g \in G$. Then Ω be a G-space under this action and $N(\Omega)$ be the G-neutrosophic space, where $N(\Omega) = \{0,1,I,1+I\}$.

Then clearly $N(\Omega)$ is a pseudo neutrosophic space.

Theorem 9: Every pseudo neutrosophic space is a neutrosophic space but the converse is not true in general.

Example 15: In example (1), $N(\Omega)$ is a neutrosophic space but it is not pseudo neutrosophic space because $\{e,y\}$, $\{x,xy\}$ and $\{x^2,x^2y\}$ are proper subsets which are G-spaces.

Definition 20: Let $N(\Omega)$ be a neutrosophic space and $N(\Omega_1)$ be a neutrosophic subspace of $N(\Omega)$. Then $N(\Omega_1)$ is called pseudo neutrosophic subspace of $N(\Omega)$ if $N(\Omega_1)$ does not contain a proper subset of Ω which is a G-subspace of Ω.

Example 16: In example (1), $\{e,y\}$, $\{Ix,Ixy\}$ etc are pseudo neutrosophic subspaces of $N(\Omega)$ but $\{e,y,Ix,Ixy\}$ is not pseudo neutrosophic subspace of $N(\Omega)$ as $\{e,y\}$ is a proper G-subspace of Ω.

Theorem 10: All pseudo neutrosophic subspaces are neutrosophic subspaces but the converse is not true in general.

Example 17: In example (1), $\{e,y,Ix,Ixy\}$ is a neutrosophic subspace of $N(\Omega)$ but it is not pseudo neutrosophic subspace of $N(\Omega)$.

Theorem 11: A neutrosophic space $N(\Omega)$ has neutrosophic subspaces as well as pseudo neutrosophic subspaces.

Proof : The proof is obvious.

Theorem 12: A transitive neutrosophic subspace is always a pseudo neutrosophic subspace but the converse is not true in general.

Proof: A transitive neutrosophic subspace is a neutrosophic orbit and hence neutrosophic orbit does not contain any other subspace and so pseudo neutrosophic subspace.

The converse of the above theorem does not holds in general. For instance let see the following example.

Example 18: In example 1 , I, Iy, Ix, Ixy is a pseudo neutrosophic subspace of $N \ \Omega$ but it is not transitive.

Theorem 13: All transitive pseudo neutrosophic subspaces are always neutrosophic orbits.

Proof: The proof is followed from by definition.

Definition 21: The pseudo property in a pseudo neutrosophic subspace is called ideal property.

Theorem 14: The transitive property implies ideal property but the converse is not true.

Proof: Let us suppose that $N \ \Omega_1$ be a transitive neutrosophic subspace of $N \ \Omega$. Then by following above theorem, $N \ \Omega_1$ is pseudo neutrosophic subspace of $N \ \Omega$ and hence transitivity implies ideal property.
The converse of the above theorem is not holds.

Example 19: In example 1 , I, Iy, Ix, Ixy is a pseudo neutrosophic subspace of $N \ \Omega$ but it is not transitive.

Theorem 15: The ideal property and transitivity both implies to each other in neutrosophic orbits.

Proof: The proof is straightforward.

Definition 22: A neutrosophic space $N \ \Omega$ is called ideal space or simply if all of its proper neutrosophic subspaces are pseudo neutrosophic subspaces.

Example 20: In example 14 , the neutrosophic space $N \; \Omega$ is an ideal space because $0,1$, $I, 1+I$ are only proper neutrosophic subspaces which are pseudo neutrosophic subspaces of $N \; \Omega$.

Theorem 16: Every ideal space is trivially a neutrosophic space but the converse is not true.

For converse, we take the following example

Example 21: In example 1 , $N \; \Omega$ is a neutrosophic space but it is not an ideal space.

Theorem 17: A neutrosophic space $N \; \Omega$ is an ideal space If Ω is transitive G -space.

Theorem 18: Let $N \; \Omega$ be a neutrosophic space, then $N \; \Omega$ is pseudo neutrosophic space if and only if $N \; \Omega$ is an ideal space.

Proof: Suppose that $N \; \Omega$ is a pseudo neutrosophic space and hence by definition all proper neutrosophic subspaces are also pseudo neutrosophic subspaces. Thus $N \; \Omega$ is an ideal space.

Conversely suppose that $N \; \Omega$ is an ideal space and so all the proper neutrosophic subspaces are pseudo neutrosophic subspaces and hence $N \; \Omega$ does not contain any proper set which is G -subspace and consequently $N \; \Omega$ is a pseudo neutrosophic space.

Theorem 19: If the neutrosophic orbits are only the neutrosophic proper subspaces of $N \; \Omega$, then $N \; \Omega$ is an ideal space.

Proof: The proof is obvious.

Theorem 20: A neutrosophic space $N \; \Omega$ is an ideal space if $\left| NO_{N \; \Omega} \; G \right| = 2$

Theorem 21: A neutrosophic space $N \; \Omega$ is ideal space if all of its proper neutrosophic subspaces are neutrosophic orbits.

6. Conclusions

The main theme of this paper is the extension of neutrosophy to group action and G-spaces to form G-neutrosophic spaces. Our aim is to see the newly generated structures and finding their links to the old versions in a logical manner. Fortunately enough, we have found some new type of algebraic structures here, such as ideal space, Pseudo spaces. Pure parts of neutrosophy and their corresponding properties and theorems are discussed in detail with a sufficient supply of examples.

[1] D. S. Dummit, Richard M. Foote, Abstract Algebra, 3rd Ed., John Viley & Sons Inc (2004).

[2] Florentin Smarandache, A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic. Rehoboth: American Research Press, (1999).

[3] W. B. Vasantha Kandasamy & Florentin Smarandache, Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures, 219 p., Hexis, 2006.

[4] W. B. Vasantha Kandasamy & Florentin Smarandache, N-Algebraic Structures and S-N-Algebraic Structures, 209 pp., Hexis, Phoenix, 2006.

[5] W. B. Vasantha Kandasamy & Florentin Smarandache, Basic Neutrosophic Algebraic Structures and their Applications to Fuzzy and Neutrosophic Models, Hexis, 149 pp., 2004.